Eisenstein Series, Crystals, and Ice

Benjamin Brubaker, Daniel Bump, and Solomon Friedberg

Automorphic forms are generalizations of periodic functions; they are functions on a group that are invariant under a discrete subgroup. A natural way to arrange this invariance is by averaging. Eisenstein series are an important class of functions obtained in this way. It is possible to give explicit formulas for their Fourier coefficients. Such formulas can provide clues to deep connections with other fields. As an example, Langlands's study of Eisenstein series inspired his far-reaching conjectures that dictate the role of automorphic forms in modern number theory.

In this article, we present two new explicit formulas for the Fourier coefficients of (certain) Eisenstein series, each given in terms of a combinatorial model: crystal graphs and square ice. Crystal graphs encode important data associated to Lie group representations, whereas ice models arise in the study of statistical mechanics. Both will be described from scratch in subsequent sections.

We were led to these surprising combinatorial connections by studying Eisenstein series not just on a group but more generally on a family of covers of the group. We will present formulas for their Fourier coefficients that hold even in this generality. In the simplest case, the Fourier coefficients of Eisenstein series are described in terms of symmetric functions known as Schur polynomials, so that is where our story begins.

Benjamin Brubaker is associate professor of mathematics at the Massachusetts Institute of Technology. His email address is `brubaker@math.mit.edu`.

Daniel Bump is professor of mathematics at Stanford University. His email address is `bump@math.stanford.edu`.

Solomon Friedberg is professor of mathematics at Boston College. His email address is `friedber@bc.edu`.

This work was supported by NSF grants DMS-0844185, DMS-1001079, and DMS-1001326, and NSA grant H98230-10-1-0183.

Schur Polynomials

The symmetric group on n letters, S_n, acts on the ring of polynomials $\mathbb{Z}[x_1, \dots, x_n]$ by permuting the variables. A polynomial is *symmetric* if it is invariant under this action. Classical examples are the familiar elementary symmetric functions

$$e_j = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j}.$$

Since the property of being symmetric is preserved by sums and products, the symmetric polynomials make up a subring Λ_n of $\mathbb{Z}[x_1, \dots, x_n]$. The e_j, $1 \le j \le n$, generate this subring.

Since Λ_n is also an abelian group under polynomial addition, it is natural to seek a set that generates Λ_n as an abelian group. One such set is given by the Schur polynomials (first considered by Jacobi), which are indexed by partitions. A *partition* of a positive integer k is a nonincreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $k = \sum \lambda_i$; necessarily only a finite number of terms in the sequence are nonzero. Partitions are added componentwise. If $\lambda = (\lambda_i)$ is a partition with $\lambda_i = 0$ for $i > n$, let $\rho = (n-1, n-2, \dots, 0, \dots)$, and let

$$a_{\lambda+\rho} = \det(x_i^{\lambda_j+n-j})_{1 \le i,j \le n}.$$

Then a_ρ divides $a_{\lambda+\rho}$, and the quotient $s_\lambda := a_{\lambda+\rho}/a_\rho$ is the Schur polynomial. It is a homogeneous, symmetric polynomial of degree k. For example, we have

$$s_{(k,0)}(x_1, x_2) = x_1^k + x_1^{k-1}x_2 + \cdots + x_1 x_2^{k-1} + x_2^k \tag{1}$$

$$s_{(2,1,0)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2 x_3 + x_2 x_3^2. \tag{2}$$

The s_λ, running over all partitions λ with $\lambda_i = 0$ for $i > n$, form a basis for Λ_n. Schur showed that these polynomials describe the characters of representations of the symmetric and general linear groups. (See Macdonald [17] for more details.) As we will see in subsequent sections, these characters are connected to the Fourier coefficients of Eisenstein series.

Eisenstein Series on SL(2)

Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ denote the complex upper half plane. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformation:

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

It is of interest to find functions that are *automorphic*—invariant under the action of a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. The modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is of particular importance. One may create a family of automorphic functions on Γ by averaging. To this end, for each $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define the *unnormalized Eisenstein series*

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s,$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}.$$ [1]

Note that we must quotient out by the subgroup Γ_∞ since this is an infinite group that stabilizes the imaginary part of z. The definition makes clear that the Eisenstein series is automorphic—$E(\gamma(z), s) = E(z,s)$ for all $\gamma \in \Gamma$. Using the identity $\mathrm{Im}(\gamma z) = y/|cz+d|^2$, we may reparameterize the sum in terms of integer pairs (c,d). Indeed, each pair of relatively prime integers (c,d) is the bottom row of a matrix in Γ, and two matrices γ_1 and $\gamma_2 \in \Gamma$ have the same bottom row if and only if $\gamma_1\gamma_2^{-1} \in \Gamma_\infty$. Thus the Eisenstein series may be expressed in the form

$$E(z,s) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}, \tag{3}$$

from which one may deduce that the series converges absolutely for $\mathrm{Re}(s) > 1$.

The series $E(z,s)$ has many spectacular analytic properties. To describe them, define the *normalized Eisenstein series*,

$$E^*(z,s) = \tfrac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)E(z,s), \tag{4}$$

where $\zeta(s)$ is the Riemann zeta function and $\Gamma(s)$ is the gamma function. One can show that $E^*(z,s)$ has analytic continuation to a meromorphic function for $s \in \mathbb{C}$ and satisfies the functional equation $E^*(z,s) = E^*(z,1-s)$. This may be proved by spectral methods, as $E(z,s)$ is an eigenfunction of the Laplace-Beltrami operator on $\mathcal{H}$.

This fact has far-reaching consequences for the theory of automorphic forms. As an illustration in our present case, observe that the invariance under $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ implies that $E^*(z+1,s) = E^*(z,s)$. Hence the Eisenstein series admits a Fourier series with respect to the real variable x as follows:

$$E^*(z,s) = \sum_{r=-\infty}^{\infty} a(r,y,s)e^{2\pi i n x},$$

where

$$a(r,y,s) = \int_0^1 E^*(x+iy,s)e^{-2\pi i r x}\,dx.$$

In the special case $r = 0$, one can show that

$$a(0,y,s) = y^s\xi(2s) + y^{1-s}\xi(2-2s),$$

where $\xi(2s) = \pi^{-s}\Gamma(s)\zeta(2s)$. Because $a(0,y,s)$ inherits the analytic properties of the Fourier series, the analytic continuation and functional equation of the Riemann zeta function follow.

What about the remaining Fourier coefficients? A calculation (see, for example [7], Section 1.6) shows that if $r \neq 0$, then

$$a(r,y,s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)y^{1/2}K_{s-1/2}(2\pi|r|y)$$

where

$$\sigma_{1-2s}(r) = \sum_{m|r} m^{1-2s}$$

and K denotes a K-Bessel function.

Let us shift s to $s+\frac{1}{2}$ and examine the "arithmetic parts" of the nonconstant Fourier coefficients of $E^*(z,s+\frac{1}{2})$:

$$a(r) \stackrel{\mathrm{def}}{=} |r|^s\sigma_{-2s}(|r|).$$

They are *multiplicative*. That is, if $\gcd(r_1,r_2) = 1$, then $a(r_1r_2) = a(r_1)a(r_2)$. Thus they are completely determined by their values at prime powers $r = p^k$. Moreover,

$$a(p^k) = p^{ks} + p^{(k-2)s} + \cdots + p^{-ks}.$$

A fundamental theme of automorphic forms identifies these coefficients with *values of characters* of a representation. Let V denote the standard representation of $\mathrm{SL}_2(\mathbb{C})$ and let $\vee^k V$ denote the kth symmetric power. Thus if $A \in \mathrm{SL}_2(\mathbb{C})$ has eigenvalues α, β, then $\vee^k A$ has eigenvalues $\alpha^k, \alpha^{k-1}\beta, \ldots, \alpha\beta^{k-1}, \beta^k$. The character χ_k of the representation $\vee^k V$ is given by

$$\chi_k(A) = \mathrm{tr}(\vee^k(A)) = \sum_{k_1+k_2=k} \alpha^{k_1}\beta^{k_2}.$$

Comparing this with our earlier expression for the arithmetic piece $a(p^k)$, we find

$$a(p^k) = \chi_k\left(\begin{pmatrix} p^s & \\ & p^{-s} \end{pmatrix}\right). \tag{5}$$

[1] *In the first line of this article, we described automorphic forms as functions on groups, but here we've defined $E(z,s)$ as a function on the upper half plane $\mathcal{H}$. The resolution of this apparent discrepancy is that $\mathcal{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ where $\mathrm{SO}_2(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \;\middle|\; \theta \in [0,2\pi) \right\}$. Indeed, $\mathrm{SL}_2(\mathbb{R})$ acts transitively on the point i by linear fractional transformation with stabilizer $\mathrm{SO}_2(\mathbb{R})$.*

Notice that $a(p^k)$ is thus the Schur polynomial in (1) evaluated at $(x_1, x_2) = (p^s, p^{-s})$:

$$a(p^k) = s_{(k,0)}(p^s, p^{-s}). \tag{6}$$

This identity has substantial generalizations. Indeed, one can define Eisenstein series analogous to $E(z,s)$ for any reductive group G. In this generality, the notion of Fourier coefficient is replaced by that of Whittaker coefficient. The Casselman-Shalika formula [8], first proved for $\mathrm{GL}(n)$ by Shintani [18], states that the values on prime powers of these coefficients may be captured by characters of a representation. For $\mathrm{GL}(n)$, these characters are expressed in terms of Schur polynomials. For more general groups, the representation is not of the complex points of G, but rather a representation of the Langlands dual group[2] of G.

These generalizations are usually stated in a different language. The coefficients $a(p^k)$ above are expressible as integrals on groups over p-adic fields known as p-adic Whittaker functions. The local version of the Eisenstein series is an induced representation, and the Whittaker function is a p-adic integral evaluated on a canonical vector in the representation space. Similarly, one may study the Whittaker functions attached to more general Eisenstein series, corresponding to more general induced representations. These may be shown to be products of Langlands L-functions, and this observation is important in the study of those L-functions.

These constructions have been known for many years. The goal of this article is to put them in a new context, by considering a group *together with its covers*. When we do this, we find that the formula (5) and its generalizations may be reinterpreted in terms of crystal graphs, which are combinatorial structures introduced by Kashiwara in the context of representations of quantum groups. We begin by illustrating this for covers of SL(2) before discussing higher rank.

Eisenstein Series on Covers of SL(2)

The classical metaplectic group is a two-sheeted cover of a symplectic group over either the reals or a p-adic field. This group was introduced by Weil and explains the transformation formulas for theta functions. More generally, Kubota and Matsumoto, working over a number field L containing a full set of nth roots of unity, defined a family of n-sheeted covers of SL(2) (or any simply connected group) for each $n \geqslant 1$. Informally, we may think of such a cover as follows: it is an n-sheeted cover, where the sheets are indexed by the nth roots of unity. The group law requires moving between the sheets, and the nth root of unity that arises in taking the product of two group elements is computed using the arithmetic of the field L.[3] In fact, the existence of this group is closely related to the nth power reciprocity law.

For these groups, one may define an Eisenstein series $E_n(z,s)$ as an average, similar to (3). The definition is modified by adding an extra factor in the average that keeps track of the sheets of the cover. The Fourier coefficients of $E_n(z,s)$ turn out to be of great interest: they are Dirichlet series made with Gauss sums.

A *Gauss sum* is a discrete analogue of the gamma integral $\Gamma(s) = \int_0^\infty y^s e^{-y}\,\frac{dy}{y}$—a product of multiplicative and additive characters summed over the invertible elements of a finite ring. For example, if the cover degree is $n = 3$, we may take $L = \mathbb{Q}(e^{2\pi i/3})$ with ring of integers $\mathfrak{o}_L = \mathbb{Z}[e^{2\pi i/3}]$. Let $e(\cdot)$ be an additive character of L that is trivial on $\mathfrak{o}_L$ but no larger fractional ideal. Then for integers $m, c \in \mathfrak{o}_L$ with $c \neq 0$, let

$$g_3(m,c) = \sum_{\substack{t \,(\mathrm{mod}\, c)\\ \gcd(t,c)=1}} \left(\frac{t}{c}\right)_3 e(mt/c), \tag{7}$$

where the sum is over $t \in \mathfrak{o}_L$ that are invertible mod c and $(-)_3$ is the cubic residue symbol.[4] For general n and L, we may define a Gauss sum $g_n(m,c)$ made with nth power residue symbols. To do so, we must pass from the ring of integers $\mathfrak{o}_L$ to a localization $\mathfrak{o}_{L,S}$ where denominators are allowed at a finite set of places S, and some additional technicalities result.

Kubota computed the Fourier expansion of $E_n(z,s)$, whose mth coefficient is a K-Bessel function times an arithmetic part $a(m)$. In the special case $n = 3$, for $m \neq 0$

$$a(m) = \|m\|^{s-1/2} \sum_{\substack{c \in \mathfrak{o}_L\\ c \equiv 1 \,(\mathrm{mod}\, 3)}} \frac{g_3(m,c)}{\|c\|^{2s}}, \tag{8}$$

where $\|\ \|$ denotes the absolute norm. The form for general n is much the same, with an arithmetic part involving $g_n(m,c)$ in place of g_3. The series is easily seen to converge absolute for $\Re(s) > 3/4$, and since $E(z,s)$ has analytic continuation and functional equation, $a(m)$ inherits these properties as well. This series (and its generalizations) are basic objects of interest.

---

[2] *In fact, the dual group enters subtly into the computation above. The Eisenstein series $E(z,s)$ may be regarded as a function on* PGL_2*, and the Langlands dual of this group is* $\mathrm{SL}_2(\mathbb{C})$.

[3] *In more detail, let $\mathbb{A}_L$ denote the adèles, an appropriately restricted product over all completions of L. Then an n-fold metaplectic group is a central extension of* $\mathrm{SL}_2(\mathbb{A}_L)$ *by μ_n: $1 \longrightarrow \mu_n \longrightarrow \tilde{G} \longrightarrow \mathrm{SL}_2(\mathbb{A}_L) \longrightarrow 1$. This extension is described by means of a two-cocycle that is constructed using the n-power Hilbert symbols of completions of L. (It is not the adelic points of an algebraic group). See* [4] *for further information.*

[4] *Thus $(t/c)_3$ is a cube root of unity and is 1 if c is a cube, and $(t_1t_2/c)_3 = (t_1/c)_3(t_2/c)_3$.*

Let us recall two properties of Gauss sums valid for any $n \geq 1$. Using the Chinese Remainder Theorem, one may show that if $\gcd(c_1, c_2) = 1$, then

(9)
$$g_n(m, c_1 c_2) = \left(\frac{c_1}{c_2}\right)_n \left(\frac{c_2}{c_1}\right)_n g_n(m, c_1) g_n(m, c_2),$$

and if $\gcd(m_1, c) = 1$, then an easy change of variables shows that for any integer m_2

$$g_n(m_1 m_2, c) = \left(\frac{m_1}{c}\right)_n^{-1} g_n(m_2, c).$$

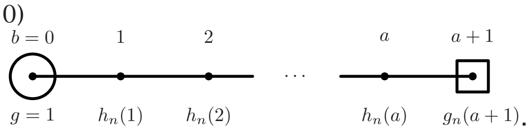
In particular, (9) shows that the Dirichlet series in (8) is not expressible as an Euler product—a product over primes—when $n > 2$. This is quite different from the situation for $n = 1, 2$ and, more generally, for Langlands L-functions. Instead, we see that to combine contributions from relatively prime c_1 and c_2, we must introduce nth roots of unity depending on arithmetic. For these reasons, we call series with such a property *twisted Euler products*. See [12] for more information and further examples.

Though not strictly multiplicative, these two properties allow us to reconstruct $g_n(m, c)$ from its values at prime powers $g_n(p^a, p^b)$ for nonnegative integers a, b. Thus we may restrict ourselves to these simpler cases in describing the Fourier coefficients.

Let us consider the coefficients $g_n(p^a, p^b)$ at a given prime p. Here a is fixed (it is the order of m at p) and b is varying. These coefficients come in three flavors. First, there is the case $b = 0$, where the coefficient is simply $1 = p^b$. Second, there are the coefficients for $1 \leq b \leq a$. The inequality $b \leq a$ makes the additive character in (7) trivial, and so this coefficient is the function

$$h_n(b) = \begin{cases} \phi(p^b) & \text{if } n | b, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi(p^b) = |(\mathfrak{o}_{L,S}/p^b \mathfrak{o}_{L,S})^\times|$ is the Euler phi function for $\mathfrak{o}_{L,S}$. Finally, there is the case $b = a+1$. In this case, the Gauss sum is always nonzero, and it is not possible to evaluate it in closed form except in special cases. We write this sum simply as $g_n(a+1)$ for short. For $b \geq a+2$, the Gauss sum is zero (which follows from expressing the sum in terms of a nontrivial character over a group). Hence the entire contribution to the p^ath Fourier coefficient can be summarized in the following diagram:

(10)


We have circled the location $b = 0$ and boxed the location $b = a + 1$ since the contributions are special at these locations, while at b such that $1 \leq b \leq a$, the contribution is $h_n(b)$. This is the most common situation. Notice that the diagram is the same for any n; it is only the functions g_n and h_n that depend on n.

For the nonmetaplectic Eisenstein series (the special case $n = 1$), we saw in the section "Eisenstein Series on SL(2)" that the coefficients at powers of p may also be described in terms of Schur polynomials. The connection to the sums of Gauss sums presented here is as follows. We work over $\mathbb{Q}$ for convenience. The residue symbol $(t/c)_1$ is trivial, and

$$h_1(a) = \phi(p^a),$$

(11)
$$g_1(a+1) = \sum_{\substack{t \,(\mathrm{mod}\, p^{a+1}) \\ \gcd(t,p)=1}} e(t/p) = -p^a.$$

Thus the arithmetic part $a(p^k)$ of the p^kth Fourier coefficient described in this section has the form

$$\begin{aligned} p^{k(s-1/2)} \Big(1 + \phi(p) p^{-2s} + \cdots + \phi(p^k) p^{-2ks} \\ - p^k p^{-2(k+1)s}\Big) \\ = (1 - p^{-2s}) s_{(k,0)}(p^{s-1/2}, p^{-(s-1/2)}). \end{aligned}$$

After sending $s \mapsto s + 1/2$ as before, this coincides with the formula (6) above. Note that the Eisenstein series defined in (4) was normalized by a zeta function, which explains the extra factor $(1 - p^{-2s})$ here.

Returning to the case of general n, the description of the Fourier coefficient as a sum of Gauss sums governed by (10) above has broad generalizations. Indeed, the underlying graph in (10) may be viewed as a crystal graph associated with a highest weight representation of SL_2. In the next section, we will discuss crystal graphs in more detail and explain how they may be used to give a generalization to covers of SL_{r+1} for any r and any cover degree n.

Eisenstein Series on Covers of SL_{r+1} and Crystal Graphs

We continue to work over a number field L containing n nth roots of unity. One can define an n-fold cover of (the adelic points of) SL_{r+1} for any r and a corresponding Eisenstein series E_n for this group. It is an average of a suitable function, this time a function of r complex variables $s_1, \ldots, s_r$, over a discrete subgroup.[5]

Fourier coefficients generalize to Whittaker coefficients. These are defined by integrating E_n against a character of U, the subgroup of upper triangular unipotent matrices of $\mathrm{SL}_{r+1}(\mathbb{A}_L)$. The necessary characters of U are indexed by r-tuples $\mathbf{m} = (m_1, \ldots, m_r)$ of elements of $\mathfrak{o}_L$. (Indeed, a character of U depends only on the r locations

[5] *There are more general Eisenstein series built from automorphic forms on lower rank groups, but we do not consider them here.*

just above the main diagonal since everything else is in $[U, U]$.) Then the Whittaker coefficients are defined by integration against this character.

The main theorem of [4] expresses the arithmetic part $a(\mathbf{m})$ of these Whittaker coefficients for $m_i \neq 0$ as $\|m_1\|^{s_1-1/2} \cdots \|m_r\|^{s_r-1/2}$ times the multiple Dirichlet series

$$\sum_{c_1,\ldots,c_r} \frac{H_n(\mathbf{m}; c_1, \ldots, c_r)}{\|c_1\|^{2s_1} \cdots \|c_r\|^{2s_r}}.$$

This is a generalization of (8). The coefficients H_n are once again twisted multiplicative, and this allows one to reduce their study to that of the coefficients $H_n(p^{\ell_1}, \ldots, p^{\ell_r}; p^{k_1}, \ldots, p^{k_r})$ attached to a given prime p of $\mathfrak{o}_L$. Here the ℓ_i and k_i are nonnegative integers. The coefficients H turn out to be built out of the functions g_n, h_n, and powers of $\|p\|$ that already appeared in the previous section for the n-fold cover of SL(2). However, the exact description is considerably more subtle. It involves the theory of crystal graphs.

To explain further, we briefly recall several important facts about finite-dimensional representations of Lie groups and their crystal graphs. A *weight* of GL_{r+1} is a rational character of the diagonal torus T of GL_{r+1}. The weights may be identified with elements of the lattice $\Lambda = \mathbb{Z}^{r+1}$: if $\mu = (\mu_1, \ldots, \mu_{r+1}) \in \Lambda$, then $\boldsymbol{t}^{\mu} := \prod t_i^{\mu_i}$ with $\boldsymbol{t} = \mathrm{diag}(t_1, \ldots, t_{r+1}) \in T$ is such a character. A weight for a representation V of the associated Lie algebra $\mathfrak{gl}_{r+1}(\mathbb{C})$ is a weight μ such that there exists a nonzero vector in V that transforms under the torus by μ; it is highest if no larger weight satisfies this property.[6] Cartan's Theorem of the Highest Weight states that every finite-dimensional irreducible complex representation of $\mathfrak{gl}_{r+1}(\mathbb{C})$ (or any complex semisimple finite-dimensional Lie algebra) has a unique highest weight vector (up to scalars) and that the highest weight classifies the representation.

The quantum group $U_q(\mathfrak{gl}_{r+1}(\mathbb{C}))$ is a deformation of the universal enveloping algebra of $\mathfrak{gl}_{r+1}(\mathbb{C})$ that is obtained when a parameter q is introduced into the relations that describe the universal enveloping algebra. (See Hong and Kang [14].) Finite-dimensional representations are once again classified by highest weight. Let λ be a *dominant* weight (that is, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{r+1}$). Then Kashiwara [15] associates with λ a *crystal graph* $\mathcal{B}_\lambda$, a directed graph whose vertices correspond to basis vectors for the representation of $U_q(\mathfrak{gl}_{r+1}(\mathbb{C}))$ with highest weight λ. The edges of this graph are colored with one color for each

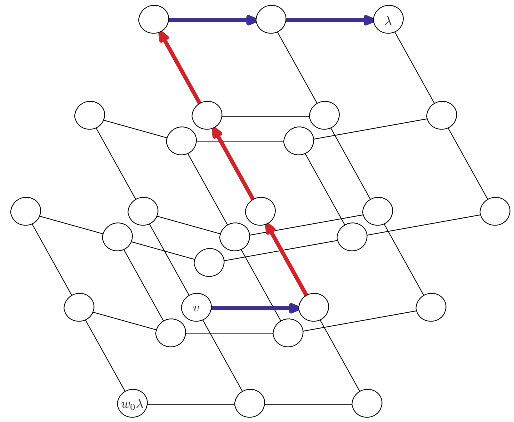


Figure 1. The crystal graph with highest weight $\lambda = (4, 2, 0)$.

simple root and describe the action of the unipotents in the Lie algebra on this basis as $q \to 0$. The crystal graph $\mathcal{B}_\lambda$ comes endowed with a map "wt" to the weight lattice Λ which is compatible with the graph structure. Walking one step along an edge of $\mathcal{B}_\lambda$ in the direction of the highest weight vector (resp. lowest weight vector) corresponds to increasing (resp. decreasing) the weight of the vertex by the simple root with which it is labeled.[7]

Figure 1 depicts a $\mathfrak{gl}_3$ crystal with highest weight $\lambda = (4, 2, 0)$ and lowest weight $w_0\lambda = (0, 2, 4)$. It is drawn so that vertices of the same weight are clustered together diagonally.

Berenstein and Zelevinsky [2] and Littelmann [16] associate paths with each vertex in $\mathcal{B}_\lambda$.[8] To do this, choose a reduced factorization of the long element w_0 of the Weyl group into simple reflections (i.e., one of minimal length). Walk the graph toward the highest weight vector in the order that the simple reflections appear in the factorization, going as far in a given direction as the graph will allow before changing to the next color. It turns out that such a factorization always leads to a path to the highest weight vector λ. The sequence $\mathrm{BZL}(v)$ of path lengths so obtained parameterizes the vertex v of $\mathcal{B}_\lambda$. (Alternatively, we could record path lengths toward the lowest weight vector $w_0\lambda$ from v.)

For example, in Figure 1 we have indicated a walk from a vertex v to the highest weight vector λ. It corresponds to the factorization of the long element $w_0 = s_1 s_2 s_1$ of the symmetric group S_3, the Weyl group of GL_3. Thus we walk along the graph in order s_1, s_2, s_1 (= blue, red, blue). The lengths of

[6] *Recall that the weights are partially ordered as follows: $\lambda \geqslant \mu$ if $\lambda - \mu$ is a nonnegative linear combination of simple roots. In terms of coordinates, $\lambda_i = \mu_i + h_i - h_{i+1}$ for each i, where the h_i are nonnegative integers and $h_0 = h_{r+2} = 0$.*

[7] *The map "wt" is such that $\sum_{v \in \mathcal{B}_\lambda} \boldsymbol{t}^{\mathrm{wt}(v)}$ is the character of an irreducible representation of $GL_{r+1}(\mathbb{C})$ whose associated Lie algebra representation has highest weight λ.*

[8] *Berenstein and Zelevinsky refer to these paths as "strings".*

the corresponding paths are 1, 3, 2, respectively, so $\mathrm{BZL}(v) = (1,3,2)$.

The main theorem of [4] computes the coefficients $H_n(p^{\ell_1},\dots,p^{\ell_r};p^{k_1},\dots,p^{k_r})$ by attaching *products of Gauss sums* to BZL sequences. Let $\lambda_{r+1} = 0$ and $\lambda_i = \ell_i + \lambda_{i+1}$ when $i \le r$, and let λ be the dominant weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{r+1})$.[9] Let ρ denote the Weyl vector, that is, half the sum of the positive roots, or in coordinates $(r, r-1, \dots, 1, 0)$.

Theorem 1. *The coefficient H_n is given by*

$$H_n(p^{\ell_1},\dots,p^{\ell_r};p^{k_1},\dots,p^{k_r}) = \sum_{\substack{v\in\mathcal{B}_{\lambda+\rho}\\ \mathrm{wt}(v)=\mu}} G_n(v), \tag{12}$$

where the function $G_n(v)$ is described below and μ is the weight related to $(k_1,\dots,k_r)$ by the condition that $\sum_{i=1}^r k_i\alpha_i = (\lambda+\rho)-\mu$ where α_i are the simple roots.

The definition of $G_n(v)$ depends on a recipe for walking the graph, so it depends on the choice of a reduced expression for w_0 in the symmetric group S_{r+1}. We will work with two different choices; these give rise to two different functions $G_n(v)$. In terms of the simple reflections s_i (recorded by their index $i \in [1,r]$), let us fix either

$$\Sigma = \Sigma_1 := (r, r-1, r, r-2, r-1, r, \dots, 1, 2, 3, \dots, r) \tag{13}$$

or

$$\Sigma = \Sigma_2 := (1,2,1,3,2,1,\dots,r,r-1,\dots,3,2,1) \tag{14}$$

and take the associated path lengths $\mathrm{BZL}(v) = (b_1,\dots,b_N)$ to the highest weight vector. (We suppress the dependence on Σ.) We then *decorate* the entries b_i as follows. The length b_i is *boxed* if the ith leg of the path is maximal (i.e., contains the entire root string). In Figure 1, with $\Sigma = \Sigma_2$, $\mathrm{BZL}(v) = (1,3,2)$, both the 1 and 2 are boxed while the 3 is not (since it is part of an s_2 root string of length 4). An entry b_i is *circled* if a corresponding leg of the path to the lowest weight vector is maximal (see [3], Ch. 3). Thus in Figure 1, the path lengths to the lowest weight vector are $(0,1,1)$, none of which are maximal. Hence none of the entries in the decorated BZL sequence $(\boxed{1}, 3, \boxed{2})$ are circled.

Then we prove that

$$G_n(v) = G_{n,\Sigma}(v) = \prod_{b_i\in\mathrm{BZL}(v)} \begin{cases} \|p\|^{b_i} & \text{if } b_i \text{ is circled (but not boxed)},\\ g_n(b_i) & \text{if } b_i \text{ is boxed (but not circled)},\\ h_n(b_i) & \text{if neither},\\ 0 & \text{if both,}\end{cases} \tag{15}$$

[9] *By fixing $\lambda_{r+1} = 0$, we parameterize representations of* $\mathrm{SL}_{r+1}(\mathbb{C})$*, the Langlands dual group of* PGL_{r+1}.

where $g_n(b)$ and $h_n(b)$ are the Gauss sum and degenerate Gauss sum, respectively, described in the previous section. Notice that this definition exactly matches the description given above and pictured in (10) in the special case of SL_2.

In Figure 1, the vertex v belongs to a weight space with multiplicity two. Again using $\Sigma = (1,2,1)$, the other vertex in the weight space containing v has decorated BZL sequence $(2, \boxed{3}, 1)$. Thus applying Theorem 1 with $G_n(v)$ as in (15), we have

$$\begin{aligned} H_n(p^2,p;p^3,p^3) &= G_n\left(\boxed{1},3,\boxed{2}\right) + G_n\left(2,\boxed{3},1\right)\\ &= g_n(1)h_n(3)g_n(2)\\ &\quad + h_n(2)g_n(3)h_n(1). \end{aligned} \tag{16}$$

Since $h_n(b) = 0$ unless n divides b, this term is nonzero only for the cover degrees $n = 1$ or 3. It is noteworthy that expressions like (16) for the function H_n in terms of Gauss sums are uniform in n.

Because we may use either Σ_1 or Σ_2 to define $G_n(v)$, these are two explicit formulas for the Whittaker coefficient. The equality of the expression in (12) for Σ_1 and Σ_2 is not formal and is established directly in [3] by an elaborate blend of number-theoretic and combinatorial arguments. It is an open problem to give a definition of $G_n(v)$ obtaining the Whittaker coefficient for an arbitrary reduced decomposition of the long element w_0 of the Weyl group.

In closing this section, we mention that there are not one but two distinct generalizations of the Casselman-Shalika formula to the metaplectic case. Chinta and Gunnells [9] and Chinta and Offen [10] show that the p-parts of the Whittaker coefficients of metaplectic Eisenstein series on covers of SL_{r+1} can also be expressed as quotients of sums over the Weyl group, directly analogous to the Weyl character formula.

The Case n = 1: Tokuyama's Deformation Formula

When $n = 1$, we are concerned with Eisenstein series on an algebraic group and not a cover. In that case, the Whittaker coefficients may be computed in two different ways. First, Theorem 1 provides an answer in terms of crystal graphs. This result holds for any $n \ge 1$. Second, the formula of Shintani [18] and Casselman and Shalika [8] (which holds only for $n = 1$) expresses the Whittaker coefficients of normalized Eisenstein series as the values of the characters of irreducible representations of $\mathrm{SL}_{r+1}(\mathbb{C})$. These characters are given by Schur polynomials s_λ, as described in the section "Schur Polynomials".

These two expressions for the Whittaker coefficients are related by the following result (see [3], Ch. 5).

Theorem 2. *Let $\boldsymbol{z} = (z_1, \dots, z_{r+1})$ and let $q = \|p\|$. For any dominant weight λ,*

$$\left[\prod_{i<j}(z_i - q^{-1}z_j)\right] s_\lambda(\boldsymbol{z}) = \sum_{\nu \in \mathcal{B}_{\rho+\lambda}} G_1(\nu) q^{-\langle \lambda+\rho-\mathrm{wt}(\nu),\rho\rangle} \boldsymbol{z}^{\mathrm{wt}(\nu)},$$

where the $G_1(\nu)$ are computed as in (15) *using the reduced word Σ_2.*

We illustrate Theorem 2 with $\lambda = (2,1,0)$, so that $\lambda + \rho = (4,2,0)$ and $\mathcal{B}_{\lambda+\rho}$ is the crystal pictured earlier. Let us compare the monomials $z_1 z_2^2 z_3^3$ appearing on both sides of the theorem for this choice of λ. The coefficient of this monomial appearing on the right-hand side is (up to a power of q) just the value of $H_n(p^2, p; p^3, p^3)$ computed in (16) in the special case $n = 1$. After simplification using (11),

$$\begin{aligned} H_1(p^2, p; p^3, p^3) &= q\phi(p^3) - q^2\phi(p^2)\phi(p) \\ &= -q^5 + 3q^4 - 2q^3. \end{aligned}$$

Since $\langle \lambda + \rho - \mathrm{wt}(\nu), \rho\rangle = 6$, these terms should be multiplied by q^{-6} to obtain the complete contribution to the monomial $z_1 z_2^2 z_3^3$ on the right-hand side.

The left-hand side is just $(z_1 - q^{-1}z_2)(z_1 - q^{-1}z_3)(z_2 - q^{-1}z_3)s_{(2,1,0)}(z_1, z_2, z_3)$, where $s_{(2,1,0)}(\boldsymbol{z})$ is given in (2). Expanding, we see that the coefficients of $z_1 z_2^2 z_3^3$ indeed match. For example, terms with q^{-3} on the left can only come from taking the term $2z_1z_2z_3$ in the Schur polynomial and multiplying by $q^{-3}z_2z_3^2$ from the product.

In general, after taking into account the normalizing factors that appear in the Casselman-Shalika formula, Theorem 2 shows that the Casselman-Shalika formula and Theorem 1 in the case $n = 1$ are equivalent.

Theorem 2 is equivalent to an earlier result of Tokuyama [19] and may be viewed as a deformation of the Weyl character formula (which results from setting $q = 1$). Tokuyama's formulation uses combinatorial arrays called Gelfand-Tsetlin patterns. We highlight the fact that the character with highest weight λ is expressed as a combinatorial sum over basis vectors of a crystal of highest weight $\lambda + \rho$.

Ice Models for Whittaker Coefficients

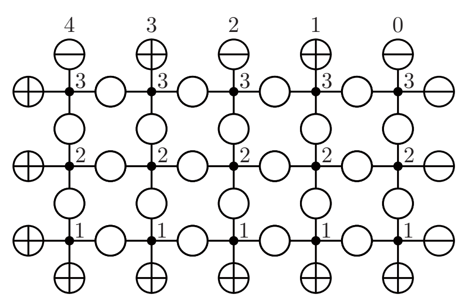
In this final section, we describe another combinatorial representation of the p-parts of Whittaker coefficients. These can be described using *square ice*, a particular example of a two-dimensional *lattice model*. We describe these in detail when $n = 1$; that is, when the Whittaker coefficients at the prime p are given in terms of Schur polynomials. An ice model description for arbitrary covers is presented in [6].

Two-dimensional lattice models arise in statistical mechanics, where they can be used to represent thin sheets of matter such as ice. Any such model has a certain set of admissible configurations called *states*, and each state is assigned a value known as a *Boltzmann weight*. A primary goal is to understand the behavior of the *partition function* of the model, the sum of the Boltzmann weights over all states.[10] Lattice models for which the partition function may be explicitly evaluated are called *exactly solved* and are of particular interest. See Baxter [1]. The study of ice models was advanced by ideas of representation theory and ultimately led to the discovery of quantum groups. See Faddeev [11] for a history.

For the application to Whittaker functions, a lattice model is given for any partition $\lambda = (\lambda_1, \dots, \lambda_{r+1})$ having $\lambda_{r+1} = 0$ as follows. Form a rectangular array of lattice points with $r+1$ rows (numbered $r+1$ to 1 from top to bottom) and $\lambda_1 + r + 1$ columns numbered 0 to $\lambda_1 + r$ from right to left. Add vertical and horizontal edges from each lattice point, so the points are embedded in a rectangular array of line segments.

Each boundary edge of this configuration is labeled with a fixed "spin" $+$ or $-$. The left and bottom edges are all assigned a $+$ spin, and the right edge spins are all $-$. The spins along the top edges correspond to $\lambda + \rho = (\lambda_1 + r, \lambda_2 + r - 1, \dots, \lambda_r + 1, 0)$ as follows: place a $-$ spin at the top of a column numbered $\lambda_i + (r + 1 - i)$ for $i \in [1, r+1]$ and place a $+$ spin above the remaining columns. The figure in (17) illustrates these boundary conditions associated with $\lambda = (2,1,0)$ so that $\lambda + \rho = (4,2,0)$, our running example.

(17)



The states of this model will be assignments of spins to the internal edges, pictured with open circles above. The only requirement on these spins is that each vertex in the grid has adjacent spins matching one of the six configurations in Figure 2 below: A model with this restriction is often called a *six-vertex* model, or *square ice*.[11] Given the

[10] *The term "partition function" should not be confused with our earlier use of "partition" of a positive integer.*

[11] *We may think of the vertices in the grid as oxygen atoms, and the six possible choices of adjacent spins are the $\binom{4}{2}$ ways of arranging two nearby hydrogen atoms on adjacent edges.*

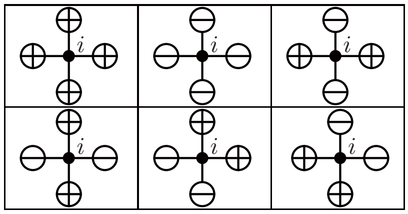
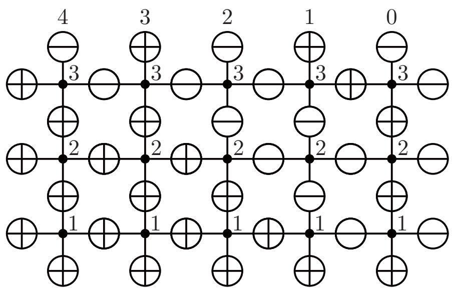


Figure 2. Six-Vertex Ice Configurations

boundary conditions as above, (18) is one such admissible filling (i.e., a state).

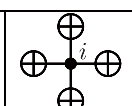
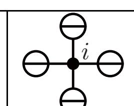
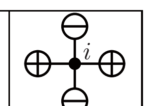
(18)

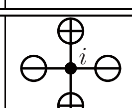
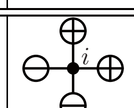
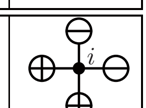


To describe the Boltzmann weight for a state, we first assign a weight to each of the six types of vertices (which is allowed to vary depending on the row in which it appears). Then the Boltzmann weight of the state is the *product* of all weights of vertices appearing in the grid. Summing the Boltzmann weight over all states with fixed boundary conditions gives the partition function for the model.

For example, choose weights for the vertices as in (19),

(19)

Ice Configuration			
Weight	1	$t_i z_i$	1

Ice Configuration			
Weight	z_i	$z_i(t_i+1)$	1

where the t_i and z_i are arbitrary parameters corresponding to the row number i.[12] Then the Boltzmann weight of the state (18) is:

$$t_3(1+t_3)z_3^3 \cdot t_2 z_2^2 \cdot z_1.$$

Setting $t_2 = t_3 = -1/q$, this is precisely equal to $G_1(v)\, q^{-\langle \lambda+\rho-\mathrm{wt}(v),\rho\rangle} \boldsymbol{z}^{\mathrm{wt}(v)}$, which appears in the right-hand side of Theorem 2, where v is the vertex pictured in the crystal graph of Figure 1. This is no accident. There is a bijection between vertices v of the crystal $B_{\lambda+\rho}$ having $G_1(v) \neq 0$ and states of the model with boundary conditions corresponding to $\lambda+\rho$ as above. See [5] for details.

Hamel and King [13] evaluated the partition function of an equivalent model and choice of Boltzmann weights by means of tableaux combinatorics and showed that it exactly equals the left-hand side of Tokuyama's theorem. In [5], we show that as long as the Boltzmann weights satisfy a single algebraic relation (which includes the case of Hamel and King), the resulting partition function may be given in terms of a Schur polynomial. We also give a different approach to these results, which we now sketch.

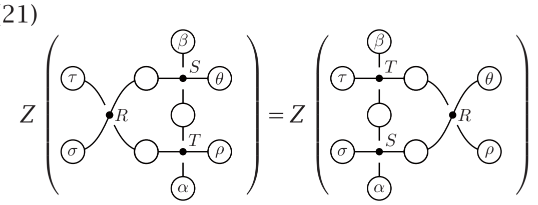
Let $\mathfrak{S}_\lambda$ denote the set of states for the model above with boundary conditions corresponding to $\lambda+\rho$. Let $Z(\mathfrak{S}_\lambda)$ be the partition function of the model with Boltzmann weights assigned according to the table in (19). We prove in [5] that

$$Z(\mathfrak{S}_\lambda) = \prod_{i<j} (t_j z_j + z_i)\, s_\lambda(z_1, \dots, z_n), \tag{20}$$

where the right-hand side has already appeared in the statement of Theorem 2. The critical step of the proof is to show that $Z(\mathfrak{S}_\lambda)\prod_{i<j}(t_j z_j + z_i)^{-1}$ is symmetric in the sense that it is unchanged if the same permutation is applied to both $(z_1, \dots, z_{r+1})$ and $(t_1, \dots, t_{r+1})$. Once this is known, it is possible to show that it is a polynomial in the z_i and t_i and, by comparing degrees, that it is independent of the t_i; finally, taking $t_i = -1$, one may invoke the Weyl character formula and conclude that it is equal to the Schur polynomial.

In order to prove the desired symmetry property we use an instance of the *Yang-Baxter equation.* In the context of a lattice model, given three fixed sets of weights R, S, and T, the Yang-Baxter equation is the identity of partition functions

(21)



for all choices of boundary spins $\pm$ for $\alpha, \beta, \sigma, \tau, \theta, \rho$. Here the R vertices have been rotated by 45° for ease in drawing the diagram. Note that both sides of this identity are sums over all states resulting from choices of the three internal edge spins indicated by empty circles above.[13] Baxter first employed the Yang-Baxter

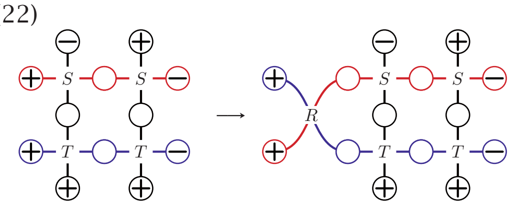
[12] *Note that these Boltzmann weights are not symmetric under the interchange of + and −, in contrast to the "field-free" situation that is often considered in the literature.*

[13] *This may be reformulated algebraically by regarding the Boltzmann weights R, S, T as giving endomorphisms of $V \otimes V$ for an abstract two-dimensional vector space V. See* [5] *for an exposition. Then the Yang-Baxter equation*

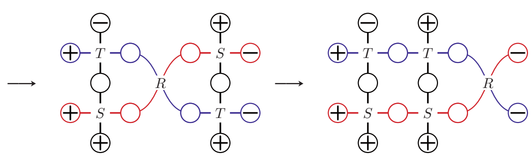
equation as a method for obtaining exactly solved models.

In the application at hand, S and T are weights given in (19) for two rows. It may be shown (cf. [5]) that there exists a choice of weights R such that the Yang-Baxter equation holds. Attach this vertex between the S and T rows thus:

(22)



This multiplies the partition function by a weight of R, which happens to be one of the linear factors in (20). Then applying the Yang-Baxter equation several times, this R-vertex may be moved rightward, leaving the partition function invariant. Picking up from (22), this looks like:



Then discarding the R-vertex divides by another Boltzmann weight of R, which is another one of the linear factors in (20). Note that S and T are interchanged, reflecting the symmetry of the Schur function in (20) and leading to a proof of that equation.

The Yang-Baxter equation can also be used to directly establish the equivalence of the two descriptions in Theorem 1 obtained from the reduced decompositions (13) and (14) when $n = 1$. See Chapter 19 of [3].

The Langlands program describes the role of automorphic forms on reductive groups in number theory. Automorphic forms on covering groups have been used to prove cases of the Langlands conjectures, but they themselves do not strictly fit into its usual formulations. Studying automorphic forms on covers reveals connections with crystals and lattice models, which are mathematical objects that first appeared in other contexts—quantum groups and mathematical physics. The exploration of these exciting connections is only beginning.

---

is the identity

$$R_{12}S_{13}T_{23} = T_{23}S_{13}R_{12},$$

where the notation R_{ij} is the endomorphism of $V \otimes V \otimes V$ in which R is applied to the ith and jth copies of V and the identity map to the kth copy, where $\{i, j, k\} = \{1, 2, 3\}$.

References

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

[2] A. Berenstein and A. Zelevinsky, Tensor product multiplicities, canonical bases and totally positive varieties, *Invent. Math* **143** (2001), 77–128.

[3] B. Brubaker, D. Bump, and S. Friedberg, *Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory*, Annals of Mathematics Studies, Vol. 175, Princeton Univ. Press, Princeton, NJ, 2011.

[4] ———, Weyl group multiple Dirichlet series, Eisenstein series and crystal bases, *Annals of Math.* **173** (2011), no. 2, 1081–1120.

[5] ———, Schur polynomials and the Yang-Baxter equation, *Comm. Math. Physics*, to appear.

[6] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and P. Gunnells, Metaplectic ice, preprint, 2010.

[7] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics, Vol. 55, Cambridge University Press, Cambridge, 1997.

[8] W. Casselman and J. Shalika, The unramified principal series of p-adic groups. II. The Whittaker function, *Compositio Math.* **41** (1980), no. 2, 207–231.

[9] G. Chinta and P. Gunnells, Constructing Weyl group multiple Dirichlet series, *J. Amer. Math. Soc.* **23** (2010), 189–215.

[10] G. Chinta and O. Offen, *A metaplectic Casselman-Shalika formula for GL_r*, preprint, 2009. `http://www.sci.ccny.cuny.edu/~chinta/publ/mwf.pdf`.

[11] L. Faddeev, History and perspectives of quantum groups, *Milan J. Math.* **74** (2006), 279–294.

[12] S. Friedberg, Euler products and twisted Euler products. In *Automorphic Forms and the Langlands Program* (L. Ji, K. Liu, S.-T. Yau, and Z.-J. Zheng, eds.), Advanced Lectures in Math, Vol. 9, 2010, 176–198. Published jointly by International Press and Higher Education Press of China.

[13] A. M. Hamel and R. C. King, Bijective proofs of shifted tableau and alternating sign matrix identities, *J. Algebraic Combin.* **25** (2007), no. 4, 417–458.

[14] J. Hong and S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics, Vol. 42, AMS, Providence, RI, 2002.

[15] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, *Comm. Math. Phys.* **133** (1990), no. 2, 249–260.

[16] P. Littelmann, Cones, crystals, and patterns, *Transform. Groups* **3** (1998), no. 2, 145–179.

[17] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[18] T. Shintani, On an explicit formula for class-1 "Whittaker functions" on GL_n over P-adic fields, *Proc. Japan Acad.* **52** (1976), no. 4, 180–182.

[19] T. Tokuyama, A generating function of strict Gelfand patterns and some formulas on characters of general linear groups, *J. Math. Soc. Japan* **40** (1988), no. 4, 671–685.