Hamilton-Connectivity of 3-Domination Critical Graphs with $\alpha = \delta + 1 \geq 5$

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Abstract: A graph G is 3-domination critical if its domination number γ is 3 and the addition of any edge decreases γ by 1. Let G be a 3-domination critical graph with toughness more than one. It was proved G is Hamilton-connected for the cases $\alpha \leq \delta$ (Discrete Mathematics 271 (2003) 1-12) and $\alpha = \delta + 2$ (European Journal of Combinatorics 23(2002) 777-784). In this paper, we show G is Hamilton-connected for the case $\alpha = \delta + 1 \geq 5$.

Key words: Domination-critical graph, Hamilton-connectivity

1. Introduction

Let $G = (V(G), E(G))$ be a graph. A graph G is said to be *t-tough* if for every cutset $S \subseteq V(G)$, $|S| \geq t\omega(G - S)$, where $\omega(G - S)$ is the number of components of $G - S$. The *toughness* of G, denoted by $\tau(G)$, is defined to be $\min\{|S|/\omega(G - S) \mid S$ is a cutset of $G\}$. Let $u, v \in V(G)$ be any two distinct vertices. We denote by $p(u, v)$ the length of a longest path connecting u and v. The *codiameter* of G, denoted by $d^*(G)$, is defined to be $\min\{p(u, v) \mid u, v \in V(G)\}$. A graph G of order n is said to be *Hamilton-connected* if $d^*(G) = n - 1$, i.e., every two distinct vertices are joined by a hamiltonian path. A graph G is called *k-domination critical*, abbreviated as *k-critical*, if $\gamma(G) = k$ and $\gamma(G + e) = k - 1$ holds for any $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G. The concept of domination critical graphs was introduced by Sumner and Blitch in [11]. Given three vertices u, v and x such that $\{u, x\}$ dominates $V(G) - \{v\}$ but not v, we will write $[u, x] \to v$. It was observed in [11] that if u, v are any two nonadjacent vertices of a 3-critical graph G, then since $\gamma(G + uv) = 2$, there exists a vertex x such that either $[u, x] \to v$ or $[v, x] \to u$. If $U, V \subseteq V(G)$ and U dominates V, that is, V is contained in the closed neighborhood of U, we write $U \succ V$; otherwise we write $U \not\succ V$. For notations not defined here, we follow [5].

It was conjectured in [10] that every connected 3-critical graph of order more than 6 has a hamiltonian path. This was proved by Wojcicka [13] who in turn conjectured that every connected 3-critical graph G with $\delta(G) \geq 2$ has a hamiltonian cycle. Wojcicka's conjecture has now been proved completely, see [8, 9, 12] or [2]. It is well known that if a graph G has a hamiltonian cycle, then $\tau(G) \geq 1$ and the converse does not hold in general. However, this is not the case when G is 3-critical. Noting that $\tau(G) < 1$ if G is a connected 3-critical graph with $\delta(G) = 1$, we see that the following theorem is a direct consequence of the validity of Wojcicka's conjecture.

Theorem 1. Let G be a connected 3-critical graph. Then G has a hamiltonian cycle if and only if $\tau(G) \geq 1$.

For Hamilton-connectivity, it is known that if a graph G is Hamilton-connected, then $\tau(G) > 1$ and the converse need not hold. However, motivated by Theorem 1, Chen et al. [5] posed the following.

Conjecture 1 (Chen et al. [5]). A connected 3-critical graph G is Hamilton-connected if and only if $\tau(G) > 1$.

In the same paper, they proved that the conjecture is true when $\alpha(G) \leq \delta(G)$.

Theorem 2 (Chen et al. [5]). Let G be a connected 3-critical graph with $\alpha(G) \leq \delta(G)$. Then G is Hamilton-connected if and only if $\tau(G) > 1$.

Let G be a 3-connected 3-critical graph. It is shown in [6] that $\tau(G) \geq 1$ and $\tau(G) = 1$ if and only if G belongs to a special infinite family $\mathcal{G}$ described in [6]. Since $\alpha(G) = \delta(G) = 3$ for each $G \in \mathcal{G}$, it is easy to obtain that $\tau(G) > 1$ if $\alpha(G) \geq \delta(G) + 1$.

In [7], Chen et al. showed that the conjecture holds when $\alpha(G) = \delta(G) + 2$.

Theorem 3 (Chen et al. [7]). Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 2$. Then G is Hamilton-connected.

By a result of Favaron et al. [8] that $\alpha(G) \leq \delta(G) + 2$ for any connected 3-critical graph G, we can see the conjecture has only one case $\alpha(G) = \delta(G) + 1$ unsolved. In this paper, we will show that the conjecture is true when $\alpha(G) = \delta(G) + 1 \geq 5$. The main result of this paper is the following.

Theorem 4. Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 \geq 5$. Then G is Hamilton-connected.

Noting that $\tau(G) > 1$ implies $\delta(G) \geq 3$, we can see that the conjecture is still open for the case $\alpha(G) = \delta(G) + 1 = 4$.

Now, we restate a result due to Chen et al. for later use.

Theorem 5 (Chen et al. [3]). Let G be a 3-connected 3-critical graph of order n. Then $d^*(G) \geq n-2$.

2. Properties of Maximum Independent Set

In order to prove Theorem 4, we need to use a classical tool — closure operation in hamiltonian theory. In 1976, Bondy and Chvátal defined a (Hamilton-connected) closure operation of a graph.

Theorem 6 (Bondy and Chvátal [1]). Let G be a graph of order n. Let a and b be nonadjacent vertices of G such that $d(a)+d(b) \geq n+1$. Then for any two distinct vertices x, y, $p(x,y) = n-1$ in G if and only if $p(x,y) = n-1$ in $G+ab$.

Now, given a graph G of order n, repeat the following recursive operation, named Bondy-Chvátal closure operation, as long as possible: For each pair of nonadjacent vertices a and b, if $d(a)+d(b) \geq n+1$, then add the edge ab to G. We denote by $cl(G)$ the resulting graph and call it the Bondy-Chvátal (Hamilton-connected) closure of G. By Theorem 6 we get the following.

Theorem 7 (Bondy and Chvátal [1]). Let G be a graph of order n. Then for any two distinct vertices x, y, $p(x,y) = n-1$ in G if and only if $p(x,y) = n-1$ in $cl(G)$.

Let G be a 3-critical graph of order n, $\alpha(G) = \delta(G)+1$ and $v_0 \in V(G)$ with $d(v_0) = \delta(G) = k \geq 3$. Suppose $N(v_0) = \{v_1, \ldots, v_k\}$ and $I = \{v_0, w_1, \ldots, w_k\}$ is an independent set. In this section, we will give some properties of I in G and $G^* = cl(G)$.

The following lemma restates a lemma due to Sumner and Blitch [11], which has proven to be of considerable use in dealing with 3-critical graphs. In [11] they considered the case $l \geq 4$, which guarantees $P(U) \cap U = \emptyset$. For the cases $l = 2$ and $l = 3$, Lemma 2.1 can be easily verified since G is a 3-critical graph.

Lemma 2.1. Let G be a connected 3-critical graph and U an independent set of $l \geq 2$ vertices. Then there exist an ordering $u_1, u_2, \cdots, u_l$ of the vertices of U and a sequence $P(U) = (y_1, y_2, \cdots, y_{l-1})$ of $l-1$ distinct vertices such that $[u_i, y_i] \to u_{i+1}$, $1 \leq i \leq l-1$.

The next lemma is a useful consequence of Lemma 2.1.

Lemma 2.2 (Favaron et al. [8]). Let U be an independent set of $l \geq 3$ vertices of a 3-critical graph G such that $U \cup \{v\}$ is independent for some $v \notin U$. Then the sequence $P(U)$ defined in Lemma 2.1 is contained in $N(v)$.

Since I is an independent set of order at least 4, by Lemmas 2.1 and 2.2, we may assume without loss of generality that

$[w_i, v_i] \to w_{i+1}$ for $1 \le i \le k-1$. (2-1)

By (2-1), it is easy to obtain the following.

$v_j v_{j+1} \in E(G)$ for $1 \le j \le k-2$. (2-2)

Lemma 2.3. If $w_i v_k \notin E(G)$ with $i \neq 1$, then $G[N(v_0) - \{v_{i-1}, v_k\}]$ is a clique. If $w_1 v_k \notin E(G)$, then $G[N(v_0) - \{v_k\}]$ is a clique.

Proof. Let $v_l, v_m \in N(v_0) - \{v_{i-1}, v_k\}$ with $l \le m-1$. If $l = m-1$, then $v_l v_m \in E(G)$ by (2-2). If $l \le m-2$, then since $w_{l+1} w_{m+1} \notin E(G)$, there is some vertex z such that $[w_{l+1}, z] \to w_{m+1}$ or $[w_{m+1}, z] \to w_{l+1}$. Since $k \ge 3$, by Lemma 2.2 we have $z \in N(v_0)$. Since $w_i v_k \notin E(G)$, we have $z \neq v_k$. By (2-1), either $[w_{l+1}, v_m] \to w_{m+1}$ or $[w_{m+1}, v_l] \to w_{l+1}$. In both cases, we have $v_l v_m \in E(G)$ and hence $G[N(v_0) - \{v_{i-1}, v_k\}]$ is a clique. As for the latter part, the proof is similar. ∎

Lemma 2.4. If $w_i v_k \notin E(G)$ with $i \neq 1$, then $[w_1, v_{j-1}] \to w_j$ for $j \ge 3$ and $j \neq i$.

Proof. Since $w_1 w_j \notin E(G)$, by Lemma 2.2, there is some $z \in N(v_0)$ such that $[w_1, z] \to w_{j+1}$ or $[w_{j+1}, z] \to w_1$. By (2-1) and the assumption, we can see that $[w_j, z] \to w_1$ is impossible for any $z \in N(v_0)$ and hence $[w_1, v_{j-1}] \to w_j$. ∎

Lemma 2.5. If $[v_0, z] \to w_i$ for some i with $1 \le i \le k-1$, then $z \notin N(v_0)$ and if $[v_0, v_l] \to w_k$ for some $v_l \in N(v_0)$, then $l = k-1$.

Proof. If $i = 1$ and $z \in N(v_0)$, then $z = v_k$ by (2-1). Thus, we have $\{v_2, v_k\} \succ V(G)$ by Lemma 2.3, a contradiction. If $i \ge 2$ and $z \in N(v_0)$, then by (2-1) we have $z = v_{i-1}$ or v_k and $N(v_0) - \{v_{i-1}, v_i, v_k\} \subseteq N(w_i)$. If $z = v_{i-1}$, then $w_i v_k \notin E(G)$ for otherwise $\{v_{i-1}, w_i\} \succ V(G)$. Since $[w_i, v_i] \to w_{i+1}$, $v_i v_k \in E(G)$. By Lemma 2.4, we have $[w_1, v_i] \to w_{i+1}$, which implies $v_i w_i \in E(G)$. Thus by Lemma 2.3, we have $\{v_{i-1}, v_i\} \succ V(G)$, a contradiction. If $z = v_k$ and $i \neq 2$, then by Lemma 2.3 we have $\{v_{i-2}, v_k\} \succ V(G)$, a contradiction. If $z = v_k$ and $i = 2$, then by Lemma 2.4 we have $[w_1, v_2] \to w_3$, which implies $v_2 w_2 \in E(G)$ and hence $\{v_2, v_k\} \succ V(G)$ by Lemma 2.3, also a contradiction. Thus, $z \notin N(v_0)$.

If $[v_0, v_l] \to w_k$ for some $v_l \in N(v_0)$, then by (2-1), we have $l = k-1$ or k. If $l = k$, then by Lemma 2.3, we have $\{v_{k-2}, v_k\} \succ V(G)$, a contradiction. ∎

Lemma 2.6. If $[v_0, v_{k-1}] \to w_k$, then $N(v_k) \cap \{v_1, \ldots, v_{k-1}, w_k\} = \emptyset$ and $\{w_1, \ldots, w_{k-1}\} \subseteq N(v_k)$.

Proof. By (2-1), we have $N(v_0) - \{v_{k-1}, v_k\} \subseteq N(w_k)$. If $w_k v_k \in E(G)$, then since $[v_0, v_{k-1}] \to w_k$, we have $\{v_{k-1}, w_k\} \succ V(G)$ and hence $w_k v_k \notin E(G)$. By Lemma 2.3, $G[N(v_0) - \{v_{k-1}, v_k\}]$ is a clique. Thus, if $v_{k-1} v_k \in E(G)$, then $\{v_{k-1}, v_1\} \succ V(G)$ and if $v_i v_k \in E(G)$ for some i with $1 \le i \le k-2$, then $\{v_{k-1}, v_i\} \succ V(G)$, a contradiction. Since $N(v_k) \cap \{v_1, \ldots, v_{k-1}\} = \emptyset$, by (2-1) we have $\{w_1, \ldots, w_{k-1}\} \subseteq N(v_k)$. ∎

Lemma 2.7. If $[v_0, v_{k-1}] \to w_k$, then $G[N(v_0)-\{v_k\}]$ is a clique and $N(w_k)\cap N(v_k) = \emptyset$.

Proof. By Lemma 2.6, $v_k w_k \notin E(G)$. By Lemma 2.3, $G[N(v_0)-\{v_{k-1}, v_k\}]$ is a clique. By (2-1), $v_{k-2}v_{k-1} \in E(G)$. For $1 \le i \le k-3$, there is some $z \in N(v_0)$ such that $[w_{i+1}, z] \to w_k$ or $[w_k, z] \to w_{i+1}$ by Lemma 2.2. By (2-1) and Lemma 2.6, we can see that $\{w_{i+1}, v_k\} \not\succ v_i$ and $\{w_k, v_k\} \not\succ v_{k-1}$, which implies $z \ne v_k$ and hence $z = v_i$ or v_{k-1}. In both cases, we have $v_i v_{k-1} \in E(G)$, which implies $G[N(v_0)-\{v_k\}]$ is a clique. If $N(w_k)\cap N(v_k) \ne \emptyset$, then since $[v_0, v_{k-1}] \to w_k$ and $G[N(v_0)-\{v_k\}]$ is a clique, we can see that $\{v_{k-1}, z\} \succ V(G)$ for any $z \in N(w_k)\cap N(v_k)$, a contradiction. ∎

Lemma 2.8. If $k \ge 4$, $[v_0, v_{k-1}] \to w_k$ and for each w_i with $1 \le i \le k-1$, there is no vertex z such that $[v_0, z] \to w_i$, then $N^*[w_1] = N_{G^*}[w_1] = V(G)$.

Proof. Let $U = V(G)-(I \cup N(v_0))$, $N(w_1)\cap U = U_1$ and $U_2 = U - U_1$. In order to prove the result, we need the following claims.

Claim 2.1. $N(w_i)\cap N(v_i)\cap U \ne \emptyset$ for $1 \le i \le k-2$.

Proof. By the assumption, there is some vertex z such that $[w_{i+1}, z] \to v_0$. Obviously $z \in U$. By (2-1), we have $z \in N(w_i)\cap N(v_i)$ and hence $z \in N(w_i)\cap N(v_i)\cap U$. ∎

By Lemmas 2.4 and 2.6, we have $[w_1, v_i] \to w_{i+1}$ for $2 \le i \le k-2$ and hence

$$w_i v_i \in E(G) \text{ for } 2 \le i \le k-2. \tag{2-3}$$

Claim 2.2. $d(w_2) \ge \delta + 1$ and if $d(w_2) = \delta + 1$, then $d(v_2) \ge n - \delta$.

Proof. By the assumption, we may assume $[w_3, z] \to v_0$, which implies $z \in N(v_2)\cap N(w_2)\cap U$. If $d(w_2) = \delta$, then $N_U(w_2) = \{z\}$ by (2-3). Since $[w_3, z] \to v_0$, by (2-1) and Lemma 2.7 we have $V(G)-\{w_3, v_k\} \subseteq N[v_2]$. By Lemma 2.6, $w_3 v_k \in E(G)$. Thus, $\{v_2, w_3\} \succ V(G)$, a contradiction. Since $k \ge 4$ and $[w_2, v_2] \to w_3$, by (2-1) and Claim 2.1, we have $|N(w_2)\cap N(v_2)| \ge 2$. By (2-3), $w_2 v_2 \in E(G)$. Thus, we have $d(w_2) + d(v_2) \ge n + 1$ and the conclusion follows. ∎

Claim 2.3. For any $u \in N_U(w_k)$, either $uw_2 \in E(G)$ or $uw_3 \in E(G)$.

Proof. Suppose $u \in N_U(w_k)$ and $w_2, w_3 \notin N(u)$. By Lemma 2.2, there is some vertex $z \in N(v_0)$ such that $[w_3, z] \to u$ or $[u, z] \to w_3$. If $[u, z] \to w_3$, then we must have $z = v_2$, which is impossible since $\{u, v_2\} \not\succ v_k$ by Lemmas 2.6 and 2.7. If $[w_3, z] \to u$, then since $[w_2, v_2] \to w_3$ and $uw_2 \notin E(G)$, we have $z \ne v_2$. By (2-1) and Lemma 2.6, we can see $z \in N(v_0) - \{v_2\}$ is also impossible, a contradiction. ∎

Claim 2.4. $v_{k-1} \in N^*(w_k)$.

Proof. Since $[v_0, v_{k-1}] \to w_k$, by Lemma 2.7 we have $d(v_{k-1}) = n - 3$. Noting that

$d(w_k) \geq \delta \geq 4$, we have $d(v_{k-1}) + d(w_k) \geq n + 1$ and hence $v_{k-1} \in N^*(w_k)$. ▮

Claim 2.5. If $d(w_2) = \delta + 1$ and $d(w_3) = \delta$, then $v_k \in N^*(w_k)$.

Proof. Let $N(w_k) \cap U = U_3$ and $U_4 = U - U_3$. By (2-1) and Lemma 2.6, we have $v_{k-1}, v_k \notin N(w_k)$ and hence $|U_3| \geq 2$. By the assumption, there are some $z_i \in U$ such that $[w_i, z_i] \to v_0$ for $i = 1, 2$. If $z_1 \neq z_2$, then $d_U(w_3) \geq 2$. If $k = 4$, then $w_3v_3 \in E(G)$ by the assumption and if $k \geq 5$, then $w_3v_3 \in E(G)$ by (2-3). By (2-1) and Lemma 2.6, $N(v_0) - \{v_2, v_3\} \subseteq N(w_3)$. Thus we have $d(w_3) \geq \delta + 1$ and hence we may assume $z_1 = z_2 = u_1$. Obviously, $u_1 \in U_3$. Since $d(w_2) = \delta + 1$ and $d(w_3) = \delta$, by Claim 2.3, we have $|U_3| = 2$ and $N_U(w_2) = U_3$. Since $[w_2, u_1] \to v_0$, $v_{k-1} \in N(w_2) \cap N(u_1)$ and $w_2u_1 \in E(G)$, we have $d(u_1)+d(w_2) \geq n$, which implies $d(u_1) \geq n-\delta-1$. We now show $[w_k, v_k] \to v_{k-1}$. If $U_4 = \emptyset$, then by (2-1) and Lemma 2.6, $[w_k, v_k] \to v_{k-1}$. If $U_4 \neq \emptyset$, then since $u_1w_3 \in E(G)$ and $d(w_3) = \delta$, we have $N(w_3) \cap U_4 = \emptyset$. For any $u \in U_4$, by Lemma 2.2, there is some vertex $z \in N(v_0)$ such that $[u, z] \to w_3$ or $[w_3, z] \to u$. If $[w_3, z] \to u$, then since $[w_2, v_2] \to w_3$ and $u \notin N(w_2)$, we have $z \neq v_2$. By (2-1) and Lemma 2.6, $z \notin N(v_0) - \{v_2\}$, a contradiction. If $[u, z] \to w_3$, then by (2-1) and Lemma 2.6, $z = v_2$. Since $v_2v_k \notin E(G)$ by Lemma 2.6, we have $v_ku \in E(G)$ and hence $U_4 \subseteq N(v_k)$. Thus, $[w_k, v_k] \to v_{k-1}$. Since $d(v_{k-1}) = n - 3$, $d(v_2) \geq n - \delta$ by Claim 2.2 and $d(u_1) \geq n - \delta - 1$, we have $v_{k-1}, v_2, u_1 \in N^*(v_k)$. By Claim 2.4, $v_{k-1} \in N^*(w_k)$. By Lemmas 2.6 and 2.7, $v_{k-1}, v_2, u_1 \notin N(v_k)$. Thus, we have $d^*(w_k) + d^*(v_k) \geq n + 1$ and hence $v_k \in N^*(w_k)$. ▮

Claim 2.6. For any $u \in U_2$, we have $[u, v_1] \to w_1$.

Proof. Since $uw_1 \notin E(G)$, there exists some vertex z such that $[w_1, z] \to u$ or $[u, z] \to w_1$. In order to dominate v_0, we have $z \in N[v_0]$. Thus by (2-1) and Lemma 2.6, it is easy to see $[w_1, z] \to u$ is impossible. If $[u, z] \to w_1$, then by the assumption we have $z \neq v_0$. By (2-1) and Lemma 2.6, we have $z = v_1$, that is, $[u, v_1] \to w_1$. ▮

Claim 2.7. For any $u \in U_2$, $N(v_0) \subseteq N(u)$.

Proof. Since $[w_1, v_1] \to w_2$ and $u \in U_2$, we have $v_1 \in N(u)$. By Lemmas 2.4 and 2.6, we have $v_i \in N(u)$ for $2 \leq i \leq k-2$. By Lemma 2.6 and Claim 2.6, we have $v_k \in N(u)$. We now show $v_{k-1} \in N(u)$. Since $w_1w_k \notin E(G)$, by Lemma 2.2, there exists some vertex $z \in N(v_0)$ such that $[w_1, z] \to w_k$ or $[w_k, z] \to w_1$. By (2-1) and Lemma 2.6, we can see $[w_k, z] \to w_1$ is impossible. Thus we have $[w_1, z] \to w_k$. By Claim 2.6 we have $w_1v_1 \notin E(G)$. By Lemma 2.6, we have $z \neq v_k$ since $\{w_1, v_k\} \not\to v_1$. By (2-1), we have $z = v_{k-1}$ which implies $v_{k-1} \in N(u)$. ▮

Claim 2.8. If $U_2 \neq \emptyset$, then $N_U(w_k) \subseteq N(w_1) \cap N(w_2)$.

Proof. Let $u \in N_U(w_k)$ and $w \in \{w_1, w_2\}$. If $uw \notin E(G)$, then there is some vertex z such that $[u, z] \to w$ or $[w, z] \to u$. If $[w, z] \to u$, then $z \in N(v_0)$. By Claim 2.6,

$v_1w_1 \notin E(G)$, which implies $[w_2, v_1] \to u$ cannot occur. Thus, by (2-1) and Lemma 2.6 we see that $[w, z] \to u$ is impossible. If $[u, z] \to w$, then by the assumption, $z \neq v_0$. By Lemma 2.6, $z \neq v_k$. If $z \in N(v_0) - \{v_k\}$, then $\{u, z\} \not\to v_k$ by Lemmas 2.6 and 2.7. Thus, $z \notin N[v_0]$, a contradiction. ∎

We first show that $w_1v_1 \in E(G^*)$.

If $w_1v_1 \in E(G)$, then $w_1v_1 \in E(G^*)$. If $\delta \geq 5$, then by Lemma 2.7, Claim 2.1 and $[w_1, v_1] \to w_2$, we have $d(w_1) + d(v_1) \geq n + 1$ and hence $w_1v_1 \in E(G^*)$. Thus, we may assume that $w_1v_1 \notin E(G)$ and $\delta = 4$.

If $|N(w_1) \cap N(v_1) \cap U| \geq 2$, then by Lemma 2.7 and $[w_1, v_1] \to w_2$, we have $d(w_1) + d(v_1) \geq n + 1$ and hence $w_1v_1 \in E(G^*)$. Thus by Claim 2.1 we may assume

$$N(w_1) \cap N(v_1) \cap U = \{u_1\}. \tag{2-4}$$

By the assumption, we let $[w_1, z] \to v_0$. If $z \neq u_1$, then $z \in U_2$ by (2-4). This is impossible since $\{w_1, z\} \not\to w_k$ by Claim 2.8 and hence we have

$$[w_1, u_1] \to v_0. \tag{2-5}$$

If $U_2 \neq \emptyset$, we let $u \in U_2$. If $u' \in U_2$ and $uu' \notin E(G)$, then there is some vertex z such that $[u, z] \to u'$ or $[u', z] \to u$. By symmetry we may assume $[u, z] \to u'$. By Claim 2.7, $z \notin N(v_0)$. If $z = v_0$, then $\{u, z\} \not\to w_1$, a contradiction. Hence U_2 is a clique. If $u' \in U_1$ and $uu' \notin E(G)$, then by Claim 2.6 we have $u' \in N(v_1)$, which implies $u' = u_1$ by (2-4). By (2-5), $u_1u \in E(G)$. Thus, $U \subseteq N[u]$ for any $u \in U_2$. By Claim 2.6, $U_2 \subseteq N(w_2)$. Thus by Claim 2.7, we have $d(u) \geq n - \delta - 1$. If $d(w_1) \geq \delta + 2$, then $uw_1 \in E(G^*)$, which implies $w_1v_1 \in E(G^*)$. If $d(w_1) \leq \delta + 1$, then by (2-1) and Lemma 2.6 we have $|U_1| \leq 2$. By Lemma 2.6 and the assumption, we have $d_U(w_k) \geq 2$. Thus by Claim 2.8 we have $U_1 = N_U(w_k) \subseteq N(w_2)$ and hence $U \subseteq N(w_2)$. In this case, we have $[v_1, w_2] \to w_1$. By Lemma 2.7, Claim 2.7 and (2-4), $|N(v_1) \cap N(w_2)| \geq 4$. Thus we have $v_1w_2 \in E(G^*)$ and hence $w_1v_1 \in E(G^*)$.

If $U_2 = \emptyset$, then since $w_1v_1 \notin E(G)$, there is some vertex z such that $[w_1, z] \to v_1$ or $[v_1, z] \to w_1$. If $[w_1, z] \to v_1$, then $z \neq v_0$ and hence $z \in N(v_0)$. By Lemma 2.7, $z = v_k$. This is impossible since $\{w_1, v_k\} \not\to w_k$ by Lemma 2.6. Thus we have $[v_1, z] \to w_1$. Since $U_2 = \emptyset$ and $N(v_0) - \{v_1\} \subseteq N(w_1)$, we have $z \in \{w_2, \ldots, w_k\}$. In this case, $z = w_2$, that is, $[w_2, v_1] \to w_1$. By (2-5), $u_1w_2 \in E(G)$. Thus by (2-4), we have $U \subseteq N(w_2)$. By (2-1) and Lemmas 2.4 and 2.6, $v_2, v_3, v_4 \in N(w_1) \cap N(w_2)$. Thus, if $|U| \geq 4$, then $d(w_1) + d(w_2) \geq n + 1$, which implies $w_1w_2 \in E(G^*)$ and hence $w_1v_1 \in E(G^*)$. If $|U| \leq 3$, then $n \leq 12$. After an easy but tedious check, we can show $w_1v_1 \in E(G^*)$.

Next, we show $U \subseteq N^*(w_1)$. If $U_2 = \emptyset$, then $U \subseteq N(w_1) \subseteq N^*(w_1)$ and hence we assume $U_2 \neq \emptyset$. Let $u \in U_2$. Suppose $u' \in V(G) - N[v_0]$ and $u' \notin N^*(u)$. Obviously, $uu' \notin E(G)$ and hence there is some z such that $[u', z] \to u$ or $[u, z] \to u'$. If $[u', z] \to u$, then $z \notin N(v_0)$ by Claim 2.7 and hence $z = v_0$. In this case, $u' \in U$.

Since $[v_0, v_{k-1}] \to w_k$, $v_{k-1} \in N(u')$. By Claim 2.6, $v_1u' \in E(G)$. Thus we have $d(u') \geq n - \delta - 1$. By the assumption, there exists some z' such that $[w_1, z'] \to v_0$. By Lemma 2.7 and Claim 2.7, $z' \in U_1$ and hence $N_{U_1}(u) \neq \emptyset$. By Claim 2.6, $w_2 \in N(u)$. Thus, by Claim 2.7 we have $d(u) \geq \delta+2$, which implies $u' \in N^*(u)$ and hence $[u', z] \to u$ is impossible. Thus we always have $[u, z] \to u'$. By Claim 2.8, $w_k \notin N(u)$. Thus we have $z \neq v_0$ since $\{u, v_0\} \not\to \{w_1, w_k\}$ and hence $z \in N(v_0)$. If $V(G) - N[v_0]$ contains δ vertices, say $u'_1, u'_2, \ldots, u'_k$, that are not adjacent to u in G^*, then there are $z_{u'_i} \in N(v_0)$ such that $[u, z_{u'_i}] \to u'_i$ for $1 \leq i \leq k$. Clearly, if $i \neq j$, then $z_{u'_i} \neq z_{u'_j}$ since $u'_i \neq u'_j$. This is impossible since $\{u, v_{k-1}\} \not\to w_k$ and $\{u, v_k\} \not\to w_k$. Therefore, $V(G) - N[v_0]$ contains at most $\delta - 1$ vertices that are not adjacent to u in G^* and hence $d^*(u) \geq n - \delta - 1$ since $N(v_0) \subseteq N(u)$ by Claim 2.7. By Claim 2.6, $w_1v_1 \notin E(G)$. By Lemma 2.6 and the assumption, $d_U(w_k) \geq 2$ which implies $d_U(w_1) \geq 2$ by Claim 2.8. Thus by (2-1) and Lemma 2.6 we have $d(w_1) \geq \delta + 1$ and hence $d^*(w_1) \geq \delta + 2$ since $w_1v_1 \in E(G^*)$. This implies $d^*(w_1) + d^*(u) \geq n + 1$ and thus $U \subseteq N^*(w_1)$.

Finally, we show $N^*[w_1] = V(G)$. Since $w_1v_1 \in E(G^*)$ and $U \subseteq N^*(w_1)$, by (2-1), we have $d^*(w_1) \geq n - \delta - 1$. By Claim 2.2, $d(w_2) \geq \delta + 1$. If $d(w_2) \geq \delta + 2$, then by Claim 2.4, we have $w_2, w_k \in N^*(w_1)$, which implies $d^*(w_1) \geq n - \delta + 1$ and hence $N^*[w_1] = V(G)$. If $d(w_2) = \delta + 1$ and $d(w_3) \geq \delta + 1$, then by Claim 2.2 we have $d^*(w_3) \geq \delta + 2$. Thus $w_3, w_2 \in N^*(w_1)$ and hence $N^*[w_1] = V(G)$. If $d(w_2) = \delta + 1$ and $d(w_3) = \delta$, then $d^*(w_k) \geq \delta+2$ by Claims 2.4 and 2.5. Thus, $w_k, w_2 \in N^*(w_1)$ and hence $N^*[w_1] = V(G)$. ∎

3. Some Lemmas

Let G be a graph of order n, and x, y vertices of G such that the longest (x, y)-path is of length $n - 2$. Let $P = P_{xy}$ be an (x, y)-path of length $n - 2$ and suppose the orientation of P is from x to y. We denote by x_P the only vertex not in P and let $d(x_P) = k \geq 2$ with

$N(x_P) = X = \{x_1, x_2, \ldots, x_k\}$,	indices following the orientation of P;
$A = X^+ = \{a_1, a_2, \ldots, a_s\}$,	where $a_i = x_i^+$, $x_i^+ \in V(P)$ and $s \geq k - 1$;
$B = X^- = \{b_t, b_{t+1}, \ldots, b_k\}$,	where $b_i = x_i^-$, $x_i^- \in V(P)$ and $t \leq 2$; and
$P_i = a_i\overrightarrow{P}b_{i+1}$,	where $1 \leq i \leq k - 1$.

Furthermore, we let $P_0 = x\overrightarrow{P}b_1$ if $x \notin X$ and $P_k = a_k\overrightarrow{P}y$ if $y \notin X$. In this section, we will establish some lemmas. It is worth noting that all lemmas in this section except the last one do not depend on the 3-critical property of G.

Definition. A vertex $v \in P_i$ $(1 \leq i \leq k)$ is called an A-vertex if $G[V(P_i) \cup \{x_{i+1}\}]$ contains a hamiltonian (v, x_{i+1})-path, and $v \in P_i$ $(0 \leq i \leq k-1)$ a B-vertex if $G[V(P_i) \cup \{x_i\}]$ contains a hamiltonian (x_i, v)-path, where $x_{k+1} = y$ and $x_0 = x$.

From the definition, we can see that each a_i is an A-vertex and each b_i is a B-vertex. Let $u_i \in P_i$ be an A-vertex and Q_i a given hamiltonian (u_i, x_{i+1})-path in $G[V(P_i) \cup \{x_{i+1}\}]$. Suppose the orientation of Q_i is from u_i to x_{i+1}. We have the following two lemmas.

Lemma 3.1. If $u_i \in P_i$ and $u_j \in P_j$ are two A-vertices (B-vertices, respectively) with $i \neq j$, then $x_P u_i \notin E(G)$ and $u_i u_j \notin E(G)$. In particular, both $A \cup \{x_P\}$ and $B \cup \{x_P\}$ are independent sets.

Proof. If $x_P u_i \in E(G)$, then $x\overrightarrow{P}x_i x_P u_i \overrightarrow{Q_i} x_{i+1} \overrightarrow{P} y$ is a hamiltonian (x, y)-path. Assume $i < j$. If $u_i u_j \in E(G)$, then the (x, y)-path $x\overrightarrow{P}x_i x_P x_j \overleftarrow{P} x_{i+1} \overleftarrow{Q_i} u_i u_j \overrightarrow{Q_j} x_{j+1} \overrightarrow{P} y$ is hamiltonian, a contradiction. ∎

Lemma 3.2. Let $u_i \in P_i$, $u_j \in P_j$ be A-vertices with $i < j$, $Q = u_i \overrightarrow{Q_i} x_{i+1} \overrightarrow{P} x_j$ and $R = u_j \overrightarrow{Q_j} x_{j+1} \overrightarrow{P} y$. If $v \in N_Q(u_i)$, then $v^- \notin N(u_j)$ and if $v \in N(u_i) \cap (x\overrightarrow{P}x_i \cup R)$, then $v^+ \notin N(u_j)$. In particular, let $a_i, a_j \in A$ with $i < j$ and $v \in N(a_i)$, then $v^- \notin N(a_j)$ if $v \in a_i \overrightarrow{P} x_j$ and $v^+ \notin N(a_j)$ if $v \in x\overrightarrow{P}x_i \cup a_j \overrightarrow{P} y$.

Proof. If $v \in N_Q(u_i)$ and $v^- \in N(u_j)$, then the (x, y)-path $x\overrightarrow{P}x_i x_P x_j \overleftarrow{Q} v u_i \overrightarrow{Q} v^- u_j \overrightarrow{R} y$ is hamiltonian, a contradiction. As for the latter case, the proof is similar. ∎

By symmetry of A and B, Lemma 3.2 still holds if we exchange A and B.

Lemma 3.3. Let $u, v \in a_i \overrightarrow{P} b_j$ with $j \geq i + 1$ and $G[a_i \overrightarrow{P} b_j]$ contain a hamiltonian (u, v)-path Q. Suppose that $w \in x\overrightarrow{P}x_i \cup x_j \overrightarrow{P} y$ and $uw \in E(G)$. Then $w^- v \notin E(G)$ if $w^- \in x\overrightarrow{P}x_i \cup x_j \overrightarrow{P} y$, and $w^+ v \notin E(G)$ if $w^+ \in x\overrightarrow{P}x_i \cup x_j \overrightarrow{P} y$. In particular, let $a_i \in A$ and $b_j \in B$ with $j \geq i + 1$. Suppose that $v \in x\overrightarrow{P}x_i \cup x_j \overrightarrow{P} y$ and $a_i v \in E(G)$. Then $v^- b_j \notin E(G)$ if $v^- \in x\overrightarrow{P}x_i \cup x_j \overrightarrow{P} y$ and $v^+ b_j \notin E(G)$ if $v^+ \in x\overrightarrow{P}x_i \cup x_j \overrightarrow{P} y$.

Proof. Suppose that $w \in x\overrightarrow{P}x_i$. If $w^- \in x\overrightarrow{P}x_i$ and $w^- v \in E(G)$, then the (x, y)-path $x\overrightarrow{P}w^- v \overleftarrow{Q} u w \overrightarrow{P} x_i x_P x_j \overrightarrow{P} y$ is hamiltonian, and if $w^+ \in x\overrightarrow{P}x_i$ and $w^+ v \in E(G)$, then the (x, y)-path $x\overrightarrow{P} w u \overrightarrow{Q} v w^+ \overrightarrow{P} x_i x_P x_j \overrightarrow{P} y$ is hamiltonian, a contradiction. As for the case $w \in x_j \overrightarrow{P} y$, the proof is similar. ∎

Lemma 3.4. Let $u, u^+ \in V(P_i)$. If $u^+ a_l \in E(G)$ for some $l \geq i + 1$, then $b_j u \notin E(G)$ for all $j \leq i$.

Proof. If $b_j u \in E(G)$ for some $j \leq i$, then the (x, y)-path $x\overrightarrow{P} b_j u \overleftarrow{P} x_j x_P x_l \overleftarrow{P} u^+ a_l \overrightarrow{P} y$ is hamiltonian, a contradiction. ∎

Lemma 3.5. Let $z \in V(G) - N[x_P]$. If $|N(z) \cap A| \geq 2$, then $z^- z^+ \notin E(G)$.

Proof. Let $a_l, a_m \in N(z)$ with $l < m$ and $z \in P_j$. If $z^- z^+ \in E(G)$, then the (x, y)-path $x\overrightarrow{P} z^- z^+ \overrightarrow{P} x_l x_P x_m \overleftarrow{P} a_l z a_m \overrightarrow{P} y$ is hamiltonian if $j < l$, $x\overrightarrow{P} x_l x_P x_m \overleftarrow{P} z^+ z^- \overleftarrow{P} a_l z a_m \overrightarrow{P} y$ is hamiltonian if $l \leq j < m$, and $x\overrightarrow{P} x_l x_P x_m \overleftarrow{P} a_l z a_m \overrightarrow{P} z^- z^+ \overrightarrow{P} y$ is hamiltonian if $m \leq j$, a contradiction. ∎

Lemma 3.6. Let $z, z^- \in P_i$, $w, w^- \in P_j$ with $i, j \geq 1$ and $k \geq 4$. If $|A - N(z)| \leq 1$ and $A \subseteq N(w)$, then $z^-w^- \notin E(G)$.

Proof. Suppose to the contrary $z^-w^- \in E(G)$. If $i = j$ and $w \in x\overrightarrow{P}z$, then $a_iz \notin E(G)$ for otherwise w is an A-vertex, which contradicts Lemma 3.1 since $A \subseteq N(w)$. Hence we have $A - \{a_i\} \subseteq N(z)$. Noting that $A \subseteq N(w)$ and $k \geq 4$, we have $w \neq z^-$ by Lemma 3.2. Thus, the (x, y)-path $x\overrightarrow{P}w^-z^-\overleftarrow{P}wa_2\overrightarrow{P}x_3x_Px_2\overleftarrow{P}za_3\overrightarrow{P}y$ is hamiltonian if $i = 1$, $x\overrightarrow{P}x_1x_Px_3\overleftarrow{P}za_1\overrightarrow{P}w^-z^-\overleftarrow{P}wa_3\overrightarrow{P}y$ is hamiltonian if $i = 2$, and $x\overrightarrow{P}x_1x_Px_2\overleftarrow{P}a_1w\overrightarrow{P}z^-w^-\overleftarrow{P}a_2z\overrightarrow{P}y$ is hamiltonian if $i \geq 3$, a contradiction. If $i = j$ and $z \in x\overrightarrow{P}w$, then since $a_iw \in E(G)$, z is an A-vertex, which contradicts Lemma 3.1 since $|A - N(z)| \leq 1$. If $i \neq j$, then since $a_jw \in E(G)$, w^- is an A-vertex. Since $z^-w^- \in E(G)$, by Lemma 3.1, $za_i \notin E(G)$. Thus, $x\overrightarrow{P}x_ix_Px_j\overleftarrow{P}za_j\overrightarrow{P}w^-z^-\overleftarrow{P}a_iw\overrightarrow{P}y$ is a hamiltonian (x, y)-path if $i < j$, and $x\overrightarrow{P}x_jx_Px_i\overleftarrow{P}wa_i\overrightarrow{P}z^-w^-\overleftarrow{P}a_jz\overrightarrow{P}y$ is a hamiltonian (x, y)-path if $i > j$, also a contradiction. ∎

Lemma 3.7. Let $z^-, z \in P_i$, $w^-, w \in P_j$ with $i, j \geq 1$ and $k \geq 4$. If $|A \cup B - N(z)| \leq 1$ and $|A - N(w)| \leq 1$, then $w^-z^- \notin E(G)$.

Proof. We first show the following claim.

Claim 3.1. Let $u^-, u \in P_l$, $v^-, v \in P_m$ and $h \neq l, m$. If $u^-v^- \in E(G)$, then either $ua_h \notin E(G)$ or $vb_{h+1} \notin E(G)$.

Proof. Assume without loss of generality $v \in u\overrightarrow{P}y$. If $ua_h, vb_{h+1} \in E(G)$, then $u \neq v^-$ by Lemma 3.3. Thus the (x, y)-path $x\overrightarrow{P}x_hx_Px_{h+1}\overrightarrow{P}u^-v^-\overleftarrow{P}ua_h\overrightarrow{P}b_{h+1}v\overrightarrow{P}y$ is hamiltonian if $h < l$, $x\overrightarrow{P}u^-v^-\overleftarrow{P}x_{h+1}x_Px_h\overleftarrow{P}ua_h\overrightarrow{P}b_{h+1}v\overrightarrow{P}y$ is hamiltonian if $l < h < m$, and $x\overrightarrow{P}u^-v^-\overleftarrow{P}ua_h\overrightarrow{P}b_{h+1}v\overrightarrow{P}x_hx_Px_{h+1}\overrightarrow{P}y$ is hamiltonian if $m < h$, a contradiction. ∎

By Lemma 3.6, we may assume $B \subseteq N(z)$. If $w^-z^- \in E(G)$, then by Claim 3.1, $a_lw \notin E(G)$ for $l \neq i, j$. Noting $k \geq 4$ and $|A - N(w)| \leq 1$, we have $i \neq j$ and $wa_i, wa_j \in E(G)$. Since $wa_j \in E(G)$, w^- is an A-vertex. If $za_i \in E(G)$, then z^- is also an A-vertex which contradicts Lemma 3.1 since $i \neq j$ and $w^-z^- \in E(G)$. Hence, $za_i \notin E(G)$, which implies $za_j \in E(G)$ since $|A \cup B - N(z)| \leq 1$. If $j < k$, then $w^-\overleftarrow{P}a_jw\overrightarrow{P}b_{j+1}$ is a hamiltonian path in $G[V(P_j)]$, which contradicts Lemma 3.3 since $w^-z^-, zb_{j+1} \in E(G)$, and hence we have $i < j$ and $j = k$ by Lemma 3.3. In this case, the (x, y)-path $x\overrightarrow{P}x_ix_Px_j\overleftarrow{P}za_j\overrightarrow{P}w^-z^-\overleftarrow{P}a_iw\overrightarrow{P}y$ is hamiltonian, a contradiction. ∎

Lemma 3.8 (Chen et al. [4]). Let $z \in V(P) - X$ and $v \in A \cup B$. If $d(x_P) = k \geq 4$ and $A \cup B - \{v\} \subseteq N(z)$, then $A \cup \{z^+\}$ is an independent set if $z^+ \in V(P)$ and $B \cup \{z^-\}$ is an independent set if $z^- \in V(P)$.

Lemma 3.9 (Chen et al. [5]). Let $u, v \notin V(P_i)$ and $\{u, v\} \succ V(P_i)$. If $ua_i, vb_{i+1} \in E(G)$, where $b_{k+1} = y$ if $i = k$, then there is some $w \in V(P_i)$ such that $uw, vw^+ \in E(G)$.

Let $z \in P_j$ and $[a_i, z] \to x_P$. We have the following five lemmas (3.10-3.14).

Lemma 3.10. If $2 \le i \le j$ and $z^+ \in V(P)$, then $A \cup \{x_P, z^+\}$ is an independent set.

Proof. Since $za_1 \in E(G)$, we have $a_l z^+ \notin E(G)$ for $2 \le l \le j$ by Lemma 3.2. If $a_1 z^+ \in E(G)$ or $a_l z^+ \in E(G)$ for some $l \ge j+1$, then by Lemmas 3.3 or 3.4 we have $b_2 z \notin E(G)$ and hence $b_2 a_i \in E(G)$. By Lemma 3.9, there is some $w \in P_1$ such that $wz, w^+ a_i \in E(G)$. Thus, the (x, y)-path $x\overrightarrow{P}x_1 x_P x_i \overleftarrow{P} w^+ a_i \overrightarrow{P} zw \overleftarrow{P} a_1 z^+ \overrightarrow{P} y$ is hamiltonian if $a_1 z^+ \in E(G)$, and $x\overrightarrow{P}wz\overleftarrow{P}a_i w^+ \overrightarrow{P} x_i x_P x_l \overleftarrow{P} z^+ a_l \overrightarrow{P} y$ is hamiltonian if $a_l z^+ \in E(G)$ for some $l \ge j+1$, a contradiction. If $z \in B$, then $z = b_{j+1}$. By Lemma 3.1 we have $a_1 b_{j+1}, b_2 a_i \in E(G)$. By Lemma 3.9, there is some $w \in P_1$ such that $wb_{j+1}, w^+ a_i \in E(G)$, which contradicts Lemma 3.3. Thus, $z \notin B$ and hence $z^+ x_P \notin E(G)$, which implies $A \cup \{x_P, z^+\}$ is an independent set. ∎

Lemma 3.11. If $2 \le i \le j$ and $|A| \ge 3$, then $B \cup \{z^-, x_P\}$ is an independent set.

Proof. Since $A - \{a_i\} \subseteq N(z)$ and $2 \le i \le j$, we have $b_l z^- \notin E(G)$ for $l \ne 1, j+1$ by Lemma 3.3. If $b_1 z^- \in E(G)$ or $z^- b_{j+1} \in E(G)$, then by Lemmas 3.2 or 3.1, we have $b_2 \notin N(z)$. Since $[a_i, z] \to x_P$, we have $b_2 a_i \in E(G)$. By Lemma 3.9, there is some $u \in P_1$ such that $uz, u^+ a_i \in E(G)$. Thus the (x, y)-path $x\overrightarrow{P}b_1 z^- \overleftarrow{P} a_i u^+ \overrightarrow{P} x_i x_P x_1 \overrightarrow{P} uz \overrightarrow{P} y$ is hamiltonian if $b_1 z^- \in E(G)$, and $x\overrightarrow{P}uz\overrightarrow{P}b_{j+1} z^- \overleftarrow{P} a_i u^+ \overrightarrow{P} x_i x_P x_{j+1} \overrightarrow{P} y$ is hamiltonian if $b_{j+1} z^- \in E(G)$, a contradiction. Since $|A| \ge 3$ and $[a_i, z] \to x_P$, by Lemma 3.1 we have $z \notin A$ which implies $z^- x_P \notin E(G)$. Thus, by Lemma 3.1 we can see that $B \cup \{z^-, x_P\}$ is an independent set. ∎

Lemma 3.12. If $j+1 < i$, then $A \cup \{z^+, x_P\}$ is an independent set.

Proof. Since $a_{j+1} z \in E(G)$, by Lemma 3.2 we have $a_l z^+ \notin E(G)$ for all l with $l \ne j+1$. If $a_{j+1} z^+ \in E(G)$, then by Lemma 3.3 we have $b_{j+2} z \notin E(G)$ and hence $a_i b_{j+2} \in E(G)$. By Lemma 3.9, there is some $u \in P_{j+1}$ such that $uz, u^+ a_i \in E(G)$. Thus, the (x, y)-path $x\overrightarrow{P}zu\overleftarrow{P}a_{j+1} z^+ \overrightarrow{P} x_{j+1} x_P x_i \overleftarrow{P} u^+ a_i \overrightarrow{P} y$ is hamiltonian, a contradiction. If $z \in B$, then $z = b_{j+1}$. Since $[a_i, z] \to x_P$ and $j+1 < i$, there is some $u \in P_{j+1}$ such that $uz, u^+ a_i \in E(G)$, which contradicts Lemma 3.4. Hence $z \notin B$ which implies $z^+ x_P \notin E(G)$. Thus, $A \cup \{z^+, x_P\}$ is an independent set by Lemma 3.1. ∎

Lemma 3.13. Let $|A| \ge 3$. If $j+1 < i$ and $z^- \in V(P)$, then $B \cup \{z^-, x_P\}$ is an independent set.

Proof. Since $a_{j+1} z \in E(G)$, we have $b_l z^- \notin E(G)$ for $l \ne j+1$ by Lemmas 3.3 and 3.4. If $b_{j+1} z^- \in E(G)$, then z is a B-vertex. By Lemma 3.1 we have $zb_{j+2} \notin E(G)$, which implies $a_i b_{j+2} \in E(G)$. By Lemma 3.9, there is some $w \in P_{j+1}$ such that $zw, w^+ a_i \in E(G)$. Thus, the (x, y)-path $x\overrightarrow{P}z^- b_{j+1} \overleftarrow{P} zw \overleftarrow{P} x_{j+1} x_P x_i \overleftarrow{P} w^+ a_i \overrightarrow{P} y$ is hamiltonian, a contradiction. Since $|A| \ge 3$ and $[a_i, z] \to x_P$, we have $z \notin A$ by Lemma 3.1 and hence $z^- x_P \notin E(G)$. Thus, $B \cup \{z^-, x_P\}$ is an independent set. ∎

The following two lemmas can be extracted from [5]: Lemma 3.14 is extracted from the Case 2 of Lemma 2.8(2) and Lemma 3.15 from Lemma 2.9 in [5].

Lemma 3.14 (Chen et al. [5]). If $j = i - 1 \geq 1$, $d(x_P) = k \geq 4$ and $\{x, y\} \subseteq N(x_Q)$ for any longest (x, y)-path Q, then $B \cup \{z^-, x_P\}$ is an independent set.

Lemma 3.15 (Chen et al. [5]). Suppose that P is a longest (x, y)-path such that $|X \cap \{x, y\}|$ is as small as possible and that for this path, $d(x_P) = k \geq 4$. If G is 3-critical, then there exists an independent set I such that either $\{x_P\} \cup A \subseteq I$ or $\{x_P\} \cup B \subseteq I$ and $|I| \geq k + 1$. ∎

4. Proof of Theorem 4

Let G be a 3-connected 3-critical graph with $\alpha(G) = \delta(G) + 1 \geq 5$. If G is not Hamilton-connected, then by Theorem 5, there are two vertices $x, y \in V(G)$ such that $p(x, y) = n - 2$. Among all the longest (x, y)-paths, we choose P such that $|\{x, y\} \cap N(x_P)|$ is as small as possible. Choose an orientation of P such that $|A| \geq |B|$. Assume without loss of generality that the orientation is from x to y. We still use the notations given in Section 3.

Since $\alpha(G) = \delta(G)+1 \geq 5$, by the choice of P and Lemma 3.15, $d(x_P) = k = \delta \geq 4$. We first show the following claims.

Claim 4.1. Let $z \in P_j$ and $[a_i, z] \rightarrow x_P$. If $|A| = k$ and $j = i-1 \geq 1$, then $B \cup \{z^-, x_P\}$ is an independent set.

Proof. Let $U = N[x_P] \cup A$. By Lemmas 2.1 and 2.2, we may assume that $[a_{i_l}, x_{j_l}] \rightarrow a_{i_{l+1}}$ for $1 \leq l \leq k - 1$. Thus, noting that $|A| = k$, we have

$$d_U(x_l) \geq \delta \text{ for any } x_l \in N(x_P). \tag{4-1}$$

Assume $b_l \in B$ and $b_l z^- \in E(G)$. Since $A - \{a_i\} \subseteq N(z)$, by Lemma 3.3, $l \in \{1, j + 1, i + 1\}$. If $j = 1$, then $i = 2$. Since $a_3 z \in E(G)$, by Lemma 3.4, $l \neq 1$ and hence $l \in \{2, 3\}$. If $l = 2$ or 3, then by Lemma 3.2 we have $b_4 z \notin E(G)$ and hence $a_2 b_4 \in E(G)$. Since $z a_3, a_2 b_4 \in E(G)$, by Lemma 3.1 we have $|P_1| \geq 2$ and $|P_2| \geq 2$, which implies $b_2, b_3 \notin U$. Thus we have $d(x_2) \geq \delta + 1$ and $d(x_3) \geq \delta + 1$ by (4-1). If $l = 2$, then $Q = x\overrightarrow{P}z^- b_2 \overleftarrow{P} z a_3 \overrightarrow{P} b_4 a_2 \overrightarrow{P} x_3 x_P x_4 \overrightarrow{P} y$ is an (x, y)-path of length $n - 2$ with $d(x_Q) = d(x_2) \geq \delta + 1$ and if $l = 3$, then $R = x\overrightarrow{P}z^- b_3 \overleftarrow{P} a_2 b_4 \overleftarrow{P} a_3 z \overrightarrow{P} x_2 x_P x_4 \overrightarrow{P} y$ is an (x, y)-path of length $n - 2$ with $d(x_R) = d(x_3) \geq \delta + 1$. Since $\alpha(G) = \delta(G) + 1$, by Lemma 3.1 we have $y \in N(x_2)$ if $l = 2$ and $y \in N(x_3)$ if $l = 3$. If $y \neq a_k$, then $d(x_2) \geq \delta + 2$ if $l = 2$ and $d(x_3) \geq \delta + 2$ if $l = 3$, which implies $\alpha(G) \geq \delta(G) + 2$ by Lemma 3.1, a contradiction. Hence $y = a_k$. Thus, $x\overrightarrow{P}z^- b_2 \overleftarrow{P} z a_3 \overrightarrow{P} x_k x_P x_3 \overleftarrow{P} x_2 a_k$ is a hamiltonian (x, y)-path if $l = 2$ and $x\overrightarrow{P}z^- b_3 \overleftarrow{P} z a_3 \overrightarrow{P} x_k x_P x_3 a_k$ is a hamiltonian (x, y)-path if $l = 3$, a contradiction. Hence we have $j \geq 2$. Since $l \in \{1, j + 1, i + 1\}$, we have

$b_2 z \notin E(G)$ by Lemma 3.2 and hence $b_2 a_i \in E(G)$. If $l = 1$, then since $[a_i, z] \to x_P$, we have $zx_1 \in E(G)$ or $a_i x_1 \in E(G)$. Thus, $x\overrightarrow{P}b_1 z^- \overleftarrow{P} x_2 x_P x_i \overleftarrow{P} z x_1 \overrightarrow{P} b_2 a_i \overrightarrow{P} y$ is a hamiltonian (x, y)-path if $zx_1 \in E(G)$ and $x\overrightarrow{P}b_1 z^- \overleftarrow{P} a_1 z \overrightarrow{P} x_i x_P x_1 a_i \overrightarrow{P} y$ is a hamiltonian (x, y)-path if $a_i x_1 \in E(G)$. If $j + 1$, then $Q = x\overrightarrow{P}x_1 x_P x_2 \overrightarrow{P} z^- b_{j+1} \overleftarrow{P} z a_1 \overrightarrow{P} b_2 a_i \overrightarrow{P} y$ is an (x, y)-path of length $n - 2$ with $x_Q = x_{j+1}$. Since $|P_j| \geq 2$, $b_{j+1} \notin U$ which implies $d(x_{j+1}) \geq \delta + 1$ by (4-1). Since $\alpha(G) = \delta(G) + 1$, by Lemma 3.1 we have $xx_{j+1} \in E(G)$ and $x = x_1$. In this case, $xx_{j+1} x_P x_2 \overrightarrow{P} z^- b_{j+1} \overleftarrow{P} z a_1 \overrightarrow{P} b_2 a_i \overrightarrow{P} y$ is a hamiltonian (x, y)-path. If $l = i + 1$, then since $[a_i, z] \to x_P$, we have $zx_{i+1} \in E(G)$ or $a_i x_{i+1} \in E(G)$. Thus, $x\overrightarrow{P}b_2 a_i \overrightarrow{P} b_{i+1} z^- \overleftarrow{P} x_2 x_P x_i \overleftarrow{P} z x_{i+1} \overrightarrow{P} y$ in the former case and $x\overrightarrow{P}x_1 x_P x_i \overleftarrow{P} z a_1 \overrightarrow{P} z^- b_{i+1} \overleftarrow{P} a_i x_{i+1} \overrightarrow{P} y$ in the latter case, is a hamiltonian (x, y)-path, a contradiction. Therefore, $B \cup \{z^-\}$ is an independent set. On the other hand, since $k \geq 4$ and $[a_i, z] \to x_P$, by Lemma 3.1, we have $z \notin A$ and hence $z^- x_P \notin E(G)$. Thus by Lemma 3.1, $B \cup \{z^-, x_P\}$ is an independent set. ∎

Claim 4.2. Let $I = \{x_P\} \cup W$ with $|I| = k + 1 \geq 5$ be an independent set. If $W = A$ or I is obtained by one of the Lemmas 3.8 and 3.10-3.15, then $[x_P, x_l] \to w$ is impossible for any $x_l \in X$ and $w \in W$.

Proof. If $[x_P, x_l] \to w$ for some $w \in W$ and $x_l \in X$, then by Lemmas 2.5 and 2.8, W contains a vertex w' such that $V(G) \subseteq N^*[w']$. If $W = A$, then by Lemma 3.1, G^* contains a hamiltonian (x, y)-path and hence $p(x, y) = n - 1$ by Theorem 7, a contradiction. If I is obtained by one of the Lemmas 3.8 and 3.10-3.15, then by the proofs of these lemmas, we can see that G^* contains a hamiltonian (x, y)-path, which implies $p(x, y) = n - 1$ by Theorem 7, also a contradiction. ∎

If $N(x_P) \cap \{x, y\} = \emptyset$, then $|A| = |B| = k$. By Lemmas 2.1 and 2.2, we may assume $[a_{i_l}, x_{j_l}] \to a_{i_{l+1}}$ for $1 \leq l \leq k - 1$. Since $k \geq 4$, by Lemma 2.5 there is some a_i with $i \geq 2$ and a vertex $z \in V(G) - N[x_P]$ such that $[x_P, z] \to a_i$ or $[a_i, z] \to x_P$. If $[x_P, z] \to a_i$, then $\alpha \geq \delta + 2$ by Lemma 3.8 and if $[a_i, z] \to x_P$, then $\alpha \geq \delta + 2$ by Lemmas 3.10-3.14 and Claim 4.1, a contradiction. Thus, $|N(x_P) \cap \{x, y\}| \geq 1$. By the choice of the orientation of P, we have $x = x_1$.

Claim 4.3. For any $a_i \in A$ and any $z \in V(G) - N[x_P]$, $[x_P, z] \to a_i$ is impossible.

Proof. Suppose to the contrary there is some $z \in V(G) - N[x_P]$ such that $[x_P, z] \to a_i$. Since $x = x_1$, by Lemma 3.8, $B \cup \{x_P, z^-\}$ is an independent set, and if $|A| = k - 1$, then $A \cup \{x_P, z^+\}$ is also an independent set. Noting that $A \cup \{x_P\}$ or $A \cup \{x_P, z^+\}$ is a maximum independent set and $k \geq 4$, by Claim 4.2, there are some $a_j \in A$ with $j \neq 1, i$ and $w \in V(G) - N[x_P]$ such that $[x_P, w] \to a_j$ or $[a_j, w] \to x_P$. In both cases, we have $w \neq z$ and $|A - N(w)| \leq 1$. By Lemma 3.8 or Lemmas 3.11, 3.13, 3.14 and Claim 4.1, $B \cup \{x_P, w^-\}$ is an independent set. By Lemma 3.7, $w^- z^- \notin E(G)$. Thus, $B \cup \{x_P, z^-, w^-\}$ is an independent set of order $k + 2$, a contradiction. ∎

If $|A| = k-1$, then Lemma 3.15 and the symmetry of A and B, we may assume that G contains an independent set I such that $A \cup \{x_P\} \subseteq I$ and $|I| = k+1$. If $|A| = k$, then $A \cup \{x_P\}$ is a maximum independent set. Thus, by Claim 4.2, $[x_P, x_l] \rightarrow a$ is impossible for any $a \in A$ and $x_l \in X$. Since $A \cup \{x_P\}$ is an independent set by Lemma 3.1 and G is 3-critical, by Claim 4.3 we may assume in the following proof that $[a_i, z_i] \rightarrow x_P$ for all $a_i \in A$.

We now consider the following two cases separately.

Case 1. $|N(x_P) \cap \{x, y\}| = 1$

Let $w \in P_i$ and $wa_i \in E(G)$. If $a_i \overrightarrow{P} w \nsubseteq N[a_i]$, say, $v \in a_i \overrightarrow{P} w$ is the last vertex that is not adjacent to a_i along $a_i \overrightarrow{P} w$, then since $wa_i \in E(G)$, v is an A-vertex. Thus, $A \cup \{x_P, v\}$ is an independent set of order $k+2$ by Lemma 3.1 and hence we have

$$a_i \overrightarrow{P} w \subseteq N[a_i] \text{ if } w \in P_i \text{ and } wa_i \in E(G). \tag{4-2}$$

Since $\alpha = \delta + 1$, by Lemmas 3.10-3.14 and Claim 4.1, we have $z_i \in P_{i-1}$ or $z_i = y$ for $2 \leq i \leq k$. If there are two vertices z_i and z_j such that $z_i \in P_{i-1}$ and $z_j \in P_{j-1}$, then both $B \cup \{x_P, z_i^-\}$ and $B \cup \{x_P, z_j^-\}$ are independent sets by Claim 4.1. Since $a_{i-1}z_i, a_{j-1}z_j \in E(G)$, z_i^- and z_j^- are A-vertices and hence $z_i^- z_j^- \notin E(G)$ by Lemma 3.1, which implies $B \cup \{x_P, z_i^-, z_j^-\}$ is an independent set of order $k+2$, a contradiction. Thus, noting that $k \geq 4$, there exist at least two vertices z_i, z_j with $i, j \neq 1$ such that $z_i = z_j = y$, which implies $A \subseteq N(y)$ and $B \cup \{y^-\}$ is an independent set by Lemma 3.11. If there is some z_i with $i \geq 2$ such that $z_i \neq y$, then $z^- y^- \notin E(G)$ by Lemma 3.6 and hence $B \cup \{x_P, z_i^-, y^-\}$ is an independent set of order $k+2$, a contradiction. Thus, we have $z_i = y$ for $2 \leq i \leq k$. By (4-2), $P_k \subseteq N[a_k]$, which implies each vertex of $P_k - \{y\}$ is an A-vertex. Let $z_1 \in P_j$. If $z_1 \neq y$, then $j \leq k-1$. Since $a_{j+1}z_1 \in E(G)$, we have $b_l z_1^- \notin E(G)$ for $l \neq j+1$ by Lemmas 3.3 and 3.4. Since $z_1 a_k, a_1 y \in E(G)$ and $[a_1, z_1] \rightarrow x_P$, by Lemma 3.9 there is some vertex $w \in P_k$ such that $wz_1, w^+ a_1 \in E(G)$, which implies $z_1^- b_{j+1} \in E(G)$ by Lemma 3.3. By Lemma 3.6, $z_1^- y^- \notin E(G)$ and hence $B \cup \{x_P, z_1^-, y^-\}$ is an independent set of order $k+2$, a contradiction. Thus, $z_1 = y$ and hence we have

$$z_i = y \text{ for } 1 \leq i \leq k. \tag{4-3}$$

Since $A \subseteq N(y)$, by Lemma 3.1, we have $y \neq a_k$ and hence $y^- x_P \notin E(G)$. If there is some $z \in V(G) - N[x_P]$ such that $[x_P, z] \rightarrow y^-$, then $z \neq y$. By Lemma 3.8, $A \cup \{x_P, z^+\}$ is an independent set of order $k+2$, a contradiction. Since $B \cup \{y^-, x_P\}$ is a maximum independent set, by Claim 4.2, there is no vertex $x_l \in X$ such that $[x_P, x_l] \rightarrow y^-$. Thus, there is some vertex $z \in P_i$ such that $[y^-, z] \rightarrow x_P$. If $z \neq y$, then since $a_k y \in E(G)$, all vertices of $a_k \overrightarrow{P} y^-$ are A-vertices by (4-2), which implies $z \notin P_k$ since otherwise $\{y^-, z\} \not\rightarrow A - \{a_k\}$ by Lemma 3.1. Since y^- is an A-vertex, we have $A - \{a_k\} \subseteq N(z)$, which implies $b_l z^- \notin E(G)$ for $l \neq i+1$. If $z^- b_{i+1} \in E(G)$, then z is a B-vertex. Thus, noting that $B \cup \{y^-\}$ is an independent set, we can see $\{y^-, z\} \not\succ B - \{b_{i+1}\}$, a contradiction. Thus we have $z^- b_l \notin E(G)$ for $2 \leq l \leq k$. Since y^- is an A-vertex, $k \geq 4$ and $[y^-, z] \to x_P$, we have $z \notin A$ and hence $z^- x_P \notin E(G)$. By Lemma 3.6, $y^- z^- \notin E(G)$. Thus, $B \cup \{x_P, y^-, z^-\}$ is an independent set of order $k+2$, also a contradiction. Thus we have $z = y$, that is,

$$[y, y^-] \to x_P. \tag{4-4}$$

By Lemma 3.1, (4-2) and (4-3), $P_k \subseteq N[y]$. By Lemma 3.11, (4-3) and (4-4), $A \cup B \subseteq N(y)$. For $1 \leq i \leq k-1$, if there is some $u \in P_i$ such that $uy \notin E(G)$, then $u^+, u^- \in P_i$ since $A \cup B \subseteq N(y)$. By (4-3), $A \subseteq N(u)$. By Lemma 3.5, we have $u^- u^+ \notin E(G)$. By Lemma 3.6, $u^- y^- \notin E(G)$. If $u^+ y^- \in E(G)$, then the (x, y)-path $x\overrightarrow{P}x_i x_P x_k \overleftarrow{P} u^+ y^- \overleftarrow{P} a_k u \overleftarrow{P} a_i y$ is hamiltonian and hence $u^+ y^- \notin E(G)$. By Lemma 3.3, $u^- b_l, u^+ b_l \notin E(G)$ for $l \neq i+1$, which implies $B \cup \{x_P, u^-, u^+, y^-\} - \{b_{i+1}\}$ is an independent set of order $k+2$, a contradiction. Thus, we have $P_i \subseteq N[y]$ for $1 \leq i \leq k-1$ and hence $\{x_P, y\} \succ V(G)$, a contradiction.

Case 2. $|N(x_P) \cap \{x, y\}| = 2$

In this case, we let $z_2 \in P_i$.

Suppose $i = 1$, $l \geq 3$ and $z_l \in P_j$. Assume $z_l \neq z_2$. If $j \neq 1$, then $z_2^- z_l^- \notin E(G)$ for otherwise the (x, y)-path $x x_P x_2 \overleftarrow{P} z_2 a_1 \overrightarrow{P} z_2^- z_l^- \overleftarrow{P} a_2 z_l \overrightarrow{P} y$ is hamiltonian. If $j = 1$ and $z_2^- z_l^- \in E(G)$, then z_l is an A-vertex if $z_l \in x\overrightarrow{P}z_2$ and z_2 an A-vertex if $z_2 \in x\overrightarrow{P}z_l$. By Lemma 3.1, $z_l a_2, z_2 a_l \notin E(G)$, which is impossible since $[a_2, z_2] \to x_P$ and $[a_l, z_l] \to x_P$. Thus, $z_2^- z_l^- \notin E(G)$ and hence $B \cup \{x_P, z_2^-, z_l^-\}$ is an independent set of order $k+2$ by Lemmas 3.11, 3.13 and 3.14. Therefore, we have

$$z_l = z_2 \text{ for } 3 \leq l \leq k-1 \text{ if } i = 1. \tag{4-5}$$

If $i \geq 2$, then $A \cup \{x_P, z_2^+\}$ is an independent set by Lemma 3.10. If $i = 1$, then by (4-5) and Lemma 3.12, $A \cup \{x_P, z_2^+\}$ is an independent set. By Lemmas 3.11 and 3.14, $B \cup \{x_P, z_2^-\}$ is an independent set. Thus, both $B \cup \{x_P, z_2^-\}$ and $A \cup \{x_P, z_2^+\}$ are independent sets.

If there is some $w \in V(G) - N[x_P]$ such that $[x_P, w] \to z_2^+$ ($[x_P, w] \to z_2^-$, respectively), then $w \neq z_2$. By Lemma 3.8, $B \cup \{x_P, w^-\}$ is an independent set. By Lemma 3.7 we have $z_2^- w^- \notin E(G)$ and hence $B \cup \{x_P, w^-, z_2^-\}$ is an independent set of order $k+2$, a contradiction. Thus, noting that both $B \cup \{x_P, z_2^-\}$ and $A \cup \{x_P, z_2^+\}$ are maximum independent sets, by Claim 4.2, we may assume $[z_2^+, w_1] \to x_P$ and $[z_2^-, w_2] \to x_P$.

Let $w_1 \in P_j$. If $w_1 \neq z_2$, then since $k \geq 4$, $A \cup \{z_2^+\}$ is an independent set and $[z_2^+, w_1] \to x_P$, we have $w_1 \notin A$, which implies $w_1^- x_P \notin E(G)$, and $A \subseteq N(w_1)$, which implies $w_1^- b_l \notin E(G)$ for $l \neq j+1$ by Lemma 3.3. If $w_1^- b_{j+1} \in E(G)$, then w_1 is a B-vertex. Thus by Lemma 3.1 we have $B - \{b_{j+1}\} \subseteq N(z_2^+)$. If $j = 2$, then since $k \geq 4$, there is some l with $l \neq 2, i$ such that $z_2 a_l \in E(G)$, which implies $z_2^+ b_{l+1} \notin E(G)$ by Lemma 3.3, a contradiction. If $j \neq 2$, then by Lemma 3.5 we have $z_2^+ z_2^- \notin E(G)$, which implies $w_1 z_2^- \in E(G)$. Since $a_j z_2 \in E(G)$, by Lemma 3.3 we have $i = j$. Thus, since $k \geq 4$, there is some l with $l \neq 2, j$ such that $z_2 a_l \in E(G)$, which implies $z_2^+ b_{l+1} \notin E(G)$ by Lemma 3.3, also a contradiction. Hence, $B \cup \{x_P, w_1^-\}$ is an independent set. By Lemma 3.6, $z_2^- w_1^- \notin E(G)$. Thus by Lemma 3.1, $B \cup \{x_P, z_2^-, w_1^-\}$ is an independent set of order $k+2$, a contradiction. Hence we have $w_1 = z_2$, that is,

$$[z_2^+, z_2] \to x_P. \tag{4-6}$$

If $w_2 \neq z_2$, then since $B \cup \{z_2^-, x_P\}$ is an independent set, we have $B \subseteq N(w_2)$. By (4-6), we have $A \subseteq N(z_2) \in E(G)$, which implies z_2^- is an A-vertex. Thus, $A - \{a_i\} \subseteq N(w_2)$, which implies $|A \cup B - N(w_2)| \leq 1$. By Lemmas 3.7 and 3.8, we can see that $B \cup \{x_P, z_2^-, w_2^-\}$ is an independent set of order $k+2$, a contradiction. Hence we have $w_2 = z_2$, that is,

$$[z_2^-, z_2] \to x_P. \tag{4-7}$$

By (4-6) and (4-7), $A \cup B \subseteq N(z_2)$. If there is some vertex $v \in a_i \overrightarrow{P} z_2$ such that $v a_i \notin E(G)$ and $v^+ a_i \in E(G)$, then v is an A-vertex. If $v z_2^+ \in E(G)$, then z_2 is an A-vertex, which contradicts Lemma 3.1. Thus, $A \cup \{x_P, v, z_2^+\}$ is an independent set of order $k+2$, a contradiction. Noting that $z_2 \in N(a_i)$, we have $a_i \overrightarrow{P} z_2 \subseteq N[a_i]$. By symmetry, we have $z_2 \overrightarrow{P} b_{i+1} \subseteq N[b_{i+1}]$. If $N(z_2^+) \cap a_i \overrightarrow{P} z_2^- \neq \emptyset$, then since $a_i \overrightarrow{P} z_2 \subseteq N[a_i]$, z_2 is A-vertex and if $N(z_2^-) \cap z_2^+ \overrightarrow{P} b_{i+1} \neq \emptyset$, then since $z_2 \overrightarrow{P} b_{i+1} \subseteq N[b_{i+1}]$, z_2 is a B-vertex, which contradicts Lemma 3.1 since $A \cup B \subseteq N(z_2)$. Thus, we have

$$N(z_2^+) \cap a_i \overrightarrow{P} z_2^- = \emptyset \text{ and } N(z_2^-) \cap z_2^+ \overrightarrow{P} b_{i+1} = \emptyset. \tag{4-8}$$

Assume $z_1 \in P_j$ and $z_1 \neq z_2$. Since $[a_1, z_1] \to x_P$ and $k \geq 4$, by Lemma 3.1 we have $z_1 \notin A$, which implies $z_1^- x_P \notin E(G)$. If $j \neq k-1$, then since $z_1 a_{j+1} \in E(G)$, we have $b_l z_1^- \notin E(G)$ for $l \neq j+1$ by Lemmas 3.3 and 3.4. If $b_{j+1} z_1^- \in E(G)$, then z_1 is a B-vertex. Thus, by Lemmas 3.1 and 3.9, there is some vertex $w \in P_{k-1}$ such that $w^+ a_1, z_1 w \in E(G)$, which contradicts Lemma 3.3. Hence, $B \cup \{x_P, z_1^-\}$ is an independent set. If $j = k-1$, then $i \neq k-1$ for otherwise $\{a_1, z_1\} \not\to z_2^+$ if $z_1 \in a_{k-1} \overrightarrow{P} z_2^-$ by Lemma 3.10 and (4-8), and $\{a_1, z_1\} \not\to z_2^-$ if $z_1 \in z_2^+ \overrightarrow{P} b_k$ by (4-8) and Lemma 3.1 since z_2^- is an A-vertex. Since $a_2 z_1 \in E(G)$, we have $b_l z_1^- \notin E(G)$ for $l \neq 2, k$ by Lemma 3.3. If $b_2 z_1^- \in E(G)$, then $b_3 z_1 \notin E(G)$ by Lemma 3.2 which implies $a_1 b_3 \in E(G)$. Since $[a_1, z_1] \to x_P$, we can see that either $a_1 x_3 \in E(G)$ or $z_1 x_3 \in E(G)$. Thus, the (x, y)-path $x x_P x_2 \overrightarrow{P} x_3 a_1 \overrightarrow{P} b_2 z_1^- \overleftarrow{P} a_3 z_1 \overrightarrow{P} y$ is hamiltonian in the former case, and $x x_P x_2 \overrightarrow{P} b_3 a_1 \overrightarrow{P} b_2 z_1^- \overleftarrow{P} x_3 z_1 \overrightarrow{P} y$ is hamiltonian in the latter case, a contradiction. If $z_1^- b_k \in E(G)$, then z_1 is a B-vertex. By (4-8), z_2^+ is a B-vertex, which implies $z_2^+ z_1 \notin E(G)$ by Lemma 3.1 and hence $\{a_1, z_1\} \not\to z_2^+$, a contradiction. Thus, $B \cup \{x_P, z_1^-\}$ is an independent set. By (4-6) and (4-7), we have $A \cup B \subseteq N(z_2)$, which implies $z_1^- z_2^- \notin E(G)$ by Lemma 3.7. Thus, $B \cup \{x_P, z_1^-, z_2^-\}$ is an independent set of order $k+2$ and hence we have $z_1 = z_2$. By (4-5), we have $z_l = z_2$ for $l \geq 3$ if $i = 1$. If $i \geq 2$ and there is some z_l with $l \geq 3$ such that $z_l \neq z_2$, then $B \cup \{x_P, z_l^-\}$ is an independent set by Lemmas 3.11, 3.13 and 3.14. By (4-6), $A \subseteq N(z_2)$ and hence $z_2^- z_l^- \notin E(G)$ by Lemma 3.6. Thus, $B \cup \{x_P, z_2^-, z_l^-\}$ is an independent set of order $k+2$, a contradiction. Thus we have

$$z_l = z_2 \text{ for } l \neq 2. \tag{4-9}$$

By (4-6), (4-7) and (4-8), we have $P_i \subseteq N[z_2]$ and $A \cup B \subseteq N(z_2)$. Let $l \neq i$. If there is some $u \in P_l$ such that $uz_2 \notin E(G)$, then $u^+, u^- \notin N(x_P)$ and $A \subseteq N(u)$ by (4-9). By Lemma 3.3, $b_m u^+, b_m u^- \notin E(G)$ for $m \neq l+1$. By Lemma 3.5, $u^+u^- \notin E(G)$. By Lemma 3.7, $u^- z_2^- \notin E(G)$. If $u^+ z_2^- \in E(G)$, then the (x,y)-path $x\overrightarrow{P}x_l x_P x_i \overleftarrow{P} u^+ z_2^- \overleftarrow{P} a_i u \overleftarrow{P} a_l z_2 \overrightarrow{P} y$ is hamiltonian if $l < i$ and if $l > i$, then $x\overrightarrow{P}x_i x_P x_l \overleftarrow{P} z_2 a_l \overrightarrow{P} u a_i \overrightarrow{P} z_2^- u^+ \overrightarrow{P} y$ is hamiltonian, a contradiction. Thus, we have $u^+ z_2^- \notin E(G)$, which implies $B \cup \{x_P, u^+, u^-, z_2^-\} - \{b_{l+1}\}$ is an independent set of order $k+2$, a contradiction. Therefore, we have $P_l \subseteq N[z_2]$ for $l \neq i$, which implies $\{x_P, z_2\} \succ V(G)$, a contradiction.

The proof of Theorem 4 is complete. ∎

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