Henig Efficiency of a Multi-criterion Supply-Demand Network Equilibrium Model

T.C. Edwin Cheng[a,1] Yunan Wu[a,b]

[a] Department of Logistics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

[b] Department of Mathematics, Chongqing Normal University, Chongqing, China

Abstract This paper addresses Henig efficiency of a multi-product network equilibrium model based on Wardrop's principle. We show that in both the single and multiple criteria cases, such proper efficiency can be recast as a vector variational inequality. In the multiple criteria case, we derive a sufficient and a necessary condition for Henig efficiency in terms of a vector variational inequality by using the Gerstewitz's function.

1 Introduction

Consider a supply-demand network that comprises some manufacturers and retailers, as well as some distributing centers or warehouses. For each pair of manufacturer (an origin) and retailer (a destination), there may exist many paths connecting them. Assume that we know the supply and demand between each origin-destination (OD) pair. The network is considered as functioning properly if all the demands are satisfied and all the suppliers choose one of the paths leading from the point of origin to the point of destination at the minimum cost. Decision-making problems in management science and operations research frequently require that decisions are made based on optimizing several criteria. Vector optimization provides a systemic approach to addressing these problems. Hence, the cost may comprise multiple criteria, which embraces tariffs, fuels, time and other relevant cost factors. Such a phenomenon results when the network follows a natural law known as the user-optimizing principle or the Wardrop's equilibrium principle (Wardrop (1952)). This principle asserts that the traffic flow along a path joining an OD pair is positive only if the cost for this path is the minimum possible amongst all paths joining the same OD pair. Examples of traffic flow networks that follow the Wardrop's equilibrium principle are telephone networks and the Internet.

[1]Corresponding author. E-Mail: lgtcheng@polyu.edu.hk.

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After Wardrop, many scholars have studied this kind of network equilibrium model (see, for example, Patriksson and Labbe (2002)). Until only recently, all these equilibrium models were based on a single criterion. The assumptions that the network users choose their paths based on a single criterion may not be reasonable under all circumstances. It is however more reasonable to assume that no user will choose a path that incurs both a higher cost and a longer delay than some other path. In other words, a vector equilibrium should be sought based on the principle that the traffic flow along a path joining an OD pair is positive only if the vector cost of this path is the minimum possible amongst all paths joining the same OD pair. That is, the cost function is a vector-valued one. Recently, equilibrium models based on multi-criterion or a vector cost function have been proposed, such as Chen and Yen (1993), and Chen, Goh and Yang (1999), among others. The original Wardrop's equilibrium principle also applies to the case of a network involving multiple products. Such models were considered by Nagurney and Dong (2002) and Nagurney (2000).

In many multi-criterion decision-making problems, the common practice is to obtain the set of efficient decisions, i.e., decisions that are not dominated by any others. Kuhn and Tucker, and later Geoffrion, observed that a subset of efficient set may be "improper". Practically, this means that points in the subset cannot be satisfactorily characterized by a scalar minimization problem, even if the decision set is convex. So, the concept of *proper efficiency* was introduced by Kuhn-Tucker (1951), Geoffrion (1968), and modified and formulated into a more general framework by Borwein (1977), Benson (1979), Henig (1982), and Borwein and Zhuang (1993), among many other researchers. The motivation for introducing proper efficiency is that it enables one to eliminate certain anomalous efficient decisions and to prove the existence of equivalent scalar problems whose solutions produce most of the efficient decisions at least, namely the proper ones. It has been amply demonstrated that proper efficiency is a natural concept in vector optimization.

In our paper we combine the above three aspects by considering a kind of proper efficiency – Henig efficiency of a multi-product network equilibrium model with a vector-valued cost function. We establish a sufficient and a necessary condition for a Henig equilibrium pattern flow for a multi-product network equilibrium problem in terms of vector variational inequalities for the single criterion case and the multiple criteria case.

The organization of the paper is as follows. In Section 2, the relation between Henig efficiency of a multi-product network equilibrium model with a single criterion and a vector variational inequality is established. In Section 3, we deduce a sufficient and a necessary condition for Henig efficiency of a multi-product network equilibrium model with multiple criteria in terms of a vector variational inequality by using Gerstewitz's scalarization function, which has never been considered in the literature. We conclude the paper in Section 4.

2 Henig efficiency of a multi-product network equilibrium model

First of all, we introduce some notations about Henig efficiency. Let Y be a real normed space ordered by a closed convex cone $M \subset Y$ with nonempty interior $int\ M$. We denote the ordering as follows:

$$x \leqslant y \quad \text{iff} \quad y - x \in M;$$

$$x < y \quad \text{iff} \quad y - x \in int\ M.$$

A nonempty convex subset P of the convex cone M is called a base of M if $M = cone(P)$ and $0 \notin cl(P)$, where $cl(P)$ is the closure of P and $cone(P)$ is the cone hull of P, i.e.,

$$cone(P) := \cup\{\lambda a : \ \lambda \geqslant 0, \ a \in P\}.$$

Denote the closed unit ball of Y by U_M. If M has a base P, let

$$\delta_M := inf\ \{\|a\| : \ a \in P\}$$

and

$$M_\varepsilon(P) := cl(cone(P + \varepsilon U_M)), \quad \forall\ 0 < \varepsilon < \delta_M.$$

By Gong (2001), we know that for any $0 < \varepsilon < \delta_M$, $M_\varepsilon(P)$ is a closed convex pointed cone and

$$int\ M \subset M \setminus \{0\} \subset int\ M_\varepsilon(P).$$

If $0 < \varepsilon < \varepsilon' < \delta_M$, then

$$M_\varepsilon(P) \subset cone(P + \varepsilon' U_M) \subset M_{\varepsilon'}(P).$$

A point $e^* \in E \subset Y$ is said to be an efficient point of E if $e - e^* \notin -P \setminus \{0\}$ for any $e \in E$. By $Eff\ E$ we denote the set of all the efficient points of E. We also need to introduce the concept of Henig efficient points of a set E. A point $e^* \in E \subset Y$ is said to be a Henig efficient point of E if $e - e^* \notin -int\ M_\varepsilon(P)$ for any $e \in E$ and $e \neq e^*$. We denote the set of all the Henig efficient points of E by $Henig\ E$.

We consider a supply-demand network in which there are q products to traverse in the network with a typical product denoted by j. Consider a general network $G = [N, A, I]$, where N denotes the set of nodes representing manufacturers and retailers, as well as distributing centers and warehouses, and A the set of directed arcs. Let $a \in A$ denote an arc connecting a pair of nodes. Let I denote the set of all the OD pairs associated with each pair of manufacturer and retailer, and $|I| = l$. We denote by K_i the set of paths that connect an OD pair $i \in I$ associated with a given pair of manufacturer and retailer and let $m = \sum\limits_{i \in I} |K_i|$. Let $k \in K_i$ denote a path, assumed to be acyclic, consisting of a sequence of arcs connecting an OD pair i.

For a path $k \in K_i$, let v_k^j denote the flow of product j on path k. A path flow v_k^j induces a flow v_a^j of product j on an arc $a \in A$ given by:

$$v_a^j = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} v_k^j,$$

where

$$\Delta = [\delta_{ak}] \in R^{|A| \times m}$$

is the arc path incidence matrix, with

$$\delta_{ak} = \begin{cases} 1, & \text{if } a \in k \\ 0, & \text{otherwise.} \end{cases}$$

A vector $v^j = (v_k^j : \ k \in K_i, \ i \in I)$ such that $v_k^j \geqslant 0, \ \forall k \in K_i, \ i \in I, \ j = 1, 2, \cdots, q$, is said to be a flow of product j on the network and $v = (v^1, v^2, \cdots, v^q)^T$ is called a flow of the network. Let there also be given a vector of demands $d = (d_i^j : \ i \in I, \ j = 1, 2, \cdots, q)$. Each component d_i^j indicates the demand of the OD pair i for product j, that is, the quantity of product j that needs to go from the manufacturer to the retailer associated with the OD pair i. We say that a flow of the network v satisfies the demands if

$$\sum_{k \in K_i} v_k^j = d_i^j, \quad \forall i \in I, \ j = 1, 2, \cdots, q.$$

Then, the set $D = \{v : \ \sum\limits_{k \in K_i} v_k^j = d_i^j, \ \forall i \in I, \ j = 1, 2, \cdots, q\}$ is the feasible set. D is clearly a convex set. In fact, for any $v, \ u \in D$ and any $\lambda \in [0, 1]$, we know

$$\sum_{k \in K_i} v_k^j = d_i^j, \quad \sum_{k \in K_i} u_k^j = d_i^j, \quad \forall i \in I, \ j = 1, 2, \cdots, q.$$

Thus, we have

$$\sum_{k \in K_i} \lambda v_k^j = \lambda d_i^j, \quad \sum_{k \in K_i} (1 - \lambda) u_k^j = (1 - \lambda) d_i^j.$$

Hence,

$$\sum_{k \in K_i} (\lambda v_k^j + (1 - \lambda) u_k^j) = d_i^j, \quad \forall i \in I, \ j = 1, 2, \cdots, q.$$

That is, $\lambda v + (1 - \lambda) u \in D$. Therefore, D is convex.

The function $c_a^j(v) : R^{q \times m} \to R_+$ is interpreted as the cost of product j on an arc $a \in A$. Then the cost function of product j on a path k $(k \in K_i, \ i \in I)$ depending on the flow of the network is defined by the formula

$$c_k^j(v) = \sum_{a \in k} c_a^j(v).$$

Then, the vector function $c^j(v) = (c^j_k(v) : \ k \in K_i, \ i \in I)$ and $c(v) = (c^1(v), c^2(v), \cdots, c^q(v))^T$ are called the cost function of product j on the network and the cost function of the network, respectively.

For each $i \in I$, we define the minimum cost function of product j for the OD pair i by putting

$$m^j_i(v) = \min_{k \in K_i} c^j_k(v).$$

Set $m_i(v) = (m^1_i(v), m^2_i(v), \cdots, m^q_i(v))^T$ and group the $q \times m$ matrix v into a q-dimensional column vector v_k ($\forall \ k \in K_i, \ i \in I$) with components $v_k = (v^1_k, v^2_k, \cdots, v^q_k)^T$, where $v = (v_k : \ k \in K_i, \ i \in I)$. Also, group the vector $c(v)$ into a q-dimensional column vector $c_k(v), \ k \in K_i, \ i \in I$, with components $c_k(v) = (c^1_k(v), c^2_k(v), \cdots, c^q_k(v))^T$, where $c(v) = (c_k(v) : \ k \in K_i, \ i \in I)$. For the q-dimensional Euclidean space R^q, by $\leqslant$ we denote the ordering induced by R^q_+ :

$$x \leqslant y \quad \text{iff} \quad y - x \in R^q_+;$$

$$x < y \quad \text{iff} \quad y - x \in int \ R^q_+,$$

where $int \ R^q_+$ is the interior of R^q_+. The ordering $\geqslant$ and $>$ are defined similarly.

Applying Wardrop's equilibrium principle (Wardrop (1952)), we see that the equilibrium principle (user-optimizing principle) in the generalized context of a multi-product supply-demand network equilibrium problem takes on the following form.

Definition 2.1. A vector $v \in D$ is called an equilibrium pattern flow iff

$$c_k(v) - m_i(v) \begin{cases} = 0 & \text{if } v_k \in R^q_+ \setminus \{0\} \\ \geqslant 0 & \text{if } v_k = 0. \end{cases} \tag{2.1}$$

for each $i \in I$ and each $k \in K_i$.

The above equilibrium principle involves no explicit optimization concept because the network users act independently, in a noncooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, condition (2.1) means that only those paths connecting an OD pair that have minimal user travel costs in terms of vector ordering will be used. Otherwise, the network users could improve upon their situations by switching to a path with a lower cost. That is, for any OD pair of manufacturer and retailer i, if the transportation cost of all the products on a path $k \in K_i$ is greater than the minimum cost of the OD pair i in terms of vector ordering, then the flow of all the products on k is zero.

For the sake of convenience, the equilibrium condition (2.1) can be expressed in the following equivalent form.

Proposition 2.1. (see Cheng and Wu (2005)) The network equilibrium condition (2.1) is equivalent to the following statement:

$$c_r(v) - c_k(v) \in R^q_+ \setminus \{0\} \Rightarrow v_r = 0, \tag{2.2}$$

for each $i \in I$ and any $k, r \in K_i$.

It seems that the left side of (2.2) is defined in a way similar to the definition of strong efficiency (see Liu and Gong (2000)). We know weak efficiency and strong efficiency are two kinds of extremal efficiency in vector optimization. Optimality conditions of these kinds of efficiency are not complete for vector optimization theory. In vector optimization problems, several notions of proper efficiency have been proposed, in order to rule out some situations (tolerated by the definition of efficiency) that are hardly meaningful. Proper efficiency has been introduced in order to get rid of anomalous efficient points.

Next, we introduce a kind of proper efficiency–Henig efficiency of a network equilibrium model. For simplicity, we replace R^q_+ with the notation H. By B and U_H, we denote the base of H and the closed unit ball of R^q, respectively. Thus,

$$H_\varepsilon(B) = cl(cone(B + \varepsilon U_H)), \quad \forall\, 0 < \varepsilon < \delta_H,$$

where $\delta_H = inf\ \{\|a\| :\ a \in B\}$.

Definition 2.2. A vector $v \in D$ is called a Henig equilibrium pattern flow iff the following statement holds:

$$c_r(v) - c_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}) \Rightarrow v_r = 0,$$

for each $i \in I$, any $k, r \in K_i$ and some $0 < \varepsilon < \delta_H$.

The cost functions are asymmetric in the model. Such cost functions are very important from an application point of view since they allow for asymmetric cost interactions in the network. However, given the asymmetry of the cost functions, one cannot compute the solution to the network equilibrium problem using standard optimization algorithms. Indeed, variational inequality theory provides a feasible approach to studying such problems.

Variational inequality theory is a powerful tool in the qualitative analysis of equilibria theory (see, for example, Nagurney (1999)). Now, let us review the concept of vector variational inequality. For a vector space $R^{q\times q}$, the vector variational inequality is:

$$\text{To find } \bar{x} \in B \text{ such that } \langle T(\bar{x}), x-\bar{x}\rangle \in R^{q\times q}_{+}, \quad \forall\, x \in B,$$

where $T:\ X \to L(X, R^{q\times q})$, $L(X, R^{q\times q})$ is the set of all the linear operators from X into $R^{q\times q}$, B is a convex subset of X and X is an abstract space.

Next, we establish a sufficient and a necessary conditions for Henig efficiency of a network equilibrium problem with multiple products in terms of vector variational inequality problems. Specifically, we wish to prove the following two theorems.

Theorem 2.1. Let a vector flow $v \in D$ be a Henig equilibrium pattern flow. Then v is a solution to the vector variational inequality: to find $v \in D$ such that

$$\langle c(v), (u-v)^T\rangle \notin -int\ R^{q\times q}_{+}, \quad \forall\, u \in D.$$

Proof. Let a vector flow $v \in D$ be a Henig equilibrium pattern flow for a multi-product supply-demand network equilibrium problem. By Definition 2.2, we have the following statement:

$$c_r(v) - c_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}) \Rightarrow v_r = 0,$$

for each $i \in I$ and any $k, r \in K_i$.

For any $u \in D$, we have

$$\begin{aligned}
&\langle c(v), (u-v)^T\rangle \\
=&(c_1(v), c_2(v), \cdot\cdot\cdot, c_m(v))(u_1-v_1, u_2-v_2, \cdot\cdot\cdot, u_m-v_m)^T \\
=&\sum_{t=1}^{m} c_t(v)(u_t-v_t)^T \\
=&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t(v)(u_t-v_t)^T].
\end{aligned}$$

We know $c_t(v)(u_t-v_t)^T$ is a $q\times q$ matrix whose components are $c_t^\alpha(v)(u_t^\beta - v_t^\beta)$, where $\alpha, \beta = 1, 2, \cdot\cdot\cdot, q$. Hence, $\langle c(v), (u-v)^T\rangle$ is also a $q\times q$ matrix whose components are $\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^\alpha(v)(u_t^\beta - v_t^\beta)]$, where $\alpha, \beta = 1, 2, \cdot\cdot\cdot, q$.

Set

$$J_i(v) := \{\bar{r} \in K_i :\ c_{\bar{r}}(v) \in Henig\{c_k(v) :\ k \in K_i\}\} \subset K_i.$$

Then, for any $\bar{r} \in J_i(v) \subset K_i$,

$$c_k(v) - c_{\bar{r}}(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}), \quad \forall\, k \in K_i,\ i \in I \text{ and } k \neq \bar{r}.$$

By Definition 2.2, we have $v_k = 0$ for any $k \in K_i,\ i \in I$ and $k \neq \bar{r}$. Thus, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\alpha}(v)(u_t^{\beta} - v_t^{\beta})] \\
=&\sum_{i=1}^{l}[\sum_{t\in K_i\setminus\{\bar{r}\}} c_t^{\alpha}(v)(u_t^{\beta} - v_t^{\beta}) + c_{\bar{r}}^{\alpha}(v)(u_{\bar{r}}^{\beta} - v_{\bar{r}}^{\beta})] \\
=&\sum_{i=1}^{l}[\sum_{t\in K_i\setminus\{\bar{r}\}} c_t^{\alpha}(v)u_t^{\beta} + c_{\bar{r}}^{\alpha}(v)(u_{\bar{r}}^{\beta} - v_{\bar{r}}^{\beta})] \\
=&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\alpha}(v)u_t^{\beta} - c_{\bar{r}}^{\alpha}(v)v_{\bar{r}}^{\beta}] \\
\geqslant&\sum_{i=1}^{l}[m_i^{\alpha}(v)\sum_{t\in K_i} u_t^{\beta} - c_{\bar{r}}^{\alpha}(v)v_{\bar{r}}^{\beta}] \\
=&\sum_{i=1}^{l}[m_i^{\alpha}(v)d_i^{\beta} - c_{\bar{r}}^{\alpha}(v)v_{\bar{r}}^{\beta}].
\end{aligned}
$$

Since $v \in D$, by $v_k = 0$ for any $k \in K_i,\ i \in I$ and $k \neq \bar{r}$ we know

$$\sum_{t\in K_i} v_k^{\beta} = \sum_{t\in K_i\setminus\{\bar{r}\}} v_k^{\beta} + v_{\bar{r}}^{\beta} = v_{\bar{r}}^{\beta} = d_i^{\beta}.$$

Hence, we have

$$
\begin{aligned}
&\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\alpha}(v)(u_t^{\beta} - v_t^{\beta})] \\
\geqslant&\sum_{i=1}^{l}[m_i^{\alpha}(v)d_i^{\beta} - c_{\bar{r}}^{\alpha}(v)d_i^{\beta}] \\
=&\sum_{i=1}^{l} d_i^{\beta}[m_i^{\alpha}(v) - c_{\bar{r}}^{\alpha}(v)]. \qquad (2.3)
\end{aligned}
$$

For any $\bar{r} \in J_i(v)$, by Liu and Gong (2000), we know $c_{\bar{r}}(v) \in Eff\{c_k(v):\ k \in K_i\}$. That is,

$$c_k(v) - c_{\bar{r}}(v) \notin -H \setminus \{0\}), \quad \forall\ k \in K_i,\ i \in I.$$

It means that there exists an $\bar{\alpha} \in \{1, 2, \cdots, q\}$ such that

$$c_k^{\bar{\alpha}}(v) - c_{\bar{r}}^{\bar{\alpha}}(v) \geqslant 0, \quad \forall\ k \in K_i,\ i \in I.$$

That is,

$$m_i^{\bar{\alpha}}(v) = c_{\bar{r}}^{\bar{\alpha}}(v).$$

Hence, from (2.3), we derive that there exists an $\bar{\alpha} \in \{1, 2, \cdots, q\}$ such that

$$\sum_{i=1}^{l}[\sum_{t\in K_i} c_t^{\bar{\alpha}}(v)(u_t^{\beta} - v_t^{\beta})] \geqslant 0.$$

Thus, we obtain

$$\langle c(v), (u-v)^T\rangle \notin -int\ R_+^{q\times q}, \quad \forall\ u \in D.$$

The proof is completed. $\square$

Theorem 2.2. A vector flow $v \in D$ is a Henig equilibrium pattern flow if v is a solution to the vector variational inequality: to find $v \in D$ such that

$$\langle c(v), (u-v)^T\rangle \in R_+^{q\times q}, \quad \forall\ u \in D.$$

Proof. Suppose that $v \in D$ is a solution to the following variational inequality:

$$\langle c(v), (u-v)^T\rangle \in R_+^{q\times q}, \quad \forall\ u \in D.$$

Also, assume that $c_r(v) - c_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\})$ for any $i \in I$ and $k, r \in K_i$. We want to deduce that $v_r = 0$.

We consider the vector u whose components are such that

$$u_t = \begin{cases} v_t & \text{if } t \neq r, k \\ 0 & \text{if } t = r \\ v_r + v_k & \text{if } t = k. \end{cases}$$

Since $v \in D$, i.e., $\sum\limits_{t\in K_i} v_t^j = d_i^j$ for any $i \in I$ and any $j = 1, 2, \cdots, q$, we have

$$\begin{aligned}\sum_{t\in K_i} u_t^j &= \sum_{t\in K_i\setminus\{r,k\}} u_t^j + u_r^j + u_k^j \\ &= \sum_{t\in K_i\setminus\{r,k\}} v_t^j + 0 + v_r^j + v_k^j \\ &= \sum_{t\in K_i} v_t^j \\ &= d_i^j.\end{aligned}$$

So, $u \in D$. By the above proof, we know

$$\begin{aligned}&\langle c(v), (u-v)^T\rangle \\ =&\sum_{t=1}^{m} c_t(v)(u_t - v_t)^T \\ =&\sum_{t\neq r,k} c_t(v)(v_t - v_t)^T - c_r(v)v_r^T + c_k(v)v_r^T \\ =&(c_k(v) - c_r(v))v_r^T \in R_+^{q\times q}.\end{aligned}$$

It is easy to see that $\langle c(v), (u-v)^T \rangle$ is a $q \times q$ matrix whose components are $(c_k^\alpha(v) - c_r^\alpha(v))v_r^\beta$, where $\alpha, \beta = 1, 2, \cdots, q$. So, we obtain

$$(c_k^\alpha(v) - c_r^\alpha(v))v_r^\beta \geqslant 0, \quad \forall\ \alpha, \beta = 1, 2, \cdots, q. \tag{2.4}$$

If $v_r \neq 0$, there exists a $\bar{\beta} \in \{1, 2, \cdots, q\}$ such that $v_r^{\bar{\beta}} > 0$. Since $c_r(v) - c_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\})$, we know

$$c_r(v) - c_k(v) \notin -int\ H_\varepsilon(B)$$

and

$$c_r(v) - c_k(v) \neq 0. \tag{2.5}$$

By $-H \setminus \{0\} \subset -int\ H_\varepsilon(B)$, we have $c_r(v) - c_k(v) \notin -H \setminus \{0\}$. Combining with (2.5), it holds that

$$c_r(v) - c_k(v) \notin -H.$$

That is, there too exists an $\bar{\alpha} \in \{1, 2, \cdots, q\}$ such that $c_k^{\bar{\alpha}}(v) - c_r^{\bar{\alpha}}(v) < 0$. Hence,

$$(c_k^{\bar{\alpha}}(v) - c_r^{\bar{\alpha}}(v))v_r^{\bar{\beta}} < 0.$$

It is a contradiction to (2.4). Therefore, $v_r = 0$.

We complete the proof. □

3 Network equilibrium problem with multi-product products and multi-criterion

The assumption that the network users choose their paths based on a single criterion may not be reasonable on all occasions. For example, if a path has a rough surface or is noted for its unsafe road conditions such as ice in winter, users may pay more attention to transportation time than cost. However, on a general road, they would rather incur less transportation cost than spend more time. It is more reasonable to assume that no user will choose a path that incurs both a higher cost and a longer delay than some other path. Therefore, the cost function is a vector-valued one. In Chen and Yen (1993), a multi-criterion traffic equilibrium model was proposed, but no attempt was made to solve the equilibrium problem. Other papers that have considered multi-criterion equilibrium models are Chen, Goh and Yang (1999), and Yang and Goh (1997), among others.

Let Z be a Hausdorff topological vector space ordered by a pointed, closed convex cone $S \subset Z$ with nonempty interior $int\ S$. For the network $G = [N, A, I]$, if we define the cost function of product j on an arc $a \in A$ as a vector-valued function of the flow v : $C_a^j(v)$: $R^{q \times m} \to Z$ and $C_a^j(v) \geqslant 0$, then the cost function $C_k^j(v)$ of product j

on a path $k \in K_i,\ i \in I$, is also a vector-valued function, which is defined as above: $C_k^j(v) = \sum\limits_{a \in k} C_a^j(v)$. The vector-valued function $C^j(v) = (C_k^j(v):\ k \in K_i,\ i \in I) \in Z^m$ and $C_k(v) = (C_k^1(v), C_k^2(v), \cdots, C_k^q(v))^T \in Z^q$ are the cost function of product j in the network and the cost function on the path $k \in K_i,\ i \in I$, respectively. Then, the vector-valued cost function of the network is $C(v) = (C^1(v), C^2(v), \cdots, C^q(v))^T \in Z^{q \times m}$ or $C(v) = (C_k(v):\ k \in K_i,\ i \in I)$.

In this section we consider Z as a finite-dimensional Euclidean space R^p with the special ordering cone $S = R_+^p$, which is more realistic than an abstract topological vector space from a practical viewpoint. Also, we replace $R_+^{q \times p}$ with the notation L. By T and U_L we denote the base of L and the closed unit ball of $R^{q \times p}$, respectively. Thus,

$$L_\varepsilon(T) = cl(cone(T + \varepsilon U_L)), \quad \forall\ 0 < \varepsilon < \delta_L,$$

where $\delta_L = inf\ \{\|b\|:\ b \in T\}$. Now we can generalize Wardrop's equilibrium principle to a multi-product supply-demand network equilibrium problem with a vector-valued cost function with respect to Henig efficiency.

Definition 3.1. A vector $v \in D$ is said to be a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function iff

$$C_r(v) - C_k(v) \notin -(int\ L_\varepsilon(T) \cup \{0\}) \Rightarrow v_r = 0,$$

for each $i \in I$, any $k, r \in K_i$ and some $0 < \varepsilon < \delta_L$.

A useful approach to analyzing the vector-valued problem is to reduce it to a scalarized problem. In general, the linear scalarization method appears to be popular. But such kind of methods rely heavily on some underlying convexity assumptions, which are hardly valid for many real problems. In our paper, by using Gerstewitz's function (see Chen, Goh and Yang (1999)), we develop another scalarization method for the vector-valued Wardrop's network equilibrium problem without any convexity assumptions.

Definition 3.2. Given a fixed $e \in int\ R_+^p$, the Gerstewitz's function $\xi_e:\ R^p \to R$ is defined by:

$$\xi_e(y) = min\{\lambda \in R:\ y \in \lambda e - R_+^p\}, \quad \forall\ y \in R^p.$$

We note that this function was used in Gerth and Weidner (1990) to establish a useful non-convex separation theorem. Obviously, there are some salient properties of this function that we will use later.

Lemma 3.1. (see Chen and Yang (2002)) Let $e \in int\ R^p_+$, then ξ is positively homogeneous and subadditive on R^p. That is, for any $\mu \geqslant 0,\ y, z \in R^p$,

$$\xi_e(\mu y) = \mu \xi_e(y);$$

$$\xi_e(y+z) \leqslant \xi_e(y) + \xi_e(z).$$

Lemma 3.2. (also see Chen and Yang (2002)) Let $e \in int\ R^p_+$. For each $\eta \in R$ and $y \in R^p$, we have the following results:

(i) $\xi_e\,(y) < \eta \Leftrightarrow y \in \eta e - int\ R^p_+$;

(ii) $\xi_e\,(y) \leqslant \eta \Leftrightarrow y \in \eta e - R^p_+$;

(iii) $\xi_e\,(y) \geqslant \eta \Leftrightarrow y \notin \eta e - int\ R^p_+$;

(iv) $\xi_e\,(y) > \eta \Leftrightarrow y \notin \eta e - R^p_+$;

(v) $\xi_e\,(y) = \eta \Leftrightarrow y \in \eta e - \partial R^p_+$, where ∂R^p_+ is the topological boundary of R^p_+.

Lemma 3.3. (see Cheng and Wu (2005)) Let $e \in int\ R^p_+$. For any $y \in R^p$, we have

$$\xi_e(-y) \geqslant -\xi_e(y).$$

Then, for any $\eta \in R$,

$$\xi_e(-\eta y) \geqslant -\xi_e(\eta y).$$

Lemma 3.4. (also see Cheng and Wu (2005)) For an $e \in int\ R^p_+$ and $\eta \in R$,

$$\xi_e(-\eta e) = -\xi_e(\eta e) = -\eta.$$

Similar to Section 2, we want to find an equivalence relation between Henig efficiency of a vector-valued network equilibrium problem with multiple products and a vector variational inequality. Since we have proved that the necessary and sufficient conditions for a vector flow $v \in D$ to be an equilibrium pattern flow in a scalar-valued network equilibrium problem are that it is a solution to a vector variational inequality in Theorem 2.1, by applying the Gerstewitz's function, we suppose that the equivalence relation must hold between the vector-valued network equilibrium problem and the vector variational inequality without any convexity assumptions. We prove this result in the following.

First, we denote

$$\xi_e \circ C^j_k(v) = \xi_e(C^j_k(v)) = min\{\lambda \in R:\ C^j_k(v) \in \lambda e - R^p_+\},$$

for any $v \in D,\ k \in K_i,\ i \in I, j = 1, 2, \cdots, q$;

$$\xi_e \circ C_k(v) = (\xi_e \circ C_k^j(v) :\ j = 1, 2, \cdots, q)^T \in R^q;$$

and

$$\xi_e(v) = \xi_e \circ C(v) = (\xi_e \circ C_k(v) :\ k \in K_i,\ i \in I) \in R^{q\times m}.$$

Definition 3.3. A vector $v \in D$ is said to be an ξ_e-Henig equilibrium pattern flow in a vector-valued network equilibrium problem with multiple products if there exists an $e \in int\ R^p_+$ such that

$$\xi_e \circ C_r(v) - \xi_e \circ C_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}) \Rightarrow v_r = 0,$$

for each $i \in I$, any $k, r \in K_i$ and some $0 < \varepsilon < \delta_H$.

Set $C_k^j(v) :\ R^{q\times m} \to R^p_+$ in the following form:

$$C_k^j(v) = f_k^j(v)k_0, \quad \forall\ k \in K_i,\ i \in I \text{ and } j \in \{1, 2, \cdots, q\}. \tag{3.1}$$

where $f_k^j(v) :\ R^{q\times m} \to R_+$ and $k_0 \in int\ R^p_+$. It is realistic from a practical viewpoint since the transportation cost function is made up of elementary costs. We see that k_0 is a vector of elementary costs, i.e., it is vector-valued, and each $C_k^j(v)$ is its real-valued multiple, i.e., the multiple $f_k^j(v)$ is a real-valued function of flow v.

Now we will scalarize the vector-valued network equilibrium problem with multiple products. It is important to note that we do not require any convexity assumptions since we use Gerstewitz's function in our scalarization method.

Theorem 3.1. Let $C_k^j(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. A vector $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function if and only if v is an ξ_{k_0}-Henig equilibrium pattern flow in a vector-valued network equilibrium problem with multiple products.

Proof. Necessity: Let $v \in D$ be a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function. That is, for some $0 < \varepsilon < \delta_L$,

$$C_r(v) - C_k(v) \notin -(int\ L_\varepsilon(T) \cup \{0\}) \Rightarrow v_r = 0, \tag{3.2}$$

for each $i \in I$ and any $k, r \in K_i$. Next we will prove for a $j \in \{1, 2, \cdots, q\}$,

$$\left.\begin{array}{l} \xi_{k_0}(C_k^j(v)) - \xi_{k_0}(C_r^j(v)) < 0 \\ C_r(v) - C_k(v) \neq 0 \end{array}\right\} \Rightarrow C_r(v) - C_k(v) \notin -(int\ L_\varepsilon(T) \cup \{0\}).$$

By (3.1) and Lemma 3.4, we obtain

$$\xi_{k_0}(C_k^j(v)) - \xi_{k_0}(C_r^j(v)) = \xi_{k_0}(C_k^j(v) - C_r^j(v)) = f_k^j(v) - f_r^j(v).$$

Hence, we get

$$\xi_{k_0}(C_k^j(v)) - \xi_{k_0}(C_r^j(v)) < 0 \Rightarrow \xi_{k_0}(C_k^j(v) - C_r^j(v)) < 0.$$

By Lemma 3.2, we get

$$C_k^j(v) - C_r^j(v) \in -int\ R_+^p.$$

Thus,

$$C_k(v) - C_r(v) \notin R_+^{q\times p} = L. \tag{3.3}$$

If not, it holds that

$$C_k^j(v) - C_r^j(v) \in R_+^p, \quad \forall\ j = 1, 2, \cdots, q.$$

We assume that for any $\varepsilon \in (0, \delta_L)$,

$$C_r(v) - C_k(v) \in -int\ L_\varepsilon(T). \tag{3.4}$$

We know that $L_\varepsilon(T) = cone(T + \varepsilon U)$. By Gong (2001), $L \setminus \{0\} \subset L_\varepsilon(T) \subset L_{\varepsilon'}(T)$, if $0 < \varepsilon < \varepsilon^{'} < \delta_L$. Hence, by the arbitrariness of $\varepsilon \in (0, \delta_L)$ in (3.4), we deduce

$$C_r(v) - C_k(v) \in -L \setminus \{0\}.$$

That is,

$$C_k(v) - C_r(v) \in L \setminus \{0\}.$$

It is a contradiction to (3.3). So, we see that for some $\varepsilon \in (0, \delta_L)$,

$$C_r(v) - C_k(v) \notin -(int\ L_\varepsilon(T) \cup \{0\}).$$

Thus, by (3.2) it holds that for an $e \in int\ R_+^p$ and a $j \in \{1, 2, \cdots, q\}$,

$$\begin{array}{l} \xi_{k_0}(C_k^j(v)) - \xi_{k_0}(C_r^j(v)) < 0, \\ C_r(v) - C_k(v) \neq 0 \end{array} \} \Rightarrow v_r = 0,$$

for each $i \in I$ and any $k, r \in K_i$.

If v is not an ξ_{k_0}-Henig equilibrium pattern flow for a vector-valued network equilibrium problem with multiple products, then there exists an $\bar{i} \in I$ and a pair of $\bar{k}, \bar{r} \in K_{\bar{i}}$ satisfying $\xi_{k_0} \circ C_{\bar{r}}(v) - \xi_{k_0} \circ C_{\bar{k}}(v) \notin -(int\ H_\varepsilon(B) \cup \{0\})$ such that $v_{\bar{r}} \neq 0$. By $\xi_{k_0} \circ C_{\bar{r}}(v) - \xi_{k_0} \circ C_{\bar{k}}(v) \notin -(int\ H_\varepsilon(B) \cup \{0\})$, we know that $C_{\bar{r}}(v) - C_{\bar{k}}(v) \neq 0$ and

$$\xi_{k_0} \circ C_{\bar{r}}(v) - \xi_{k_0} \circ C_{\bar{k}}(v) \notin -int\ H_\varepsilon(B).$$

Since $H \setminus \{0\} \subset int\ H_\varepsilon(B)$, we obtain

$$\xi_{k_0} \circ C_{\bar{r}}(v) - \xi_{k_0} \circ C_{\bar{k}}(v) \notin -H \setminus \{0\}.$$

Since $\xi_{k_0} \circ C_{\bar{r}}(v) - \xi_{k_0} \circ C_{\bar{k}}(v) \neq 0$, we see that there exists $\bar{j}$ such that

$$\xi_{k_0}(C_{\bar{r}}^{\bar{j}}(v)) - \xi_{k_0}(C_{\bar{k}}^{\bar{j}}(v)) > 0.$$

Combining with $C_{\bar{r}}(v) - C_{\bar{k}}(v) \neq 0$, we obtain $v_{\bar{r}} = 0$. It is a contradiction. So v is also an ξ_{k_0}-Henig equilibrium pattern flow in a vector-valued network equilibrium problem with multiple products.

Sufficiency: Suppose that v is an ξ_{k_0}-Henig equilibrium pattern flow for a vector-valued network equilibrium problem with multiple products. That is, for some $\varepsilon \in (0, \delta_H)$,

$$\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}) \Rightarrow v_r = 0,$$

for each $i \in I$ and any $k, r \in K_i$. Next, we will prove that for a $j \in \{1, 2, \cdots, q\}$,

$$\left.\begin{array}{l} \xi_{k_0}(C_r^j(v)) - \xi_{k_0}(C_k^j(v)) > 0 \\ C_r(v) - C_k(v) \neq 0 \end{array}\right\} \Rightarrow \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}).$$

It is easy to see that $\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \neq 0$. So, we only need to prove $\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \notin -int\ H_\varepsilon(B)$. We assume that for any $\varepsilon \in (0, \delta_H)$, $\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \in -int\ H_\varepsilon(B)$. By an analogous analysis with the proof of *Necessary*, we derive

$$\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \in -H \setminus \{0\}.$$

That is, for any $j \in \{1, 2, \cdots, q\}$, it holds that

$$\xi_{k_0}(C_r^j(v)) - \xi_{k_0}(C_k^j(v)) \leqslant 0.$$

It is a contradiction to $\xi_{k_0}(C_r^j(v)) - \xi_{k_0}(C_k^j(v)) > 0$. Thus, we obtain that for some $\varepsilon \in (0, \delta_H)$,

$$\xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \notin -int\ H_\varepsilon(B).$$

Hence, we deduce that for a $j \in \{1, 2, \cdots, q\}$,

$$\left.\begin{array}{l} \xi_{k_0}(C_r^j(v)) - \xi_{k_0}(C_k^j(v)) > 0 \\ C_r(v) - C_k(v) \neq 0 \end{array}\right\} \Rightarrow \xi_{k_0} \circ C_r(v) - \xi_{k_0} \circ C_k(v) \notin -(int\ H_\varepsilon(B) \cup \{0\}).$$

Thus, we also deduce that for a $j \in \{1, 2, \cdots, q\}$,

$$\left.\begin{array}{l} \xi_{k_0}(C_r^j(v)) - \xi_{k_0}(C_k^j(v)) > 0 \\ C_r(v) - C_k(v) \neq 0 \end{array}\right\} \Rightarrow v_r = 0 \tag{3.5}$$

for each $i \in I$ and any $k, r \in K_i$.

If for any $i \in I$ and $k, r \in K_i$, $C_r(v) - C_k(v) \notin -(int\ L_\varepsilon(T) \cup \{0\})$, we want to derive $v_r = 0$. By $C_r(v) - C_k(v) \notin -(int\ L_\varepsilon(T) \cup \{0\})$, we know $C_r(v) - C_k(v) \neq 0$ and

$$C_r(v) - C_k(v) \notin -int\ L_\varepsilon(T).$$

Since $int\ L \subset L \setminus \{0\} \setminus int\ L_\varepsilon(T)$, we obtain

$$C_r(v) - C_k(v) \notin -L.$$

That is, there exists a $\bar{j} \in \{1, 2, \cdots, q\}$ such that

$$C_r^{\bar{j}}(v) - C_k^{\bar{j}}(v) \notin -R_+^p.$$

Thus, by Lemma 3.2, we obtain

$$\xi_{k_0}(C_r^{\bar{j}}(v) - C_k^{\bar{j}}(v)) > 0.$$

By (3.1) and Lemma 3.4, it holds that

$$\xi_{k_0}(C_r^{\bar{j}}(v)) - \xi_{k_0}(C_k^{\bar{j}}(v)) > 0.$$

Hence, by (3.5), we know $v_r = 0$. Therefore, $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function. □

From Theorem 2.1, we obtain the following two corollaries.

Corollary 3.1. Let $C_k^j(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. If a vector flow $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function, then v is a solution to a vector variational inequality: to find $v \in D$ such that

$$\langle \xi_{k_0}(v), (u - v)^T \rangle \notin -int\ R_+^{q \times q}, \quad \forall\ u \in D.$$

Corollary 3.2. Let $C_k^j(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. A vector flow $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function if v is a solution to a vector variational inequality: to find $v \in D$ such that

$$\langle \xi_{k_0}(v), (u - v)^T \rangle \in R_+^{q \times q}, \quad \forall\ u \in D.$$

We know that the Gerstewitz's function is difficult to compute. So the best way to proceed is to convert two vector variational inequality above to the following vector forms: to find $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \notin -int\ (R^p_+)^{q\times q}, \quad \forall\ u \in D.$$

and to find $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \in (R^p_+)^{q\times q}, \quad \forall\ u \in D.$$

In Cheng and Wu (2005), they proved the following theorem.

Theorem 3.2. Let $C^j_k(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. $v \in D$ is a solution to the following vector variational inequality:

finding $v \in D$ such that

$$\langle \xi_{k_0}(v), (u-v)^T \rangle \in R^{q\times q}_+, \quad \forall\ u \in D,$$

if and only if v is also a solution to vector variational inequality:

$$\langle C(v), (u-v)^T \rangle \in (R^p_+)^{q\times q}, \quad \forall\ u \in D.$$

Now we prove another equivalent relation.

Theorem 3.3. If $v \in D$ is the solution to the vector variational inequality: finding $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \notin -int\ (R^p_+)^{q\times q}, \quad \forall\ u \in D,$$

then v is also a solution to the following vector variational inequality: finding $v \in D$ such that for an $e \in int\ R^p_+$,

$$\langle \xi_e(v), (u-v)^T \rangle \notin -int\ R^{q\times q}_+, \quad \forall\ u \in D.$$

Proof. For any $u \in D$, we know that $\langle C(v), (u-v)^T \rangle = \sum_{t=1}^{m} C_t(v)(u_t - v_t)^T$. By the above proof, it is a $q \times q$ matrix whose components are $\sum_{t=1}^{m} [C^\alpha_t(v)(u^\beta_t - v^\beta_t)] \in R^p$, where $\alpha, \beta = 1, 2, \cdots, q$. Since $\langle C(v), (u-v)^T \rangle \notin -int\ (R^p_+)^{q\times q}$, we obtain that there exist $\bar{\alpha},\ \bar{\beta} \in \{1, 2, \cdots, q\}$ such that

$$\sum_{t=1}^{m} [C^{\bar{\alpha}}_t(v)(u^{\bar{\beta}}_t - v^{\bar{\beta}}_t)] \notin -int\ R^p_+, \quad \forall\ u \in D.$$

By Lemma 3.2 we obtain

$$\xi_e(\sum_{t=1}^{m}[C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})]) \geqslant 0, \quad \forall\, u \in D.$$

By Lemma 3.1, it holds that

$$\sum_{t=1}^{m}[\xi_e(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))] \geqslant 0, \quad \forall\, u \in D.$$

For any given $u \in D$, set

$$N_1 = \{t \in \{1, 2, \cdots, m\} : \ u_t^{\bar{\beta}} - v_t^{\bar{\beta}} \geqslant 0\}$$

and

$$N_2 = \{t \in \{1, 2, \cdots, m\} : \ u_t^{\bar{\beta}} - v_t^{\bar{\beta}} < 0\}.$$

Therefore, $|N_1| + |N_2| = m$, and the following formula also holds by Lemma 3.1 and Lemma 3.3:

$$\begin{aligned}
&\sum_{t=1}^{m}[\xi_e(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))]\\
=&\sum_{t\in N_1}[\xi_e(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))] + \sum_{t\in N_2}[\xi_e(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))]\\
\geqslant&\sum_{t\in N_1}[(\xi_e(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})] - \sum_{t\in N_2}[(\xi_e(C_t^{\bar{\alpha}}(v)))(v_t^{\bar{\beta}} - u_t^{\bar{\beta}})]\\
=&\sum_{t\in N_1}[(\xi_e(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})] + \sum_{t\in N_2}[(\xi_e(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})]\\
=&\sum_{t=1}^{m}[(\xi_e(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})]\\
\geqslant&\ 0. \qquad\qquad\qquad\qquad\qquad\qquad (3.6)
\end{aligned}$$

We also know for an $e \in int\ R_+^p$,

$$\langle \xi_e(v), (u - v)^T \rangle = \sum_{t=1}^{m}[(\xi_e \circ C_t(v))(u_t - v_t)^T].$$

It is also a $q \times q$ matrix whose components are $\sum_{t=1}^{m}[(\xi_e(C_t^{\alpha}(v)))(u_t^{\beta} - v_t^{\beta})] \in R,\ \alpha, \beta = 1, 2, \cdots, q$. We assume that there exists a $\bar{u} \in D$ such that $\langle \xi_e(v), (\bar{u} - v)^T \rangle \in -int\ R_+^{q\times q}$, i.e., for any $\alpha,\ \beta \in \{1, 2, \cdots, q\}$ we have

$$\sum_{t=1}^{m}[(\xi_e(C_t^{\alpha}(v)))(\bar{u}_t^{\beta} - v_t^{\beta})] < 0.$$

This contadicts (3.6). Therefore, we obtain

$$\langle \xi_e(v), (u-v)^T \rangle \notin -int\ R_+^{q\times q}, \quad \forall\ u \in D. \qquad \square$$

Next, we deduce Theorem 3.4.

Theorem 3.4. Let $C_k^j(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. If $v \in D$ is a solution to the vector variational inequality: finding $v \in D$ such that

$$\langle \xi_{k_0}(v), (u-v)^T \rangle \notin -int\ R_+^{q\times q}, \quad \forall\ u \in D,$$

then v is also a solution to the following vector variational inequality:

$$\langle C(v), (u-v)^T \rangle \notin -int\ (R_+^p)^{q\times q}, \quad \forall\ u \in D.$$

Proof. For any $u \in D$, since $\langle \xi_{k_0}(v), (u-v)^T \rangle$ is a $q \times q$ matrix whose components are $\sum_{t=1}^{m}[(\xi_{k_0}(C_t^{\alpha}(v)))(u_t^{\beta} - v_t^{\beta})]$ $(\alpha, \beta = 1, 2, \cdots, q)$, by $\langle \xi_{k_0}(v), (u-v)^T \rangle \notin -int\ R_+^{q\times q}$, we know that there exist $\bar{\alpha}, \bar{\beta} \in \{1, 2, \cdots, q\}$ such that

$$\sum_{t=1}^{m}[(\xi_{k_0}(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})] \geqslant 0, \quad \forall\ u \in D.$$

Similar to the proof above, we group the set $\{1, 2, \cdots, q\}$ into two parts, N_1 and N_2, where

$$N_1 = \{t \in \{1, 2, \cdots, m\} :\ u_t^{\bar{\beta}} - v_t^{\bar{\beta}} \geqslant 0\},$$

and

$$N_2 = \{t \in \{1, 2, \cdots, m\} :\ u_t^{\bar{\beta}} - v_t^{\bar{\beta}} < 0\},$$

for any given $u \in D$. Then, by Lemma 3.1 and Lemma 3.3,

$$\begin{aligned}
&\sum_{t=1}^{m}[(\xi_{k_0}(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})] \\
=&\sum_{t\in N_1}[(\xi_{k_0}(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})] + \sum_{t\in N_2}[(\xi_{k_0}(C_t^{\bar{\alpha}}(v)))(u_t^{\bar{\beta}} - v_t^{\bar{\beta}})] \\
=&\sum_{t\in N_1}[\xi_{k_0}(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))] + \sum_{t\in N_2}[-\xi_{k_0}(C_t^{\bar{\alpha}}(v)(v_t^{\bar{\beta}} - u_t^{\bar{\beta}}))] \\
\leqslant&\sum_{t\in N_1}[\xi_{k_0}(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))] + \sum_{t\in N_2}[\xi_{k_0}(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))] \\
=&\sum_{t=1}^{m}[\xi_{k_0}(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}} - v_t^{\bar{\beta}}))].
\end{aligned}$$

Therefore,
$$\sum_{t=1}^{m}[\xi_{k_0}(C_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}}-v_t^{\bar{\beta}}))]\geqslant 0,\quad \forall\ u\in D.$$
By $C_t^{\alpha}(v)=f_t^{\alpha}(v)k_0$ and Lemma 3.4, for all $u\in D$, we get
$$\begin{aligned}&\sum_{t=1}^{m}[\xi_{k_0}(f_t^{\bar{\alpha}}(v)k_0(u_t^{\bar{\beta}}-v_t^{\bar{\beta}}))]\\ =&\sum_{t=1}^{m}[f_t^{\bar{\alpha}}(v)(u_t^{\bar{\beta}}-v_t^{\bar{\beta}})]\\ \geqslant&\ 0.\end{aligned}\tag{3.7}$$
If there exists a $\bar{u}\in D$ such that
$$\langle C(v),(\bar{u}-v)^T\rangle\in -int\ (R_+^p)^{q\times q},$$
i.e., for any $\alpha,\ \beta\in\{1,2,\cdots,q\}$, we obtain that the component of $\langle C(v),(\bar{u}-v)^T\rangle$ belongs to $-int\ R_+^p$, i.e.,
$$\sum_{t=1}^{m}[C_t^{\alpha}(v)(\bar{u}_t^{\beta}-v_t^{\beta})]\in -int\ R_+^p.$$
By Lemma 3.2,
$$\xi_{k_0}(\sum_{t=1}^{m}[C_t^{\alpha}(v)(\bar{u}_t^{\beta}-v_t^{\beta})])<0.$$
By $C_t^{\alpha}(v)=f_t^{\alpha}(v)k_0$ and Lemma 3.4, we obtain
$$\begin{aligned}&\xi_{k_0}(\sum_{t=1}^{m}[C_t^{\alpha}(v)(\bar{u}_t^{\beta}-v_t^{\beta})])\\ =&\xi_{k_0}(k_0\sum_{t=1}^{m}(f_t^{\alpha}(v)(\bar{u}_t^{\beta}-v_t^{\beta}))\\ =&\sum_{t=1}^{m}(f_t^{\alpha}(v)(\bar{u}_t^{\beta}-v_t^{\beta}))\\ <&\ 0.\end{aligned}$$
This is a contradiction to (3.7). Therefore, for any $u\in D$,
$$\langle C(v),(u-v)^T\rangle\notin -int\ (R_+^p)^{q\times q}.$$
The proof is completed. $\square$

Corollary 3.3. Let $C_k^j(v)$ be defined as (3.1) for all $k\in K_i,\ i\in I$ and $j\in\{1,2,\cdots,q\}$. $v\in D$ is a solution to the vector variational inequality: finding $v\in D$ such that
$$\langle\xi_{k_0}(v),(u-v)^T\rangle\notin -int\ R_+^{q\times q},\quad \forall\ u\in D,$$

if and only if v is also a solution to the following vector variational inequality:

$$\langle C(v), (u-v)^T \rangle \notin -int\ (R^p_+)^{q\times q}, \quad \forall\ u \in D.$$

Combining with Corollary 3.1, Corollary 3.2 and Theorem 3.2 above, we have derived the following Theorem 3.5 and Theorem 3.6.

Theorem 3.5. Let $C^j_k(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. If a vector flow $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function, then v is a solution to a vector variational inequality: to find $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \notin -int\ (R^p_+)^{q\times q}, \quad \forall\ u \in D.$$

Theorem 3.6. Let $C^j_k(v)$ be defined as (3.1) for all $k \in K_i,\ i \in I$ and $j \in \{1, 2, \cdots, q\}$. A vector flow $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function if v is a solution to a vector variational inequality: to find $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \in (R^p_+)^{q\times q}, \quad \forall\ u \in D.$$

It is instructive to summarize all of the relations we have derived so far. We denote them by (i)$\sim$(vi):

(i) $v \in D$ is an ξ_{k_0}-Henig equilibrium pattern flow in a vector-valued network equilibrium problem with multiple products;

(ii) $v \in D$ is a Henig equilibrium pattern flow in the generalized context of a multi-product supply-demand network equilibrium problem with a vector-valued cost function;

(iii) $v \in D$ is a solution to the vector variational inequality: finding $v \in D$ such that

$$\langle \xi_{k_0}(v), (u-v)^T \rangle \notin -int\ R^{q\times q}_+, \quad \forall\ u \in D;$$

(iv) $v \in D$ is a solution to the vector variational inequality: finding $v \in D$ such that

$$\langle \xi_{k_0}(v), (u-v)^T \rangle \in R^{q\times q}_+, \quad \forall\ u \in D;$$

(v) v is a solution to a vector variational inequality: to find $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \notin -int\ (R^p_+)^{q\times q}, \quad \forall\ u \in D;$$

(vi) $v \in D$ is a solution to a vector variational inequality: to find $v \in D$ such that

$$\langle C(v), (u-v)^T \rangle \in (R_+^p)^{q\times q}, \quad \forall\ u \in D.$$

Then, we have the following relations:

$$\begin{array}{ccc}
 & \text{(iii)} \underset{\text{Corollary 3.3}}{\Longleftrightarrow} \text{(v)} & \\
\text{Theorem 2.1} \Uparrow & & \Uparrow \text{Theorem 3.5} \\
 & \text{(i)} \underset{\text{Theorem 3.1}}{\Longleftrightarrow} \text{(ii)} & \\
\text{Theorem 2.2} \Uparrow & & \Uparrow \text{Theorem 3.6} \\
 & \text{(iv)} \underset{\text{Theorem 3.2}}{\Longleftrightarrow} \text{(vi)} &
\end{array}$$

4 Conclusions

The focus of the paper is to decide on the delivery paths of shipping q products between manufacturers and retailers in a supply-demand network based on a single criterion and multiple criteria, respectively. Based on Wardrop's equilibrium principle, we have considered Henig efficiency of a scalar multi-product network equilibrium model and a vector one for the supply-demand network. We have also derived a sufficient and a necessary condition for a Henig equilibrium pattern flow for multi-product network equilibrium models in terms of vector variational inequalities when the cost function is defined in a certain form. Given that vector variational inequalities have been proven useful for algorithm design, these results provide a viable approach to solving Henig efficiency of a multi-product network equilibrium problem.

In this paper we have established a sufficient and a necessary condition for a Henig equilibrium pattern flow for multi-product network equilibrium models in terms of vector variational inequalities. We have not been able to derive a condition that are both necessary and sufficient. It is worth noting that there exists no such result in the literature. That is, the question of a solution to what kind of vector variation inequalities is also a Henig equilibrium pattern flow for multi-product network equilibrium models is yet to be answered.

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