Some Result on Triple Fiber Bundle

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Abstract— The aim of this paper to introduce and and study new concepts of the Triple fiber bundle, Triple local Serre fibration, Triple path lifting property, and prove that the Triple Hurewicz (Serre) fiber bundle over paracompact base space is a Triple Hurewicz (Serre) fibration.

Keywords— c-topology, Tri-fiber bundle, Tri- Local Serre fibration, Tri-path lifting property, Tri-bundle property.

1   INTRODACTION

Fiber bundle and more general fibrations are basic objects of study in many areas of mathematics [9][17]. A continuous map is called Hurewicz fibration if it has the covering homotopy property for all topological space and called Serre fibration if it has the covering homotopy property for the CW complex space [1][12][2], Serre fibration are more appropriate for many purposes [11], every Hurewicz fibration is serre fibration but the converse is not true[14]. A fiber bundle (is also called locally trivial fiber bundle) over B and fiber F (where B and F are topological space) is a topological space E togather with a continuous map $f : E \rightarrow B$ such that for every $b \in B$ there exist an open neighborhood U of b for which $f^{-1}(U)$ is homeomorphic to $U \times F$ and restriction of f to $f^{-1}(U)$ is the standard projection onto first term $f : U \times F \rightarrow U$ [6][3]. Fibrations need not to be fiber bundles. In fact, the definition of fibrations is less stringent than that of fiber bundles, and it is not difficult to slightly modify fiber bundles to obtain fibrations which is not locally trivial[10]. A fiber bundle is regular Hurwicz (Serre) fibration if the base space is paracompact[1][4]. In addition, mixed (co) fibration and mixed Hurewicz (co) fibration which introuduce by [2][4][3] . Also, the mixed Serre (co) fibration was study by [18]. In this paper we introduce new concept Triple fiber bundle which is generalization of fiber bundle. Although, we introduce new concept Triple structure of triple Serre fibration [13].

2- PRELIMINARIES

Definition 2.1 [16] let (X, τ) be a topological space and let $\{U_\alpha \subset x\}_{\alpha \in I}$ be an open cover, then refinement of this open cover is a set of open subset $\{V_\beta \subset X\}_{\beta \in J}$ which is still an open cover in itself and such that for each $\beta \in J$ there exists $\alpha \in I$ with $V_\beta \subset U_\alpha$.

Definition 2.2 [7] A Hausdorff space X is paracompact if each open covering of X has an open neighborhood finite refinement.

Definition 2.3 [8] A topological space is called shrinking space if every open cover is a shrinking. A shrinking of an open cover is another open cover by the same indexing set, with hold the property that the closure of any open set lies inside the corresponding original open set, and the fact are know that every paracompact space is shrinking space.[see also 16]

Theorem 2.4 [7]   (P. Urysohn)
Let A, B be any pair of disjoint closed sets in a normal space X,
Then there exist a continuous map $f: X \to I$, called Urysohn function for A, B such that,
$f(x) = 0$ for $x \in A.$
$f(x) = 1$ for $x \in B.$

Definition 2.5 [7] Let (E , p ,B) be a fiber structure, let X be any space, and let $h_t : X \to B$ be a homotopy, the map $h_t^*: X \to E$ be a homotopy covering h_t. We say that h_t^* is stationary with h_t if for each $x_0 \in X$ such that $h_t(x_0)$ is constant as a function of t, the function $h_t^*(x_0)$ is also constant.

Definition 2.6 [7][12][15]   Let $p : E \to B$ be a continuous map of topological spaces,

(1) A fiber structure is called *a fibration* for a class $\Re$ of spaces if the covering homotopy property (C.H.P) is hold.
(2) A fiber structure for a class $\Re$ of all spaces is called a *Hurewicz fibration*.
(3) p has the covering homotopy property (C.H.P) with respect to a CW-complex space X is called *Serre Fibration*.
(4) The fibration *is (regular)* if h_t^* can always be selected to be is *stationary* with h_t.

Definition 2.7 [5][14]
 the map $f : E \to B$ is called *fiber bundle* , *with fiber* F if it satisfies the following properties:

1) $f^{-1}(b_0) = F$
2) $f : E \to B$ is surjective
3) For every point $y \in B$ there is an open neighborhood $U_y \subset B$ and a "fiber preserving homeomorphism" $\psi_{U_x}: f^{-1}(U_x) \to U_x \times F$, that is a homeomorphism making the following diagram commute:
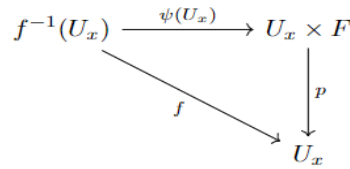
Fig. 1. : Fiber bundle

Corollary 2.8 [10]
If $f : E \rightarrow X$ is a fiber bundle over paracompact space X then f is fibration.

Definition 2.9[18] Let Y be a CW-space, $f_1 : E_1 \rightarrow Y$, $f_2 : E_2 \rightarrow Y$ and $\alpha : E_2 \rightarrow E_1$ are maps of spaces such that $f_1 o\ \alpha = f_2$, let $E_i = \{E_1, E_2\}$, $f_i = \{f_1, f_2\}$, where $i = 1,2$. The quartic $\{E_i, f_i, Y, \alpha\}$ has the mixed covering homotopy property (M-CHP) w.r.t a CW-space X iff given a map $k : X \rightarrow E_2$ and a homotopy $h_t : X \rightarrow Y$ such that $f_2 o\ k = h_0$, then there exists a homotopy $g_t : X \rightarrow E_1$ such that
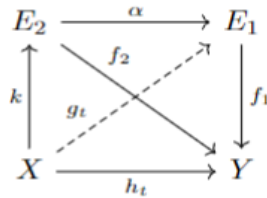1) $f_1 o g_t = h_t$.
2) $\alpha\ ok = g_0$,

Fig. 2. M-Serre Fibration

M-fiber space is called M-Serre fibration, if it has the (MCHP) with respect to a CW-space Y, show Fig.2 above .

Definition 2.10 [13]
1) Let E_1, E_2, E_3, X be any four topological spaces, let $E_i = \{E_1, E_2, E_3\}$, $f_i = \{f_1, f_2, f_3\}$, $\alpha_{ij} = \{\alpha_{21}, \alpha_{32}, \alpha_{31}\}$ where $f_1 = E_1 \rightarrow X$, $f_2 = E_2 \rightarrow X$, $f_3 = E_3 \rightarrow X$ are three maps, and $\alpha_{21} = E_2 \rightarrow E_1$, $\alpha_{32} : E_3 \rightarrow E_2$, $\alpha_{31} : E_3 \rightarrow E_1$ such that $\alpha_{21} o\ \alpha_{32} = \alpha_{31}$, $f_1 o\ \alpha_{21} = f_2$, $f_2 o\ \alpha_{32} = f_3$ then $(E_i, f_i, X, \alpha_{ij})$ is a Triple fiber space (Tri-fiber space), show Fig.3   below. If $E_1 = E_2 = E_3 = E$, α_{ij} = identity $f_1 = f_2 = f_3 = f$ then (E, f, X) is the usual fiber space.
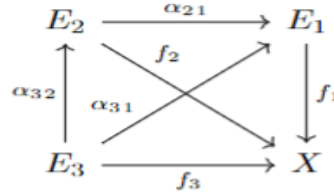
Fig. 3. Triple-fiber space

2) Let $\{E_i, f_i, X, \alpha_{ij}\}$, where $i > j$, be a Tri-fiber space and let $x_0 \in X$. Then $f = \{f_i^{-1}(x_0)\}$ is the Tri-fiber over x_0.

Definition 2.11 [13] Let Y be a CW-complex space $f_1 : E_1 \rightarrow Y$, $f_2 : E_2 \rightarrow Y$, $f_3 : E_3 \rightarrow Y$, $\alpha_{21} : E_2 \rightarrow E_1$, $\alpha_{32} : E_3 \rightarrow E_2$, and $\alpha_{31} : E_3 \rightarrow E_1$ are maps of spaces such that

$\alpha_{21} o\ \alpha_{32} = \alpha_{31}$, $\quad f_1 o\ \alpha_{21} = f_2$, $\quad f_2 o\ \alpha_{32} = f_3$, let $E_i = \{E_1, E_2, E_3\}$ $f_i = \{f_1, f_2, f_3\}$, $\alpha_{ij} = \{\alpha_{21}, \alpha_{32}, \alpha_{31}\}$ where $i = 1,2,3$, $j = 1,2$ and $i > j$,(see Fig. below), the quartic $(E_i, f_i, Y, \alpha_{ij})$ has the Triple covering homotopy property (Tri-CHP) w.r.t a CW-space X iff given a map $k_i : X \rightarrow E_i$ and a homotopy $\quad h_t : X \rightarrow Y$ such that $f_i o\ k_{\ i} = h_0$, then

there exists a homotopy $\qquad L_t^i : X \rightarrow E_i \qquad\qquad$ such that (1) $f_i o\ L_t^i = h_t$, (2) $k_{\ i} = L_0^i$
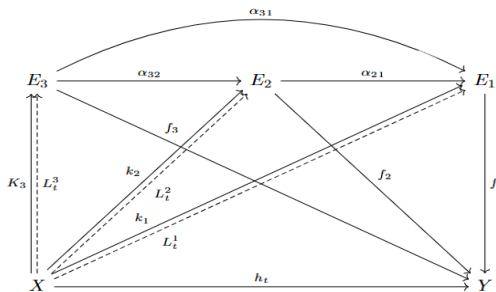

Fig. 4. Tri Serre fibration

The Tri-fiber space is called Tri-Serre fibration if it has the (Tri-CHP) with respect to all CW-complex Y.

3-Triple Fiber Bundle

Definition 3.1. Let B be a base space with basepoint $b_0 \in B$,and $f_i : E_i \rightarrow B$ be continuous maps where $i = 1,2,3$, $f_i = \{f_1, f_2, f_3\}$ and $E_i = \{E_1, E_2, E_3\}$,

The maps $f_i : E_i \to B$ are called Tri-fiber bundle , *with fiber F_i* if it satisfies the following properties:

1) $f_i^{-1}(b_0) = F_i$
2) $f_i : E_i \to B$ is surjective
3) For every point $y \in B$ there is an open neighborhood $U_y \subset B$ and a "fiber preserving homeomorphism" $\psi_{i_{U_x}}: f_i^{-1}(U_x) \to U_x \times F_i$, that is a homeomorphism making the following diagram commute:
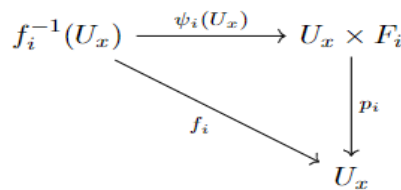

Fig. 5. Triple fiber bundle

Definition 3.2 Let $(E_i , p_i , B, \alpha_{ij})$ be Tri-fiber structure where $i, j = 1,2,3$ and α_{ij} is continuous maps where $i > j$, let X be any space, and let $h_t : X \to B$ be a homotopy, the maps $L_t^i : X \to E_i$ be a homotopy covering h_t. We say that L_t^i is stationary with h_t if for each $x_0 \in X$ such that $h_t(x_0)$ is constant as a function of t, the function $L_t^i(x_0)$ is also constant.

Definition 3.3 The Tri-Serre fibration is (regular) if L_t^i can always be selected to be is stationary with h_t.

Definition 3.4 A Tri-fiber structure $(E_i , p_i , B, \alpha_{ij})$ where $i, j = 1,2,3$, α_{ij} is continuous maps where $i > j$ and
X is CW complex space is called Triple local (regular) Serre fibration if for each $b \in B$ has a neighborhood U such that $(p_i^{-1}(U), p_i|p_i^{-1}(U), U, \alpha_{ij})$ is Tri- (regular) Serre fibration.

Theorem 3.5 Let the base B is paracompact, then Tri-fiber structure $(E_i , p_i , B, \alpha_{ij})$ where $i, j = 1,2,3$, is a Tri-(regular) Serre fibration if and only if it is Tri-local (regular) Serre fibration.

Proof. It is clear that every Tri-(regular) Serre fibration $(E_i , p_i , B, \alpha_{ij})$ is Tri-local (regular) Serre fibration, so only the converse requires proof.
Since B is paracompact , the covering of B by the neighborhood U_i of the definition has a neighborhood-finite refinement $\{V_{i_\beta}\}$ which we shrink to get $\{\overline{W}_{i_\beta}\}$. [Since every paracompact space is normal space[7]], so For each index β, let $c_{i_\beta}: X \to I$ be a

Urysohn function such that $c_{i_\beta}\left(\overline{W}_{i_\beta}\right) = 1$ and $c_{i_\beta}\left(V^c{}_{i_\beta}\right) = 0$. The covering of B by the open sets $U_{i_\beta} = c^{-1}{}_{i_\beta}[I - \{0\}]$ satisfy the requirement of theorem.

Corollary 3.6 If $f_i : E_i \to B$ are Tri-fiber bundle over paracompact space B then f_i is Tri-(Serre) fibration.

Proof: Since Tri-fiber bundles have local product structure ($U \times F_i \to U$), thorem 3.5 implies that a Tri-fiber bundle are Tri-(Serre) fibration.

4-Triple Path Lifting Property

In this section we introduce the triple path lifting property (Tri-PLP) with the some result and properties.

Definition 4.1.[7][9] Let M, N be topological spaces, and denoted by N^M the set of all continuous maps of M into N. for each pair of sets $A \subseteq M$, $B \subseteq N$, let $(A, B) = \{f \in N^M | f(A) \subseteq B\}$. The compact open topology (c-topology) in N^M is that having as sub basis all sets (A, V),where $A \subseteq M$ is compact $B \subseteq N$ is open.

Definition 4.2[7] A path in X is continuous mapping unit interval I into X, rather than a continuous image of I in X ; that is, a path in X is an element of X^I. The path $\alpha \in X^I$ is said to start at the point $\alpha(0) \in X$ and to end at the point $\alpha(1) \in X$; a closed path or loop at $x_0 \in X$ is path starting and ending at x_0.
   constant path $\eta(I) = x_0$ is called the null path or loop at x_0.

Definition 4.3[12] Let $p: E \to B$ be a map. Then p is said to have path lifting property (abbreviated PLP) if for any $e \in E$ and any path $f: I \to B$ with $f(0) = p(e)$, then there exist a path $\tilde{f} : I \to E$ such that $\tilde{f}(0) = e$,
$p\tilde{f} = f$, and $\tilde{f}$ depends continuously on e and f.
   For a precise definition, let $B^I: (f : I \to B)$ and let $\Omega_p \subset E \times B^I$ be
a subspace of the product space $E \times B^I$
Defiend by,
$$\Omega_p = \{(e, f) \in E \times B^I \,|p(e) = f(0)\}.$$
   Define a map
$$q: E^I \to \Omega_p$$
By taking $q(\tilde{f}) = (\tilde{f}(0), p\tilde{f})$ for each $\tilde{f}: I \to$ E in E^I.
  A lifting function for a fiber structure (E, p, B) is continuous map $\lambda : \Omega_p \to E^I$, such that $\lambda(e, f)(0) = e$, and $p \circ \lambda(e, f)(t) = f(t)$ for each $(e, f) \in \Omega_p$ and $t \in I$. If λ lifts constant path to constant paths, then it is called a regular lifting function for $p: E \to B$ and the triple (E, p, B) is called a regular fiber space.

Then the map $p: E \to B$ is said to have the PLP $q\lambda$ is the identity map on Ω_p. It is well-known that a map $p: E \to B$ has the PLP iff it has the ACHP (i.e., it is has Hurewicz fibration).

Definition 4.4: let $(E_i, f_i, Y, \alpha_{ij})$ be Tri-fiber structure, where $i = 1,2,3$, $j = 1,2$ $i > j$, and X be CW complex space , and $Y^I = \{\omega : I \to Y\}$, $\Omega_{f_i} \subseteq E_i \times Y^I$ be subspaces

$\Omega_{f_i} = \{(e_i, \omega) \in E_i \times Y^I | f_i(e_i) = \omega(0)\}$.

A Tri-lifting functions for $(E_i, f_i, Y, \alpha_{ij})$ is continues maps $\lambda_i : \Omega_{f_i} \to E_i^I$ such that $\lambda_i(e_i, \omega)(0) = e$ and $f_i o \lambda_i(e_i, \omega)(t) = \omega(t)$, for each $(e_i, \omega) \in \Omega_{f_i}$, and $t \in I$

Thus $\lambda_i = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\Omega_{f_i} = \{\Omega_{f_1}, \Omega_{f_2}, \Omega_{f_3}\}$, where $\lambda_1 : \Omega_{f_1} \to E_1^I$, $\lambda_2 : \Omega_{f_2} \to E_2^I$, $\lambda_3 : \Omega_{f_3} \to E_3^I$ defined as

$\lambda_1(e_1, \omega)(0) = e_1$, $f_1 o \lambda_1(e_1, \omega)(t) = \omega(t)$,
$\lambda_2(e_2, \omega)(0) = e_2$, $f_2 o \lambda_2(e_2, \omega)(t) = \omega(t)$,
$\lambda_3(e_3, \omega)(0) = e_3$, $f_3 o \lambda_3(e_3, \omega)(t) = \omega(t)$,

Thus a Tri-lifting functions there for associates with each $e_i \in E_i$, and each path ω in Y starting at $f_i(e_i)$,

A path $\lambda_1(e_1, \omega)$ in E_1 , $\lambda_2(e_2, \omega)$ in E_2 and $\lambda_3(e_3, \omega)$ in E_3 starting at e_1, e_2 and e_3 , and is Tri-cover of ω since the c-topology used in E^I, the continuity of λ is equivalent to that of associated $\widehat{\lambda}: \Omega_{f_i} \times I \to E_i$.

Tri- regular Serre If λ_i lifts constant paths to constant paths, then it called a regular lifting functions for $f_i : E_i \to Y$ and the quartic $(E_i, f_i, Y, \alpha_{ij})$ is called Tri-regular fiber space.

Now, define a map

$$q_i: E_i^I \to \Omega_{f_i}$$

By taking $q(\alpha_i) = (\alpha_i(0), f\alpha_i)$ for each $\alpha_i: I \to E_i$ in E_i^I.

Then $f_i : E_i \to Y$ is said to have Tri-PLP if $q_i \lambda_i$ are the identity maps on Ω_{f_i}.

Note : We can see the mixed path lifting property in [18].

Example 4.5 : Let $(E_i, f_i, Y, \alpha_{ij})$ be the Tri-Serre fibration ,and let $Y^I = \{\omega : I \to Y\}$ and $f_1(\omega) = \omega(1)$, $f_2(\omega') = \omega'(1)$ and $f_3(\omega'') = \omega''(1)$ where $\omega, \omega', \omega'' : I \to Y$. A lifting functions

$\lambda_1 : \Omega_{f_1} \to E_1^I$ for $f_1 : E_1 \to Y$,
$\lambda_2 : \Omega_{f_2} \to E_2^I$ for $f_2 : E_2 \to Y$,
$\lambda_3 : \Omega_{f_3} \to E_3^I$ for $f_3 : E_3 \to Y$.

Are defined as follows:

$$\lambda_1(\delta,\omega)(t)(s) = \begin{cases} \delta\left(\frac{6s}{1+t}\right) & if & 0 \le s \le \frac{1+t}{\delta} \\ \omega(6s - t - 1) & if & \frac{1+t}{\delta} \le s \le \frac{2+t}{\delta} \end{cases}$$

$$\lambda_2(\delta'\omega')(t)(s) = \begin{cases} \delta'(6s - t - 2) & if & \frac{2+t}{\delta} \le s \le \frac{3+t}{\delta} \\ \omega'(6s - t - 3) & if & \frac{3+t}{\delta} \le s \le \frac{4+t}{\delta} \end{cases}$$

$$\lambda_3(\delta''\omega'')(t)(s) = \begin{cases} \delta''(6s - t - 3) & if & \frac{4+t}{\delta} \le s \le \frac{5+t}{\delta} \\ \omega''\left(\frac{6s-t-5}{1-t}\right) & if & \frac{5+t}{\delta} \le s \le 1 \end{cases}$$

Notice that the particular λ_1, λ_2 and λ_3 are not regular because they are not constants.

Theorem 4.6 The Tri-Serre structure $(E_i, f_i, X, \alpha_{ij})$ where $i, j = 1,2,3$ and $i > j$ is Tri-Serre fibration if and only if f_i has T-lifting functions.

Proof. If f_i is Tri-Serre fibration, let $Y = \Omega_{f_i}$, let $g_i : \Omega_{f_i} \to E_i$, $h_t : \Omega_{f_i} \to X$ define by $g_i(e_i, \omega)(0) = e_i$, $h_t(e_i, \omega) = \omega(t)$, then $h_0(e_i, \omega)(0) = \omega(0) = f_i(e_i) = f_i \ o \ g_i(e_i, \omega)$,

since f_i is Tri-Serre fibration, then there exist $L_t^i : \Omega_{f_i} \to E_i$ such that
(1) $f_i \ o \ L_t^i = h_t$, (2) $g_i(e_i, \omega), = L_0^i$

Define as a lifting function $\lambda_i(e_i, \omega)(t) = L_t^i(e_i, \omega)$
i.e:

$$\lambda_1(e_1, \omega)(t) = L_t^1(e_1, \omega) \implies \lambda_1(e_1, \omega)(0) = L_0^1(e_1, \omega) = e_1$$
$$f_1 \ o \ \lambda_1(e_1, \omega)(t) = f_1 \ o \ L_t^1(e_1, \omega) \implies f_1 \ o \ \lambda_1(e_1, \omega) = h_t(e_i, \omega) = \omega(t)$$

$$\lambda_2(e_2, \omega)(t) = L_t^2(e_2, \omega) \implies \lambda_2(e_2, \omega)(0) = L_0^2(e_2, \omega) = e_2$$
$$f_2 \ o \ \lambda_2(e_2, \omega)(t) = f_2 \ o \ L_t^2(e_2, \omega) \implies f_2 \ o \ \lambda_2(e_2, \omega) = h_t(e_i, \omega) = \omega(t)$$

$$\lambda_3(e_3, \omega)(t) = L_t^3(e_3, \omega) \implies \lambda_3(e_3, \omega)(0) = L_0^3(e_3, \omega) = e_3$$
$$f_3 \ o \ \lambda_3(e_3, \omega)(t) = f_3 \ o \ L_t^3(e_3, \omega) \implies f_3 \ o \ \lambda_3(e_3, \omega) = h_t(e_i, \omega) = \omega(t)$$

Conversely:

If f_i is Tri-lifting function
Let $g_i^* : Y \to E_i$ and $h_t^* : Y \to X$ such that $f_i \ o \ g_i^* = h_0^*$, consider
$\omega_y : I \to X$ such that $\omega_y(t) = h_i^*(y)$

And define $L_t^{*i}: Y \to E_i$ as
$$L_t^{*i}(e) = \lambda_i [\, g_i^* , \omega_y \,](t)$$

Thus : $L_0^{*i} = g_i$ and $f_i \; o \; L_t^{*i} = h_t^*$
Hence L_t^{*i} are Tri-CHP of h_t with respect to Y, since Y is arbitrary

There for f_i is Tri-Serre fibration .

Refrences

[1] Armin Rainer, Wien, " Orbit Projections as Fibrations" Czechoslovak Mathematical Journal, V 59, No
(134), p 592-538, (2009).

[2]   Daher W. Al Baydli, "New types of Fibration" ,M.SC. A research Babylon University,2003.

[3] D. Al Baydali, A Rahm, "Computations in the Pre-Bloch group", Journal of Physics: Conference Series,   V1591,   No 1, P 012102, 2022.

[4] D. Al Baydali, M. M. AL Hadidi, "Bockstein homomorphism for non-prime power groups", AIP Conference Proceeding, V 2398, No 1, P 060085, 2022.

[5] Daniel S. Freed, "Fiber Bundle and Vector Bundle", University of Texas, Austin, January 21, 2017.

[6] Dmitry N. Kozlov, "Combinatorial Algebraic Topology", Algorithms and Computation in Mathematics,   Springer-Verlag Berlin Heidelberg, V 21, 2008.

[7] Dugundji, J., "Topology", Allyn and Bacon. Boston, 1966.

[8] Hart k.P., Nagata Jun-iti, Vaughan J.E., "Encyclopedia of General Topology", Elsevier, 2004.

[9] Huynh Q. Vu, "Lecture Note on Topology", University of Science, Ho Chi Minh City, 2019.

[10]   James F. Davis, Paul Kirk, "Lecture Note in Algebraic Topology", Indiana University, Blooming-ton, IN 47405,1991.

[11]   J.P.May , "A Concise in Algebraic Topology" , University of Chicago Press,1999 .

[12]   Mustafa, H.J, "Some theorems on Fibration and Cofibration",Ph.D.Thesis ,California University, Los Angeles ,1972.

[13]   Nabaa Ibrahim and Daher Al Baydli " Triple strecture of Triple Serre Fibration "AIP Conference Proceeding, 2023,Appear.

[14]   Ralph L. Cohen, "The Topology Fiber Bundles Lecture Note", Stanford University, August, 1998.

[15]   Stephen A. Mitchell, "Notes on Serre Fibration" , website link internet, August 2001.

[16]   Tyrone Cutler, "Paracompact Space Lecture Note", Bielefeld University, Germany, 2020.

[17]   W. h. Ali, D. W. Freh, J. H. Hussein, "Supra q-open set and supra q-continuous map in supra topological space", ", AIP Conference Proceeding, V 2398, No 1, P 060068, 2022.

[18]   Z. Y. Habeeb, D. Al Baydli, "MIXED SEREE FIBRATION", Wasit Journal of Pure Sciences, V 1, No 2, P 50-60, 2022.