An analytical investigation of uniformly star like class of functions via generalized koebe function

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Abstract. Recently, several of the generalizations Koebe function are introduced and investigated. In this study, a linear complex operator is investigated in terms of the generalized Koebe function and Wright function. A new geometric class, namely a uniformly starlike class, is provided. Furthermore, certain properties are discussed such as coefficient bounds, growth, distortion results, radii of convexity, starlikeness and close-to-convexity.

Keywords. Univalent function, Starlike function, Hadamard product, Wright function, Koebe function.

1 Introduction

$\mathcal{H}(\Lambda)$ the class of holomorphic functions in the unit disk $\Lambda = \{z \in \mathbb{C}: |z| < 1\}$. Let $\mathcal{H}[\mathfrak{a},\mathfrak{n}]$ be the subclass of $\mathcal{H}(\Lambda)$ involving of functions of the formula $\varphi(z) = \mathfrak{a} + \mathfrak{a}_{\mathfrak{n}} z^{\mathfrak{n}} + \mathfrak{a}_{\mathfrak{n}+1} z^{\mathfrak{n}+1} + \cdots$, and let $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H} = \mathcal{H}[1,1]$. The class $\mathfrak{A}$ of holomorphic functions is stated as:

$$\varphi(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \rho_{\mathfrak{n}} z^{\mathfrak{n}}, \tag{1}$$

which are normalized (means that $\varphi(0) = \varphi'(0) - 1 = 0$) in Λ**.**

A function φ in $\mathfrak{A}$ is said to be starlike of order δ in Λ**,**

$$\mathcal{R}e\left(\frac{z\varphi'(z)}{\varphi(z)}\right) > \delta,$$

for some$\delta(0 \leq \delta < 1)$ and for all $z \in \Lambda$.

A function φ in $\mathfrak{A}$ is said to be convex of order δ in Λ, if φ satisfies the inequality:

$$\mathcal{R}e\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > \delta\,,$$

for some $\delta(0 \leq \delta < 1)$ and for all $z \in \Lambda$.

The φ is to be convex of order δ in Λ if φ in $\mathfrak{A}$ satisfies the inequality.

$$\mathcal{R}e\{\varphi'(z)\} > \delta\,. \tag{2}$$

A function φ in $\mathfrak{A}$ is convex of order $\delta \Leftrightarrow z\varphi'(z)$ is starlike of order δ in Λ, [1].

For $-1 < \nu \leq 1$ and $\eta \geq 0$ a function $\varphi \in \mathfrak{A}$ is said to be in the class of η-parabolic starlike function denoted by $\eta - S^*(\nu)$ if

$$\mathcal{R}e\left\{\frac{z\varphi'(z)}{\varphi(z)} - \nu\right\} > \eta\left|\frac{z\varphi'(z)}{\varphi(z)} - 1\right|, \qquad z \in \Lambda$$

The class S^* of starlike function is equal by $S^* \equiv S^*(0)$. Bharti et al. [2] Defined $\eta - S^*(\nu)$ to be the class of function φ with $0 \le \eta < \infty$, and $0 \le \nu < 1$ the satisfy the condition:

$$\mathcal{R}e\left\{\frac{z\varphi'(z)}{\varphi(z)}\right\} \ge \eta\left|\frac{z\varphi'(z)}{\varphi(z)} - 1\right| + \nu.$$

The familiar wright function is given as [3]:

$$\mathrm{w}(\tau, \beta; z) = \sum_{\mathfrak{n}=0}^{\infty} \frac{z^{\mathfrak{n}}}{\mathfrak{n}!\,\Gamma(\tau\,\mathfrak{n} + \beta)}, \qquad (\tau > -1, \beta \in \mathbb{C}).$$

In [4] Moureh et at. Imposed generalization Koebe function as:

$$\mathcal{K}_{\mu,\sigma}(z) = \sum_{\mathfrak{n}=0}^{\infty} \frac{\Gamma(\mu + \sigma\mathfrak{n})}{\Gamma(\mu)\,\Gamma(\mathfrak{n}+1)}\, z^{\mathfrak{n}}, \qquad (z, \mu \in \mathbb{C};\ \ 0 < \mathcal{R}(\sigma)) \tag{3}$$

This class $\eta - S^*(\nu)$ was considered in [5]. Most investigators are interested to study and they introduced various classes of uniformly star-like and convex functions. For instance, Breaz [6], Breaz el. at [7], Stanciu and Breaz in [8], Al-Janaby et. al [9], Layth et.at [10]. Furthermore, in the last century, the use of special functions (SFs) has been intensified productively because of its significance in the Geometric Function Theory (GFT). The reason that SFs attracted researchers is their use as a tool in resolving Bieberbach's problem in 1984 by De Branges, [11]. Then, a number of important works on connections between analytic univalent and SFs have been introduced by several complex analyses such as, Mahmoud et. al [12], Atshan et. al [13], Elhaddad and Darus [14], Yan and Liu [15], Al-Janaby et. al [16], Oros [17] and Layth et. al [18].

2.1 Imposed new Wright - Koebe Operator

The normalization of the Wright function and generalized Koebe-type function given as:

$$\psi(\beta, \tau; z) = z\Gamma(\beta)\mathrm{w}(\tau, \beta; z) = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{\Gamma(\beta)}{(\mathfrak{n}-1)!\,\Gamma\big(\beta + \tau(\mathfrak{n}-1)\big)}\, z^{\mathfrak{n}}, \tag{4}$$

and,

$$\mathcal{N}_{\mu,\sigma}(z) = z\mathcal{K}_{\mu,\sigma}(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{\Gamma(\mu + \sigma(\mathfrak{n}-1))}{\Gamma(\mu)\,(\mathfrak{n}-1)!}\, z^{\mathfrak{n}}. \tag{5}$$

By using convolution principle, equation (4) and (5) , we imposed a function $\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}(z)$ given by

$$\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{\Gamma(\beta)\Gamma(\mu)}{\Gamma(\mu + \sigma(\mathfrak{n}-1))\Gamma(\beta + \tau(\mathfrak{n}-1))}\, z^{\mathfrak{n}}, \tag{6}$$

Such that

$$\mathcal{N}_{\mu,\sigma}(z) * \mathcal{Q}_{\mu,\beta}^{\sigma,\tau}(z) = \psi(\tau, \beta; z). \tag{7}$$

Therefore, from (6) we consider a new Wright-Koebe operator as: for $\varphi \in \mathfrak{A}$, and $\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}: \mathfrak{A} \to \mathfrak{A}$ such that

$$\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)=\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}(z)*\varphi(z)=z+\sum_{\mathrm{n}=2}^{\infty}\frac{\Gamma(\beta)\Gamma(\mu)}{\Gamma\big(\mu+\sigma(\mathrm{n}-1)\big)\Gamma\big(\beta+\tau(\mathrm{n}-1)\big)}\,\rho_{\mathrm{n}}z^{\mathrm{n}}. \quad (8)$$

For convenience,

$$\mathrm{H}(\beta,\mu,\sigma,\tau)=\frac{\Gamma(\beta)\Gamma(\mu)}{\Gamma(\mu+\sigma(\mathrm{n}-1))\Gamma(\beta+\tau(\mathrm{n}-1))}\ .$$

Thus,

$$\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)=z+\sum_{\mathrm{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathrm{n}}\,z^{\mathrm{n}}. \quad (9)$$

The following specific cases related to operator $\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)$ introduced by (8) are investigated.

Remark 2.1 By assumption suitable special value of the parameters, β,μ,σ,τ, as:

1- For $\mu=\sigma=\tau=\beta=1$, the operator $\mathcal{Q}_{1,1}^{1,1}\varphi(z)=\ z+\sum_{\mathrm{n}=2}^{\infty}\frac{1}{(\mathrm{n}-1)!(\mathrm{n}-1)!}\ \rho_{\mathrm{n}}\,z^{\mathrm{n}}$ is yielded.

2- For $\mu=\sigma=\tau=1,\beta=2$, the operator $\mathcal{Q}_{1,1}^{1,2}\varphi(z)=\ z+\sum_{\mathrm{n}=2}^{\infty}\frac{1}{(\mathrm{n}-1)!(\mathrm{n})!}\ \rho_{\mathrm{n}}\,z^{\mathrm{n}}$ is gained.

3- For $\mu=\ \sigma=1$, the operator $\mathcal{Q}_{1,\beta}^{1,\tau}\varphi(z)=\ z+\sum_{\mathrm{n}=2}^{\infty}\frac{\Gamma(\beta)}{(\mathrm{n}-1)!\Gamma(\tau(\mathrm{n}-1)+\beta)}\ \rho_{\mathrm{n}}\,z^{\mathrm{n}}$ linear operator in terms of normalized wright function is provided.

4- For $\tau=\beta=1$, the operator $\mathcal{Q}_{\mu,1}^{\sigma,1}\varphi(z)=z+\sum_{\mathrm{n}=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\mu+\sigma(\mathrm{n}-1))(\mathrm{n}-1)!}\ \rho_{\mathrm{n}}\,z^{\mathrm{n}}$ linear operator by internal normalized Wright function $\psi(\tau,\beta;z)$ given in (4) is obtained.

2.2 Uniformly Starlike Class

By employing Wright-Koebe operator given by (8), we establish and discuss the following geometric class $\eta-S_{\nu}^{*}(\beta,\mu,\sigma,\tau)$ namely univalent function defined as:

$$\mathcal{R}e\left\{\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}\right\}\geq\eta\left|\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}-1\right|+\nu, \quad (10).$$

where $0\leq\eta<\infty$, and $0\leq\nu<1$.

Let T represent the subclass of $\mathfrak{A}$ including of functions φ in Λ of the formula:

$$\varphi(z)=z-\sum_{\mathrm{n}=2}^{\infty}\rho_{\mathrm{n}}\,z^{\mathrm{n}}, \qquad \text{for } (\rho_{\mathrm{n}}>0\ \mathit{and}\ \ z\in\Lambda) \quad (11)$$

Also let

$$\eta-TS_{\nu}^{*}(\beta,\mu,\sigma,\tau)=S_{\nu}^{*}(\beta,\mu,\sigma,\tau)\cap T. \quad (12)$$

3. Coefficient Bounds

In this section, some interesting geometric properties of the uniformly starlike class $\eta-S_{\nu}^{*}(\beta,\mu,\sigma,\tau)$ are presented and investigated.

Theorem 3.1 A function φ defined by (11), then $\varphi\in\ \eta-S_{\nu}^{*}(\beta,\mu,\sigma,\tau)$ if and only if

$$\sum_{\mathrm{n}=2}^{\infty}[\mathrm{n}(\eta+1)-(\nu+\eta)]\,H(\beta,\mu,\sigma,\tau)\rho_{\mathrm{n}}\leq1-\nu, \quad (\ 0\leq\nu<1). \quad (13)$$

Proof. Suppose that $\varphi(z)$ given by (11) in $\eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$. So that choosing the value of z on the positive real axis, the inequality (10) readily gains

$$\frac{1-\sum_{\mathfrak{n}=2}^{\infty}\mathfrak{n}\,H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}\,z^{\mathfrak{n}-1}}{1-\sum_{\mathfrak{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}\,z^{\mathfrak{n}-1}}-\nu \geq \eta\left|\frac{\sum_{\mathfrak{n}=2}^{\infty}(\mathfrak{n}-1)H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}\,z^{\mathfrak{n}-1}}{1-\sum_{\mathfrak{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}\,z^{\mathfrak{n}-1}}\right|,$$

letting $z\to 1^-$ along the real axis, we yield

$$\frac{1-\sum_{\mathfrak{n}=2}^{\infty}\mathfrak{n}\,H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}}{1-\sum_{\mathfrak{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}}-\nu \geq \eta\,\frac{\sum_{\mathfrak{n}=2}^{\infty}(\mathfrak{n}-1)\,H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}}{1-\sum_{\mathfrak{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}},$$

then

$$\frac{(1-\nu)-\sum_{\mathfrak{n}=2}^{\infty}[\mathfrak{n}(\eta+1)-(\nu+\eta)]H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}}{1-\sum_{\mathfrak{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}} \geq 0.$$

Therefore, we yield assertion (13).

To prove converse, we consider that the inequality (13) holds right and let $|z|=1$. It suffices to prove that

$$\mathcal{R}e\left\{\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}\right\} \geq \eta\left|\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}-1\right|+\nu,$$

we gain,

$$\eta\left|\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}-1\right| - \mathcal{R}\mathrm{e}\left\{\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}-\nu\right\}$$

$$\leq (\eta+1)\left|\frac{z\left(\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)\right)'}{\mathcal{Q}_{\mu,\beta}^{\sigma,\tau}\varphi(z)}-1\right| \leq \frac{(\eta+1)\sum_{\mathfrak{n}=2}^{\infty}(\mathfrak{n}-1)H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}}{1-\sum_{\mathfrak{n}=2}^{\infty}H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}}}.$$

The last expression is bounded above by (1-ν) if

$$\sum_{\mathfrak{n}=2}^{\infty}[\mathfrak{n}(\eta+1)-(\nu+\eta)]\,H(\beta,\mu,\sigma,\tau)\rho_{\mathfrak{n}} \leq 1-\nu,$$

which is right by hypothesis. Therefore, we gain $\varphi(z) \in \eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$.

$$\varphi(z) = z - \frac{(1-\nu)}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}z^{\mathfrak{n}}, \qquad (\mathfrak{n}\geq 2).$$

Corollary 3.1 Let the function $g(z)$ be gained by (7) if $\varphi(z) \in \eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$ then

$$\rho_{\mathfrak{n}} \leq \frac{1-\nu}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}, \qquad (\mathfrak{n}\geq 2). \tag{14}$$

The result is sharp for the following function:

$$\varphi(z) = z - \frac{(1-\nu)}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}z^{\mathfrak{n}}, \qquad (\mathfrak{n}\geq 2). \tag{15}$$

4. Growth Theorems

Theorem 4.1 Let the function $\varphi(z)$ defined by (11) be in the class $\eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$, then

$$|z| + \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|^2 \geq |\,\varphi(z)| \geq |z| - \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|^2. \quad (16)$$

with equality for

$$|\,\varphi(z)| = |z| - \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|^2, \qquad (z = \pm r). \quad (17)$$

Proof: Let $\varphi(z) \in \eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$ By Theorem 3.1, and

$$\mathrm{E}(\mathrm{n}) = \big(\mathrm{n}(1+\eta) - (\nu+\eta)\big)\, H(\beta,\mu,\sigma,\tau),$$

we yield

$$\mathrm{E}(2)\sum_{\mathrm{n}=2}^{\infty}\rho_{\mathrm{n}} \leq \sum_{\mathrm{n}=2}^{\infty} E(\mathrm{n})\rho_{\mathrm{n}} \leq (1-\nu),$$

that is

$$\sum_{\mathrm{n}=2}^{\infty}\rho_{\mathrm{n}} \leq \frac{(1-\nu)}{E(2)} \;.$$

Therefore

$$|\,\varphi(z)| \geq |z| - \sum_{\mathrm{n}=2}^{\infty}\rho_{\mathrm{n}}|z^{\mathrm{n}}| \geq |z| - |z|^2\sum_{\mathrm{n}=2}^{\infty}\rho_{\mathrm{n}} \geq |z| - \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|^2.$$

Similarly, we gain

$$|\,\varphi(z)| \leq |z| + |z|^2\sum_{\mathrm{n}=2}^{\infty}\rho_{\mathrm{n}} \;\leq |z| + \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|^2.$$

Thus, outcome is sharp for $\varphi(z)$

$$|\,\varphi(z)| = |z| - \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|^2, \qquad (z = \pm r).$$

Theorem 4.2 Let the function φ(z) given by (11) be in the class $\eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$, then

$$1 + \frac{(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z| \leq \;|\,\varphi'(z)| \geq 1 - \frac{2(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z| \;. \quad (18)$$

with sharp for $\varphi(z)$

$$|\,\varphi'(z)| = 1 - \frac{2(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|\,, \qquad (z = \pm r). \quad (19)$$

Proof: Let $\varphi(z) \in \eta - TS_\nu^*(\beta,\mu,\sigma,\tau)$ By Theorem 3.1, and ,

$$\mathrm{E}(\mathrm{n}) = \big(\mathrm{n}(\eta+1) - (\nu+\eta)\big)H(\beta,\mu,\sigma,\tau),$$

we have

$$\frac{E(2)}{2}\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}\,\rho_{\mathrm{n}} \leq \sum\nolimits_{\mathrm{n}=2}^{\infty}\frac{E(\mathrm{n})}{\mathrm{n}}\mathrm{n}\,\rho_{\mathrm{n}} = \sum_{\mathrm{n}=2}^{\infty}E(\mathrm{n})\rho_{\mathrm{n}} \leq (1-\nu),$$

that is

$$\sum_{\mathrm{n}=2}^{\infty} \mathrm{n}\,\rho_{\mathrm{n}} \le \frac{2(1-\nu)}{E(2)} .$$

Therefore,

$$|\,\varphi'(z)| \ge 1-\sum_{\mathrm{n}=2}^{\infty} \mathrm{n}\,\rho_{\mathrm{n}}\,|z^{\mathrm{n}-1}| \ge 1-|z|\sum_{\mathrm{n}=2}^{\infty} \mathrm{n}\,\rho_{\mathrm{n}}$$

$$\ge 1-\frac{2(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z|.$$

Similarly, we gain

$$|\,\varphi'(z)| \le 1+|z|\sum_{\mathrm{n}=2}^{\infty} \mathrm{n}\,\rho_{\mathrm{n}} \le 1+\frac{2(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z| .$$

The outcome is sharp for $\varphi(z)$ is

$$|\,\varphi'(z)| = 1-\frac{2(1-\nu)\Gamma(\mu+2\sigma)\Gamma(\beta+2\tau)}{(2+\eta-\nu)\Gamma(\beta)\Gamma(\mu)}|z| , \qquad (z=\pm r).$$

5. Radii, of Convexity Starlikeness, and Close to Convexity

In this section, radii of convexity starlikeness, and close to convexity for functions belonging to the class $\eta - TS_{\nu}^{*}(\beta,\mu,\sigma,\tau)$ are obtained.

Theorem 5.1 Let the function φ(z) defined by (11) be in the class $\eta - TS_{\nu}^{*}(\beta,\mu,\sigma,\tau)$, then

1. $\varphi(z)$ is convex of order $\delta(0\le\delta<1)$ in $|z|<r_1$, where

$$r_1 = \inf_{\mathrm{n}\ge 2}\left\{\left[\frac{(1-\delta)\big(\mathrm{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{\mathrm{n}(\mathrm{n}-\delta)(1-\nu)}\right]^{\frac{1}{\mathrm{n}-1}}\right\}. \qquad (20)$$

2. $\varphi(z)$ is starlike of order $\delta(0\le\delta<1)$ in $|z|<r_2$, where

$$r_2 = \inf_{\mathrm{n}\ge 2}\left\{\left[\frac{(1-\delta)\big(\mathrm{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(\mathrm{n}-\delta)(1-\nu)}\right]^{\frac{1}{\mathrm{n}-1}}\right\}. \qquad (21)$$

3. φ(z) is closed to convex of order $\delta(0\le\delta<1)$ in $|z|<r_3$, where

$$r_3 = \inf_{\mathrm{n}\ge 2}\left\{\left[\frac{(1-\delta)\big(\mathrm{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{\mathrm{n}(1-\nu)}\right]^{\frac{1}{\mathrm{n}-1}}\right\}. \qquad (22)$$

Proof.(1) It is enough to show that

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \le 1-\delta, \quad \text{for } |z|<r_1, \text{ and } \delta(0\le\delta<1),$$

where r_1 is given by (20) Indeed, we find from (11) that

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \le \frac{\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}(\mathrm{n}-1)\rho_{\mathrm{n}}|z|^{\mathrm{n}-1}}{1-\sum_{\mathrm{n}=2}^{\infty}\mathrm{n}\,\rho_{\mathrm{n}}\,|z|^{\mathrm{n}-1}} .$$

Thus, we note that

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \le 1-\delta,$$

which is equivalent to

$$\sum_{\mathfrak{n}=2}^{\infty}\frac{\mathfrak{n}(\mathfrak{n}-\delta)\rho_{\mathfrak{n}}}{1-\delta}|z|^{\mathfrak{n}-1} \le 1 \;. \qquad (23)$$

However, considering Theorem 3.1 we have

$$\sum_{\mathfrak{n}=2}^{\infty}\frac{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(1-\nu)}\rho_{\mathfrak{n}} \;\le 1\,. \qquad (24)$$

Thus, φ is convex of order r_1 $\delta(0 \le \delta < 1)$ if

$$\mathfrak{n}(\mathfrak{n}-\delta)|z|^{\mathfrak{n}-1} \le \frac{(1-\delta)\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(1-\nu)},$$

that is, if

$$|z| \le \left[\frac{(1-\delta)\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{\mathfrak{n}(\mathfrak{n}-\delta)(1-\nu)}\right]^{\frac{1}{\mathfrak{n}-1}}, \qquad (\mathfrak{n} \ge 2).$$

(2) It is enough to show that

$\left|\frac{z\varphi'(z)}{\varphi(z)}-1\right| \le 1-\delta,$ for $|z| < r_2$, and $\delta(0 \le \delta < 1)$,

where r_2 is given by (21). Indeed, we find from (11) that

$$\left|\frac{z\varphi'(z)}{\varphi(z)}-1\right| \le \frac{\sum_{\mathfrak{n}=2}^{\infty}(\mathfrak{n}-1)\rho_{\mathfrak{n}}|z|^{\mathfrak{n}-1}}{1-\sum_{\mathfrak{n}=2}^{\infty}\rho_{\mathfrak{n}}\,|z|^{\mathfrak{n}-1}}.$$

Thus, we note that

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| \le 1-\delta,$$

which is equivalent to

$$\sum_{\mathfrak{n}=2}^{\infty}(\mathfrak{n}-\delta)\rho_{\mathfrak{n}}|z|^{\mathfrak{n}-1} \le 1-\delta. \qquad (25)$$

But, in view of Theorem 3.1, we obtain it by (24).

Thus, φ is starlike r_2 if

$$(\mathfrak{n}-\delta)|z|^{\mathfrak{n}-1} \le \frac{(1-\delta)\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(1-\nu)},$$

that is, if

$$|z| \le \left[\frac{(1-\delta)\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(\mathfrak{n}-\delta)(1-\nu)}\right]^{\frac{1}{\mathfrak{n}-1}}, \qquad (\mathfrak{n} \ge 2).$$

(3) It is enough to show that

$|\varphi'(z)-1| \le 1-\delta,$ for $|z| < r_3$, and δ $(0 \le \delta < 1)$.

Where r_3 is given by (21) Indeed, we find from (11) that

$$|\varphi'(z)-1| = \sum_{\mathfrak{n}=2}^{\infty}\mathfrak{n}\,\rho_{\mathfrak{n}}\,|z|^{\mathfrak{n}-1}.$$

Thus, note that

$$|\varphi'(z)-1| \le 1-\delta,$$

which is equivalent to

$$\sum_{\mathfrak{n}=2}^{\infty} \mathfrak{n}\, \rho_{\mathfrak{n}}\, |z|^{\mathfrak{n}-1} \leq 1-\delta. \qquad (26)$$

But, by Theorem 3.1, we given it by (24)

$$\sum_{\mathfrak{n}=2}^{\infty} \frac{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(1-\nu)} \rho_{\mathfrak{n}} \ \leq 1\,.$$

Therefore, φ is closed to convex of r_3 if

$$\mathfrak{n}|z|^{\mathfrak{n}-1} \leq \frac{(1-\delta)\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(1-\nu)},$$

that is, if

$$|z| \leq \left[\frac{(1-\delta)\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(\mathfrak{n}-\delta)(1-\nu)}\right]^{\frac{1}{\mathfrak{n}-1}}. \qquad (\mathfrak{n} \geq 2).$$

6.Extreme points

Theorem 6.1 Let

$$\varphi_1(z) = z,$$

$$\varphi_{\mathfrak{n}}(z) = z - \frac{(1-\nu)}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)} z^{\mathfrak{n}}, \qquad (\mathfrak{n} \geq 2).$$

Then $\varphi(z) \in \eta - TS_{\nu}^{*}(\beta,\mu,\sigma,\tau)$ if and only if it can be expressed in the following form:

$$\varphi(z) = \sum_{\mathfrak{n}=1}^{\infty} \gamma_{\mathfrak{n}} \varphi_{\mathfrak{n}},$$

where

$$\gamma_{\mathfrak{n}} \geq 0, \qquad \sum_{\mathfrak{n}=1}^{\infty} \gamma_{\mathfrak{n}} = 1.$$

Proof. Let $\varphi(z) \in \eta - TS_{\nu}^{*}(\beta,\mu,\sigma,\tau)$, by Corollary 3.1, we have

$$\rho_{\mathfrak{n}} \leq \frac{(1-\nu)}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}\,, \qquad (\mathfrak{n} \geq 2).$$

set

$$\gamma_{\mathfrak{n}} = \frac{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}{(1-\nu)} \ \rho_{\mathfrak{n}}\,, \qquad (\mathfrak{n} \geq 2).$$

$$\gamma_1 = 1 - \sum_{\mathfrak{n}=2}^{\infty} \gamma_{\mathfrak{n}}.$$

Thus, we gain

$$\varphi(z) = \sum_{\mathfrak{n}=1}^{\infty} \gamma_{\mathfrak{n}} \varphi_{\mathfrak{n}}\,(z).$$

Conversely, consider that

$$\varphi(z) = \sum_{\mathfrak{n}=1}^{\infty} \gamma_{\mathfrak{n}} \varphi_{\mathfrak{n}}\,(z),$$

$$= z - \sum_{\mathfrak{n}=2}^{\infty} \gamma_{\mathfrak{n}} \frac{(1-\nu)}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)} z^{\mathfrak{n}}.$$

Then ,form theorem 3.1, we yield

$$\sum_{\mathfrak{n}=2}^{\infty}\left([\mathfrak{n}(\eta+1)-(\nu+\eta)]H(\beta,\mu,\sigma,\tau)\frac{(1-\nu)}{\big(\mathfrak{n}(\eta+1)-(\nu+\eta)\big)H(\beta,\mu,\sigma,\tau)}\gamma_{\mathfrak{n}}\right)$$

$$= (1-\nu)\sum_{\mathfrak{n}=2}^{\infty}\gamma_{\mathfrak{n}} = (1-\nu)(1-\gamma_1) \le 1-\nu.$$

Then, $\varphi(z) \in \eta - TS_{\nu}^{*}(\beta,\mu,\sigma,\tau)$.

7. Conclusion:

In this current investigation, a discussion is presented about the new complex operator in terms of generalization of koebe type function and Write function and its inclusion in a new geometric class that is uniformly starlike class. In addition to that, a study on the coefficient bounds, growth bound, distortion bound, radii of convexity, star-likeness, and close-to-convexity are also yielded.

8. Referencs

1. P.L. Duren, *Univalent functions* Springer-Verlag, New York Berlin, Heidelberg, Tokyo, 1983.

2. R. Bharati, P. Parvatham and A. Swaminathan, "*On subclass of uniformly convex functions and corresponding class of starlike function", Tamkang J. Math*.,vol. 28, pp. 17-32, 1997.

3. E.M. Wright*:" On the coefficients of power series having exponential singularities*" *.J.Lon-don Math.Soc*. vol. 8,pp. 71-80. 1933.

4. A.H. Moureh and H.F. Al-Janaby, " Sandwich Subordinations Imposed by New Generalized Koebe-Type Operator on Holomorphic Function Class", accepted.

5. M. Darus, "Certain class of uniformly analytic functions", *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis* vol. 24, pp. 345–353, 2008.

6. D. Breaz, *A Convexity Property for an Integral Operator on the Class* $S^*(\nu)$, Journal of Inequalities and Applications, vol. 15, pp. 177-183, 2008.

7. N. Breaz and D. Breaz, M. Darus, "Convexity properties for some general integral operators on uniformly analytic functions classes", *Computers and Mathematics with Applications*, vol. 60, pp. 3105-3107, 2010.

8. L. Stanciu and D. Breaz," Some properties of a general integral operator", *Bull Iran Math Soc* vol. 40, no. 6, pp. 1433–1439, 2014.

9. H.F. Al-Janaby, F. Ghanim and M. Darus, "Some Geometric Properties of Integral Operators Proposed by Hurwitz-Lerch Zeta Function", *Journal of Physics Conf. Series* vol. 1212, pp. 1-7, 2019.

10. Layth T. Khudhuir1, Ahmed M. Ali1 and Hiba F. Al-Janaby, "A New Class of K-Uniformly Starlike Functions Imposed by Generalized Salagean's Operator*", AIP Conference Proceedings*, vol. 2398, no. 1, pp. 1-12, 2022.

11. L.D. Branges, "A proof of the Bieberbach conjecture", *Acta Mathematica*, vol. 154, no. (1-2), pp. 137-152, 1984.

12. M.S. Mahmoud, A.R.S. Juma and R.A.M. Al-Saphory, "On a subclass of meromorphic univalent functions involving hypergeometric function", *Journal of Al-Qadisiyah for Computer Science and Mathematics*, Vol.11, no. (3), pp. 12–20, 2019.

13. W.G. Atshan, A.H. Battor and A.F. Abaas, "On third-order differential subordination results for univalent analytic functions involving an operator", *Journal of Physics*, vol. 1664, pp. 1-19, 2020.

14. S. Elhaddad and M. Darus, "Coefficient estimates for a subclass of bi-univalent functions defined by qderivative operator", *Mathematics*, vol. 8, no. 3, pp. 1-14, 2020.

15. C.M. Yan and J.L. Liu, "on second -order differential subordination for certain meromorphically multivalent functions", *AIMS Mathematics*,vol. 5, no. 5 pp. 4995–5003, 2020.

16. H.F. Al-Janaby, F. Ghanim and M. Darus, "On The Third-Order Complex Differential Inequalities of ξ-Generalized-Hurwitz–Lerch Zeta Functions", *Mathematics*, vol. 8, no. 5, pp. 1-21, 2020.

17. G.I. Oros, "New Conditions for Univalence of Confluent Hypergeometric Function", *Symmetry*, 1-10, 2021.

18. T. K. Layth, A. M. Ali and H. F. Al-Janaby, "Complex Differential Implications of Linear Operator Imposed By Mittag-Leffler Type Function", *Journal of Physics Conf. Series*, vol. 1818, pp. 1-11, 2021.

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