Finite Horizon H_∞ Fixed-Lag Smoothing for Time-Varying Continuous Systems

Huanshui Zhang and David Zhang

***Abstract*—In this paper, we aim to solve the long-standing H_∞ fixed-lag smoothing problem for time-varying continuous systems. By applying a novel innovation analysis approach in an indefinite linear space, a sufficient and necessary condition for the existence of an H_∞ fixed-lag smoother is derived. The H_∞ smoother is calculated by performing the linear matrix differential equation and the integral equation.**

***Index Terms*—Continuous-time systems, fixed-lag smoothing, H ∞, innovation approach, time varying.**

I. Introduction

THE PROBLEM of estimation, including filtering, prediction and smoothing, has been one of the key research topics in the control community since the seminal paper by Wiener [12]. The Kalman filtering [5], which addresses the minimization of the filtering error covariance (usually termed as H_2 estimation), emerged as a major tool of state estimation in the 1960s.

The estimation in the sense of H_∞ has become another important method since the 1980s [7]. An H_∞ estimator is such that the ratio between the energy of estimation error and the energy of input noise is bounded by a prescribed level γ^2; see [8] for the continuous-time case and [10] for the discrete-time one, and is applicable to situations where no information on statistics of input noises is available. In the past decades, most of the works in this area were focused on the H_∞ filtering. The objective of filtering is to estimate the system states via past observations.

The H_∞ smoothing problem, in which the use of future measurement information is available, is classified into three categories [5], namely fixed-point smoothing, fixed-interval smoothing and fixed-lag smoothing. The fixed-point smoothing and fixed-interval smoothing have been shown to be identical to the H_2 smoother [8]. Of the three different smoothing problems, the fixed-lag one is the most complicated and remains the least investigated. In the case of discrete-time system, the fixed-lag smoothing problems have been studied in [11], through system augmentation and filtering, and in [13] by applying the re-organized innovation approach in an indefinite linear space. [4] considered the infinite horizon smoother design for time invariant systems. Very recently, some attention has been received [2], [3] for the H_∞ fixed-lag smoothing for continuous systems and nice results are obtained, where the so-called Meinsma's trick [1] is used to reduce a problem delay-free problem. However, it should be pointed out that the results in [2], [3] are only applicable to the *infinite horizon* H_∞ smoother for *time invariant* systems. So far, the *finite horizon* H_∞ fixed-lag smoothing in the sense of *time-varying* systems has received little attention [10]. A general solution to the problem remains to be explored.

In this paper, our purpose is to study the solution to the H_∞ fixed-lag smoothing for time-varying continuous systems. First, by constructing an indefinite linear space (Krein space) [9], the H_∞ fixed-lag smoothing is converted into an H_2 filtering problem for a system with current and delayed measurements. Secondly, applying innovation analysis in Krein space, a necessary and sufficient condition of the existence for an H_∞ smoother is derived. The H_∞ fixed-lag smoother is given in terms of linear matrix differential equation and integral equation.

Manuscript received January 16, 2003; revised January 2, 2004. The work of the H. Zhang was supported by the National Nature Science Foundation of China under Grant 60174017. This paper was recommended by Associate Editor X. Yu.

H. Zhang was with the Department of Computing, The Hong Kong Polytechnic University, Hong Kong. He is now with Shenzhen Graduate School of the Harbin Institute of Technology, Shenzhen 518085, China (e-mail: h_s_zhang@hit.edu.cn).

D. Zhang is with the Department of Computing, The Hong Kong Polytechnic University, Hong Kong (e-mail: csdzhang@comp.polyu.edu.hk).

Digital Object Identifier 10.1109/TCSII.2004.832771

II. Problem Statement

We consider the following time-varying continuous system for an H_∞ estimation problem:

$$\dot{x}(t) = \Phi(t)x(t) + \Gamma(t)u(t) \tag{2.1}$$

$$y(t) = H(t)x(t) + v(t) \tag{2.2}$$

$$z(t) = L(t)x(t) \tag{2.3}$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$, $v(t) \in R^m$, and $z(t) \in R^p$ represent the state, input noise, measurement output, measurement noise, and the signal to be estimated, respectively. It is assumed that the input and measurement noises are deterministic signals and are from $\ell_2[0, T]$ where T is the time horizon of the estimation problem under investigation.

- Given a scalar $\gamma > 0$, an integer $h \geq 0$ and the observation $\{y(s),\ s \leq t\}$, find an estimate $\check{z}(t-h \mid t)$ of $z(t-h)$, if it exists, such that the inequality in (2.4) at the bottom of the next page is satisfied, where "$'$" stands for matrix transposition and Π_0 is a given positive definite matrix which reflects the relative uncertainty of the initial state to the input and measurement noises.

Remark II.1: When $h > 0$, the problem stated in the above is termed as **finite horizon H_∞ fixed-lag smoothing**. When $h = 0$, it is the well-known **finite horizon filtering** problem. The fixed-lag smoothing is a much more complicated problem than the filtering. The latter has been well studied in previous works [8]–[11]. However, the former remains the least investigated.

III. H_∞ Fixed-Lag Smoothing

In this section, the H_∞ smoothing is converted into an H_2 smoothing problem for the time-delayed systems in an indefinite linear space. To this end, we define

$$\begin{aligned}\mathcal{J}_T &\triangleq x'(0)\Pi_0 x(0) + \int_0^T u'(s)u(s)ds + \int_0^T v'(s)v(s)ds \\ &\quad - \gamma^{-2}\int_h^T v_z'(s)v_z(s)ds \\ &= x'(0)\Pi_0 x(0) + \int_0^T u'(s)u(s)ds \\ &\quad + \int_0^T \begin{bmatrix} v(s) \\ v_z(s)\end{bmatrix}' \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^{-2}I_p\end{bmatrix}\begin{bmatrix} v(s) \\ v_z(s)\end{bmatrix} ds \end{aligned} \tag{3.1}$$

where $v_z(s) = \check{z}(s-h \mid s) - L(s-h)x(s-h)$ or

$$\check{z}(s-h \mid s) = L(s-h)x(s-h) + v_z(s) \tag{3.2}$$

and $v_z(s) = 0$ for $s < h$.

It is obvious that the estimator $\check{z}(t-h \mid t)(h \le t \le T)$ that achieves (2.4) exists if and only if [9]: 1) $\mathcal{J}_T$ has a minimum $\mathcal{J}_T^{\min}$, with, respectively, $u(\cdot)$ and $x(0)$; 2) $\check{z}(\cdot \mid \cdot)$ can be chosen such that the value at the minimum is positive for all $y(\cdot)$.

A. Equivalent H_2 Fixed-Lag Smoothing in Krein Space

To derive the estimator, we introduce a Krein-space stochastic system

$$\dot{\mathbf{x}}(t) = \Phi(t)\mathbf{x}(t) + \Gamma(t)\mathbf{u}(t), \quad \mathbf{x}(0) \tag{3.3}$$

$$\mathbf{y}(t) = H(t)\mathbf{x}(t) + \mathbf{v}(t) \tag{3.4}$$

$$\check{\mathbf{z}}(t-h \mid t) = L(t-h)\mathbf{x}(t-h) + \mathbf{v}_z(t), \quad t \ge h \tag{3.5}$$

where the initial state $\mathbf{x}(0)$, $\mathbf{u}(t)$, $\mathbf{v}(t)$, and $\mathbf{v}_z(t)$ $(t \ge h)$[1], in bold faces, are mutually uncorrelated white noises with zero means and known covariance matrices Π_0, $Q_u = I_r$, $Q_v = I_m$ and $Q_{v_z} = -\gamma^2 I_p$, respectively. Since Q_{v_z} is indefinite, (3.5) is no longer a stochastic system in Hilbert space but a stochastic system in Krein space. Readers are referred to [9] for details of the Krein space.

[1]Whenever the Krein-space elements [9] and the Euclidean space elements satisfy the same set of constraints, we shall denote them by the same letters with the former identified by bold faces and the latter by normal faces.

In (3.5), $\check{\mathbf{z}}(t-h \mid t)$ is a "fictitious" observation at time t for the state $\mathbf{x}(t-h)$, while $\mathbf{y}(t)$ in (3.4) is an observation at time t for the state $\mathbf{x}(t)$. So, the above system is in fact, a measurement time-delay system. Denote

$$\mathbf{y}_z(t) \triangleq \begin{cases} \mathbf{y}(t), & 0 \le t < h \\ \begin{bmatrix} \mathbf{y}(t) \\ \check{\mathbf{z}}(t-h \mid t)\end{bmatrix}, & t \ge h. \end{cases} \tag{3.6}$$

$\mathbf{y}_z(t)$ is an observation at time t for different states $\mathbf{x}(t)$ and $\mathbf{x}(t-h)$. So, the measurements up to time t are collected as

$$\{\mathbf{y}_z(s),\ 0 \le s \le t\}. \tag{3.7}$$

Similar to the case in Hilbert space, the innovation of the observation $\mathbf{y}_z(s)$ is defined as

$$\mathbf{w}_z(s) \triangleq \mathbf{y}_z(s) - \hat{\mathbf{y}}_z(s \mid s) \tag{3.8}$$

where $\hat{y}_z(s \mid s)$, if exists, is given by the projection of $\mathbf{y}_z(s)$ onto $\mathcal{L}\{\mathbf{y}_z(r),\ r < s\}$. Obviously, the linear space $\mathcal{L}\{\mathbf{y}_z(s),\ s < t\}$ is equivalent to $\mathcal{L}\{\mathbf{w}_z(s),\ s < t\}$ [5]. From the definition (3.8), the innovation $\mathbf{w}_z(t)$ can be rewritten directly from (3.6) and (3.4)–(3.5) as shown in (3.9)–(3.10) at the bottom of the page, where $\hat{\mathbf{x}}(t \mid t)$ and $\hat{\mathbf{x}}(t-h \mid t)$ are, respectively, the projections of $\mathbf{x}(t)$ and $\mathbf{x}(t-h)$ onto $\mathcal{L}\{\mathbf{y}_z(s),\ s < t\}$. Immediately from (3.10) and recalling [5], the innovation covariance matrix $Q_{w_z}(t) \triangleq \langle \mathbf{w}_z(t), \mathbf{w}_z(t)\rangle$ is computed by

$$\begin{aligned} Q_{w_z}(t) &= \begin{cases} Q_v, & 0 \le t < h \\ \begin{bmatrix} Q_v & 0 \\ 0 & Q_{v_z}\end{bmatrix}, & t \ge h \end{cases} \\ &= \begin{cases} I_m, & 0 \le t < h \\ \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p\end{bmatrix}, & t \ge h. \end{cases} \end{aligned} \tag{3.11}$$

By applying the above innovation process, we have the following results.

Theorem III.1: Consider the system (2.1)–(2.3) and the associated performance criterion (2.4). Then, for a given scalar $\gamma > 0$, the following hold.

$$\sup_{(x_0,u,v)\neq 0} \frac{\int_h^T \left\{[\check{z}(t-h \mid t) - z(t-h)]'[\check{z}(t-h \mid t) - z(t-h)]\right\} dt}{x'(0)\Pi_0^{-1}x(0) + \int_0^T u'(t)u(t)dt + \int_0^T v'(t)v(t)dt} < \gamma^2 \tag{2.4}$$

$$\mathbf{w}_z(t) = \begin{cases} \mathbf{y}(t) - H(t)\hat{\mathbf{x}}(t \mid t), & 0 \le t < h \\ \begin{bmatrix} \mathbf{y}(t) \\ \check{\mathbf{z}}(t-h \mid t)\end{bmatrix} - \begin{bmatrix} H(t)\hat{\mathbf{x}}(t \mid t) \\ L(t-h)\hat{\mathbf{x}}(t-h \mid t)\end{bmatrix}, & t \ge h \end{cases} \tag{3.9}$$

$$= \begin{cases} H(t)\left[\mathbf{x}(t) - \hat{\mathbf{x}}(t \mid t)\right] + \mathbf{v}(t), & 0 \le t < h \\ \begin{bmatrix} H(t)\left\{\mathbf{x}(t) - \hat{\mathbf{x}}(t \mid t)\right\} \\ L(t-h)\left\{\mathbf{x}(t-h) - \hat{\mathbf{x}}(t-h \mid t)\right\}\end{bmatrix} + \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}_z(t)\end{bmatrix}, & t \ge h \end{cases} \tag{3.10}$$

- The estimator $\check{z}(t-h \mid t)(h \leq t \leq T)$ that achieves (2.4) exists if and only if
  - $\hat{x}(t \mid t)$ exists for $0 \leq t \leq T$
  - $\hat{x}(t-h \mid t)$ exists for $h \leq t \leq T$

  where $\hat{x}(t \mid t)$ and $\hat{x}(t-h \mid t)$ are, respectively, given by the projections of $\mathbf{x}(t)$ and $\mathbf{x}(t-h)$ onto $\{\mathbf{y}_z(s),\ s<t\}$.
- In this case, an H_∞ smoother (central estimator) $\check{z}(t-h \mid t)$ is given by

$$\check{z}(t-h \mid t) = L(t-h)\hat{x}(t-h \mid t). \tag{3.12}$$

Proof: In view of (3.1) and the stochastic system (3.3)–(3.5), recall the discussion in [9, ch. 16, pp. 499–529], $\mathcal{J}_T$ has a minimum $\mathcal{J}_T^{\min}$, with, respectively, $u(\cdot)$ and $x(0)$ if and only if the innovation $w_z(t)$ $(0 \leq t \leq T)$, defined by (3.8), exists. From (3.9), it is readily known that $w_z(t)$ $(0 \leq t \leq T)$ exists implies that both $\hat{x}(t \mid t)$ for $0 \leq t \leq T$ and $\hat{x}(t-h \mid t)$ for $h \leq t \leq T$ exist.

Furthermore, if the innovation $w_z(s)$ exists, the minimum of $\mathcal{J}_T$ with respect to $\{x(0);\ u(s), 0 \leq s \leq T\}$ is given by

$$\begin{aligned}\mathcal{J}_T^{\min} &= \int_0^T w_z'(t) Q_{w_z}^{-1}(t) w_z(t)dt \\ &= \int_0^h w_z'(t)w_z(t)dt + \int_h^T w_z'(t) \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^{-2} I_g \end{bmatrix} w_z(t)dt\end{aligned} \tag{3.13}$$

where $w_z(t)$ is as defined in (3.8). By using (3.9), it follows that

$$\begin{aligned}\mathcal{J}_T^{\min} = &\int_0^T [y(t) - H(t)\hat{x}(t \mid t)]' [y(t) - H(t)\hat{x}(t \mid t)]\, dt \\ &- \gamma^{-2} \int_h^T [\check{z}(t-h \mid t) - L(t-h)\hat{x}(t-h \mid t)]' \\ &\cdot [\check{z}(t-h \mid t) - L(t-h)\hat{x}(t-h \mid t)]\, dt\end{aligned} \tag{3.14}$$

Recall that the estimator $\check{z}(t-h \mid t)$ can be chosen such that $\mathcal{J}_T^{\min} > 0$, one natural choice is that

$$\check{z}(t-h \mid t) = L(t-h)\hat{x}(t-h \mid t) \tag{3.15}$$

for all $h \leq t \leq T$, where $\hat{x}(t-h \mid t)$ is given by the projection of $\mathbf{x}(t-h)$ onto $\mathcal{L}\{\mathbf{y}_z(s),\ s<t\}$.

Remark III.1: Now, the calculation of the H_∞ fixed-lag smoother $\check{z}(t-h \mid t)$ is converted to an H_2 fixed-lag smoothing problem for the indefinite system of (3.3)–(3.5).

B. Calculation of the H_∞ Fixed-Lag Smoother $\check{z}(t-h \mid t)$

In order to compute the H_∞ estimator $\check{z}(t-h \mid t)$, we need to check if both of $\hat{x}(t \mid t)$ $(0 \leq t \leq T)$ and $\hat{x}(t-h \mid t)$ $(h \leq t \leq T)$ exist, and compute $\hat{x}(t-h \mid t)$, if exist. To this end, denote

$$P(t,\theta_1,\theta_2) \triangleq \langle \mathbf{x}(t-\theta_1),\ \mathbf{x}(t-\theta_2) - \hat{\mathbf{x}}(t-\theta_2 \mid t)\rangle \tag{3.16}$$

where $\theta_1 \geq 0, \theta_2 \geq 0$ and $\hat{\mathbf{x}}(t-\theta_2 \mid t)$ is the projection of $\mathbf{x}(t-\theta_2)$ onto $\mathcal{L}\{\mathbf{y}_z(\tau),\ \tau<t\}$), which is calculated from the projection formula as

$$\hat{\mathbf{x}}(t-\theta_2 \mid t) = \int_0^t \langle \mathbf{x}(t-\theta_1), \mathbf{w}_z(\tau)\rangle Q_{w_z}^{-1}(\tau) \langle \mathbf{x}(t-\theta_2), \mathbf{w}_z(\tau)\rangle' d\tau \tag{3.17}$$

Now, we have the following results.

Theorem III.2: The matrix $P(t,\theta_1,\theta_2)$, defined as in (3.16), satisfies the following partial differential equation and boundary conditions

$$\frac{\partial P(t,\theta_1,\theta_2)}{\partial t} + \frac{\partial P(t,\theta_1,\theta_2)}{\partial \theta_1} + \frac{\partial P(t,\theta_1,\theta_2)}{\partial \theta_2} = -K(t,\theta_1,\theta_2) \tag{3.18}$$

$$\frac{\partial P(t,\theta_1,0)}{\partial t} + \frac{\partial P(t,\theta_1,0)}{\partial \theta_1} = P(t,\theta_1,0)\Phi'(t) - K(t,\theta_1,0) \tag{3.19}$$

$$\begin{aligned}\frac{\partial P(t,0,0)}{\partial t} &= \Phi(t)P(t,0,0) + P(t,0,0)\Phi'(t) \\ &\quad - K(t,0,0) + \Gamma(t)\Gamma'(t)\end{aligned} \tag{3.20}$$

where $P(t,\theta,0) = P'(t,0,\theta)$ and

$$\begin{aligned}K(t,\theta_1,\theta_2) &= P(t,\theta_1,0)H'(t)H(t)P'(t,\theta_2,0) \\ &\quad -\gamma^{-2}P(t,\theta_1,h)L'(t-h)L(t-h)P'(t,\theta_2,h)\end{aligned} \tag{3.21}$$

with $L(t-h) = 0$ for $t<h$. In addition, the initial value $P(0,0,0) = \langle \mathbf{x}(0), \mathbf{x}(0)\rangle = \Pi_0$.

Proof: Note (3.16) is rewritten as

$$\begin{aligned}&P(t,\theta_1,\theta_2) \\ &= \langle \mathbf{x}(t-\theta_1),\ \mathbf{x}(t-\theta_2)\rangle - \langle \mathbf{x}(t-\theta_1),\ \hat{\mathbf{x}}(t-\theta_2 \mid t)\rangle \\ &= \langle \mathbf{x}(t-\theta_1), \mathbf{x}(t-\theta_2)\rangle \\ &\quad - \int_0^t \langle \mathbf{x}(t-\theta_1), \mathbf{w}_z(\tau)\rangle Q_{w_z}^{-1}(\tau)\langle \mathbf{x}(t-\theta_2), \mathbf{w}_z(\tau)\rangle' d\tau\end{aligned} \tag{3.22}$$

where $\mathbf{w}_z(\tau)$ is as in (3.10) and $Q_{w_z}(\tau)$ is as in (3.11).

Differentiation with respect to t, θ_1 and θ_2 and addition of the results yield (3.18). Setting $\theta_2 = 0$ in (3.22), differentiation with respect to t, θ_1 and addition of the results yield (3.19). Finally, setting $\theta_1 = \theta_2 = 0$ in (3.22), differentiation with respect to t and addition of the results yield (3.20).

Remark III.2: The partial differential (3.18) is nonlinear. Its solution can be reduced to the solution of a number of simultaneous linear equations such as in the case with matrix Riccati equation [6]. The most practical approach seems to discretize the problem and solve the discrete-time problem, the solution of which is well-known [6].

Theorem III.3: Consider the system (2.1)–(2.3) and the associated performance criterion (2.4). Given scalar $\gamma > 0$ and $h \geq 0$.

Then, the fixed-lag smoother $\check{z}(t-h \mid t)$ that achieves (2.4) is solvable if and only if the matrix $P(t,0,0)$ and $P(s,s+h-t,0)$ are bounded for $t-h \leq s \leq t$ over $h \leq t \leq T$, where $P(t,0,0)$ and $P(s,s+h-t,0)$ are obtained by solving (3.18) with the boundary conditions (3.19) and (3.20). In this case, the smoother $\check{z}(t-h \mid t)$ is given by

$$\check{z}(t-h \mid t) = L(t-h)\hat{x}(t-h \mid t) \tag{3.23}$$

where $\hat{x}(t-h \mid t)$ is computed as

$$\begin{aligned}\hat{x}(t-h \mid t) &= \hat{x}_0(t-h \mid t) \\ &\quad + \int_{t-h}^t P(s,s+h-t,0)H'(s)\,[y(s) - H(s)\hat{x}(s \mid s)]\, ds\end{aligned} \tag{3.24}$$

and $\hat{x}_0(t-h \mid t)$ is given by

$$\begin{aligned}\dot{\hat{x}}_0(t-h \mid t) = &\Phi(t-h)\hat{x}_0(t-h \mid t)\\ &+ P(t-h,0,0)H'(t-h)\\ &\cdot [y(t-h) - H(t-h)\hat{x}(t-h \mid t-h)],\\ &\hat{x}_0(0 \mid h) = 0. \end{aligned} \quad (3.25)$$

The estimator $\hat{x}(s \mid s)$ in (3.24) and $\hat{x}(t-h \mid t-h)$ in (3.25) is calculated by

$$\begin{aligned}\dot{\hat{x}}(t \mid t) = &\{\Phi(t) - P(t,0,0)H'(t)H(t)\}\,\hat{x}(t \mid t)\\ &+ P(t,0,0)H'(t)y(t). \end{aligned} \quad (3.26)$$

Proof: Since $\hat{x}(t \mid t)$ is given by the projection of $\mathbf{x}(t)$ onto $\mathcal{L}\{\mathbf{y}_z(s),\ \ s<t\}$ or equivalently onto $\mathcal{L}\{\mathbf{w}_z(s),\ \ s<t\}$, by using the projection formula,it follows that

$$\hat{x}(t \mid t) = \int_0^t \langle \mathbf{x}(t), \mathbf{w}_z(s)\rangle Q_{w_z}^{-1}(s) w_z(s) ds \quad (3.27)$$

where $Q_{w_z}(s)$ is as in (3.11). By differentiating both sides of the above equation with respect to t yields

$$\begin{aligned}\dot{\hat{x}}(t \mid t) = &\Phi(t)\hat{x}(t \mid t) + \Gamma(t)\int_0^t \langle \mathbf{u}(t), \mathbf{w}_z(s)\rangle Q_{w_z}^{-1}(s) w_z(s) ds\\ &+ \langle \mathbf{x}(t), \mathbf{w}_z(t)\rangle Q_{w_z}^{-1}(t) w_z(t). \end{aligned} \quad (3.28)$$

Note $\mathbf{u}(t)$ is uncorrelated with $\mathbf{w}_z(s)$ for $s<t$, we have

$$\dot{\hat{x}}(t \mid t) = \Phi(t)\hat{x}(t \mid t) + \langle \mathbf{x}(t), \mathbf{w}_z(t)\rangle Q_{w_z}^{-1}(t) w_z(t). \quad (3.29)$$

On the other hand, from (3.10), we obtained

$$\begin{aligned}&\langle \mathbf{x}(t), \mathbf{w}_z(t)\rangle\\ &= \begin{cases} \langle \mathbf{x}(t),\ \mathbf{x}(t) - \hat{\mathbf{x}}(t \mid t)\rangle\, H'(t), & 0 \le t < h\\ [\langle \mathbf{x}(t),\ \mathbf{x}(t) - \hat{\mathbf{x}}(t \mid t)\rangle\, H'(t) \quad *], & t \ge h\end{cases}\\ &= \begin{cases} P(t,0,0)H'(t), & 0 \le t < h\\ [P(t,0,0)H'(t) \quad *], & t \ge h\end{cases} \end{aligned} \quad (3.30)$$

where

$$* = \langle \mathbf{x}(t),\ \mathbf{x}(t-h) - \hat{\mathbf{x}}(t-h \mid t)\rangle\, L'(t-h)$$

and $P(t,0,0)$ is as in (3.16). By applying $\check{z}(t-h \mid t) = L(t-h)\hat{x}(t-h \mid t)$, the innovation $w_z(t)$ is given from (3.9) as

$$w_z(t) = \begin{cases} y(t) - H(t)\hat{x}(t \mid t), & 0 \le t < h\\ \begin{bmatrix} y(t) - H(t)\hat{x}(t \mid t)\\ 0\end{bmatrix}, & t \ge h.\end{cases} \quad (3.31)$$

Thus, (3.26) is readily obtained from (3.29) by using (3.30) and (3.31). Similarly, we can show that $\hat{x}(t-h \mid t)$ can be computed by (3.24) and (3.25). For the limit of space, the proof is omitted.

From Theorem 3.1 and (3.24)–(3.26), it is readily known that the fixed-lag smoother $\check{z}(t-h \mid t)$ that achieves (2.4) is solvable if and only if that the matrix $P(t,0,0)$ and $P(s,s+h-t,0)$ are bounded for $t-h \le s \le t$ over $h \le t \le T$. Thus we complete the proof of the theorem.

The matrices $P(t,0,0)$ and $P(s,s+h-t,0)$ $(s>t-h)$, which can be obtained by performing the partial differential equation (3.18) with the boundary (3.19) and (3.20), play an important role for computing the smoother H_∞. Usually, as shown in [6], the partial differential equation (3.18) was solved by converting it into a discrete-time problem or solving a equivalent problem that is with a number of simultaneous linear equations.

IV. Conclusion

Although the solution to H_∞ filtering has been well known, the H_∞ fixed-lag smoothing of the time-varying continuous-time system remains the least investigated [10]. In this paper, we have addressed the problem by applying an innovation approach. It has been shown that the H_∞ fixed-lag smoothing is equivalent to an H_2 fixed-lag smoothing for the system with current and delayed measurements in an indefinite linear space. A necessary and sufficient condition for the existence of the estimator has been derived. It is equivalent to that of two matrices, which can be obtained by performing one partial differential equation with boundary conditions, are bounded. We would point out that the complicated fixed-lag smoothing for time-varying continuous system has been converted into a Riccati type partial differential equation which maybe not easy to be given an analytical solution. However, it can be solved by numerical methods such as the finite element method.

References

[1] G. Meinsma, L. Mirkin, and Q.-C. Zhong, "H_∞ control of systems with a single delay via reduction to a one-block problem," in *Proc. IEEE Conf. Decision Control*, vol. 3, Dec. 2002, pp. 3458–3463.

[2] L. Mirkin and G. Meinsma, "When does the H_∞ fixed-lag smoothing performance saturates?," in *Proc. IFAC World Congr.*, Barcelona, Spain, 2002.

[3] L. Mirkin, "Continuous-time fixed-lag smoothing in an H_∞ setting," in *Proc. IEEE Conf. Decision Control*, vol. 4, Dec. 2001, pp. 3512–3517.

[4] P. Colaneri and A. Ferrante, "A J-spectral factorization approach for H_∞ estimation problem in discrete time," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 2108–2113, Dec. 2002.

[5] T. Kailath, A. H. Sayed, and B. Hassibi, *Linear Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1999.

[6] H. Kwakernaak, "Optimal filtering in linear systems with time delays," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 169–173, Apr. 1967.

[7] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 301–320, Apr. 1981.

[8] K. M. Nagpal and P. P. Khargonekar, "Filtering and smoothing in an H_∞ setting," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 152–166, Feb. 1991.

[9] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite Quadratic Estimation and Control: A Unified Approach to H_2 and H_∞ Theories*. Philadelphia, PA: SIAM, 1998.

[10] U. Shaked and Y. Theodor, "H_∞ optimal estimation. A tutorial," in *Proc. IEEE Conf. Decision Control*, vol. 2, Dec. 1992, pp. 2279–2286.

[11] Y. Theodor and U. Shaked, "Game theory approach to H_∞ optimal discrete-time fixed-point and fixed-lag smoothing," *IEEE Trans. Automat. Control*, vol. 39, pp. 1944–1948, Sept. 1994.

[12] N. Wiener, *Extrapolation Interpolation, and Smoothing of Stationary Time Series*. New York: Wiley, 1950.

[13] H. Zhang, L. Xie, and Y. C. Soh, "A unified approach to linear estimation for discrete-time systems-Part II: H_∞ estimation," in *Proc. IEEE Conf. Decision Control*, vol. 3, Dec. 2001, pp. 2917–2922.