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Research article

On solvability of some inverse problems for a nonlocal fourth-order parabolic equation with multiple involution

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Abstract: In this paper, the solvability of some inverse problems for a nonlocal analogue of a fourth-order parabolic equation was studied. For this purpose, a nonlocal analogue of the biharmonic operator was introduced. When defining this operator, transformations of the involution type were used. In a parallelepiped, the eigenfunctions and eigenvalues of the Dirichlet type problem for a nonlocal biharmonic operator were studied. The eigenfunctions and eigenvalues for this problem were constructed explicitly and the completeness of the system of eigenfunctions was proved. Two types of inverse problems on finding a solution to the equation and its righthand side were studied. In the two problems, both of the righthand terms depending on the spatial variable and the temporal variable were obtained by using the Fourier variable separation method or reducing it to an integral equation. The theorems for the existence and uniqueness of the solution were proved.

Keywords: inverse problem; nonlocal biharmonic operator; parabolic equation; eigenfunction; eigenvalue; Fourier method; existence of solution; uniqueness of solution
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1. Introduction

Let $Q = \Pi \times (0, T)$, where $\Pi = \{x \in \mathbb{R}^n : 0 < x_j < p_j, j = 1, \ldots, n\}$ is a parallelepiped, $p_j > 0$, and $T > 0$. Consider the mappings $S_j : \mathbb{R}^n \to \mathbb{R}^n, 1 \le j \le n$ of the type $S_j x = \left(x_1, \ldots, x_{j-1}, p_j - x_j, x_{j+1}, \ldots, x_n\right)$. Obviously, the mappings S_j are involutions, i.e., $S_j^2 = I$, where I is the identity mapping. Let us consider all possible products of mappings S_j, i.e., $S_{ij} = S_i S_i$, or $S_{ijk} = S_i S_i S_k, \ldots$. The total number of such mappings, taking into account the identity mapping $S_0 x = x$, is equal to 2^n. To number such mappings, we will use the binary number system, namely, if $0 \le i < 2^n$ in the binary number system, the representation $i \equiv (i_n \ldots i_1)_2 = i_1 + 2i_2 + \ldots + 2^{n-1} i_n$, where

$i_k = 0, 1$, is valid. Therefore, introducing the vector $\mathbf{i} = (i_1, \ldots, i_n)$, we can consider mappings of the type $\mathbf{S_i} \equiv S_1^{i_1} \ldots S_n^{i_n}$ corresponding to the index i. Using these mappings, we introduce the operator

$$L_x v(x) = \sum_{i=0}^{2^n-1} a_i \Delta^2 v(\mathbf{S_i} x),$$

where $a_0, a_1, a_2, a_3, \ldots, a_{2^n-1}$ is a set of real numbers, and Δ^2 is a biharmonic operator.

Let us consider the following problems in the domain Q.

Problem 1. Find a pair of such functions $\{u(t,x), f(x)\}$ that are smooth $u(t,x) \in C\left(\bar{Q}\right)$, $f(x) \in C\left(\bar{\Pi}\right)$, $u_t(t,x)$, $L_x u(t,x) \in C(Q)$, satisfy the equation

$$u_t(t,x) + L_x u(t,x) = f(x) g(t), \quad (t,x) \in Q, \tag{1.1}$$

the boundary conditions

$$u(t,x)|_{t\in[0,T],\ x\in\partial\Pi} = 0, \ \left.\frac{\partial^2 u(t,x)}{\partial x_i^2}\right|_{t\in[0,T],\ x\in\partial\Pi} = 0, \quad i = 1, 2, ..., n, \tag{1.2}$$

and the initial and final conditions

$$u(0,x) = \varphi(x), \quad x \in \bar{\Pi}, \tag{1.3}$$

$$u(T,x) = \psi(x), \quad x \in \bar{\Pi}, \tag{1.4}$$

where $g(t)$, $\varphi(x)$, $\psi(x)$ are the given functions.

Problem 2. Find a pair of such functions $\{u(t,x), g(t)\}$ that are smooth $u(t,x) \in C\left(\bar{Q}\right)$, $g(t) \in C[0,T]$, $u_t(t,x), L_x u(t,x) \in C(Q)$ and satisfy conditions (1.1)–(1.3) and an additional condition

$$u(t, x_0) = h(t), \quad 0 \le t \le T, \tag{1.5}$$

where $x_0 \in \Pi$ is a given fixed point, $h(t), \varphi(x)$, and $f(x)$ are the given functions, and $h(0) = \varphi(x_0)$.

Differential equations with involution are an important part of the general theory of functional differential equations. As it is known, the first works on differential equations with involution were written by Babbage [1] and continued by Carleman [2]. Various issues in the theory of differential equations with involution were studied in a series of papers written by Przeworska-Rolewicz [3–5] The case $n = 1$ is considered in several papers [6–11], where the solvability of inverse problems on finding the righthand side of differential equations with involution is studied. Thus, in the work of Nasser Al-Salti et al. [7], in the rectangular domain $\Omega = \{(t,x) : 0 < t < T, -\pi < x < \pi\}$ for the equation

$$u_t(t,x) - u_{xx}(t,x) - \varepsilon u_{xx}(t, \pi - x) = f(x), \quad (t,x) \in \Omega,$$

inverse problems for determining a pair of functions $\{u(t,x), f(x)\}$ have been studied. In this work, the authors state that redefined initial and final conditions

$$u(0,x) = \varphi(x), \ \ u(T,x) = \varphi(x), \ -\pi < x < \pi,$$

as well as one of the boundary conditions, Dirichlet, Neumann, periodic, or antiperiodic conditions, can be considered as additional conditions. The problem studied in the paper is solved using the method of separation of variables. In this case, a spectral problem arises with respect to the spatial variable for the equation with involution. For example, in the case of Dirichlet boundary conditions, we have

$$-X''(x) - \varepsilon X''(\pi - x) = \lambda X(x),\ -\pi < x < \pi, \quad X(-\pi) = X(\pi) = 0.$$

It is proven that the eigenfunctions of this problem are the functions

$$X_{1k}(x) = \cos\left(k + \frac{1}{2}\right)x,\ k \geq 0;\ \ X_{2k}(x) = \sin kx,\ k \geq 1,$$

and the corresponding eigenvalues are the numbers

$$\lambda_{1k} = (1-\varepsilon)\left(k + \frac{1}{2}\right)^2,\ k \geq 0;\ \ \lambda_{2k} = (1+\varepsilon)k,\ k \geq 1.$$

Similar studies in the case of two spatial variables for classical equations were carried out in [12–18], as well as for equations with involution in [19, 20]. The authors of this paper studied inverse problems for equations of parabolic type with an operator of the fourth and higher orders in the spatial variable (see, for example, [21–28]).

The methods used to solve the inverse problems in order to determine the right side of the equation depend on whether the function $f(x)$ or the function $g(t)$ is unknown. In the case when the function $f(x)$ is unknown, the problem under consideration is usually solved using the Fourier method, whereas in the case when the function $g(t)$ is unknown, the problem is reduced to the Volterra integral equation (see, for example, [18]).

As we have already noted, studies of direct and inverse problems for equations with involutive transformed arguments were mainly carried out for equations with one and two spatial variables. For equations with many variables, such problems are insufficiently studied. In this direction, we can only note the work of Kozhanov and Bzheumikhova [25].

When studying Problems 1 and 2, a spectral problem arises for the nonlocal operator L_x. In the one-dimensional case in [7], this problem is solved by finding a general solution to the studied equation and using the boundary conditions to determine the eigenfunctions and eigenvalues. In the two-dimensional case in [19], the corresponding spectral problem was solved by reducing it to four auxiliary problems.

In general, in the $n-$ dimensional case, $n \geq 3$, such spectral problems in a parallelepiped have not been considered. When solving this problem, to denote the summation index we used the notation in the binary number system. The transition to this system allowed us to construct eigenfunctions and eigenvalues of this problem explicitly and to prove the completeness of the systems of eigenfunctions in space $L_2(\Pi)$.

It should be also noted that the use of differential equations with involution when modeling a specific physical process of thermal diffusion in the case $n = 1$ is given in [7, 9], and in [29, 30] in the case $n = 2$, where equations with involutive transformations, which have applications in modeling of optical systems, are considered. In addition, the influence of nonlocality is graphically illustrated in [7] in the one-dimensional case.

2. Eigenfunctions and eigenvalues of a nonlocal biharmonic operator

In this section, we study the eigenfunctions and eigenvalues of a Dirichlet type problem for a nonlocal biharmonic equation.

Consider the following boundary value problem.

Problem S. Find a function $v(x) \neq 0$ from the class $v(x) \in C^2\left(\bar{\Pi}\right) \cap C^4(\Pi)$ that satisfies the equation

$$L_x v(x) = \lambda v(x), \ x \in \Pi, \tag{2.1}$$

and boundary conditions

$$v(x)|_{\partial\Pi} = \left.\frac{\partial^2 v(x)}{\partial x_i^2}\right|_{\partial\Pi} = 0, i = 1, 2, ..., n, \tag{2.2}$$

where $\lambda \in \mathbb{R}$.

If $a_0 = 1$, $a_j = 0$, $j = 1, \ldots, 2^n - 1$, then the problem coincides with the spectral problem with the Dirichlet condition for the classical biharmonic operator.

Note that in the case of a sphere, a similar spectral problem for the nonlocal Laplace operator was studied in [31]. Various boundary value problems for nonlocal harmonic and biharmonic equations are studied in [32–34].

Let us consider a set of functions

$$v_{\mathbf{k}}(x) = v_{k_1 k_2 \ldots k_n}(x) = C(n, \mathbf{p}) \prod_{j=1}^{n} \sin \frac{k_j \pi x_j}{p_j}, \tag{2.3}$$

where $\mathbf{k} = (k_1 k_2 \ldots k_n) \in \mathbb{N}^n$ and $C(n, \mathbf{p}) = 2^{n/2} \prod\limits_{j=1}^{n} \frac{1}{\sqrt{p_j}}$, $\mathbf{p} = (p_1, \ldots, p_n)$.

The following statement is proved in [35].

Lemma 2.1. The system of functions $\{v_{\mathbf{k}}(x) : \mathbf{k} \in \mathbb{N}^n\}$ is orthonormal and complete in space $L_2((0, p_1) \times (0, p_2) \times \ldots \times (0, p_n))$.

Let us introduce the following numbers $\varepsilon_{\mathbf{k}} = \sum\limits_{i=0}^{2^n-1} (-1)^{\mathbf{i}\cdot(\mathbf{k}+\mathbf{e})} a_i$, where $\mathbf{k} = (k_1, k_2, ..., k_n)$, $\mathbf{i} = (i_1, \ldots, i_n)$, $\mathbf{i}\cdot\mathbf{k} = i_1 k_1 + \ldots + i_n k_n$, and $\mathbf{e} = (1, \ldots, 1)$. Note that components of the vector $\mathbf{k} = (k_1, k_2, ..., k_n)$ can be calculated using mode 2 as in this case the $\varepsilon_{\mathbf{k}}$ value will not change. Taking this into account we can state that the equality $\mathbf{k} + \mathbf{e} = \mathbf{k}^*$ is valid, where the vector $\mathbf{k}^*$ is conjugate to $\mathbf{k}$. For example, $\mathbf{k} = (0, 1, 0, 1) \Rightarrow \mathbf{k}^* = (1, 0, 1, 0)$. Under these assumptions,

$$\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k} \bmod 2} = \sum_{i=0}^{2^n-1} (-1)^{\mathbf{i}\cdot\mathbf{k}^*} a_i.$$

Theorem 2.1. Let the conditions $\varepsilon_{\mathbf{k}} \neq 0$ be satisfied for all $\mathbf{k} \in \mathbb{N}^n$, then the system of functions $\{v_{\mathbf{k}}(x) : \mathbf{k} \in \mathbb{N}^n\}$ is a system of eigenfunctions of Problem **S**. The corresponding eigenvalues are determined by the equalities

$$\lambda_{\mathbf{k}} = \varepsilon_{\mathbf{k}} \mu_{\mathbf{k}}^2, \ \ \mu_{\mathbf{k}} = \pi^2 \sum_{j=1}^{n} \frac{k_j^2}{p_j^2}, \quad \mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n. \tag{2.4}$$

Proof. It is obvious that the functions $v_{\mathbf{k}}(x)$, by their structure, satisfy conditions (2.2) $v_{\mathbf{k}}(x)|_{\partial\Pi} = 0$, and as

$$\begin{aligned} \frac{\partial^2 v_{\mathbf{k}}}{\partial x_i^2} &= C(n, \mathbf{p}) \prod_{j=1, j\neq i}^{n} \sin \frac{k_j \pi x_j}{p_j} \cdot \frac{\partial^2}{\partial x_i^2}\left(\sin \frac{k_i \pi x_i}{p_i}\right) \\ &= -\left(\frac{k_i \pi}{p_i}\right)^2 C(n, \mathbf{p}) \prod_{j=1, j\neq i}^{n} \sin \frac{k_j \pi x_j}{p_j} \cdot \left(\sin \frac{k_i \pi x_i}{p_i}\right) = -\left(\frac{k_i \pi}{p_i}\right)^2 v_{\mathbf{k}}(x), \end{aligned}$$

they also satisfy conditions (2.2). Let us check whether Eq (2.1) is fulfilled. Applying the Laplace operator to this function, we obtain

$$\Delta v_{\mathbf{k}}(x) = -\left(\sum_{i=1}^{n}\left(\frac{k_i\pi}{p_i}\right)^2\right)v_{\mathbf{k}}(x),$$

which means that

$$\Delta^2 v_{\mathbf{k}}(x) = \mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x), \quad \mu_{\mathbf{k}} = \pi^2\left(\sum_{i=1}^{n}\frac{k_i^2}{p_i^2}\right).$$

Hence, for any $m \in \{1, 2, ..., n\}$, we get

$$\begin{aligned}
\Delta^2 v_{\mathbf{k}}(S_m x) &= \mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x_1, ..., x_{m-1}, p_m - x_m, x_{m+1}, ..., x_n)\\
&= \mu_{\mathbf{k}}^2 C(\mathbf{p}) \sin\frac{k_m\pi(p_m - x_m)}{p_m} \cdot \prod_{j=1, j\neq m}^{n} \sin\frac{k_j\pi x_j}{p_j}\\
&= \mu_{\mathbf{k}}^2 C(\mathbf{p}) (-1)^{k_m+1} \sin\frac{k_m\pi x_m}{p_m} \cdot \prod_{j=1, j\neq m}^{n} \sin\frac{k_j\pi x_j}{p_j}\\
&= \mu_{\mathbf{k}}^2 C(\mathbf{p}) (-1)^{k_m+1} \prod_{j=1}^{n} \sin\frac{k_j\pi x_j}{p_j} = (-1)^{k_m+1}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x).
\end{aligned}$$

Let $0 \leq i \leq 2^n - 1$, which corresponds to vector $\mathbf{i} = (i_1, \ldots, i_n)$. If $i_m = 1$, then

$$\Delta^2 v_{\mathbf{k}}(S_m^{i_m} x) = \mu_{\mathbf{k}}^2 C(\mathbf{p}) \sin\frac{k_m\pi(p_m - x_m)}{p_m} \cdot \prod_{j=1, j\neq m}^{n} \sin\frac{k_j\pi x_j}{p_j} = (-1)^{k_m+1}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x).$$

Obviously, this equality is also valid for the case $i_m = 0$. Thus, we get

$$\Delta^2 v_{\mathbf{k}}(S_m^{i_m} x) = (-1)^{i_m(k_m+1)}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x). \tag{2.5}$$

In the general case, the following equality holds:

$$\begin{aligned}
\Delta^2 v_{\mathbf{k}}(\mathbf{S_i} x) = \Delta^2 v_{\mathbf{k}}\left(S_1^{i_1} \ldots S_n^{i_n} x\right) &= (-1)^{i_1(k_1+1)+i_2(k_2+1)+\ldots+i_n(k_n+1)}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x)\\
&= (-1)^{\mathbf{i}\cdot(\mathbf{k}+1)}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x) = (-1)^{\mathbf{i}\cdot\mathbf{k}^*}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x),
\end{aligned}$$

where $\mathbf{k} = (k_1, k_2, ..., k_n)$, $\mathbf{i} = (i_1, \ldots, i_n)$, $\mathbf{i} \cdot \mathbf{k} = i_1 k_1 + ... + i_n k_n$ and $\mathbf{e} = (1, \ldots, 1)$.

Now, if we apply the operator L_x to the function $v_{\mathbf{k}}$, then from the previous equalities it follows that

$$\begin{aligned}
L_x v_{\mathbf{k}}(x) &= \sum_{i=0}^{2^n-1} a_i \Delta^2 v_{\mathbf{k}}(\mathbf{S_i} x) = \sum_{i=0}^{2^n-1} a_i (-1)^{\mathbf{i}\cdot\mathbf{k}^*}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x)\\
&= \mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x)\left(\sum_{i=0}^{2^n-1}(-1)^{\mathbf{i}\cdot\mathbf{k}^*}a_i\right) = \varepsilon_{\mathbf{k} \bmod 2}\mu_{\mathbf{k}}^2 v_{\mathbf{k}}(x).
\end{aligned}$$

Thus, in addition to conditions (2.2), the function $v_{\mathbf{k}}(x)$ also satisfies the equality $L_x v_{\mathbf{k}}(x) = \lambda_{\mathbf{k}} v_{\mathbf{k}}(x)$, i.e., Eq (2.1). The theorem is proved.

Remark 2.1. If the condition $\varepsilon_{\mathbf{k} \bmod 2} > 0$ is satisfied for any $\mathbf{k} \in \mathbb{N}^n$, then all eigenvalues of Problem **S** are positive.

Example 2.1. Let $n = 2$, then Eq (2.1) takes the form

$$a_0\Delta^2 v(x_1, x_2) + a_1\Delta^2 v(p_1 - x_1, x_2) + a_2\Delta^2 v(x_1, p_2 - x_2) + a_2\Delta^2 v(p_1 - x_1, p_2 - x_2) = \lambda v(x_1, x_2),$$

and the boundary conditions are written as

$$\begin{aligned} v(0, x_2) = v(p_1, x_2) = 0,\ 0 \le x_2 \le p_2;\ v(x_1, 0) = v(x_1, p_2) = 0,\ 0 \le x_1 \le p_1, \\ v_{x_1x_1}(0, x_2) = v_{x_1x_1}(p_1, x_2) = 0,\ 0 \le x_2 \le p_2;\ v_{x_2x_2}(x_1, 0) = v_{x_2x_2}(x_1, p_2) = 0,\ 0 \le x_1 \le p_1. \end{aligned}$$

Eigenfunctions are specified in accordance with (2.3) as

$$v_{(m,k)}(x) = \frac{2}{\sqrt{p_1 p_2}} \sin\frac{m\pi x_1}{p_1} \sin\frac{k\pi x_2}{p_2}, \quad m, k = 1, 2, ...,$$

and the corresponding eigenvalues are

$$\lambda_{(m,k)} = \varepsilon_{(m,k)}\pi^4\left(\frac{m^2}{p_1^2} + \frac{k^2}{p_2^2}\right)^2, \quad m, k = 1, 2, \dots,$$

where $\varepsilon_{(m,k)}$ is

$$\begin{aligned} \varepsilon_{(m,k)} &= \varepsilon_{(m,k) \bmod 2} = a_0 + (-1)^{m+1}a_1 + (-1)^{k+1}a_2 + (-1)^{k+m}a_3 \\ &= a_0 + (-1)^{m^*}a_1 + (-1)^{k^*}a_2 + (-1)^{k^*+m^*}a_3. \end{aligned}$$

More precisely, $\varepsilon_{(m,k)}$ can be written as

$$\begin{aligned} \varepsilon_{(2m-1,2k-1)} &= a_0 + a_1 + a_2 + a_3; \quad \varepsilon_{(2m-1,2k)} = a_0 + a_1 - a_2 - a_3; \\ \varepsilon_{(2m,2k-1)} &= a_0 - a_1 + a_2 - a_3; \quad \varepsilon_{(2m,2k)} = a_0 - a_1 - a_2 + a_3, \end{aligned}$$

where $m, k = 1, 2, ...$

3. Study of convergence of Fourier series

Let

$$h(x) = \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} h_{k_1...k_n} v_{k_1...k_n}(x) = \sum_{\mathbf{k}\in\mathbb{N}^n} h_{\mathbf{k}} v_{\mathbf{k}}(x) \tag{3.1}$$

be the Fourier series expansion of the function $h(x)$ by the system $\{v_{\mathbf{k}}(x) : \mathbf{k} \in \mathbb{N}^n\}$, where

$$h_{\mathbf{k}} = (h, v_{\mathbf{k}}) \equiv \int_0^{p_1} ... \int_0^{p_n} h(x_1, x_2, ..., x_n) v_{k_1...k_n}(x_1, x_2, ..., x_n)\, dx_1 ... dx_n. \tag{3.2}$$

Further, we will use the symbol C to denote an arbitrary constant whose value does not affect our conclusions.

Lemma 3.1. Let the function $h(x)$ be continuous in a closed domain $\bar{\Pi}$ and have continuous partial derivatives $\frac{\partial^j h(x)}{\partial x_1 ... \partial x_j}$, $1 \le j \le n$ in $\bar{\Pi}$. If the conditions

$$\begin{aligned} & h(0, x_2, ..., x_n) = h(p_1, x_2, ..., x_n) = 0,\ 0 \le x_j \le p_j,\ j = 2, ..., n, \\ & h(x_1, 0, ..., x_n) = h(x_1, p_2, ..., x_n) = 0,\ 0 \le x_j \le p_j,\ j = 1, ..., n, j \ne 2, \\ & ... \\ & h(x_1, ..., x_{n-1}, 0) = h(x_1, ..., x_{n-1}, p_n) = 0,\ 0 \le x_j \le p_j,\ j = 1, ..., n-1 \end{aligned} \tag{3.3}$$

are satisfied, then the number series $\sum\limits_{\mathbf{k} \in \mathbb{N}^n} |h_{\mathbf{k}}|$ converges.

Proof. If $h(x) \in C\left(\bar{\Pi}\right)$ and the function $\frac{\partial h(x)}{\partial x_1}$ is continuous, then integrating the integral in (3.2) by parts over the variable x_1 and taking into account equality (3.3), we obtain

$$h_{\mathbf{k}} = h_{k_1 ... k_n} = \frac{1}{k_1} h_{k_1 ... k_n}^{1,0,..,0},$$

where

$$\begin{aligned} & h_{k_1 ... k_n}^{1,0,..,0} = C \int_0^{p_1} ... \int_0^{p_n} \frac{\partial h(x_1, x_2, ..., x_n)}{\partial x_1} v_{k_1 ... k_n}^{1,0,..,0}(x_1, x_2, ..., x_n) dx_1 ... dx_n, \\ & v_{k_1 ... k_n}^{1,0,..,0}(x_1, x_2, ..., x_n) = \cos \frac{k_1 \pi x_1}{p_1} \cdot \prod_{j=2}^{n} \sin \frac{k_j \pi x_j}{p_j}. \end{aligned}$$

Applying this process to all $j \in \{2, 3, ..., n\}$, we get

$$h_{\mathbf{k}} = h_{k_1 ... k_n} = \frac{1}{k_1 k_2 ... k_n} h_{k_1 ... k_n}^{1,1,..,1},$$

where

$$\begin{aligned} & h_{k_1 ... k_n}^{1,1,..,1} = C \int_0^{p_1} ... \int_0^{p_n} \frac{\partial^n h(x_1, x_2, ..., x_n)}{\partial x_1 ... \partial x_n} v_{k_1 ... k_n}^{1,1,..,1}(x_1, x_2, ..., x_n) dx_1 ... dx_n, \\ & v_{k_1 ... k_n}^{1,1,..,1}(x_1, x_2, ..., x_n) = \prod_{j=1}^{n} \cos \frac{k_j \pi x_j}{p_j}. \end{aligned}$$

Using the Cauchy-Bunyakovsky inequality, we obtain

$$\sum_{\mathbf{k} \in \mathbb{N}^n} |h_{\mathbf{k}}| \le \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \left| \frac{1}{k_1 ... k_n} h_{k_1 ... k_n}^{1,1,..,1} \right| \le \sqrt{\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \frac{1}{k_1^2 ... k_n^2}} \sqrt{\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \left| h_{k_1 ... k_n}^{1,1,..,1} \right|^2}.$$

As the system $\left\{ v_{k_1 ... k_n}^{1,1,..,1}(x_1, x_2, ..., x_n) \right\}$ is orthogonal in space $L_2(\Pi)$ and $\frac{\partial^n h(x_1, x_2, ..., x_n)}{\partial x_1 ... \partial x_n} \in L_2(\Pi)$, then due to Bessel's inequality, the series $\sum\limits_{k_1=1}^{\infty} ... \sum\limits_{k_n=1}^{\infty} \left| h_{k_1 ... k_n}^{1,1,..,1} \right|^2$ converges. Moreover, the series

$$\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \frac{1}{k_1^2 ... k_n^2} = \sum_{k_1=1}^{\infty} \frac{1}{k_1^2} ... \sum_{k_n=1}^{\infty} \frac{1}{k_n^2} < \infty$$

also converges. This implies the assertion of the lemma.

The following assertion is proved in a similar way.

Lemma 3.2. Let a function $h(x)$ belong to a class $C^4\left(\bar{\Pi}\right)$ and have continuous partial derivatives of the form $\frac{\partial^{n+4}h(x)}{\partial x_1...\partial x_j^5...\partial x_n}$ for $j \in \{1,2,...,n\}$ in $\bar{\Pi}$. If the functions $h(x)$, $\frac{\partial^2 h(x)}{\partial x_j^2}$, and $\frac{\partial^4 h(x)}{\partial x_j^4}$ satisfy conditions (3.3), the number series $\sum\limits_{k_1=1}^{\infty} ... \sum\limits_{k_j=1}^{\infty} ... \sum\limits_{k_n=1}^{\infty} k_j^4 \left|h_{k_1...k_j...k_n}\right|$ converges.

If the functions $h(x)$, $\frac{\partial^2 h(x)}{\partial x_j^2}$, and $\frac{\partial^4 h(x)}{\partial x_j^4}$ satisfy conditions (3.3), the number series

$$\sum_{k_1=1}^{\infty} ... \sum_{k_j=1}^{\infty} ... \sum_{k_n=1}^{\infty} k_j^4 \left|h_{k_1...k_j...k_n}\right|$$

converges.

Proof. Let the functions $h(x)$ and $\frac{\partial^2 h(x)}{\partial x_j^2}$, $\frac{\partial^4 h(x)}{\partial x_j^4}$ satisfy conditions (3.3), then integrating the integral four times over the variable x_j in the equality

$$h_{k_1...k_n} = \frac{C}{k_1...k_n}\int_0^{p_1} ... \int_0^{p_n} \frac{\partial^n h(x_1,x_2,...,x_n)}{\partial x_1...\partial x_n} v_{k_1...k_n}^{1,1,...,1}(x_1,x_2,...,x_n)\, dx_1...dx_n,$$

we get

$$h_{k_1...k_n} = \frac{1}{k_1 k_2...k_j^5...k_n} h_{k_1k_2...k_j...k_n}^{1,1,..,5,...,1},$$

where

$$h_{k_1k_2...k_j...k_n}^{1,1,..,5,...,1} = C\int_0^{p_1} ... \int_0^{p_n} \frac{\partial^{n+4} h(x_1,x_2,...,x_n)}{\partial x_1...\partial x_j^5...\partial x_n} v_{k_1k_2...k_j...k_n}^{1,1,..,5,...,1}(x_1,x_2,...,x_n)\, dx_1...dx_n,$$

$$v_{k_1k_2...k_j...k_n}^{1,1,..,5,...,1}(x_1,x_2,...,x_n) = \prod_{j=1}^{n} \cos\frac{k_j \pi x_j}{p_j}.$$

Using the Cauchy-Bunyakovsky inequality for $\sum\limits_{k_1=1}^{\infty} ... \sum\limits_{k_n=1}^{\infty} \left|h_{k_1...k_n}\right|$, we obtain

$$\sum_{k_1=1}^{\infty} ... \sum_{k_j=1}^{\infty} ... \sum_{k_n=1}^{\infty} k_j^4 \left|h_{k_1...k_n}\right| = \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \frac{k_j^4}{k_1k_2...k_j^5...k_n}\left|h_{k_1k_2...k_j...k_n}^{1,1,..,3,...,1}\right|$$

$$\leq \sqrt{\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \frac{1}{k_1^2...k_j^2...k_n^2}}\sqrt{\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \left|h_{k_1k_2...k_j...k_n}^{1,1,..,3,...,1}\right|^2}.$$

As the system $\left\{v_{k_1...k_n}^{1,1,...,1}(x_1,x_2,...,x_n)\right\}$ is orthogonal in space $L_2(\Pi)$ and $\frac{\partial^n h(x_1,x_2,...,x_n)}{\partial x_1...\partial x_n} \in L_2(\Pi)$, then due to Bessel's inequality, the series $\sum\limits_{k_1=1}^{\infty} ... \sum\limits_{k_n=1}^{\infty} \left|h_{k_1...k_n}^{1,1,...,1}\right|^2$ converges. This implies the statement of the lemma.

Corollary 3.1. Let the conditions of Lemma 3.2 be satisfied, then the number series

$$\sum_{\mathbf{k}\in\mathbb{N}^n} \lambda_{\mathbf{k}}\left|h_{\mathbf{k}}\right| = \sum_{k_1=1}^{\infty} ... \sum_{k_j=1}^{\infty} ... \sum_{k_n=1}^{\infty} \lambda_{k_1...k_j...k_n}\left|h_{k_1...k_j...k_n}\right|$$

converges.

The proof of this assertion follows from the representation of eigenvalues in the form $\lambda_{k_1k_2...k_n} = \varepsilon_{k_1k_2...k_n}\pi^4\Big(\sum\limits_{j=1}^{n}\frac{k_j^2}{p_j^2}\Big)^2$ and the assertion of Lemma 3.2.

4. Uniqueness and existence of a solution to Problem 1

By definition, the solution to Problem 1 is expressed as functions $u(t, x), f(x)$. If such functions exist, they must be the elements of space $L_2(\Pi)$ and, therefore, they can be represented as series

$$u(t, x) = \sum_{\mathbf{k}\in\mathbb{N}^n}^{\infty} u_{\mathbf{k}}(t)v_{\mathbf{k}}(x) = \sum_{k_1=1}^{\infty}\sum_{k_n=1}^{\infty} u_{k_1\ldots k_n}(t)v_{k_1\ldots k_n}(x), \tag{4.1}$$

$$f(x) = \sum_{\mathbf{k}\in\mathbb{N}^n} f_{\mathbf{k}}v_{\mathbf{k}}(x) = \sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty} f_{\mathbf{k}}v_{\mathbf{k}}(x), \tag{4.2}$$

where $u_{\mathbf{k}}(t) = u_{k_1\ldots k_n}(t)$ and $f_{\mathbf{k}} = f_{k_1\ldots k_n}$ are the coefficients to be determined.

Substituting (4.1) and (4.2) into Eq (1.1) and equating the coefficients for $v_{\mathbf{k}}(x)$, we obtain the following problem for the coefficients $u_{\mathbf{k}}(t)$:

$$\frac{\partial u_{\mathbf{k}}(t)}{\partial t} = -\lambda_{\mathbf{k}}u_{\mathbf{k}}(t) + f_{\mathbf{k}}g(t), \tag{4.3}$$

$$u_{\mathbf{k}}(0) = \varphi_{\mathbf{k}},\ u_{\mathbf{k}}(T) = \psi_{\mathbf{k}}. \tag{4.4}$$

The general solution to Eq (4.3) is the function

$$u_{\mathbf{k}}(t) = C_{\mathbf{k}} \cdot e^{-\lambda_{\mathbf{k}}}t + f_{\mathbf{k}}\int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}g(\tau)\,d\tau, \quad \mathbf{k} = (k_1, ..., k_n) \in \mathbb{N}^n, \tag{4.5}$$

where $C_{\mathbf{k}}$ are arbitrary constants. Let us introduce the notation

$$g_{\mathbf{k}}(t) = \int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}g(\tau)\,d\tau,$$

then $u_{\mathbf{k}}(t)$ from (4.5) is represented in the form

$$u_{\mathbf{k}}(t) = C_{\mathbf{k}} \cdot e^{-\lambda_{\mathbf{k}}t} + f_{\mathbf{k}}g_{\mathbf{k}}(t), \quad \mathbf{k} = (k_1, ..., k_n) \in \mathbb{N}^n.$$

Now, we will consider two cases separately: $g(t) = 1$ and $g(t) \neq 1$.

(I) Suppose $g(t) = 1$, then

$$g_{\mathbf{k}}(t) = \int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}\,d\tau = \frac{1}{\lambda_{\mathbf{k}}}\left(1 - e^{-\lambda_{\mathbf{k}}t}\right)$$

and $u_{\mathbf{k}}(t)$ are represented as

$$u_{\mathbf{k}}(t) = C_{\mathbf{k}} \cdot e^{-\lambda_{\mathbf{k}}t} + \frac{f_{\mathbf{k}}}{\lambda_{\mathbf{k}}}\left(1 - e^{-\lambda_{\mathbf{k}}t}\right).$$

If we use condition (4.4), we obtain

$$C_{\mathbf{k}} = u_{\mathbf{k}}(0) = \varphi_{\mathbf{k}}, \quad \psi_{\mathbf{k}} = u_{\mathbf{k}}(T) = \varphi_{\mathbf{k}}e^{-\lambda_{\mathbf{k}}T} + \frac{f_{\mathbf{k}}}{\lambda_{\mathbf{k}}}\left(1 - e^{-\lambda_{\mathbf{k}}T}\right),$$

then we find

$$f_{\mathbf{k}} = \lambda_{\mathbf{k}}\frac{\psi_{\mathbf{k}} - \varphi_{\mathbf{k}}e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}. \tag{4.6}$$

The solution to problem (4.3), (4.4) is represented as

$$u_{\mathbf{k}}(t) = \frac{e^{-\lambda_{\mathbf{k}}t} - e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}\varphi_{\mathbf{k}} + \frac{1 - e^{-\lambda_{\mathbf{k}}t}}{1 - e^{-\lambda_{\mathbf{k}}T}}\psi_{\mathbf{k}}. \tag{4.7}$$

Hence, the solution to Problem 1 can be written as

$$u(t,x) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[\frac{e^{-\lambda_{\mathbf{k}}t} - e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}\varphi_{\mathbf{k}} + \frac{1 - e^{-\lambda_{\mathbf{k}}t}}{1 - e^{-\lambda_{\mathbf{k}}T}}\psi_{\mathbf{k}}\right]v_{\mathbf{k}}(x), \tag{4.8}$$

$$f(x) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\lambda_{\mathbf{k}}\frac{\psi_{\mathbf{k}} - \varphi_{\mathbf{k}}e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}v_{\mathbf{k}}(x). \tag{4.9}$$

By construction and also due to the properties of functions $v_{\mathbf{k}}(x)$, the function $u(t,x)$ from (4.8) formally satisfies conditions (1.1)–(1.4). Let us examine the smoothness of the functions $u(t,x)$ and $f(x)$.

Let the functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Corollary 3.1, then the number series

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\lambda_{\mathbf{k}}\left(|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}|\right)$$

converges. As $\lambda_{\mathbf{k}} = \varepsilon_{\mathbf{k}}\mu_{\mathbf{k}}^2$ and $\varepsilon_{\mathbf{k}} > 0$ does not tend to 0, the functions $\frac{1}{1-e^{-\lambda_{\mathbf{k}}T}}$ and $\frac{e^{-\lambda_{\mathbf{k}}T}}{1-e^{-\lambda_{\mathbf{k}}T}}$ are bounded. For the coefficients $f_{\mathbf{k}}$ from equality (4.6), we obtain

$$|f_{\mathbf{k}}| \leq \lambda_{\mathbf{k}}\frac{1}{1 - e^{-\lambda_{\mathbf{k}}T}}|\psi_{\mathbf{k}}| + \lambda_{\mathbf{k}}\frac{e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}|\varphi_{\mathbf{k}}| \leq C\left(\lambda_{\mathbf{k}}|\varphi_{\mathbf{k}}| + \lambda_{\mathbf{k}}|\psi_{\mathbf{k}}|\right).$$

Based on this estimate and the uniform boundedness of the moduli of the eigenfunctions $\left|v_{k_1\ldots k_n}(x)\right|$ from (2.3), we obtain absolute and uniform convergence of the functional series (4.9) in the closed domain $\bar{\Pi}$. Hence, the sum of this series, i.e., the function $f(x)$, is continuous in the domain $\bar{\Pi}$. For all $0 \leq t \leq T$, there are estimates

$$\left|\frac{e^{-\lambda_{\mathbf{k}}t} - e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}\right| = e^{-\lambda_{\mathbf{k}}T}\left|\frac{e^{\lambda_{\mathbf{k}}(T-t)} - 1}{1 - e^{-\lambda_{\mathbf{k}}T}}\right| \leq C, \quad \left|\frac{1 - e^{-\lambda_{\mathbf{k}}t}}{1 - e^{-\lambda_{\mathbf{k}}T}}\right| \leq C.$$

The series (4.8) satisfies the estimate

$$\left|\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[\frac{e^{-\lambda_{\mathbf{k}}t} - e^{-\lambda_{\mathbf{k}}T}}{1 - e^{-\lambda_{\mathbf{k}}T}}\varphi_{\mathbf{k}} + \frac{1 - e^{-\lambda_{\mathbf{k}}t}}{1 - e^{-\lambda_{\mathbf{k}}T}}\psi_{\mathbf{k}}\right]v_{\mathbf{k}}(x)\right|$$
$$\leq C\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}|\right] < \infty.$$

This means that this series converges absolutely and uniformly in a closed domain $\bar{Q}$ and, therefore the function $u(t,x)$, the sum of this series, belongs to the class $C(\bar{Q})$.

Differentiating series (4.8) with respect to the variable t, we obtain

$$u_t(t,x) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[-\lambda_{\mathbf{k}}\frac{e^{-\lambda_{\mathbf{k}}t}}{1 - e^{-\lambda_{\mathbf{k}}T}}\varphi_{\mathbf{k}} + \lambda_{\mathbf{k}}\frac{e^{-\lambda_{\mathbf{k}}t}}{1 - e^{-\lambda_{\mathbf{k}}T}}\psi_{\mathbf{k}}\right]v_{\mathbf{k}}(x). \tag{4.10}$$

If $f(\lambda)=\lambda e^{-\lambda t}$, then $\max_{\lambda\geq 0} f(\lambda)=f(1/t)=\frac{1}{t}e^{-1}$, which means that $\lambda e^{-\lambda t}\leq\frac{1}{\delta e}$ for $t\geq\delta$. Therefore, the series (4.10) satisfies the estimate

$$|u_t(t,x)|\leq C\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}[|\varphi_{\mathbf{k}}|+|\psi_{\mathbf{k}}|].$$

If the condition of Lemma 3.1 is satisfied, the last series converges, then for an arbitrary $\delta>0$, series (4.10) converges absolutely and uniformly in the closed domain $\bar{Q}_\delta=[\delta\leq t\leq T]\times\bar{\Pi}\subset Q$. Hence, $u_t(t,x)\in C\left(\bar{Q}_\delta\right)$. Thus, by virtue of arbitrariness of $\delta>0$, it follows that $u_t(t,x)\in C(Q)$. The inclusion $L_x u(t,x)\in C(Q)$ is proved in a similar way. Thus, functions (4.8) and (4.9) for $g(t)=1$ satisfy all the conditions of Problem 1.

If in Problem 1 homogeneous conditions (4.2) are specified, then for the coefficients $f_{k_1...k_n}$ in equality (4.6), we obtain that $f_{k_1...k_n}\equiv(f,v_{k_1k_2...k_1})=0$. Hence, the function $f(x)$ is orthogonal to all elements of the system $\{v_{\mathbf{k}}(x)\}_{\mathbf{k}\in\mathbb{N}^n}$. Due to the completeness of this system and continuity of the function $f(x)$, we obtain $f(x)\equiv 0$, $x\in\bar{\Pi}$.

Similarly, for the coefficients $u_{\mathbf{k}}(t)$ from equality (4.5) we obtain $u_{\mathbf{k}}(t)\equiv(u,v_{\mathbf{k}})=0$. Hence, the equality $u(t,x)=0$, $x\in\bar{\Pi}$ is valid for almost all $t\in[0,T]$. Due to the continuity of the function $u(t,x)$ in $\bar{Q}$, we obtain that $u(t,x)\equiv 0$, $(t,x)\in\bar{Q}$. This implies the uniqueness of the solution to Problem 1.

Thus, we proved the following assertion.

Theorem 4.1. Let the functions $\varphi(x)$ and $\psi(x)$ in Problem 1 satisfy the conditions of Corollary 3.1, $g(t)=1$ and let the coefficients a_i, $i=0,1,...,2^n-1$ be such that the conditions $\varepsilon_{\mathbf{k}}>0$ are satisfied. The solution to the problem exists, is unique, and is represented in the form of series (4.8) and (4.9).

(II) Let us study Problem 1 for the case $g(t)\neq 1$. If we search for a solution to the problem in the form of (4.1) and (4.2), then for unknown coefficients $u_{\mathbf{k}}(t)$, we obtain problem (4.3) and (4.4). Moreover, the general solution to Eq (4.3) is determined by equality (4.5). Substituting this function into condition (4.4), we have

$$\varphi_{\mathbf{k}}=u_{\mathbf{k}}(0)=C_{\mathbf{k}},\quad \psi_{\mathbf{k}}=u_{\mathbf{k}}(T)=\varphi_{\mathbf{k}}\cdot e^{-\lambda_{\mathbf{k}}T}+f_{\mathbf{k}}g_{\mathbf{k}}(T).$$

Thus, if for all $\mathbf{k}\in\mathbb{N}^n$, the condition $g_{\mathbf{k}}(T)\neq 0$, then

$$f_{\mathbf{k}}=\frac{1}{g_{\mathbf{k}}(T)}\left[\psi_{\mathbf{k}}-\varphi_{\mathbf{k}}\cdot e^{-\lambda_{\mathbf{k}}T}\right],\tag{4.11}$$

and

$$u_{k_1...k_n}(t)=\left[1-\frac{g_{k_1...k_n}(t)}{g_{k_1...k_n}(T)}e^{-\lambda_{k_1...k_n}(T-t)}\right]e^{-\lambda_{k_1...k_n}t}\varphi_{k_1...k_n}+\frac{g_{k_1...k_n}(t)}{g_{k_1...k_n}(T)}\psi_{k_1...k_n}.\tag{4.12}$$

If in Problem 1, as in the case of $g(t)=1$, homogeneous conditions (4.2) are given, we obtain $f(x)\equiv 0$, $x\in\bar{\Pi}$ and $u(t,x)\equiv 0,(t,x)\in\bar{Q}$. Therefore, if the conditions $g_{\mathbf{k}}(T)\neq 0$, $\mathbf{k}\in\mathbb{N}^n$ are satisfied, the solution to Problem 1 is unique. If for some $\mathbf{m}=(m_1,m_2,...,m_n)$ the equality $g_{\mathbf{m}}(T)=0$ holds, then the homogeneous Problem 1 has a nonzero solution. Let us show that if this condition is satisfied, a pair of functions

$$u(t,x)=f_{\mathbf{m}}g_{\mathbf{m}}(t)v_{\mathbf{m}}(x),\quad f(x)=f_{\mathbf{m}}v_{\mathbf{m}}(x)$$

will be a solution to homogeneous Problem 1, where $f_{\mathbf{m}}$ is an arbitrary constant. Indeed, applying the operators $\frac{\partial}{\partial t}$ and L_x to the function $u(t,x)$, we get

$$u_t(t,x) = f_{\mathbf{m}} \frac{\partial g_{\mathbf{m}}(t)}{\partial t} v_{\mathbf{m}}(x) = f_{\mathbf{m}} g(t) v_{\mathbf{m}}(x) - \lambda_{\mathbf{m}} f_{\mathbf{m}} g_{\mathbf{m}}(t) v_{\mathbf{m}}(x),$$
$$L_x u(t,x) = f_{\mathbf{m}} g_{\mathbf{m}}(t) L v_{\mathbf{m}}(x) = -\lambda_{\mathbf{m}} f_{\mathbf{m}} g_{\mathbf{m}}(t) v_{\mathbf{m}}(x).$$

Hence,

$$u_t(t,x) - Lu(t,x) = f_{\mathbf{m}} g(t) v_{\mathbf{m}}(x) = f(x) g(t).$$

It is obvious that this function satisfies homogeneous conditions (1.2) and (1.3). Thus, we proved the following assertion.

Theorem 4.2. If a solution to Problem 1 exists, then it is unique if, and only if, the conditions $g_{\mathbf{k}}(T) \neq 0$ are satisfied for all $\mathbf{k} \in \mathbb{N}^n$.

Regarding the existence of a solution to Problem 1 in the case $g(t) \neq 1$, the following assertion is valid.

Theorem 4.3. Let in Problem 1 the functions $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Corollary 3.1, and the coefficients a_i, $i = 0, 1, ..., 2^n - 1$ are such that the conditions $\varepsilon_{\mathbf{k}} > 0$ are satisfied. A solution to the problem exists and is represented in the form of series (4.8) and (4.9), where the coefficients $f_{\mathbf{k}}$ and $u_{\mathbf{k}}$ are respectively determined by equalities (4.11) and (4.12).

Proof. If $g(t) \in C[0,T], |g(t)| \geq g_0 \equiv C$, then according to the mean value theorem there is a point $\xi \in [0,T]$ such that

$$g_{\mathbf{k}}(T) = \int_0^T e^{-\lambda_{\mathbf{k}}(T-\tau)} g(\tau)\, d\tau = g(\xi) \int_0^T e^{-\lambda_{\mathbf{k}}(T-\tau)}\, d\tau = g(\xi) \frac{1 - e^{-\lambda_{\mathbf{k}} T}}{\lambda_{\mathbf{k}}}.$$

This gives us the following lower estimate:

$$\left| g_{k_1...k_n}(T) \right| = |g(\xi)| \frac{1 - e^{-\lambda_{k_1...k_n} T}}{\lambda_{k_1...k_n}} \geq \frac{C}{\lambda_{k_1...k_n}}. \tag{4.13}$$

Using inequality (4.13) for $f_{\mathbf{k}}$ from equality (4.11), we get

$$|f_{\mathbf{k}}| \leq \frac{1}{|g_{\mathbf{k}}(T)|} \left| \psi_{\mathbf{k}} - \varphi_{\mathbf{k}} \cdot e^{-\lambda_{\mathbf{k}} T} \right| \leq C \lambda_{\mathbf{k}} \left(|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}| \right).$$

Similarly from equality (4.12), we obtain for $u_{\mathbf{k}}(t)$,

$$|u_{\mathbf{k}}(t)| \leq \left| e^{-\lambda_{\mathbf{k}} t} - \frac{g_{\mathbf{k}}(t)}{g_{\mathbf{k}}(T)} e^{-\lambda_{\mathbf{k}} T} \right| |\varphi_{\mathbf{k}}| + \left| \frac{g_{\mathbf{k}}(t)}{g_{\mathbf{k}}(T)} \right| |\psi_{\mathbf{k}}|.$$

As the function $g(t)$ is continuous on the interval $[0,T]$, it follows that

$$|g_{\mathbf{k}}(t)| \leq \int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)} |g(\tau)|\, d\tau \leq \max_{0 \leq \tau \leq T} |g(\tau)| \int_0^T e^{-\lambda_{\mathbf{k}}(t-\tau)}\, d\tau.$$

Hence, $\left| \frac{g_{\mathbf{k}}(t)}{g_{\mathbf{k}}(T)} \right| \leq C$. Therefore, the estimate

$$|u_{\mathbf{k}}(t)| \leq C \left(|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}| \right)$$

is valid for $u_{\mathbf{k}}(t)$. The estimate

$$\left|\frac{\partial u_{\mathbf{k}}(t)}{\partial t}\right| \le C\lambda_{\mathbf{k}}\left(|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}|\right)$$

is proved in a similar way. From these estimates and the convergence of the series

$$\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\left(|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}|\right), \quad \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\lambda_{\mathbf{k}}\left(|\varphi_{\mathbf{k}}| + |\psi_{\mathbf{k}}|\right)$$

absolute and uniform convergence of series (4.8) and (4.9) follows. For the sums of these series we get $f(x) \in C\left(\bar{\Pi}\right)$ and $u(t,x) \in C\left(\bar{Q}\right)$.

The absolute and uniform convergence of the series

$$\frac{\partial u(t,x)}{\partial t} = \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\frac{\partial u_{\mathbf{k}}(t)}{\partial t}v_{\mathbf{k}}(x), \quad L_x u(t,x) = \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}u_{\mathbf{k}}(t)Lv_{\mathbf{k}}(x)$$

in an arbitrary closed domain $\bar{Q}_\delta \subset Q$ and $\delta > 0$ is proved similar to the case $g(t) = 1$. Therefore, the inclusions $\frac{\partial u(t,x)}{\partial t}$, $Lu(t,x) \in C(Q)$ are valid. The theorem is proved.

5. Uniqueness and existence of a solution to Problem 2

The main assertion regarding Problem 2 is the following theorem.

Theorem 5.1. Let $\varepsilon_{\mathbf{k}} > 0$, the function $\varphi(x)$, satisfy the conditions of Corollary 3.1, then if $f(x_0) \neq 0$, $h(t) \in C^1[0,T]$, $h(0) = \varphi(x_0)$, the solution to Problem 2 exists and is unique.

Proof. If we assume that the function $g(t)$ is known, then the solution to Problem 2 can be represented as

$$u(t,x) = \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\left[\varphi_{k_1...k_n}\cdot e^{-\lambda_{\mathbf{k}}t} + f_{\mathbf{k}}\int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}g(\tau)\,d\tau\right]v_{\mathbf{k}}(x), \tag{5.1}$$

where

$$f_{\mathbf{k}} = \int_0^{p_1}...\int_0^{p_n} f(x)v_{\mathbf{k}}(x)\,dx_1...dx_n.$$

Let us suppose that $x = x_0$ in (5.1), then

$$\begin{aligned}
h(t) &= \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\left[\varphi_{\mathbf{k}}\cdot e^{-\lambda_{\mathbf{k}}t} + f_{\mathbf{k}}\int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}g(\tau)\,d\tau\right]v_{\mathbf{k}}(x_0)\\
&= \varphi_0(t) + \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\left[f_{\mathbf{k}}\int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}g(\tau)\,d\tau\right]v_{\mathbf{k}}(x_0)\\
&= \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\varphi_{k_1...k_n}\cdot e^{-\lambda_{\mathbf{k}}t}v_{\mathbf{k}}(x_0) + \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\left[f_{\mathbf{k}}\int_0^t e^{-\lambda_{\mathbf{k}}(t-\tau)}g(\tau)\,d\tau\right]v_{\mathbf{k}}(x_0),
\end{aligned}$$

where

$$\varphi_0(t) = \sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\varphi_{\mathbf{k}}\cdot e^{-\lambda_{\mathbf{k}}t}v_{\mathbf{k}}(x_0). \tag{5.2}$$

Denote $r(t) = h(t) - \varphi_0(t)$ and

$$K(t,\tau) = \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} f_{\mathbf{k}} e^{-\lambda_{\mathbf{k}}(t-\tau)} v_{\mathbf{k}}(x_0). \tag{5.3}$$

For the function $g(t)$, we obtain the following Volterra integral equation of the first kind

$$\int_0^t K(t,\tau) g(\tau)\, d\tau = r(t). \tag{5.4}$$

Lemma 5.1. If the function $\varphi(x)$ satisfies the conditions of Corollary 3.1, then the function $\varphi_0(t)$ from (5.2) is continuous and has a continuous derivative on the interval $[0, T]$.
Proof. Let us differentiate the series (5.2)

$$\varphi_0'(t) = -\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \lambda_{\mathbf{k}} \varphi_{\mathbf{k}} \cdot e^{-\lambda_{\mathbf{k}} t} v_{\mathbf{k}}(x_0).$$

As for the points of the domain $\bar{Q}$, we have the estimate $\left| e^{-\lambda_{\mathbf{k}} t} v_{\mathbf{k}}(x_0) \right| \le C$, then

$$\left| \varphi'{}_0(t) \right| \le \sum_{k_1=1}^{\infty} \sum_{k_n=1}^{\infty} \lambda_{\mathbf{k}} \left| \varphi_{\mathbf{k}} \right|.$$

If the condition of Corollary 3.1 is satisfied, the last number series converges, then series (5.2) converges absolutely and uniformly. Therefore, the sum of this series represents a continuous function, i.e., $\varphi_0(t) \in C^1[0, T]$. The lemma is proved.

Let us study the properties of the kernel $K(t, \tau)$. Differentiating series (5.3) with respect to t, we get

$$K_t(t,\tau) = -\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \lambda_{\mathbf{k}} f_{\mathbf{k}} e^{-\lambda_{\mathbf{k}}(t-\tau)} v_{\mathbf{k}}(x_0). \tag{5.5}$$

For series (5.3) and (5.5), we obtain the estimates

$$|K(t,\tau)| \le \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} |f_{\mathbf{k}}|, \quad |K_t(t,\tau)| \le \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \lambda_{\mathbf{k}} |f_{\mathbf{k}}|.$$

If the function $f(x)$ satisfies the conditions of Corollary 3.1, then the number series

$$\sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} |f_{\mathbf{k}}|, \quad \sum_{k_1=1}^{\infty} ... \sum_{k_n=1}^{\infty} \lambda_{\mathbf{k}} |f_{\mathbf{k}}|$$

converges. The series (5.3) and (5.5) converge absolutely and uniformly in the closed domain $[0, T] \times [0, T]$. Therefore, the functions $K(t, \tau)$ and $K_t(t, \tau)$ are continuous in this domain. Further, differentiating equality (5.4), we obtain

$$K(t,t) g(t) + \int_0^t K_t(t,\tau) g(\tau)\, d\tau = r'(t). \tag{5.6}$$

As

$$K(t,t) = \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} f_{\mathbf{k}} v_{\mathbf{k}}(x_0) = f(x_0) \neq 0,$$

the equality (5.6) is an integral Volterra equation of the second kind with a continuous kernel and a continuous righthand side. According to the general theory, such an equation has a unique solution $g(t)$ from the class $C[0,T]$.

If we substitute this function into equality (5.1), then the pair of functions $\{u(t,x), g(t)\}$ satisfies all the conditions of Problem 2. The smoothness of the function and the uniqueness of the solution are proved as in the case of Problem 1. The theorem is proved.

6. Conclusions

In this paper, the solvability of some inverse problems for a nonlocal analogue of the fourth-order parabolic equation is studied. The nonlocal operator is introduced using involutive mappings. Unlike previous works of the authors, in this paper the problems are studied in the n-dimensional case. The considered problems are solved by applying the Fourier method and reducing them to the Volterra integral equation. In this case, a spectral problem arises for the nonlocal analogue of the biharmonic operator. The eigenfunctions and eigenvalues for this problem are found explicitly and the completeness of the system of eigenfunctions is proved. Solutions to the main problems are constructed in the form of series using a system of eigenfunctions. Further, it is planned to continue the study of inverse problems for high order differential equations with involutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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