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ORIGINAL PAPER

Real interpolation of variable martingale Hardy spaces and BMO spaces

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Abstract

In this paper, we mainly consider the real interpolation spaces for variable Lebesgue spaces defined by the decreasing rearrangement function and for the corresponding martingale Hardy spaces. Let $0 < q \leq \infty$ and $0 < \theta < 1$. Our three main results are the following:

$$(\mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} = \mathcal{L}_{p(\cdot)/(1-\theta),q}(\mathbb{R}^n),$$
$$(\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q} = \mathcal{H}^s_{p(\cdot)/(1-\theta),q}(\Omega)$$

and

$$(\mathcal{H}^s_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q} = \mathcal{H}^s_{p(\cdot)/(1-\theta),q}(\Omega),$$

where the variable exponent $p(\cdot)$ is a measurable function.

Keywords Variable exponent Lebesgue spaces · Martingale Hardy spaces · BMO spaces · Real interpolation

Mathematics Subject Classification 60G42 · 60G46

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1 Introduction

In the past three decades, as a generalization of the classical Lebesgue spaces $L_p(\mathbb{R}^n)$, the variable exponent Lebesgue spaces $L_{p(\cdot)}(\mathbb{R}^n)$ have attracted much attention. Here, we refer the interested readers to the monographs [4, 5] for more information. Kempka and Vybíral investigated the Lorentz spaces $L_{p(\cdot),q}(\mathbb{R}^n)$ and $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with variable exponents in [19]. Recently, a new kind of variable Lebesgue spaces and variable Lorentz spaces, which are defined by rearrangement functions, came into sight: Kokilashvili et al. [20] introduced the variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$; Ephremidze et al. [7] studied the variable Lorentz spaces $\mathfrak{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The classical martingale theory was systematically studied by Garsia [9], Long [21], Weisz [27] and many others. With the development of variable Lebesgue spaces in harmonic analysis, variable martingale spaces have gained a steadily increasing interest. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $p(\cdot)$ be a measurable function on Ω. Similar to $L_{p(\cdot)}(\mathbb{R}^n)$ and $L_{p(\cdot),q}(\mathbb{R}^n)$, we can define $L_{p(\cdot)}(\Omega)$ and $L_{p(\cdot),q}(\Omega)$ (see the definitions of these spaces in Sect. 2.1). Aoyama [1] proved the boundedness of the Doob maximal operator on $L_{p(\cdot)}(\Omega)$ as $p(\cdot)$ meets certain condition, which was pointed out to be quite strong by Nakai and Sadasue [22]. Jiao et al. [17] introduced variable martingale Hardy spaces associated with $L_{p(\cdot)}(\Omega)$. Very recently, Jiao et al. provided a relatively complete research on variable Hardy–Lorentz spaces relative to $L_{p(\cdot),q}(\Omega)$ in [14]. Now let $p(\cdot)$ be a variable exponent on [0, 1]. With the emergence of variable Lebesgue spaces defined by the rearrangement function, Jiao et al. [16] introduced new variable martingale Hardy spaces associated with $\mathcal{L}_{p(\cdot)}(\Omega)$ and Zeng [28] investigated variable martingale Hardy–Lorentz spaces relative to $\mathcal{L}_{p(\cdot),q}(\Omega)$ (for the definitions of these spaces, see Sects. 2.2 and 2.3). The readers may consult the articles [11, 15] for more results about variable martingale spaces.

The real interpolation of variable Lebesgue spaces defined on $\mathbb{R}^n$ has been studied in several papers. In [19], the authors proved that the Lorentz spaces $L_{p(\cdot),q}(\mathbb{R}^n)$ serve as the intermediate spaces between $L_{p(\cdot)}(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ via the real interpolation. To be precise,

$$(L_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} = L_{p(\cdot)/(1-\theta),q}(\mathbb{R}^n), \quad 0 < \theta < 1, 0 < q \leq \infty.$$

Very recently, the real interpolation between variable Hardy spaces $H_{p(\cdot)}(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ was investigated in [18, 29], where the authors obtained that, if $p(\cdot)$ satisfies the log-Hölder continuous condition, then

$$(H_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} = H_{p(\cdot)/(1-\theta),q}(\mathbb{R}^n), \quad 0 < \theta < 1, 0 < q \leq \infty.$$

This extends Fefferman's corresponding interpolation results on the classical $H_p(\mathbb{R}^n)$ spaces in [8]. The real interpolation for martingale Hardy spaces is also fruitful. Weisz [27] firstly studied the real interpolation spaces of martingale Hardy spaces. Indeed, he proved that

$$(H_p^s(\Omega), H_\infty^s(\Omega))_{\theta,q} = H^s_{p/(1-\theta),q}(\Omega), \quad 0 < \theta < 1, 0 < q \leq \infty.$$

Recently, Jiao et al. [13] extended Weisz's result above to variable martingale setting, that is, they achieved that

$$(H^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q} = H^s_{p(\cdot)/(1-\theta),q}(\Omega), \quad 0 < \theta < 1, 0 < q \leq \infty.$$

With the help of a new sharp maximal function and a new BMO space, Weisz identified the real interpolation spaces between martingale Hardy and BMO spaces in [26]. For more results about the real interpolation of martingale Hardy spaces, we refer to [12, 23–25].

Based on the works of Jiao et al. [16] and Zeng [28], our main purpose of this article is to establish the real interpolation spaces between variable martingale Hardy spaces $\mathcal{H}^s_{p(\cdot)}(\Omega)$ defined in [16] and martingale BMO spaces $BMO_2(\Omega)$. To this end, we firstly identify the interpolation spaces between variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ spaces as variable Lorentz $\mathcal{L}_{p(\cdot),q}(\mathbb{R}^n)$ spaces, and then via formulating the real interpolation spaces between variable martingale Hardy spaces and applying the sharp maximal functions of martingales, we further prove that the real interpolation spaces between variable martingale Hardy and BMO spaces are just the variable martingale Hardy–Lorentz spaces. More precisely, we obtain the following results:

$$\begin{aligned}(\mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} &= \mathcal{L}_{p(\cdot)/(1-\theta),q}(\mathbb{R}^n), \quad 0 < \theta < 1, 0 < q \leq \infty,\\ (\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q} &= \mathcal{H}^s_{p(\cdot)/(1-\theta),q}(\Omega), \quad 0 < \theta < 1, 0 < q \leq \infty\end{aligned}$$

and

$$(\mathcal{H}^s_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q} = \mathcal{H}^s_{p(\cdot)/(1-\theta),q}(\Omega), \quad 0 < \theta < 1, 0 < q \leq \infty;$$

see the definitions in Sect. 2.

This paper is organized as follows. In Sect. 2, we present the necessary background and some basic facts that will be used later. Section 3 is devoted to establishing the real interpolation between $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$ spaces and $L_\infty(\mathbb{R}^n)$ spaces. In Sect. 4, we aim at identifying the real interpolation spaces between variable martingale Hardy spaces. Finally, we formulate the real interpolation between variable martingale Hardy spaces and martingale BMO spaces in Sect. 5.

Now, let us make some conventions to end this section. In the whole article, we use $\mathbb{Z}$ and $\mathbb{N}$ to denote the integer set and nonnegative integer set, respectively. We denote by C an absolute positive constant that is independent of the main parameters but whose value may differ from line to line, and denote by $C_{p(\cdot)}$ the constant depending only on $p(\cdot)$. The symbol $a \lesssim b$ stands for the inequality $a \leq Cb$ or $a \leq C_{p(\cdot)}b$. If we write $a \approx b$, it means that $a \lesssim b \lesssim a$. The characteristic function of a measurable set A is written as χ_A.

2 Preliminaries

In this section, we mainly provide some preparations for the follow-up work. We divide this section into four subsections. Throughout the paper, we always assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. Let (R, μ) be a complete measure space. In this section, (R, μ) could be $(\Omega, \mathcal{F}, \mathbb{P})$ or the Euclidean space $(\mathbb{R}^n, m)$ $(n \geq 1)$, where m denotes the Lebesgue measure. In Sect. 2.1, we give the definitions of variable Lebesgue spaces $L_{p(\cdot)}(R)$ and variable Lorentz spaces $L_{p(\cdot),q}(R)$. In Sect. 2.2, we introduce the variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(R)$, the variable Lorentz spaces $\mathcal{L}_{p(\cdot),q}(R)$ and $\mathfrak{L}_{p(\cdot),q}(R)$, which are defined by rearrangement function, and give some useful properties about these spaces. To our surprise, the variable Lorentz space $\mathcal{L}_{p(\cdot),q}(\Omega)$ is equivalent to the classical Lorentz space $L_{p(0),q}(\Omega)$ if $p(\cdot)$ satisfies the locally log-Hölder condition. In Sect. 2.3, we introduce the variable martingale Hardy spaces and variable martingale Hardy–Lorentz spaces associated with variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(\Omega)$ and variable Lorentz spaces $\mathcal{L}_{p(\cdot),q}(\Omega)$, respectively. Finally, we recall some basic notations and results about real interpolation in Sect. 2.4.

2.1 Variable Lebesgue spaces $L_{p(\cdot)}(R)$ and variable Lorentz spaces $L_{p(\cdot),q}(R)$

The so-called variable exponent function (or simply variable exponent) on R is a measurable function $p(\cdot) : R \to (0, \infty)$. Let $\mathcal{P}(R)$ be the collection of all variable exponents on R. For a variable exponent $p(\cdot) \in \mathcal{P}(R)$ and a set $E \subset R$, we denote

$$p_-(E) := \operatorname*{ess\,inf}_{x \in E} p(x), \quad p_+(E) := \operatorname*{ess\,sup}_{x \in E} p(x),$$

and

$$p_- := p_-(R), \quad p_+ := p_+(R), \quad \underline{p} := \min\{1, p_-\}.$$

For simplicity, we adopt the following notation in the sequel:

$$\mathcal{B}(R) = \{p(\cdot) \in \mathcal{P}(R) : 0 < p_- \leq p_+ < \infty\}.$$

Moreover, we recall the definition of locally log-Hölder continuous condition for the variable exponent $p(\cdot)$ defined on $\mathbb{R}^n$.

Definition 2.1 ([4, *Definition 2.2*]) Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we say that $p(\cdot)$ is locally log-Hölder continuous if there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$, $|x-y| < \frac{1}{2}$,

$$\big|p(x) - p(y)\big| \leq \frac{C}{-\ln|x-y|}. \tag{2.1}$$

Remark 2.2 Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. It was proved in [4, Proposition 2.3] that $p(\cdot)$ satisfies the locally log-Hölder continuous condition is equivalent to $\frac{1}{p(\cdot)}$ satisfies the locally log-Hölder continuous condition.

Now the definitions of variable Lebesgue spaces $L_{p(\cdot)}(R)$ and variable Lorentz spaces $L_{p(\cdot),q}(R)$ are given as follows.

Definition 2.3 Given $p(\cdot) \in \mathcal{B}(R)$, the variable Lebesgue space $L_{p(\cdot)}(R)$ is defined to be the collection of all measurable functions f on R such that $\|f\|_{L_{p(\cdot)}(R)} < \infty$, where

$$\|f\|_{L_{p(\cdot)}(R)} = \inf \left\{ \lambda > 0 : \rho(f/\lambda) \triangleq \int_R \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}\mu \leq 1 \right\}.$$

Definition 2.4 Let $p(\cdot) \in \mathcal{B}(R)$ and $0 < q \leq \infty$. Then the variable Lorentz space $L_{p(\cdot),q}(R)$ is defined as the set of all measurable functions f on R such that $\|f\|_{L_{p(\cdot),q}(R)} < \infty$, where

$$\|f\|_{L_{p(\cdot),q}(R)} = \begin{cases} \left(\int_0^\infty \lambda^q \|\chi_{\{x \in R : |f(x)| > \lambda\}}\|^q_{L_{p(\cdot)}(R)} \frac{d\lambda}{\lambda} \right)^{1/q}, & \text{if } \ 0 < q < \infty; \\ \sup_{\lambda > 0} \lambda \|\chi_{\{x \in R : |f(x)| > \lambda\}}\|_{L_{p(\cdot)}(R)}, & \text{if } \ q = \infty. \end{cases}$$

If $p(\cdot) \equiv p$ $(0 < p < \infty)$, then the variable Lebesgue space $L_{p(\cdot)}(R)$ and variable Lorentz space $L_{p(\cdot),q}(R)$ reduce to the classical Lebesgue space $L_p(R)$ and Lorentz space $L_{p,q}(R)$, respectively.

2.2 Variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(R)$ and variable Lorentz spaces $\mathcal{L}_{p(\cdot),q}(R)$, $\mathfrak{L}_{p(\cdot),q}(R)$

Let f be a measurable function defined on (R, μ). The distribution function of f is given by

$$d_f(s) := \mu\{x \in R : |f(x)| > s\}, \quad s \in [0, \infty).$$

While the decreasing rearrangement function of f is defined as

$$f^*(t) := \inf \{s \geq 0 : d_f(s) \leq t\}, \quad t \in [0, \infty).$$

In addition, as the maximal function of f^*, the function f^{**} is defined by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \mathrm{d}s, \quad t > 0.$$

It is well known that f^{**} is non-negative, non-increasing and continuous on $(0, \infty)$. Moreover,

$$f^*(t) \leq f^{**}(t) \quad \text{and} \quad \left(\sum_{k \in \mathbb{Z}} f_k \right)^{**}(t) \leq \sum_{k \in \mathbb{Z}} f_k^{**}(t), \ t > 0. \tag{2.2}$$

We refer to [2] for these properties.

Next, we use the rearrangement function to define respectively the variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(R)$ and variable Lorentz spaces $\mathcal{L}_{p(\cdot),q}(R)$.

Definition 2.5 Let $p(\cdot) \in \mathcal{B}([0,\infty))$. We define the variable Lebesgue space $\mathcal{L}_{p(\cdot)}(R)$ as the collection of all measurable functions f on R such that

$$\varrho_{p(\cdot)}(f) := \int\limits_0^\infty f^*(t)^{p(t)} \mathrm{d}t < \infty.$$

This is a quasi-Banach space with respect to the quasi-norm

$$\|f\|_{\mathcal{L}_{p(\cdot)}(R)} = \inf\{\lambda > 0 : \varrho(f/\lambda) \le 1\}.$$

If $R = \Omega$, then $f^*(t) = 0$ for $t \ge 1$, so it is enough to suppose that $p(\cdot) \in \mathcal{B}([0,1])$ in this case. It is clear that $\|f\|_{\mathcal{L}_{p(\cdot)}(R)} = \|f^*\|_{L_{p(\cdot)}([0,\infty))}$.

Definition 2.6 Let $p(\cdot) \in \mathcal{B}([0,\infty))$ and $0 < q \le \infty$. Then the variable Lorentz space $\mathcal{L}_{p(\cdot),q}(R)$ is defined as the set of all measurable functions f on R such that $\|f\|_{\mathcal{L}_{p(\cdot),q}(R)} < \infty$, where

$$\|f\|_{\mathcal{L}_{p(\cdot),q}(R)} = \begin{cases} \left(\int\limits_0^\infty \lambda^q \|\chi_{\{x\in R:|f(x)|>\lambda\}}\|^q_{\mathcal{L}_{p(\cdot)}(R)} \frac{d\lambda}{\lambda}\right)^{1/q}, & \text{if } \ 0<q<\infty; \\ \sup_{\lambda>0} \lambda \|\chi_{\{x\in R:|f(x)|>\lambda\}}\|_{\mathcal{L}_{p(\cdot)}(R)}, & \text{if } \ q=\infty. \end{cases}$$

Obviously, when $p(\cdot) \equiv p$ $(0 < p < \infty)$, the variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}(R)$ and variable Lorentz spaces $\mathcal{L}_{p(\cdot),q}(R)$ respectively go back to the classical Lebesgue spaces $L_p(R)$ and Lorentz spaces $L_{p,q}(R)$.

The following two useful lemmas can be founded in [16] and [28], respectively.

Lemma 2.7 *Given $p(\cdot) \in \mathcal{B}([0,\infty))$ and $f \in \mathcal{L}_{p(\cdot)}(R)$, we have*

$$\left\| |f|^t \right\|_{\mathcal{L}_{p(\cdot)}(R)} = \|f\|^t_{\mathcal{L}_{tp(\cdot)}(R)}, \quad t > 0. \tag{2.3}$$

Lemma 2.8 *Let $p(\cdot) \in \mathcal{B}([0,\infty))$ and $0 < q \le \infty$. If $f \in \mathcal{L}_{p(\cdot),q}(R)$, then*

$$\|f\|_{\mathcal{L}_{p(\cdot),q}(R)} \approx \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{x\in R:|f(x)|>2^k\}}\|^q_{\mathcal{L}_{p(\cdot)}(R)}\right)^{\frac{1}{q}}, & \text{if } \ 0<q<\infty; \\ \sup_{k\in\mathbb{Z}} 2^k \|\chi_{\{x\in R:|f(x)|>2^k\}}\|_{\mathcal{L}_{p(\cdot)}(R)}, & \text{if } \ q=\infty. \end{cases} \tag{2.4}$$

The following result tells us that $\|\cdot\|_{\mathcal{L}_{p(\cdot)}(R)}$ is a b-norm $(0 < b < \underline{p})$ in some sense if $p(\cdot) \in \mathcal{B}([0,\infty))$ is locally log-Hölder continuous.

Proposition 2.9 *If $p(\cdot) \in \mathcal{B}([0,\infty))$ satisfies the locally log-Hölder continuous condition, $f_k \in \mathcal{L}_{p(\cdot)}(R)$ $(k \in \mathbb{Z})$ and $0 < b < \underline{p}$, then*

$$\left\| \sum_{k\in\mathbb{Z}} f_k \right\|_{\mathcal{L}_{p(\cdot)}(R)}^b \lesssim \sum_{k\in\mathbb{Z}} \| f_k \|_{\mathcal{L}_{p(\cdot)}(R)}^b.$$

Proof According to (2.3) and (2.2), we find that

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} f_k \right\|_{\mathcal{L}_{p(\cdot)}(R)}^b &\leq \left\| \sum_{k\in\mathbb{Z}} |f_k| \right\|_{\mathcal{L}_{p(\cdot)}(R)}^b \\
&= \left\| \left(\sum_{k\in\mathbb{Z}} |f_k| \right)^b \right\|_{\mathcal{L}_{p(\cdot)/b}(R)} \leq \left\| \sum_{k\in\mathbb{Z}} |f_k|^b \right\|_{\mathcal{L}_{p(\cdot)/b}(R)} \\
&= \left\| \left(\sum_{k\in\mathbb{Z}} |f_k|^b \right)^* \right\|_{L_{p(\cdot)/b}([0,\infty))} \leq \left\| \sum_{k\in\mathbb{Z}} \left(|f_k|^b\right)^{**} \right\|_{L_{p(\cdot)/b}([0,\infty))}.
\end{aligned}$$

Since $p(\cdot)/b > 1$, $\|\cdot\|_{L_{p(\cdot)/b}([0,\infty))}$ is a norm. Moreover, it follows from [6, Theorem 3.1] that

$$\left\| \left(|f_k|^b\right)^{**} \right\|_{L_{p(\cdot)/b}([0,\infty))} \lesssim \left\| \left(|f_k|^b\right)^{*} \right\|_{L_{p(\cdot)/b}([0,\infty))}.$$

Again, by (2.3), we deduce that

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} f_k \right\|_{\mathcal{L}_{p(\cdot)}(R)}^b &\leq \sum_{k\in\mathbb{Z}} \left\| \left(|f_k|^b\right)^{**} \right\|_{L_{p(\cdot)/b}([0,\infty))} \\
&\lesssim \sum_{k\in\mathbb{Z}} \left\| \left(|f_k|^b\right)^{*} \right\|_{L_{p(\cdot)/b}([0,\infty))} \\
&= \sum_{k\in\mathbb{Z}} \left\| |f_k|^b \right\|_{\mathcal{L}_{p(\cdot)/b}(R)} = \sum_{k\in\mathbb{Z}} \| f_k \|_{\mathcal{L}_{p(\cdot)}(R)}^b.
\end{aligned}$$

The proof is complete. □

Inspired by Ephremidze et al. [7], Zeng [28] gave another way to define the variable Lorentz space $\mathfrak{L}_{p(\cdot),q}(\Omega)$ as follows. Furthermore, the author in [28] showed that $\mathcal{L}_{p(\cdot),q}(\Omega) = \mathfrak{L}_{p(\cdot),q}(\Omega)$ with equivalent quasi-norms if $p(\cdot)$ satisfies the locally log-Hölder continuous condition.

Definition 2.10 Let $p(\cdot) \in \mathcal{B}([0,1])$ and $0 < q \leq \infty$. Then we define the variable Lorentz space $\mathfrak{L}_{p(\cdot),q}(\Omega)$ as the space of all measurable functions f on Ω such that $\|f\|_{\mathfrak{L}_{p(\cdot),q}(\Omega)} < \infty$, where

$$\|f\|_{\mathfrak{L}_{p(\cdot),q}(\Omega)} = \begin{cases} \left(\int\limits_0^1 t^{\frac{q}{p(t)}} f^*(t)^q \frac{dt}{t}\right)^{1/q} = \left\|t^{\frac{1}{p(t)}} f^*(t)\right\|_{L_q([0,1],\frac{dt}{t})}, & \text{if} \quad 0 < q < \infty; \\ \sup_{0 \leq t \leq 1} t^{\frac{1}{p(t)}} f^*(t), & \text{if} \quad q = \infty. \end{cases}$$

Lemma 2.11 [28, Proposition 2.15] *Let* $p(\cdot) \in \mathcal{B}([0,1])$ *satisfy the locally log-Hölder continuous condition and let* $0 < q \leq \infty$. *If* $f \in \mathcal{L}_{p(\cdot),q}(\Omega)$, *then*

$$\|f\|_{\mathcal{L}_{p(\cdot),q}(\Omega)} \approx \|f\|_{\mathfrak{L}_{p(\cdot),q}(\Omega)},$$

that is, $\mathcal{L}_{p(\cdot),q}(\Omega) = \mathfrak{L}_{p(\cdot),q}(\Omega)$ *with equivalent quasi-norms.*

Based on this, we further prove that $\mathcal{L}_{p(\cdot),q}(\Omega) = L_{p(0),q}(\Omega)$ with equivalent quasi-norms under the same condition as in the lemma above.

Proposition 2.12 *Let* $p(\cdot) \in \mathcal{B}([0,1])$ *satisfy the locally log-Hölder continuous condition and* $0 < q \leq \infty$. *If* $f \in \mathcal{L}_{p(\cdot),q}(\Omega)$, *then*

$$\|f\|_{\mathcal{L}_{p(\cdot),q}(\Omega)} \approx \|f\|_{L_{p(0),q}(\Omega)},$$

namely, $\mathcal{L}_{p(\cdot),q}(\Omega) = L_{p(0),q}(\Omega)$ *with equivalent quasi-norms.*

Proof From Lemma 2.11, it suffices to show that $\|f\|_{\mathfrak{L}_{p(\cdot),q}(\Omega)} \approx \|f\|_{L_{p(0),q}(\Omega)}$. Since $p(\cdot) \in \mathcal{B}([0,1])$ satisfies the locally log-Hölder continuous condition, we assert that

$$t^{\frac{1}{p(t)}} \approx t^{\frac{1}{p(0)}}, \qquad t \in [0,1]. \tag{2.5}$$

Indeed, if $t \in [1/2, 1]$, then

$$2^{-\frac{1}{p_-}} \leq \left(\frac{1}{2}\right)^{\frac{1}{p(t)}} \leq t^{\frac{1}{p(t)}} \leq t^{\frac{1}{p(t)} - \frac{1}{p(0)}} \leq t^{-\frac{1}{p(0)}} \leq 2^{\frac{1}{p(0)}},$$

which means that (2.5) holds for $t \in [1/2, 1]$.

By (2.1), it is easy to see that

$$\left|p(t) - p(0)\right| \leq \frac{C}{-\ln t}, \qquad t \in [0, 1/2). \tag{2.6}$$

Now let us consider the functions

$$g(t) = t^{\frac{1}{p(t)} - \frac{1}{p(0)}}, \quad t \in [0, 1/2)$$

and

$$h(t) = \left(\frac{1}{p(t)} - \frac{1}{p(0)}\right) \ln t, \quad t \in [0, 1/2).$$

Then $g(t) = \exp h(t),\ t \in [0, 1/2)$. From Remark 2.2 and (2.6), it is clear that $h(t)$ is bounded on $[0, 1/2)$. Hence, there exist two positive constants C and C' such that $C \leq g(t) \leq C'$, $t \in [0, 1/2)$. Thus, (2.5) holds. Consequently,

$$\begin{aligned}\|f\|_{\mathfrak{L}_{p(\cdot),q}(\Omega)} &= \left\| t^{\frac{1}{p(t)}} f^*(t) \right\|_{L_q\left([0,1], \frac{\mathrm{d}t}{t}\right)} \\ &\approx \left\| t^{\frac{1}{p(0)}} f^*(t) \right\|_{L_q\left([0,1], \frac{\mathrm{d}t}{t}\right)} = \|f\|_{L_{p(0),q}(\Omega)}\end{aligned}$$

for $0 < q < \infty$ and

$$\|f\|_{\mathfrak{L}_{p(\cdot),\infty}(\Omega)} = \sup_{0\leq t\leq 1} t^{\frac{1}{p(t)}} f^*(t) \approx \sup_{0\leq t\leq 1} t^{\frac{1}{p(0)}} f^*(t) = \|f\|_{L_{p(0),\infty}(\Omega)}.$$

Now the proof is complete. □

2.3 Variable martingale Hardy spaces

In this subsection, we give some basic notions and notations for martingales. We refer the interested readers to monographs [9, 21, 27] for further study. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_n)_{n\geq 0}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma\big(\bigcup_{n\geq 0} \mathcal{F}_n\big)$. The expectation operator and the conditional expectation operator relative to $\mathcal{F}_n$ are written as $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f = (f_n)_{n\geq 0}$ of adapted and integrable functions is said to be a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$ if $\mathbb{E}_n(f_{n+1}) = f_n$ for all $n \geq 0$. For a martingale $f = (f_n)_{n\geq 0}$, the martingale differences are given by $d_n f = f_n - f_{n-1}$ (with the convention $d_0 f = 0$ and $f_{-1} = 0$). Let $\mathcal{T}$ be the set of all stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. For a martingale $f = (f_n)_{n\geq 0}$ and a stopping time $\tau \in \mathcal{T}$, we denote the stopped martingale by $f^\tau = (f_n^\tau)_{n\geq 0} = (f_{n\wedge\tau})_{n\geq 0}$, where $a \wedge b = \min(a, b)$. Let $p(\cdot) \in \mathcal{B}\big([0, 1]\big)$. If $f_n \in \mathcal{L}_{p(\cdot)}(\Omega)$ for every $n \geq 0$, f is called an $\mathcal{L}_{p(\cdot)}$-martingale. In this case, we set

$$\|f\|_{\mathcal{L}_{p(\cdot)}(\Omega)} := \sup_{n\geq 0} \|f_n\|_{\mathcal{L}_{p(\cdot)}(\Omega)}.$$

If $\|f\|_{\mathcal{L}_{p(\cdot)}(\Omega)} < \infty$, f is called a bounded $\mathcal{L}_{p(\cdot)}$-martingale and denote it by $f \in \mathcal{L}_{p(\cdot)}(\Omega)$.

The maximal function, the square function and the conditional square function of a martingale $f = (f_n)_{n\geq 0}$ are defined respectively as follows:

$$M_m f = \sup_{n\leq m} |f_n|, \quad M(f) = \sup_{n\geq 0} |f_n|;$$

$$S_m(f) = \left(\sum_{n=0}^{m} |d_n f|^2\right)^{\frac{1}{2}}, \quad S(f) = \left(\sum_{n=0}^{\infty} |d_n f|^2\right)^{\frac{1}{2}};$$
$$s_m(f) = \left(\sum_{n=0}^{m} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}}, \quad s(f) = \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}}.$$

Denote by Γ the set of all sequences $(\lambda_n)_{n\geq 0}$ of non-negative, non-decreasing and adapted functions with $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Let $p(\cdot) \in \mathcal{B}([0,1])$ and let $0 < q \leq \infty$. Denote variable exponent $p(\cdot)$ or $(p(\cdot), q)$ by θ. Define the variable martingale Hardy spaces associated with the variable Lebesgue spaces $\mathcal{L}_\theta(\Omega)$ as follows:

$$\mathcal{H}_\theta^s(\Omega) := \left\{f = (f_n)_{n\geq 0} : \|f\|_{\mathcal{H}_\theta^s(\Omega)} = \left\|s(f)\right\|_{\mathcal{L}_\theta(\Omega)} < \infty\right\};$$
$$\mathcal{H}_\theta^S(\Omega) := \left\{f = (f_n)_{n\geq 0} : \|f\|_{\mathcal{H}_\theta^S(\Omega)} = \left\|S(f)\right\|_{\mathcal{L}_\theta(\Omega)} < \infty\right\};$$
$$\mathcal{H}_\theta^M(\Omega) := \left\{f = (f_n)_{n\geq 0} : \|f\|_{\mathcal{H}_\theta^M(\Omega)} = \left\|M(f)\right\|_{\mathcal{L}_\theta(\Omega)} < \infty\right\};$$
$$\mathcal{P}_\theta(\Omega) := \left\{f = (f_n)_{n\geq 0} : \exists(\lambda_n)_{n\geq 0} \in \Gamma, \text{ s.t. } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in \mathcal{L}_\theta(\Omega)\right\},$$
$$\|f\|_{\mathcal{P}_\theta(\Omega)} = \inf_{(\lambda_n)_{n\geq 0}\in\Gamma} \|\lambda_\infty\|_{\mathcal{L}_\theta(\Omega)};$$
$$\mathcal{Q}_\theta(\Omega) := \left\{f = (f_n)_{n\geq 0} : \exists(\lambda_n)_{n\geq 0} \in \Gamma, \text{ s.t. } |S_n(f)| \leq \lambda_{n-1}, \lambda_\infty \in \mathcal{L}_\theta(\Omega)\right\},$$
$$\|f\|_{\mathcal{Q}_\theta(\Omega)} = \inf_{(\lambda_n)_{n\geq 0}\in\Gamma} \|\lambda_\infty\|_{\mathcal{L}_\theta(\Omega)}.$$

In [16, Theorem 2.22], the authors obtained the Doob's inequality for variable martingale spaces $\mathcal{L}_{p(\cdot)}(\Omega)$.

Lemma 2.13 *Let $p(\cdot) \in \mathcal{B}([0,1])$ satisfy the locally log-Hölder continuous condition. Then there exists a positive constant $C_{p(\cdot)}$ such that, for every martingale $f \in \mathcal{L}_{p(\cdot)}(\Omega)$,*

$$\|Mf\|_{\mathcal{L}_{p(\cdot)}(\Omega)} \leq C_{p(\cdot)}\|f\|_{\mathcal{L}_{p(\cdot)}(\Omega)}, \quad p_- > 1.$$

To establish the interpolation theorems, we need the atomic characterizations of variable martingale Hardy–Lorentz spaces. First, let us recall the definition of an atom.

Definition 2.14 Let $p(\cdot) \in \mathcal{B}([0,1])$. A measurable function a is said to be a $(1, p(\cdot), \infty)$-atom (or $(2, p(\cdot), \infty)$-atom, $(3, p(\cdot), \infty)$-atom, respectively), if there exists a stopping time $\tau \in \mathcal{T}$ such that

(i) $a_n := \mathbb{E}_n a = 0$, if $n \leq \tau$;
(ii) $\left\|s(a)\right\|_\infty$ (or $\left\|S(a)\right\|_\infty$, $\left\|M(a)\right\|_\infty$, respectively) $\leq \left\|\chi_{\{\tau<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)}^{-1}$.

Definition 2.15 Given $p(\cdot) \in \mathcal{B}([0,1])$ and $0 < q \leq \infty$. Assume that $i = 1, 2$ or 3. Denote by $\mathcal{H}_{p(\cdot),q}^{at,i,\infty}(\Omega)$ the space of all martingales $f = (f_n)_{n\geq 0}$ such that, for any

$n \geq 0$

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \quad \text{a.e.}, \tag{2.7}$$

where $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(i, p(\cdot), \infty)$-atoms with respect to the stopping time sequence $(\tau_k)_{k\in\mathbb{Z}}$ and $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}(\Omega)}$ $(k \in \mathbb{Z})$. Endow this space with the following quasi-norm

$$\|f\|_{\mathcal{H}^{at,i,\infty}_{p(\cdot),q}(\Omega)} := \inf \|\{\mu_k\}_{k\in\mathbb{Z}}\|_{\ell_q},$$

where the infimum is taken over all the decompositions of f by the form (2.7).

Lemma 2.16 [28, Theorem 3.3] *Let* $p(\cdot) \in \mathcal{B}([0, 1])$ *and* $0 < q \leq \infty$. *Then*

$$\mathcal{H}^s_{p(\cdot),q}(\Omega) = \mathcal{H}^{at,1,\infty}_{p(\cdot),q}(\Omega)$$

with equivalent quasi-norms.

Lemma 2.17 [28, Theorem 3.4] *Let* $p(\cdot) \in \mathcal{B}([0, 1])$ *and* $0 < q \leq \infty$. *Then*

$$\mathcal{Q}_{p(\cdot),q}(\Omega) = \mathcal{H}^{at,2,\infty}_{p(\cdot),q}(\Omega), \quad \mathcal{P}_{p(\cdot),q}(\Omega) = \mathcal{H}^{at,3,\infty}_{p(\cdot),q}(\Omega),$$

with equivalent quasi-norms.

2.4 Real interpolation

In this subsection, we collect some basic concepts and results about real interpolation theory. For the details, we refer to the monographs [2, 3]. Let (Y_0, Y_1) be a compatible couple of quasi-normed spaces, namely, Y_0 and Y_1 can be embedded continuously into a topological vector space Y. Define the sum of Y_0 and Y_1 as

$$Y_0 + Y_1 := \big\{f \in Y : f = f_0 + f_1,\ f_i \in Y_i,\ i = 0, 1\big\}.$$

For any $t \in (0, \infty)$ and $f \in Y_0 + Y_1$, the Peetre K-functional is defined by

$$K(t, f, Y_0, Y_1) := \inf_{f=f_0+f_1} \big\{\|f_0\|_{Y_0} + t\|f_1\|_{Y_1}\big\}.$$

For every $0 < \theta < 1$, $0 < q \leq \infty$, define the real interpolation space $(Y_0, Y_1)_{\theta,q}$ as the set of all functions $f \in Y_0 + Y_1$ such that

$$\|f\|_{(Y_0,Y_1)_{\theta,q}} := \begin{cases} \left(\int_0^\infty [t^{-\theta} K(t, f, Y_0, Y_1)]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{t>0} t^{-\theta} K(t, f, Y_0, Y_1), & q = \infty, \end{cases}$$

is finite. We adopt the conventions $(Y_0, Y_1)_{0,q} = Y_0$ and $(Y_0, Y_1)_{1,q} = Y_1$ for each $0 < q \leq \infty$. Now we give two basic properties of K and $(Y_0, Y_1)_{\theta,q}$, which may be used in the sequel.

1. $|f| \leq |g| \Rightarrow K(t, f, Y_0, Y_1) \leq K(t, g, Y_0, Y_1)$;
2. $Y_0 \hookrightarrow \tilde{Y}_0 \Rightarrow (Y_0, Y_1)_{\theta,q} \hookrightarrow (\tilde{Y}_0, Y_1)_{\theta,q}$, where $\hookrightarrow$ denotes the continuous embedding relationship.

We take the following reiteration lemma and Wolff's lemma from [2].

Lemma 2.18 *Suppose that (Y_0, Y_1) is a compatible couple of quasi-normed spaces. Let $0 \leq \theta_0, \theta_1 \leq 1, 0 < q_0, q_1 \leq \infty$ and let $Z_i = (Y_0, Y_1)_{\theta_i,q_i}$ $(i = 0, 1)$. If $\theta_0 \neq \theta_1, 0 < \eta < 1$ and $0 < q \leq \infty$, then*

$$(Z_0, Z_1)_{\eta,q} = (Y_0, Y_1)_{\theta,q},$$

where $\theta = (1-\eta)\theta_0 + \eta\theta_1$. In addition, if Y_0 and Y_1 are complete and $0 < \theta_0 = \theta_1 = \theta < 1$, then

$$\big((Y_0, Y_1)_{\theta,q_0}, (Y_0, Y_1)_{\theta,q_1}\big)_{\eta,q} = (Y_0, Y_1)_{\theta,q}, \quad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}.$$

Lemma 2.19 *Suppose that Y_1, Y_2, Y_3 and Y_4 are four quasi-normed spaces continuously embedded in some quasi-normed space. Let $0 < \lambda, \mu < 1$ and $0 < p, q \leq \infty$. If*

$$Y_2 = (Y_1, Y_3)_{\lambda,p}, \quad Y_3 = (Y_2, Y_4)_{\mu,q},$$

then

$$Y_2 = (Y_1, Y_4)_{\eta,p}, \quad Y_3 = (Y_1, Y_4)_{\theta,q},$$

where

$$\eta = \frac{\lambda\mu}{1-\lambda+\lambda\mu}, \quad \theta = \frac{\mu}{1-\lambda+\lambda\mu}.$$

3 Interpolation between $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$

Before we formulate the interpolation between variable martingale Hardy spaces, we need to establish the real interpolation between $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ spaces. The main result of this section is stated as follows.

Theorem 3.1 *Let $p(\cdot) \in \mathcal{B}([0,\infty))$ be locally log-Hölder continuous and let $0 < \theta < 1, 0 < q \leq \infty$. Put*

$$\frac{1}{\tilde{p}(\cdot)} = \frac{1-\theta}{p(\cdot)}.$$

Then

$$(\mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} = \mathcal{L}_{\tilde{p}(\cdot),q}(\mathbb{R}^n).$$

Remark 3.2 With only minor modifications to the following proof of Theorem 3.1, we can easily find that Theorem 3.1 is also true for general measure spaces (R, μ).

To prove the theorem above, we need the following technical lemma.

Lemma 3.3 *For any $t \in (0, \infty)$ and $f \in \mathcal{L}_{p(\cdot)}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, we have*

$$K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n)) \geq t f_*(t),$$

where

$$f_*(t) = \sup\left\{\lambda > 0 : \|\chi_{\{x\in\mathbb{R}^n:|f(x)|\geq\lambda\}}\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} \geq t\right\}.$$

Proof We show the above inequality in two steps.

Step 1: In this step, we show that

$$K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n)) \geq K(t, f^*, L_{p(\cdot)}([0,\infty)), L_\infty([0,\infty))), \quad t > 0. \tag{3.1}$$

Let $f = f_0 + f_1$ with $f_0 \in \mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$, $f_1 \in L_\infty(\mathbb{R}^n)$. Then we have $f_0^* \in L_{p(\cdot)}([0,\infty))$ and

$$f^*(s) \leq f_0^*(s) + f_1^*(0) = f_0^*(s) + \|f_1\|_{L_\infty(\mathbb{R}^n)}, \quad s > 0.$$

From the property of the K-functional, we obtain

$$\begin{aligned} K(t, f^*, L_{p(\cdot)}([0,\infty)), L_\infty([0,\infty))) &\leq \|f_0^*\|_{L_{p(\cdot)}([0,\infty))} + t\|f_1\|_{L_\infty(\mathbb{R}^n)} \\ &= \|f_0\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\|f_1\|_{L_\infty(\mathbb{R}^n)}. \end{aligned}$$

Taking the infimum over all decompositions $f = f_0 + f_1 \in \mathcal{L}_{p(\cdot)}(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$, we get (3.1).

Step 2: By Step 1, it is enough to prove that

$$K(t, f^*, L_{p(\cdot)}([0,\infty)), L_\infty([0,\infty))) \geq t f_*(t), \quad t > 0. \tag{3.2}$$

We can show easily (see also the proof of [19, Theorem 4.1]) that

$$\begin{aligned} &K(t, f^*, L_{p(\cdot)}([0,\infty)), L_\infty([0,\infty))) \\ &= \inf_{\mu>0}\left\{\left\|\left(f^* - \mu\right)_+\right\|_{L_{p(\cdot)}([0,\infty))} + t\left\|\min\left(f^*, \mu\right)\right\|_{L_\infty([0,\infty))}\right\}. \end{aligned} \tag{3.3}$$

First, we note that, for every $g \in L_{p(\cdot)}([0,\infty))$,

$$\|g\|_{L_{p(\cdot),\infty}([0,\infty))} \leq \|g\|_{L_{p(\cdot)}([0,\infty))}.$$

In fact, Definition 2.4 gives that

$$\begin{aligned}\|g\|_{L_{p(\cdot),\infty}([0,\infty))} &= \sup_{\lambda>0} \lambda \|\chi_{\{|g|>\lambda\}}\|_{L_{p(\cdot)}([0,\infty))} \leq \sup_{\lambda>0} \lambda \big\| |g|/\lambda \big\|_{L_{p(\cdot)}([0,\infty))} \\ &= \|g\|_{L_{p(\cdot)}([0,\infty))}.\end{aligned}$$

Furthermore, by (3.3), one can conclude that

$$\begin{aligned}&K(t, f^*, L_{p(\cdot)}([0,\infty)), L_\infty([0,\infty))) \\ &\geq \inf_{\mu>0} \Big\{ \big\| \big(f^* - \mu\big)_+ \big\|_{L_{p(\cdot),\infty}([0,\infty))} + t \big\| \min\big(f^*, \mu\big) \big\|_{L_\infty([0,\infty))} \Big\} \\ &= \inf_{\mu>0} \Big\{ \sup_{\lambda>0} \lambda \big\| \chi_{\{f^*>\lambda+\mu\}} \big\|_{L_{p(\cdot)}([0,\infty))} + t \big\| \min\big(f^*, \mu\big) \big\|_{L_\infty([0,\infty))} \Big\}.\end{aligned} \tag{3.4}$$

Obviously, $f_*(t) \leq \|f\|_{L_\infty(\mathbb{R}^n)}$ for any $t > 0$. For a given $t > 0$, divide the interval $(0, \infty)$ into the subintervals

$$(0, f_*(t)), [f_*(t), \|f\|_{L_\infty(\mathbb{R}^n)}], (\|f\|_{L_\infty(\mathbb{R}^n)}, \infty),$$

when $f \in L_\infty(\mathbb{R}^n)$ and into the subintervals

$$(0, f_*(t)), [f_*(t), \infty),$$

when $f \notin L_\infty(\mathbb{R}^n)$. Further on, we estimate the last infimum of (3.4) on each subinterval.

For the estimation on the first interval $(0, f_*(t))$, denote

$$h(\lambda) = \|\chi_{\{x\in\mathbb{R}^n:|f(x)|\geq\lambda\}}\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)}.$$

Then we have

$$f_*(t) = \sup\big\{\lambda > 0 : h(\lambda) \geq t\big\}.$$

Moreover,

$$\begin{aligned}&\inf_{0<\mu<f_*(t)} \Big\{ \sup_{\lambda>0} \lambda \big\| \chi_{\{f^*>\lambda+\mu\}} \big\|_{L_{p(\cdot)}([0,\infty))} + t \big\| \min\big(f^*, \mu\big) \big\|_{L_\infty([0,\infty))} \Big\} \\ &= \inf_{0<\mu<f_*(t)} \Big\{ \sup_{\lambda>0} \lambda \big\| \chi_{\{f^*>\lambda+\mu\}} \big\|_{L_{p(\cdot)}([0,\infty))} + t\mu \Big\}.\end{aligned}$$

Note that $\{f^* > s\} = [0, d_f(s))$; see [10, Proposition 1.4.5(3)]. Since $\chi_E^* = \chi_{[0,m(E))}$ for any measurable set $E \subset \mathbb{R}^n$, we have

$$\begin{aligned}
\left\|\chi_{\{f>s\}}\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} &= \left\|\chi_{[0,m(\{f>s\}))}\right\|_{L_{p(\cdot)}([0,\infty))} \\
&= \left\|\chi_{[0,d_f(s))}\right\|_{L_{p(\cdot)}([0,\infty))} = \left\|\chi_{\{f^*>s\}}\right\|_{L_{p(\cdot)}([0,\infty))}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\inf_{0<\mu<f_*(t)} \left\{ \sup_{\lambda>0} \lambda \left\|\chi_{\{f^*>\lambda+\mu\}}\right\|_{L_{p(\cdot)}([0,\infty))} + t \left\| \min\left(f^*, \mu\right)\right\|_{L_\infty([0,\infty))} \right\} \\
&= \inf_{0<\mu<f_*(t)} \left\{ \sup_{\lambda>0} \lambda \left\|\chi_{\{f>\lambda+\mu\}}\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\mu \right\}.
\end{aligned}$$

Since h is left-continuous, we conclude that $h(f_*(t)) \geq t$ for all $t > 0$. Moreover, we can take all $0 < \lambda < f_*(t) - \mu$ as $0 < \mu < f_*(t)$. Hence,

$$\begin{aligned}
&\inf_{0<\mu<f_*(t)} \left\{ \sup_{\lambda>0} \lambda \left\|\chi_{\{f^*>\lambda+\mu\}}\right\|_{L_{p(\cdot)}([0,\infty))} + t \left\| \min\left(f^*, \mu\right)\right\|_{L_\infty([0,\infty))} \right\} \\
&\geq \inf_{0<\mu<f_*(t)} \left\{ \left(f_*(t) - \mu\right) \left\|\chi_{\{f\geq f_*(t)\}}\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\mu \right\} \\
&= \inf_{0<\mu<f_*(t)} \left\{ \left(f_*(t) - \mu\right) h(f_*(t)) + t\mu \right\} \\
&\geq \inf_{0<\mu<f_*(t)} \left\{ \left(f_*(t) - \mu\right) t + t\mu \right\} \\
&= t f_*(t).
\end{aligned} \tag{3.5}$$

Now suppose that $f \notin L_\infty(\mathbb{R}^n)$. Since $\lim_{t\to 0+0} f^*(t) = \infty$, we estimate (3.4) further by

$$\begin{aligned}
&\inf_{f_*(t)\leq\mu<\infty} \left\{ \sup_{\lambda>0} \lambda \left\|\chi_{\{f^*>\lambda+\mu\}}\right\|_{L_{p(\cdot)}([0,\infty))} + t \left\| \min\left(f^*, \mu\right)\right\|_{L_\infty([0,\infty))} \right\} \\
&= \inf_{f_*(t)\leq\mu<\infty} \left\{ \sup_{\lambda>0} \lambda \left\|\chi_{\{f^*>\lambda+\mu\}}\right\|_{L_{p(\cdot)}([0,\infty))} + t\mu \right\} \\
&\geq \inf_{f_*(t)\leq\mu<\infty} \{t\mu\} = t f_*(t).
\end{aligned} \tag{3.6}$$

Now suppose that $f \in L_\infty(\mathbb{R}^n)$. The same estimation holds on the second interval $[f_*(t), \|f\|_{L_\infty([0,\infty))}]$ as in (3.6), because $f^*(0) = \|f\|_{L_\infty(\mathbb{R}^n)} \geq \mu$ and f^* is right-continuous.

Finally, on the last interval,

$$\begin{aligned}
&\inf_{\|f\|_{L_\infty(\mathbb{R}^n)}<\mu<\infty} \left\{ \sup_{\lambda>0} \lambda \left\|\chi_{\{f^*>\lambda+\mu\}}\right\|_{L_{p(\cdot)}([0,\infty))} + t \left\| \min\left(f^*, \mu\right)\right\|_{L_\infty([0,\infty))} \right\} \\
&= \inf_{\|f\|_{L_\infty(\mathbb{R}^n)}<\mu<\infty} \left\{ t \left\| f^* \right\|_{L_\infty([0,\infty))} \right\}
\end{aligned}$$

$$= t\|f\|_{L_\infty(\mathbb{R}^n)} \geq t f_*(t).$$

Combining this with (3.2), (3.4), (3.5) and (3.6), we finish the proof of Lemma 3.3. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1 **Step 1:** In this step, we shall prove

$$(\mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} \hookrightarrow \mathcal{L}_{\tilde{p}(\cdot),q}(\mathbb{R}^n).$$

For this, it suffices to show that

$$\begin{aligned}
&\int_0^\infty \lambda^q \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|^q_{\mathcal{L}_{\tilde{p}(\cdot)}(\mathbb{R}^n)} \frac{\mathrm{d}\lambda}{\lambda} \\
&\quad \lesssim \int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{3.7}$$

By (2.3), we find that

$$\begin{aligned}
\int_0^\infty \lambda^q \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|^q_{\mathcal{L}_{\tilde{p}(\cdot)}(\mathbb{R}^n)} \frac{\mathrm{d}\lambda}{\lambda} &\leq \int_0^\infty \lambda^q h(\lambda)^{(1-\theta)q} \frac{\mathrm{d}\lambda}{\lambda} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{k(1-\theta)q} f_*(2^k)^q,
\end{aligned}$$

where the last inequality is referred to [19, p. 948]. On the other hand, it follows from Lemma 3.3 that

$$\begin{aligned}
\int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t} &\geq \int_0^\infty t^{-\theta q} t^q f_*(t)^q \frac{\mathrm{d}t}{t} \\
&\gtrsim \sum_{k\in\mathbb{Z}} 2^{k(1-\theta)q} f_*(2^k)^q,
\end{aligned}$$

see also [19, p. 948] for the details. Thus (3.7) is valid.

Step 2: In this step, let us verify the reverse embedding, that is,

$$\mathcal{L}_{\tilde{p}(\cdot),q}(\mathbb{R}^n) \hookrightarrow (\mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q}.$$

By (2.4), it is enough to check the following inequality:

$$\int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t}$$

$$\lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^k\}}\|^q_{\mathcal{L}_{\tilde{p}(\cdot)}(\mathbb{R}^n)}. \tag{3.8}$$

To this end, we begin with a reformulation of $K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))$. Similarly to (3.3),

$$\begin{aligned}
&K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n)) \\
&= \inf_{f_0+f_1} \left\{\|f_0\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\|f_1\|_{L_\infty(\mathbb{R}^n)}\right\} \\
&= \inf_{\mu>0} \left\{\left\|(|f|-\mu)_+\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\left\|\min(|f|,\mu)\right\|_{L_\infty(\mathbb{R}^n)}\right\} \\
&\le \inf_{\mu>0} \left\{\left\||f|\chi_{\{|f|>\mu\}}\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\mu\right\} \\
&= \inf_{\mu>0} \left\{\left\|\sum_{i=0}^\infty |f|\chi_{\{2^i\mu<|f|\le 2^{i+1}\mu\}}\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\mu\right\} \\
&\lesssim \inf_{\mu>0} \left\{\left\|\sum_{i=0}^\infty 2^i\mu\chi_{\{|f|>2^i\mu\}}\right\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} + t\mu\right\}.
\end{aligned}$$

By Proposition 2.9, we have

$$\begin{aligned}
&K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n)) \\
&\lesssim \inf_{\mu>0} \left\{\left[\sum_{i=0}^\infty (2^i\mu)^b \|\chi_{\{|f|>2^i\mu\}}\|^b_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)}\right]^{\frac{1}{b}} + t\mu\right\} \\
&= \inf_{\mu>0} \left\{\left[\sum_{i=0}^\infty (2^i)^b \|\chi_{\{|f|>2^i\mu\}}\|^b_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)}\right]^{\frac{1}{b}} \mu + t\mu\right\} \\
&= \inf_{\mu>0} \left\{\left[\sum_{i=0}^\infty (2^i)^b g(2^i\mu)^b\right]^{\frac{1}{b}} \mu + t\mu\right\},
\end{aligned}$$

where

$$0 < b < \underline{p} \quad \text{and} \quad g(\lambda) = \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\|_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)}, \ \lambda > 0.$$

For fixed $t > 0$, we choose $\mu = \mu(t)$ by

$$\mu(t) := \inf\left\{\mu > 0 : \left[\sum_{i=0}^\infty (2^i)^b g(2^i\mu)^b\right]^{\frac{1}{b}} \le t\right\}.$$

Since $g(\cdot)$ is right-continuous, we have

$$\left[\sum_{i=0}^{\infty}(2^i)^b g(2^i\mu(t))^b\right]^{\frac{1}{b}} \le t.$$

Thus,

$$K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n)) \lesssim t\mu(t). \tag{3.9}$$

Applying (3.9), one can deduce that

$$\begin{aligned}
&\int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty t^{-\theta q} t^q \mu(t)^q \frac{\mathrm{d}t}{t} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \int_{\{t:2^k<\mu(t)\le 2^{k+1}\}} t^{(1-\theta)q} \frac{\mathrm{d}t}{t} \\
&\le \sum_{k\in\mathbb{Z}} 2^{kq} \int_{\{t:2^k<\mu(t)\}} t^{(1-\theta)q} \frac{\mathrm{d}t}{t}.
\end{aligned}$$

Since

$$\left[\sum_{i=0}^{\infty}(2^i)^b g(2^{i+k})^b\right]^{\frac{1}{b}} \ge t \quad \text{if} \quad \mu(t) > 2^k,$$

we conclude that

$$\begin{aligned}
&\int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t} \\
&\le \sum_{k\in\mathbb{Z}} 2^{kq} \int_0^{[\sum_{i=0}^{\infty}(2^i)^b g(2^{i+k})^b]^{\frac{1}{b}}} t^{(1-\theta)q} \frac{\mathrm{d}t}{t} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \left[\sum_{i=0}^{\infty}(2^i)^b g(2^{i+k})^b\right]^{\frac{(1-\theta)q}{b}}.
\end{aligned}$$

Set $l = i + k$. If $(1-\theta)q \le b$, then

$$\begin{aligned}\int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t} &\lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \sum_{i=0}^\infty 2^{i(1-\theta)q} g(2^{i+k})^{(1-\theta)q} \\ &= \sum_{l\in\mathbb{Z}} \sum_{i=0}^\infty 2^{(l-i)q} 2^{i(1-\theta)q} g(2^l)^{(1-\theta)q} \\ &= \sum_{l\in\mathbb{Z}} 2^{lq} g(2^l)^{(1-\theta)q} \sum_{i=0}^\infty 2^{-i\theta q} \\ &\lesssim \sum_{l\in\mathbb{Z}} 2^{lq} g(2^l)^{(1-\theta)q}.\end{aligned}$$

If $(1-\theta)q > b$, then we set $r = \frac{(1-\theta)q}{b}$ and choose $\delta \in \left(0, \frac{\theta}{1-\theta}\right)$. A similar approach combined with Hölder's inequality gives that

$$\begin{aligned}&\int_0^\infty t^{-\theta q} K(t, f, \mathcal{L}_{p(\cdot)}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))^q \frac{\mathrm{d}t}{t} \\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \left[\sum_{i=0}^\infty 2^{-\delta ib} \cdot 2^{(\delta+1)ib} g(2^l)^b\right]^r \\ &\le \sum_{k\in\mathbb{Z}} 2^{kq} \left(\sum_{i=0}^\infty 2^{-\delta ibr'}\right)^{\frac{r}{r'}} \sum_{i=0}^\infty 2^{(\delta+1)i(1-\theta)q} g(2^l)^{(1-\theta)q} \\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \sum_{i=0}^\infty 2^{(\delta+1)i(1-\theta)q} g(2^l)^{(1-\theta)q} \\ &= \sum_{l\in\mathbb{Z}} 2^{lq} g(2^l)^{(1-\theta)q} \sum_{i=0}^\infty 2^{[\delta(1-\theta)-\theta]iq} \\ &\lesssim \sum_{l\in\mathbb{Z}} 2^{lq} g(2^l)^{(1-\theta)q}.\end{aligned}$$

On the other hand, using (2.3), we find that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^k\}}\|^q_{\mathcal{L}_{\tilde{p}(\cdot)}(\mathbb{R}^n)} &= \sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^k\}}\|^{(1-\theta)q}_{\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)} \\ &= \sum_{k\in\mathbb{Z}} 2^{kq} g(2^k)^{(1-\theta)q}.\end{aligned}$$

Hence, (3.8) holds. The proof is complete. □

4 Interpolation between variable martingale Hardy spaces

This section is devoted to identifying the interpolation spaces between variable martingale Hardy spaces. The following theorem is the main result of this section.

Theorem 4.1 *Let $p(\cdot) \in \mathcal{B}([0,1])$ satisfy the locally log-Hölder continuous condition, $0 < \theta < 1, 0 < q \leq \infty$. Put*

$$\frac{1}{\tilde{p}(\cdot)} = \frac{1-\theta}{p(\cdot)}. \tag{4.1}$$

Then

$$(\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q} = \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega),$$
$$(\mathcal{P}_{p(\cdot)}(\Omega), P_\infty(\Omega))_{\theta,q} = \mathcal{P}_{\tilde{p}(\cdot),q}(\Omega),$$

and

$$(\mathcal{Q}_{p(\cdot)}(\Omega), Q_\infty(\Omega))_{\theta,q} = \mathcal{Q}_{\tilde{p}(\cdot),q}(\Omega).$$

Remark 4.2 According to Proposition 2.12, we can see that the real interpolation spaces between variable martingale Hardy $\mathcal{H}^s_{p(\cdot)}(\Omega)$ and $H^s_\infty(\Omega)$ are just the classical Hardy–Lorentz martingale spaces $H^s_{\tilde{p}(0),q}(\Omega)$. The same holds for the other two types of spaces.

To prove this theorem, we state the following lemma first.

Lemma 4.3 *Let $p(\cdot) \in \mathcal{B}([0,1])$ satisfy the locally log-Hölder continuous condition, $0 < \theta < 1, 0 < q \leq \infty$ and $\tilde{p}(\cdot)$ be defined as (4.1). Then, for every $f \in \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega)$ and $t > 0$, we have*

$$K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega)) \lesssim \inf_{\lambda>0} \left\{ \left[\sum_{i=0}^{\infty} 2^{ib} \|\chi_{\{s(f)>2^i\lambda\}}\|^b_{\mathcal{L}_{p(\cdot)}(\Omega)} \right]^{\frac{1}{b}} \lambda + t\lambda \right\},$$

where $0 < b < \underline{p}$.

Proof Assume that $f \in \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega)$. It follows from Lemma 2.16 that f can be decomposed as

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k \quad \text{a.e.},$$

where $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(1, \tilde{p}(\cdot), \infty)$-atoms with respect to the stopping time sequence $(\tau_k)_{k\in\mathbb{Z}}$ and $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{\tilde{p}(\cdot)}(\Omega)}$ $(k \in \mathbb{Z})$. Moreover, by the definition of $(1, \tilde{p}(\cdot), \infty)$-atom a^k, we find that,

$$\mu_k s(a^k) \leq 3 \cdot 2^k \chi_{\{\tau_k<\infty\}}, \quad k \in \mathbb{Z}.$$

For any $\lambda > 0$, there exists a $k_0 \in \mathbb{Z}$ such that $2^{k_0} < \lambda \leq 2^{k_0+1}$. Denote

$$g = \sum_{k=k_0+1}^{\infty} \mu_k a^k \quad \text{and} \quad h = \sum_{k=-\infty}^{k_0} \mu_k a^k.$$

Then

$$\begin{aligned}
&K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega)) \\
&\leq \inf_{\lambda>0} \left\{\|g\|_{\mathcal{H}^s_{p(\cdot)}(\Omega)} + t\|h\|_{H^s_\infty(\Omega)}\right\} \\
&\leq \inf_{\lambda>0} \left\{\left\|\sum_{k=k_0+1}^{\infty} \mu_k s(a^k)\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)} + t\left\|\sum_{k=-\infty}^{k_0} \mu_k s(a^k)\right\|_{L_\infty(\Omega)}\right\} \\
&\lesssim \inf_{\lambda>0} \left\{\left\|\sum_{k=k_0+1}^{\infty} 2^k \chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)} + t\sum_{k=-\infty}^{k_0} 2^k\right\} \\
&\lesssim \inf_{\lambda>0} \left\{\left\|\sum_{k=k_0+1}^{\infty} 2^k \chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)} + t2^{k_0}\right\}.
\end{aligned}$$

Note that

$$\sum_{k=k_0+1}^{\infty} 2^k \chi_{\{\tau_k<\infty\}} \leq 2s(f)\chi_{\{s(f)>2^{k_0+1}\}}$$

(see [13, Lemma 3.5]). Therefore,

$$\begin{aligned}
&K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega)) \\
&\lesssim \inf_{\lambda>0} \left\{\|s(f)\chi_{\{s(f)>\lambda\}}\|_{\mathcal{L}_{p(\cdot)}(\Omega)} + t\lambda\right\} \\
&= \inf_{\lambda>0} \left\{\left\|s(f)\sum_{i=0}^{\infty} \chi_{\{2^i\lambda<s(f)\leq 2^{i+1}\lambda\}}\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)} + t\lambda\right\} \\
&\lesssim \inf_{\lambda>0} \left\{\left\|\sum_{i=0}^{\infty} 2^i\lambda\chi_{\{2^i\lambda<s(f)\}}\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)} + t\lambda\right\}.
\end{aligned}$$

We obtain by Proposition 2.9 that

$$K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega)) \lesssim \inf_{\lambda>0} \left\{\left[\sum_{i=0}^{\infty} 2^{ib}\|\chi_{\{s(f)>2^i\lambda\}}\|^b_{\mathcal{L}_{p(\cdot)}(\Omega)}\right]^{\frac{1}{b}} \lambda + t\lambda\right\},$$

where $0 < b < \underline{p}$. This finishes the proof. □

Remark 4.4 Similarly, by Lemma 2.17, one can deduce that

$$K(t, f, \mathcal{P}_{p(\cdot)}(\Omega), P_\infty(\Omega)) \lesssim \inf_{\lambda>0} \left\{ \left[\sum_{i=0}^{\infty} 2^{ib} \|\chi_{\{\lambda_\infty > 2^i\lambda\}}\|^b_{\mathcal{L}_{p(\cdot)}(\Omega)} \right]^{\frac{1}{b}} \lambda + t\lambda \right\},$$

and

$$K(t, f, \mathcal{Q}_{p(\cdot)}(\Omega), Q_\infty(\Omega)) \lesssim \inf_{\lambda>0} \left\{ \left[\sum_{i=0}^{\infty} 2^{ib} \|\chi_{\{\lambda_\infty > 2^i\lambda\}}\|^b_{\mathcal{L}_{p(\cdot)}(\Omega)} \right]^{\frac{1}{b}} \lambda + t\lambda \right\},$$

where $0 < b < \underline{p}$ and $(\lambda_n) \in \Gamma$ with $\lim_{n\to\infty} \lambda_n = \lambda_\infty$.

Proof of Theorem 4.1 We are going to show the theorem for $\mathcal{H}^s_{p(\cdot)}(\Omega)$ only, since the proofs for the other two spaces are similar. Firstly, let us prove

$$(\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q} \hookrightarrow \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega).$$

Consider the operator $T : f \to s(f)$. Note that both $T : \mathcal{H}^s_{p(\cdot)}(\Omega) \to \mathcal{L}_{p(\cdot)}(\Omega)$ and $H^s_\infty(\Omega) \to L_\infty(\Omega)$ are bounded. It follows from the interpolation theorem and Theorem 3.1 that

$$T : (\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q} \to (\mathcal{L}_{p(\cdot)}(\Omega), L_\infty(\Omega))_{\theta,q} = \mathcal{L}_{\tilde{p}(\cdot),q}(\Omega)$$

is bounded as well, which means that

$$\|f\|_{\mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega)} = \|s(f)\|_{\mathcal{L}_{\tilde{p}(\cdot),q}(\Omega)} \lesssim \|f\|_{(\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q}}.$$

Conversely, we show that

$$\mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega) \hookrightarrow (\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q},$$

or, equivalently,

$$\int_0^\infty t^{-\theta q} K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))^q \frac{\mathrm{d}t}{t} \lesssim \sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{x\in\Omega: s(f)(x) > 2^k\}}\|^q_{\mathcal{L}_{\tilde{p}(\cdot)}(\Omega)}. \tag{4.2}$$

Let $f \in \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega)$. Applying Lemma 4.3, we have, for every $t > 0$,

$$K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega)) \lesssim \inf_{\lambda>0} \left\{ \left[\sum_{i=0}^\infty 2^{ib} \|\chi_{\{s(f)>2^i\lambda\}}\|^b_{\mathcal{L}_{p(\cdot)}(\Omega)} \right]^{\frac{1}{b}} \lambda + t\lambda \right\},$$

where $0 < b < \underline{p}$. Denote $g(\lambda) = \|\chi_{\{s(f)>\lambda\}}\|_{\mathcal{L}_{p(\cdot)}(\Omega)}$. Then

$$K(t, f, \mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega)) \lesssim \inf_{\lambda>0} \left\{ \left[\sum_{i=0}^\infty 2^{ib} g(2^i\lambda)^b \right]^{\frac{1}{b}} \lambda + t\lambda \right\}.$$

Now an argument similar to Step 2 of the proof in Theorem 3.1 allows us to prove (4.2) as well as the theorem. □

Remark 4.5 If $p(\cdot) \equiv p$ $(0 < p < \infty)$, then the theorem above reduces to [27, Theorem 5.11].

5 Interpolation between variable martingale Hardy spaces and BMO spaces

In this section, we aim at formulating the real interpolation between variable martingale Hardy spaces and martingale BMO spaces. Recall that, for any $r \in [1, \infty)$, the space $BMO_r(\Omega)$ is defined to be the collection of all martingales $f \in L_r(\Omega)$ such that

$$\|f\|_{BMO_r(\Omega)} := \sup_{n\geq 0} \left\| \left(\mathbb{E}_n \left(|f - f_n|^r \right) \right)^{\frac{1}{r}} \right\|_{L_\infty(\Omega)} < \infty.$$

To be precise, we mainly obtain the following theorem.

Theorem 5.1 *Let $p(\cdot) \in \mathcal{B}([0, 1])$ satisfy the locally log-Hölder continuous condition and let $0 < q \leq \infty$, $0 < \theta < 1$ with $\theta + p_- > 1$. Then*

$$(\mathcal{H}^s_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q} = \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega),$$

where $\tilde{p}(\cdot)$ was defined in (4.1).

Remark 5.2 According to Proposition 2.12, we can see that the real interpolation spaces between variable martingale Hardy $\mathcal{H}^s_{p(\cdot)}(\Omega)$ and $BMO_2(\Omega)$ are just the classical Hardy–Lorentz martingale spaces $H^s_{\tilde{p}(0),q}(\Omega)$.

In order to prove the theorem above, we also introduce the sharp maximal function f^s_r and BMO space $BMO^s_r(\Omega)$, which were developed by Weisz [27]. For any $r > 0$, the sharp maximal function f^s_r of a martingale $f = (f_n)_{n\geq 0}$ is defined by

$$f^s_r := \sup_{n\geq 0} \left[\mathbb{E}_n \left(s^2(f) - s^2_n(f) \right)^{\frac{r}{2}} \right]^{\frac{1}{r}}$$

and $BMO_r^s(\Omega)$ denotes the set of all martingales $f=(f_n)_{n\geq 0}$ for which

$$\|f\|_{BMO_r^s(\Omega)}:=\sup_{n\geq 0}\left\|\left(\mathbb{E}_n\left(s^2(f)-s_n^2(f)\right)^{\frac{r}{2}}\right)^{\frac{1}{r}}\right\|_{L_\infty(\Omega)}<\infty.$$

Obviously, $\|f\|_{BMO_r^s(\Omega)}=\|f_r^s\|_{L_\infty(\Omega)}$. It was proved in [27, Theorem 2.50] that, for every $r>0$,

$$BMO_r^s(\Omega)=BMO_2(\Omega) \tag{5.1}$$

with equivalent quasi-norms.

Before proving Theorem 5.1, we first give the following technical results.

Proposition 5.3 *Let $p(\cdot)\in\mathcal{B}([0,1])$ satisfy the locally log-Hölder continuous condition and let $r\in(0,p_-)$. Then*

$$\|f_r^s\|_{\mathcal{L}_{p(\cdot)}(\Omega)}\lesssim\|f\|_{\mathcal{H}_{p(\cdot)}^s(\Omega)}.$$

Proof Note that

$$f_r^s=\sup_{n\geq 0}\left[\mathbb{E}_n\left(s^2(f)-s_n^2(f)\right)^{\frac{r}{2}}\right]^{\frac{1}{r}}\leq\left[\sup_{n\geq 0}\mathbb{E}_n\left(s^r(f)\right)\right]^{\frac{1}{r}}=\left(M\left(s^r(f)\right)\right)^{\frac{1}{r}}.$$

Since $r\in(0,p_-)$, it follows immediately from (2.3) and Lemma 2.13 that

$$\begin{aligned}\|f_r^s\|_{\mathcal{L}_{p(\cdot)}(\Omega)}&\leq\left\|\left(M\left(s^r(f)\right)\right)^{\frac{1}{r}}\right\|_{\mathcal{L}_{p(\cdot)}(\Omega)}=\left\|M\left(s^r(f)\right)\right\|_{\mathcal{L}_{p(\cdot)/r}(\Omega)}^{\frac{1}{r}}\\&\lesssim\left\|s^r(f)\right\|_{\mathcal{L}_{p(\cdot)/r}(\Omega)}^{\frac{1}{r}}=\|f\|_{\mathcal{H}_{p(\cdot)}^s(\Omega)},\end{aligned}$$

which completes the proof. □

Proposition 5.4 *Let $p(\cdot)\in\mathcal{B}([0,1])$ satisfy the locally log-Hölder continuous condition and let $q\in(0,\infty]$. If $r\in[1,\infty)$, then there exists a positive constant C such that, for every martingale f,*

$$\|f\|_{\mathcal{H}_{p(\cdot),q}^s(\Omega)}\leq C\|f_r^s\|_{\mathcal{L}_{p(\cdot),q}(\Omega)}.$$

To prove this proposition, we need the following lemma.

Lemma 5.5 [23, Lemma 1] *If $r\in[1,\infty)$, then, for any martingale $f=(f_n)_{n\geq 0}$, we have*

$$s(f)^*(t)\leq 4(f_r^s)^*(t/2)+s(f)^*(2t),\quad t>0.$$

Now we provide the proof of Proposition 5.4.

Proof of Proposition 5.4 Since $p(\cdot) \in \mathcal{B}([0,1])$ satisfies the locally log-Hölder continuous condition, by Proposition 2.12, we know that the quasi-norm $\|\cdot\|_{\mathcal{L}_{p(\cdot),q}(\Omega)}$ is equivalent to the quasi-norm $\|\cdot\|_{L_{p(0),q}(\Omega)}$. Hence, to prove Proposition 5.4, it suffices to show that

$$\|s(f)\|_{L_{p(0),q}(\Omega)} \lesssim \|f_r^s\|_{L_{p(0),q}(\Omega)}. \tag{5.2}$$

Recall that $\|s(f)\|_{L_{p,\infty}(\Omega)} \lesssim \|f_r^s\|_{L_{p,\infty}(\Omega)}$ for any $0 < r, p < \infty$ (see [24, Theorem 2.3]) and $\|s(f)\|_{L_{p,q}(\Omega)} \lesssim \|f_r^s\|_{L_{p,q}(\Omega)}$ for any $0 < p < \infty$, $1 \le q, r < \infty$ (see [23, Theorem 1]). It remains to verify (5.2) for the case $0 < q < 1$. Applying Lemma 5.5, we find that

$$\begin{aligned}
\|s(f)\|^q_{L_{p(0),q}(\Omega)} &= \int_0^\infty \left[t^{\frac{1}{p(0)}-\frac{1}{q}} s(f)^*(t)\right]^q \mathrm{d}t \\
&\le \int_0^\infty \left[t^{\frac{1}{p(0)}-\frac{1}{q}} \left(4(f_r^s)^*(t/2) + s(f)^*(2t)\right)\right]^q \mathrm{d}t \\
&\le 4^q \int_0^\infty \left[t^{\frac{1}{p(0)}-\frac{1}{q}} (f_r^s)^*(t/2)\right]^q \mathrm{d}t + \int_0^\infty \left[t^{\frac{1}{p(0)}-\frac{1}{q}} s(f)^*(2t)\right]^q \mathrm{d}t \\
&= 4^q \cdot 2^{\frac{q}{p(0)}} \int_0^\infty \left[t^{\frac{1}{p(0)}-\frac{1}{q}} (f_r^s)^*(t)\right]^q \mathrm{d}t + \left(\frac{1}{2}\right)^{\frac{q}{p(0)}} \int_0^\infty \left[t^{\frac{1}{p(0)}-\frac{1}{q}} s(f)^*(t)\right]^q \mathrm{d}t \\
&= 4^q \cdot 2^{\frac{q}{p(0)}} \|f_r^s\|^q_{L_{p(0),q}(\Omega)} + \left(\frac{1}{2}\right)^{\frac{q}{p(0)}} \|s(f)\|^q_{L_{p(0),q}(\Omega)},
\end{aligned}$$

which implies that (5.2) is valid for $0 < q < 1$. Now the proof is complete. $\square$

Based on the results above, we are ready to prove Theorem 5.1.

Proof of Theorem 5.1 Let us prove this theorem in two steps.

Step 1: In this step, we verify Theorem 5.1 for the case $p_- > 1$. From (5.1) and the boundedness of the Doob maximal operator on $L_\infty(\Omega)$, it follows that

$$\begin{aligned}
\|f\|_{BMO_2(\Omega)} \lesssim \|f\|_{BMO_1^s(\Omega)} = \|f_1^s\|_{L_\infty(\Omega)} &\le \left\|\sup_{n\ge 0} \mathbb{E}_n s(f)\right\|_{L_\infty(\Omega)} \\
&\lesssim \|s(f)\|_{L_\infty(\Omega)} = \|f\|_{H_\infty^s(\Omega)}.
\end{aligned}$$

Combining this with Theorem 4.1, we deduce that

$$\|f\|_{(\mathcal{H}^s_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q}} \lesssim \|f\|_{(\mathcal{H}^s_{p(\cdot)}(\Omega), H^s_\infty(\Omega))_{\theta,q}} = \|f\|_{\mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega)},$$

where $\tilde{p}(\cdot)$ was defined in (4.1).

Conversely, for any $r \in [1, p_-)$, consider the operator $T_r^s : f \to f_r^s$. Obviously, T_r^s is sublinear. By (5.1), it is easy to see that $T_r^s : BMO_2(\Omega) \to L_\infty(\Omega)$ is bounded. Since $r < p_-$, we deduce from Proposition 5.3 that $T_r^s : \mathcal{H}_{p(\cdot)}^s(\Omega) \to \mathcal{L}_{p(\cdot)}(\Omega)$ is bounded as well. Applying Theorem 3.1, we find that

$$T_r^s : (\mathcal{H}_{p(\cdot)}^s(\Omega), BMO_2(\Omega))_{\theta,q} \to (\mathcal{L}_{p(\cdot)}(\Omega), L_\infty(\Omega))_{\theta,q} = \mathcal{L}_{\tilde{p}(\cdot),q}(\Omega)$$

is bounded. In other words,

$$\|f_r^s\|_{\mathcal{L}_{\tilde{p}(\cdot),q}(\Omega)} \lesssim \|f\|_{(\mathcal{H}_{p(\cdot)}^s(\Omega), BMO_2(\Omega))_{\theta,q}}.$$

Since $r \geq 1$, from this and Proposition 5.4, one can conclude that

$$\|f\|_{\mathcal{H}_{\tilde{p}(\cdot),q}^s(\Omega)} \lesssim \|f\|_{(\mathcal{H}_{p(\cdot)}^s(\Omega), BMO_2(\Omega))_{\theta,q}}.$$

Consequently, $(\mathcal{H}_{p(\cdot)}^s(\Omega), BMO_2(\Omega))_{\theta,q} = \mathcal{H}_{\tilde{p}(\cdot),q}^s(\Omega)$ for the case $p_- > 1$, which finishes the proof of Step 1.

Step 2: In this step, we show Theorem 5.1 for the case $p_- \leq 1$ with $\theta + p_- > 1$. Since $\theta + p_- > 1$, there exists $\lambda \in (0, 1)$ such that $p_- > 1 - \theta\lambda$. Let $\eta \in (0, 1)$ with $\eta = \lambda\theta$ and let $0 < q, \bar{q} \leq \infty$. It then follows from Lemma 2.18 and Theorem 4.1 that

$$\begin{aligned}(\mathcal{H}_{p(\cdot)}^s(\Omega), \mathcal{H}_{p(\cdot)/(1-\theta),q}^s(\Omega))_{\lambda,\bar{q}} &= \big(\mathcal{H}_{p(\cdot)}^s(\Omega), (\mathcal{H}_{p(\cdot)}^s(\Omega), H_\infty^s(\Omega))_{\theta,q}\big)_{\lambda,\bar{q}} \\ &= (\mathcal{H}_{p(\cdot)}^s(\Omega), H_\infty^s(\Omega))_{\eta,\bar{q}} = \mathcal{H}_{p(\cdot)/(1-\eta),\bar{q}}^s(\Omega).\end{aligned}$$

Moreover, we can find a number $\beta \in (1, \infty)$ such that $\frac{1}{p_-} < \beta < \frac{1}{1-\eta}$, and choose $\alpha \in (0, 1)$ such that $1 - \alpha = \beta(1 - \eta)$. We know that $1 - \eta < p_-$, thus $\beta p_- > 1$. By Step 1 of this proof, we have

$$\mathcal{H}_{p(\cdot)/(1-\eta),\bar{q}}^s(\Omega) = (\mathcal{H}_{\beta p(\cdot)}^s(\Omega), BMO_2(\Omega))_{\alpha,\bar{q}}.$$

Take $\mu \in (0, 1)$ satisfying $(1 - \mu)(1 - \alpha) = \beta(1 - \theta)$. Using the preceding equality and Lemma 2.18, we further get

$$\begin{aligned}(\mathcal{H}_{p(\cdot)/(1-\eta),\bar{q}}^s(\Omega), BMO_2(\Omega))_{\mu,q} &= \big((\mathcal{H}_{\beta p(\cdot)}^s(\Omega), BMO_2(\Omega))_{\alpha,\bar{q}}, BMO_2(\Omega)\big)_{\mu,q} \\ &= (\mathcal{H}_{\beta p(\cdot)}^s(\Omega), BMO_2(\Omega))_{(1-\mu)\alpha+\mu,q} \\ &= \mathcal{H}_{\beta p(\cdot)/(1-\mu)(1-\alpha),q}^s(\Omega) = \mathcal{H}_{p(\cdot)/(1-\theta),q}^s(\Omega).\end{aligned}$$

Now, set

$$Y_1 = \mathcal{H}_{p(\cdot)}^s(\Omega), \qquad Y_2 = \mathcal{H}_{p(\cdot)/(1-\eta),\bar{q}}^s(\Omega),$$

and

$$Y_3 = \mathcal{H}_{p(\cdot)/(1-\theta),q}^s(\Omega), \qquad Y_4 = BMO_2(\Omega).$$

From the argument above, it follows immediately that

$$Y_2 = (Y_1, Y_3)_{\lambda,\bar{q}}, \quad \text{and} \quad Y_3 = (Y_2, Y_4)_{\mu,q}.$$

Therefore, applying Lemma 2.19, we obtain

$$Y_3 = \mathcal{H}^s_{p(\cdot)/(1-\theta),q}(\Omega) = (Y_1, Y_4)_{\theta,q} = (\mathcal{H}^s_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q},$$

that is, $(\mathcal{H}^s_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q} = \mathcal{H}^s_{\tilde{p}(\cdot),q}(\Omega)$. Now the proof is complete. □

Recall that the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is said to be regular, if for every $n \geq 0$ and $A \in \mathcal{F}_n$, there exists $B \in \mathcal{F}_{n-1}$ such that

$$A \subset B \quad \text{and} \quad \mathbb{P}(B) \leq K\mathbb{P}(A),$$

where K is a positive constant independent of n and the choices A and B; see [21]. From [16, Theorem 5.4], it follows that, if $(\mathcal{F}_n)_{n\geq 0}$ is regular and $p(\cdot) \in \mathcal{B}([0, 1])$ satisfies the locally log-Hölder continuous condition, then,

$$\mathcal{H}^s_{p(\cdot)}(\Omega) = \mathcal{H}^S_{p(\cdot)}(\Omega) = \mathcal{H}^M_{p(\cdot)}(\Omega) = \mathcal{P}_{p(\cdot)}(\Omega) = \mathcal{Q}_{p(\cdot)}(\Omega),$$

with equivalent quasi-norms. Furthermore, for any $q \in (0, \infty]$,

$$\mathcal{H}^s_{p(\cdot),q}(\Omega) = \mathcal{H}^S_{p(\cdot),q}(\Omega) = \mathcal{H}^M_{p(\cdot),q}(\Omega) = \mathcal{P}_{p(\cdot),q}(\Omega) = \mathcal{Q}_{p(\cdot),q}(\Omega),$$

with equivalent quasi-norms; see [28, Theorem 4.4]. A combination of these results and Theorem 5.1 immediately yields the following corollary.

Corollary 5.6 *Let $p(\cdot) \in \mathcal{B}([0, 1])$ satisfy the locally log-Hölder continuous condition and let $0 < q \leq \infty$, $0 < \theta < 1$ with $\theta + p_- > 1$. If $(\mathcal{F}_n)_{n\geq 0}$ is regular, then*

$$(X_{p(\cdot)}(\Omega), BMO_2(\Omega))_{\theta,q} = X_{\tilde{p}(\cdot),q}(\Omega),$$

where $\tilde{p}(\cdot)$ was defined in (4.1). Here X denotes one of the spaces $\mathcal{H}^S$, $\mathcal{H}^M$, $\mathcal{P}$ and $\mathcal{Q}$.

Applying Lemma 2.18 together with Theorem 5.1 and Corollary 5.6, one can further obtain the real interpolation between variable martingale Hardy–Lorentz spaces and martingale BMO spaces.

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