Methodology for adapting the parameters of a fuzzy system using the extended Kalman filter

A. Javier Barragán Piña[1] José M. Andújar Márquez[1] Mariano J. Aznar Torres[2]
Agustín Jiménez Avello[3] Basil M. Al-Hadithi[3]

[1]DIESIA - Universidad de Huelva, Spain, {antonio.barragan, andujar}@diesia.uhu.es
[2]MICSEII - Universidad de Huelva, Spain, marianojose.aznar@alu.uhu.es
[3]DISAM - Universidad Politécnica de Madrid, Spain, {agustin.jimenez, basil.alhadithi}@upm.es

Abstract

When we try to analyze and to control a system whose model was obtained only based on input/output data, accuracy is essential in the model. On the other hand, to make the procedure practical, the modeling stage must be computationally efficient. In this regard, this paper presents the application of extended Kalman filter for the parametric adaptation of a fuzzy model.

Keywords: Kalman filter, estimation, fuzzy system, modeling.

1. Introduction

The Kalman filter has been used with fuzzy logic in various applications, such as the extraction of rules from a given rule base [1], parameters optimization of defuzzification mechanisms that are based on both Gaussian and polynomic distributions [2] or in optimization of consequents of Takagi-Sugeno models [3]. In 2002, Simon introduced the use of Kalman filter for adjusting the parameters of a fuzzy model [4], assuming that antecedents were membership functions of triangular type, and using its center of gravity to perform the adaptation process. However, the complexity of the calculation for others types of membership functions has meant that this proposal has not been widespread so far.

In this paper we present a methodology for use of the extended Kalman filter (EKF) to estimate the adaptive parameters of a general fuzzy model, i.e., with no constraint in size of input/output vectors, neither in type or distribution of the membership functions used in the definition of fuzzy sets of the model. So, authors try to use the excellent features of Kalman filter to obtain fuzzy models of unknown systems from input/output data, and also allowing its application in real-time [5].

This article is organized as follows: section 2 presents the fuzzy modeling problem in a completely general form, and sets the notation used throughout this article. Section 3 is devoted to formal presentation of the extended Kalman filter and its use for modeling systems. Later, in section 4 we solve the calculation of derivatives for use of extended Kalman filter to the problem of fuzzy modeling, keeping the generality of the problem and considering that are included in the process of modeling adaptive parameters of both antecedents and consequents of the rules. Finally, section 5 presents some conclusions and future work.

2. Problem Formulation

Since that building an appropriate model is a fundamental step for subsequent application of different techniques of both analysis and design, we have chosen to perform a fuzzy model with Takagi-Sugeno type, completely general. As is known, TS models are universal approximators, and they can achieve high accuracy with a small number of rules [6, 7, 8, 9].

Let n be the number of input variables and m the number of output variables of the system to model; a discrete fuzzy model Multiple Input Multiple Output - MIMO - can be represented by the following set of rules [10, 11, 12, 13]:

$$R^{(l,i)}: \text{ If } x_1(k) \text{ is } A^l_{1i} \text{ and} \ldots \text{and } x_n(k) \text{ is } A^l_{ni}$$
$$\text{Then } y^l_i(k) = a^l_{0i} + \sum_{j=1}^{n} a^l_{ji} x_j(k),$$

where $l = 1..M$ is the index of the rule and M_i the number of rules that model the evolution of the i-th system output, $y_i(k)$. The a^l_{ij}, $j = 0..n$, elements represent the set of adaptive parameters of the consequents of the rules, thus they must be determined by the process of system modeling.

Note that in using the above representation, dynamics of each output can be modeled by a different number of rules, which facilitates the reduction of the total number of rules needed to model correctly a complex system, and, therefore, facilitates the modeling process by reducing the number of model parameters.

If input vector is extended in a coordinate [14, 15] by $\tilde{x}_0 = 1$, extended vector $\tilde{\mathbf{x}}$ takes the form:

$$\tilde{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)^{\mathrm{T}} = (1, x_1, \ldots, x_n)^{\mathrm{T}} \quad (1)$$

Using the weighted average as a method of aggregation and the extension of the state vector given in (1) , the output y_i generated by the set of rules

$R^{(l,i)}$, can be calculated by [16, 17]:

$$y_i(k) = h_i(\mathbf{x}(k)) = \sum_{j=0}^{n} a_{ji}(\mathbf{x})\tilde{x}_j(k), \qquad (2)$$

being $a_{ji}(\mathbf{x})$ variables coefficients [18] defined by (3), where $w_i^l(\mathbf{x})$ is calculated by (4) and represents the degree of activation of the rules of fuzzy model:

$$a_{ji}(\mathbf{x}) = \frac{\sum_{l=1}^{M_i} w_i^l(\mathbf{x}) a_{ji}^l}{\sum_{l=1}^{M_i} w_i^l(\mathbf{x})} \qquad (3)$$

$$w_i^l(\mathbf{x}) = \prod_{j=1}^{n} \mu_{ji}^l(x_j(k), \boldsymbol{\sigma}_{ji}^l) \qquad (4)$$

$\mu_{ji}^l(x_j(k), \boldsymbol{\sigma}_{ji}^l)$ represents the j-th membership function of the l rule for the i-th model output, which determines the fuzzy set A_{ji}^l. The $\boldsymbol{\sigma}_{ji}^l$ elements represent the set of adaptive parameters of this membership function, so these values, with the adaptive parameters of the consequents of the rules, a_{ji}^l, shall be determined according to estimation algorithm to achieve an appropriate system model.

3. Extended Kalman Filter

Kalman filter was developed by Rudolph E. Kalman [19, 20] and allows to construct an optimal observer for linear systems in presence of white noise both in model and in measures. Subsequently, the Kalman filter was adapted for use in nonlinear systems via extended Kalman filter [21], if the system supports linearized models around any working point. Although the extended Kalman filter is not optimal, since it is based on a linear approximation of a model and its accuracy depends heavily on the goodness of such approximations, is a powerful tool for estimation in environments with noise.

If we consider a nonlinear discrete system as follows:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{f}\left(\mathbf{x}(k), \mathbf{u}(k)\right), \mathbf{v}(k) \\ \mathbf{y}(k) &= \mathbf{g}\left(\mathbf{x}(k)\right) + \mathbf{e}(k), \end{aligned}$$

where $\mathbf{v}(k)$ and $\mathbf{e}(k)$ are white noises that represent uncertainty both in the model of equation of state and in output, respectively.

Being the Jacobian matrices of the system:

$$\boldsymbol{\Phi}\left(\mathbf{x}(k), \mathbf{u}(k)\right) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}(k), \mathbf{u}=\mathbf{u}(k)} \qquad (5)$$

$$\boldsymbol{\Gamma}\left(\mathbf{x}(k), \mathbf{u}(k)\right) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}(k), \mathbf{u}=\mathbf{u}(k)} \qquad (6)$$

and

$$\mathbf{C}\left(\mathbf{x}(k)\right) = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}(k)}, \qquad (7)$$

the Extended Kalman filter can be solved by iterative application of following set of equations [22]:

$$\mathbf{P}(k|k) = \boldsymbol{\Phi}(k)\mathbf{P}(k|k-1)\boldsymbol{\Phi}^{\mathrm{T}}(k) + \mathbf{R}_v \qquad (8)$$

$$\begin{aligned} \mathbf{K}(k) = &\left(\boldsymbol{\Phi}(k)\mathbf{P}(k|k)\mathbf{C}^{\mathrm{T}}(k) + \mathbf{R}_{ve}\right) \\ &\left(\mathbf{C}(k)\mathbf{P}(k|k)\mathbf{C}^{\mathrm{T}}(k) + \mathbf{R}_e\right)^{-1} \end{aligned} \qquad (9)$$

$$\begin{aligned} \hat{\mathbf{x}}(k+1|k) = &\boldsymbol{\Phi}(k)\hat{\mathbf{x}}(k|k-1) + \boldsymbol{\Gamma}(k)\mathbf{u}(k) \\ &+\mathbf{K}(k)\left(\mathbf{y}(k) - \mathbf{C}(k)\hat{\mathbf{x}}(k|k-1)\right) \end{aligned} \qquad (10)$$

$$\begin{aligned} \mathbf{P}(k+1|k) = &\boldsymbol{\Phi}(k)\mathbf{P}(k|k)\boldsymbol{\Phi}^{\mathrm{T}}(k) + \mathbf{R}_v \\ &-\mathbf{K}(k)\left(\mathbf{C}(k)\mathbf{P}(k|k)\boldsymbol{\Phi}^{\mathrm{T}}(k) + \mathbf{R}_{ve}^{\mathrm{T}}\right), \end{aligned} \qquad (11)$$

where $\hat{\mathbf{x}}(\cdot)$ is the estimate of state vector, and $\mathbf{R}_v$, $\mathbf{R}_{ve}$ and $\mathbf{R}_e$ are the noise covariance matrices, estimated from the hope operator, $E(\cdot)$:

$$\mathbf{R}_v = E\left(\mathbf{v}(k)\mathbf{v}^{\mathrm{T}}(k)\right)$$

$$\mathbf{R}_{ve} = E\left(\mathbf{v}(k)\mathbf{e}^{\mathrm{T}}(k)\right)$$

$$\mathbf{R}_e = E\left(\mathbf{e}(k)\mathbf{e}^{\mathrm{T}}(k)\right)$$

The iterative process starts with an initial estimate of state vector $\hat{\mathbf{x}}(0) = \mathbf{m}_0 = E\left(\mathbf{x}(0)\right)$ and $\mathbf{P}(0) = E\left((\mathbf{x}(0) - \mathbf{m}_0)(\mathbf{x}(0) - \mathbf{m}_0)^{\mathrm{T}}\right)$ being known $\mathbf{x}(0|-1)$, $\mathbf{u}(0)$ and $\mathbf{y}(0)$, and it is evolving in-line with respect the system, obtaining a solution that minimizes both estimation error and its covariance matrix for the linearization obtained at each instant.

4. Application of the Extended Kalman Filter to fuzzy modeling

A so interesting application of extended Kalman filter is the adaptive identification of parameters in nonlinear systems, which allows the in-line obtaining of the adaptive parameters set of a discrete nonlinear model with noise presence and in a pseudo-optimal way (is optimal in linear systems). The identification of a TS fuzzy model can be seen as the obtaining of parameters of a nonlinear model, so the Kalman filter can be applied using the extended algorithm for estimating these parameters.

First we must raise the problem of estimation by extended Kalman filter. For this we have to build a system whose states depend directly on the parameters to be estimated, then we apply recursively from (8) to (11).

Let $\mathbf{p}(k)$ be the set of adaptive parameters of a fuzzy system, and $\mathbf{y}(k)$ the set of outputs of this fuzzy system, the system represented in (12) allows to obtain these parameters using the extended Kalman filter. The diagram that allows the use of extended Kalman filter for the in-line estimation of a fuzzy model is shown in Figure 1.

$$\begin{aligned} \mathbf{p}(k+1) &= \mathbf{p}(k) \\ \mathbf{y}(k) &= \mathbf{h}(\mathbf{x}(k), \mathbf{p}(k)) + \mathbf{e}(k) \end{aligned} \qquad (12)$$

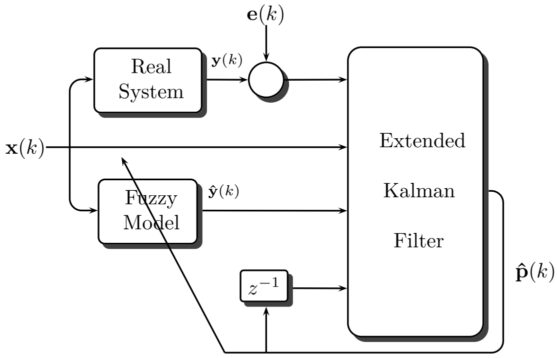

Figure 1: Fuzzy modeling using the extended Kalman filter.

Signal $\mathbf{e}(k)$ is the uncertainty of the measurement of the output system and is represented by a white noise whose covariance is determined by $\mathbf{R}_e$.

Thus, the first thing to do is the calculation of Jacobian matrices of the system using (5), (6) and (7). In applying these expressions on (12) we obtain:

$$\mathbf{\Phi}\left(\mathbf{p}(k)\right) = \mathbf{I} \tag{13}$$

$$\mathbf{\Gamma}\left(\mathbf{p}(k)\right) = \mathbf{0} \tag{14}$$

$$\mathbf{C}\left(\mathbf{p}(k)\right) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{p}}\right|_{\mathbf{p}=\hat{\mathbf{p}}(k)}, \tag{15}$$

being $\hat{\mathbf{p}}(k)$ the current estimation of the parameters vector of the fuzzy model.

Note that, given formulation exposed in section 2, the estimation problem is to determine the values of the adaptive parameters of both antecedents, $\boldsymbol{\sigma}_{ji}^{l}$, and consequents, a_{ji}^{l}, of rules. Therefore, for a TS fuzzy model, the expression $\mathbf{h}(\mathbf{x}(k), \mathbf{p}(k))$ corresponds to (2), and (15) must be obtained from the derivative of this expression with respect to each of adaptive parameters of the fuzzy model.

As can be seen in (2) and (3), function $\mathbf{h}(\cdot)$ is linear with respect the set of adaptive parameter of consequents, a_{ji}^{l}, so:

$$\frac{\partial h_i}{\partial a_{JI}^{L}} = \begin{cases} \dfrac{w_I^L \tilde{x}_J}{\sum\limits_{l=1}^{M_I} w_I^l} & \text{if } i = I \\ 0 & \text{if } i \neq I, \end{cases}$$

where L, J and I determine the particular parameter a_{JI}^{L} of the possibleset of consequent parameters. Moreover, for each parameters set of the membership function of antecedent, we obtain:

$$\frac{\partial h_i}{\partial \boldsymbol{\sigma}_{JI}^{L}} = \sum_{j=0}^{n} \frac{\partial \left(\dfrac{\sum\limits_{l=1}^{M_i} w_i^l a_{ji}^l}{\sum\limits_{l=1}^{M_i} w_i^l} \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} \tilde{x}_j \tag{16}$$

Only the I-th output depends of the $\boldsymbol{\sigma}_{JI}^{L}$ parameter, thus,

$$\frac{\partial h_i}{\partial \boldsymbol{\sigma}_{JI}^{L}} = 0 \text{ if } i \neq I. \tag{17}$$

Developing the partial derivate of (16), and considering (17):

$$\frac{\partial \left(\dfrac{\sum\limits_{l=1}^{M_I} w_I^l a_{jI}^l}{\sum\limits_{l=1}^{M_I} w_I^l} \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} = \frac{\dfrac{\partial \left(\sum\limits_{l=1}^{M_I} w_I^l a_{jI}^l \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} \sum\limits_{l=1}^{M_I} w_I^l - \dfrac{\partial \left(\sum\limits_{l=1}^{M_I} w_I^l \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} \sum\limits_{l=1}^{M_I} w_I^l a_{jI}^l}{\left(\sum\limits_{l=1}^{M_I} w_I^l \right)^2}, \tag{18}$$

Considering that $\boldsymbol{\sigma}_{JI}^{L}$ parameter is only present in L rule of I-th output, is easy to deduce that

$$\frac{\partial \left(\sum\limits_{l=1}^{M_I} w_I^l a_{jI}^l \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} = \frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^{L}} a_{jI}^{L}, \tag{19}$$

and working in a similar way,

$$\frac{\partial \left(\sum\limits_{l=1}^{M_I} w_I^l \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} = \frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^{L}}. \tag{20}$$

Replacing (19) and (20) in (18):

$$\frac{\partial \left(\dfrac{\sum\limits_{l=1}^{M_I} w_I^l a_{jI}^l}{\sum\limits_{l=1}^{M_I} w_I^l} \right)}{\partial \boldsymbol{\sigma}_{JI}^{L}} = \frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^{L}} \left(\frac{a_{jI}^{L} \sum\limits_{l=1}^{M_I} w_I^l - \sum\limits_{l=1}^{M_I} w_I^l a_{jI}^l}{\left(\sum\limits_{l=1}^{M_I} w_I^l \right)^2} \right) = \frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^{L}} \left(\frac{\sum\limits_{l=1}^{M_I} w_I^l (a_{jI}^{L} - a_{jI}^{l})}{\left(\sum\limits_{l=1}^{M_I} w_I^l \right)^2} \right)$$

Replacing the above expression in (16), we obtain the final expression to calculate $\dfrac{\partial h_i}{\partial \boldsymbol{\sigma}_{JI}^{L}}$ when $i = I$:

$$\frac{\partial h_I}{\partial \boldsymbol{\sigma}_{JI}^{L}} = \frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^{L}} \sum_{j=0}^{n} \left(\frac{\sum\limits_{l=1}^{M_I} w_I^l (a_{jI}^{L} - a_{jI}^{l})}{\left(\sum\limits_{l=1}^{M_I} w_I^l \right)^2} \right) \tilde{x}_j \tag{21}$$

Finally, to conclude the calculation of (21) is necessary determine the derivative of the degree of activation from the rules of the fuzzy model, w_i^l, with respect to each of the parameters of the antecedents. Obviously, this calculation is dependent on the type of membership function that is used for each antecedent. However, is possible to express in a general form:

$$\frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^L} = \frac{\partial}{\partial \boldsymbol{\sigma}_{JI}^L}\left(\prod_{q=1}^{n} \mu_{qI}^L(x_q(k), \boldsymbol{\sigma}_{qI}^L)\right),$$

or a more developed form:

$$\frac{\partial w_I^L}{\partial \boldsymbol{\sigma}_{JI}^L} = \frac{\partial \mu_{JI}^L(x_J(k), \boldsymbol{\sigma}_{JI}^L)}{\partial \boldsymbol{\sigma}_{JI}^L}\prod_{q=1,q\neq J}^{n} \mu_{qI}^L(x_q(k), \boldsymbol{\sigma}_{qI}^L) \quad (22)$$

Note that $\frac{\partial \mu_{JI}^L(x_J(k), \boldsymbol{\sigma}_{JI}^L)}{\partial \boldsymbol{\sigma}_{JI}^L}$ in (22) represents the derivative of the membership function that is defined by the parameters set $\boldsymbol{\sigma}_{JI}^L$, thus, the calculation of this derivative depends on the type of membership function used and it can be performed from the expression that defines it. For example, for a Gaussian membership function

$$\mu_{Gaussian}[c, \beta](x) = e^{-\frac{(x-c)^2}{\beta^2}},$$

where $\boldsymbol{\sigma}$ is the vector $[c, \beta]$,

$$\frac{\partial \mu[c, \beta](x)}{\partial c} = \frac{2(x-c)}{\beta^2}\mu_{Gaussian}[c, \beta](x),$$

and

$$\frac{\partial \mu[c, \beta](x)}{\partial \beta} = \frac{2(x-c)^2}{\beta^3}\mu_{Gaussian}[c, \beta](x).$$

Note it is not necessary that the membership functions are differentiable, but it is enough to be piecewise differentiable. Piecewise membership functions could provide a jump discontinuity in its derivative, however, since the set of singular points is a null set. In numerical implementations this means that it is possible to suppose that its derivative is a point infinitesimally close to the right, to the left, or as the average of these derivatives.

5. Conclusions

First results obtained by authors to apply the extended Kalman filter in estimation of adaptive parameters of a fuzzy system completely general, i.e. without restrictions on the size of the input or output vectors, or the type or distribution of membership functions used in the definition of fuzzy sets of the model has been presented in this paper.

This article presents the solution of this problem in a theoretical level from the state models obtained by authors in previous works. In a forthcoming work will present the realization of a theoretical algorithm developed in this work, along with several examples of application.

Future work seeks to make practical applications, which will be compared for the purposes of accuracy and computational efficiency in a reliable way to assess the use of extended Kalman filter in parametric adaptation of fuzzy systems obtained from input/output data of a unknown system.

Acknowledgment

The present work is a contribution of the DPI2010-17123 project, funded by the Spanish Ministry for Education and Science and by the European Regional Development Union.

References

[1] Wang Liang and Yen John. Extracting fuzzy rules for system modeling using a hybrid of genetic algorithms and Kalman filter. *Fuzzy Sets and Systems*, 101(3):353–362, February 1999.

[2] Tao Jiang and Yao Tang Li. Generalized defuzzification strategies and their parameter learning procedures. *IEEE Transactions on Fuzzy Systems*, 4(1):64–71, August 1996.

[3] Pramath Ramaswamy, Martin Riese, Robert M. Edwards, and Kwang Youl Lee. Two approaches for automating the tuning process of fuzzy logic controllers. In *32nd IEEE Conference on Decision and Control. Part 2 (of 4)*, San Antonio, TX, USA, December 1993.

[4] Dan J. Simon. Training fuzzy systems with the extended Kalman filter. *Fuzzy Sets and Systems*, 132(2):189–199, 2002.

[5] Agustín Jiménez, Basil M. Al-Hadithi, and Fernando Matía. An optimal T-S model for the estimation and identification of nonlinear functions. *WSEAS Trans. Sys. Ctrl.*, 3(10):897–906, January 2008.

[6] Bart Kosko. Fuzzy systems as universal approximators. *IEEE Transactions on Computers*, 43(11):1329–1333, November 1994.

[7] Li-Xin Wang. Fuzzy systems are universal approximators. In *IEEE International Conference on Fuzzy Systems*, pages 1163–1170, San Diego, CA, USA, 1992.

[8] Hao Ying. Sufficient conditions on uniform approximation of multivariate functions by general Takagi–Sugeno fuzzy systems with linear rule consequent. *IEEE Transactions on Systems, Man, and Cybernetics—Part A: Systems and Humans*, 28(4):515–520, July 1998.

[9] Ke Zeng, Nai-Yao Zhang, and Wen-Li Xu. A comparative study on sufficient conditions for Takagi–Sugeno fuzzy systems as universal approximators. *IEEE Transactions on Fuzzy Systems*, 8(6):773–780, December 2000.

[10] Robert Babuška. Fuzzy modeling – a control engineering perspective. In *Proceedings of FUZZ-IEEE/IFES'95*, volume 4, pages 1897–1902, Yokohama, Japan, March 1995.

[11] Robert Babuška, Magne Setnes, Uzay Kaymak, and Hans R. van Nauta Lemke. Rule base simplification with similarity measures. In *Proceedings of the 5th IEEE International Conference on Fuzzy Systems*, volume 3, pages 1642–1647, New Orleans, LA, September 1996.

[12] Hung T. Nguyen, Michio Sugeno, Richard M. Tong, and Ronald R. Yager. *Theoretical aspects of fuzzy control*. John Wiley Sons, 1995.

[13] T. Takagi and M. Sugeno. Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, 15(1):116–132, 1985.

[14] José Manuel Andújar and Antonio Javier Barragán. A methodology to design stable nonlinear fuzzy control systems. *Fuzzy Sets and Systems*, 154(2):157–181, September 2005.

[15] José Manuel Andújar, Antonio Javier Barragán, and Manuel Emilio Gegúndez. A general and formal methodology for designing stable nonlinear fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 17(5):1081–1091, October 2009.

[16] L. X. Wang. *Adaptive fuzzy systems and control*. Prentice Hall, New Jersey, 1994.

[17] Li-Xin Wang. *A course in fuzzy systems and control*. Prentice Hall, New Yersey, USA, 1997.

[18] L.K. Wong, F.H.F. Leung, and P.K.S. Tam. Stability design of TS model based fuzzy systems. In *IEEE International Conference on Fuzzy Systems*, volume 1, pages 83–86, Barcelona, Spain, July 1997.

[19] Rudolf E. Kalman. A new approach to linear filtering and prediction problems. *Transactions on ASME-Journal of Basic Engineering*, 82(series D):35–45, 1960.

[20] Rudolf E. Kalman. New methods in Wiener filtering theory. In John L. Bogdanoff and Frank Kozin, editors, *1st Symposium On Engineering Applications of Random Function Theory and Probability*, New York, 1963. John Wiley and Sons.

[21] Peter S. Maybeck. *Stochastic models, estimation, and control*, volume 141 of *Mathematics in Science and Engineering*. Academyc Press, 1979.

[22] Mohinder S. Grewal and Angus P. Andrews. *Kalman Filtering: Theory and Practice Using MATLAB*. John Wiley & Sons, Inc., 2nd edition, 2001.