The U(5)–O(6) transition in the Interacting Boson Model and the E(5) critical point symmetry

J.M. Arias[1], C.E. Alonso[1], A. Vitturi[2], J.E. García-Ramos[3], J. Dukelsky[4], and A. Frank[5]

[1] *Departamento de Física Atómica, Molecular y Nuclear, Facultad de Física Universidad de Sevilla, Apartado 1065, 41080 Sevilla, Spain*

[2] *Dipartimento di Fisica Galileo Galilei, Via Marzolo 8 35131 Padova, Italy*

[3] *Departamento de Física Aplicada, Universidad de Huelva, 21071 Huelva, Spain*

[4] *Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain*

[5] *Instituto de Ciencias Nucleares, UNAM, Circuito Exterior C.U., México D.F., 04510 México*

Abstract

The relation of the recently proposed E(5) critical point symmetry with the interacting boson model is investigated. The large-N limit of the interacting boson model at the critical point in the transition from U(5) to O(6) is obtained by solving the Richardson equations. It is shown explicitly that this algebraic calculation leads to the same results as the solution of the Bohr differential equation with a β^4 potential.

PACS numbers: 21.60.Fw, 21.10.Re

The study of phase transitions is one of the most exciting topics in Physics. Recently the concept of critical point symmetry has been proposed by Iachello [1]. These kind of symmetries apply when a quantal system undergoes transitions between traditional dynamical symmetries. In Ref. [1] the particular case of the Bohr Hamiltonian [2] in Nuclear Physics was worked out. In this case, in the situation in which the potential energy surface in the β-γ plane is γ-independent and the dependence in the β degree of freedom can be modeled by an infinite square well, the so called E(5) symmetry appears. This situation is expected to be realized in actual nuclei when they undergo a transition from spherical to γ-unstable deformed shapes. The E(5) symmetry is obtained within the formalism based on the Bohr hamiltonian, but it has also been used in connection with the Interacting Boson Model (IBM) [3]. Although this is not the form it was originally proposed [1], it has been in fact argued that moving from the spherical to the γ-unstable deformed case within the IBM one should reobtain, at the critical point in the transition, the predictions of the E(5) symmetry. This correspondence is supposed to be valid in the limit of large number N of bosons, but the calculations with the IBM should provide predictions for finite N as stated in Ref. [4]. In this letter, on one hand we calculate exactly the large N limit of the IBM at the critical point in the transition from U(5) (spherical case) to O(6) (deformed γ-unstable case). On the other hand, we solve the Bohr differential equation for a β^4 potential. Both calculations lead to the same results and are not close to those obtained by solving the Bohr equation for an infinite square well (E(5) symmetry). We also show with two schematic examples that the corrections arising from the finite number of bosons are important. With this in mind, the IBM calculations still provide a tool for including corrections due to the finite number of bosons.

In Ref. [1] the Bohr Hamiltonian is considered for the case of a γ independent potential, described by an infinite square well in the β variable. In that case, the hamiltonian is separable in both variables and if we set

$$\Psi(\beta, \gamma, \theta_i) = f(\beta)\Phi(\gamma, \theta_i) \tag{1}$$

where θ_i stands for the three Euler angles, the Schrödinger equation can be split in two equations. The solutions of the (γ, θ_i) part were studied in Ref. [5] and tabulated in Ref. [6]. Iachello solved the β part and found that the $f(\beta)$ functions are related to Bessel functions. The main results are illustrated in Table I and Fig. 1 of Ref. [1]. These results are obtained from a geometrical picture and we would like to investigate its relation with the interacting boson model.

The geometrical interpretation of the abstract IBM hamiltonian can be obtained by introducing a coherent state [7–9] which allows to associate to it a geometrical shape in terms of the deformation variables (β, γ). The basic idea of this formalism is to consider that the pure quadrupole states are globally described by a boson condensate of the form

$$|g; N, \beta, \gamma\rangle = \frac{1}{\sqrt{N!}}(\Gamma_g^\dagger)^N |0\rangle \quad , \tag{2}$$

where the basic boson is given by

$$\Gamma_g^\dagger = \frac{1}{\sqrt{1+\beta^2}}\left[s^\dagger + \beta\cos\gamma d_0^\dagger + \frac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)\right] \quad , \tag{3}$$

which depends on the β and γ shape variables. The energy surface is defined as

$$E_N(\beta,\gamma) = \langle g; N, \beta, \gamma|\hat{H}|g; N, \beta, \gamma\rangle \quad , \tag{4}$$

where $\hat{H}$ is the IBM hamiltonian. Here we are interested in the case in which the hamiltonian undergoes a transition from U(5) to O(6) and, consequently, the corresponding potential energy surfaces are γ-independent.

In order to investigate the geometrical limit of the IBM in the transitional class going from U(5) (spherical) to O(6) (deformed γ-unstable) the most general (up to two-body terms) IBM hamiltonian is,

$$\hat{H} = \varepsilon_d \hat{n}_d + \kappa_0 \hat{P}^\dagger \hat{P} + \kappa_1 \hat{L}\cdot\hat{L} + \kappa_2 \hat{Q}^{\chi=0}\cdot\hat{Q}^{\chi=0} + \kappa_3 \hat{T}_3\cdot\hat{T}_3 + \kappa_4 \hat{T}_4\cdot\hat{T}_4 \tag{5}$$

where $\hat{n}_d$ is the d boson number operator, and

$$\hat{P}^\dagger = \frac{1}{2}\,(d^\dagger \cdot d^\dagger - s^\dagger \cdot s^\dagger), \tag{6}$$

$$\hat{L} = \sqrt{10}(d^\dagger \times \tilde{d})^{(1)}, \tag{7}$$

$$\hat{Q}^{\chi=0} = (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)}, \tag{8}$$

$$\hat{T}_3 = (d^\dagger \times \tilde{d})^{(3)}, \tag{9}$$

$$\hat{T}_4 = (d^\dagger \times \tilde{d})^{(4)}. \tag{10}$$

The scalar product is defined as $\hat{T}_L \cdot \hat{T}_L = \sum_M (-1)^M \hat{T}_{LM} \hat{T}_{L-M}$, where $\hat{T}_{LM}$ corresponds to the M component of the operator $\hat{T}_L$. The operators $\tilde{d}_m = (-1)^m d_{-m}$ and $\tilde{s} = s$ are introduced to ensure the correct tensorial character under spatial rotations. The corresponding energy surface is obtained from Eq. (4)

$$E(N,\beta) = \frac{N}{1+\beta^2}\left[5\kappa_2 + \beta^2(\varepsilon_d + 6\kappa_1 + \kappa_2 + \frac{7}{5}\kappa_3 + \frac{9}{5}\kappa_4)\right] + \frac{N(N-1)}{(1+\beta^2)^2}\left[\frac{(1-\beta^2)^2}{4}\kappa_0 + 4\beta^2\kappa_2 + \frac{18}{35}\beta^4\kappa_4\right]. \tag{11}$$

The condition to find the critical point is

$$\left(d^2E(N,\beta)/d\beta^2\right)_{\beta=0} = 0 \tag{12}$$

and gives the following relation among the hamiltonian parameters

$$\varepsilon_d = -6\kappa_1 + 4\kappa_2 - \frac{7}{5}\kappa_3 - \frac{9}{5}\kappa_4 + (N-1)(\kappa_0 - 4\kappa_2). \tag{13}$$

Thus the most general energy surface at the critical point in the U(5)–O(6) phase transition is

$$E^{crit}(N,\beta) = 5N\kappa_2 + N(N-1)\left[\frac{\kappa_0}{4} + \left(\kappa_0 - 4\kappa_2 + \frac{18}{35}\kappa_4\right)\frac{\beta^4}{(1+\beta^2)^2}\right]. \tag{14}$$

These expressions are consistent with those obtained in Ref. [10] for a slightly different hamiltonian. Note that (14) completely defines the form of the potential up to a scale and an energy translation. The expansion of this critical energy surface around $\beta = 0$ is

$$E^{crit}(N,\beta) \approx 5\kappa_2 N + \frac{\kappa_0}{4}N(N-1) + N(N-1)\left(\kappa_0 - 4\kappa_2 + \frac{18}{35}\kappa_4\right)\left[\beta^4 - 2\beta^6 + \ldots\right]. \tag{15}$$

whose leading term is β^4. Alternatively, one can carry out the transformation $\beta^2/(1+\beta^2) \rightarrow \bar{\beta}^2$ and finds $\bar{\beta}^4$ as the critical potential.

In order to make some calculations to illustrate the large N limit in the IBM at the critical point in the U(5)–O(6) phase transition and the corresponding finite N corrections, we propose two schematic transitional hamiltonians. The first one is

$$\hat{H}_I = x\hat{n}_d + \frac{1-x}{N-1}\hat{P}^\dagger\hat{P} \ . \tag{16}$$

The corresponding energy surface is obtained from Eq. (11) with $\varepsilon_d = x$, $\kappa_0 = \frac{1-x}{N-1}$ and all the rest of the parameters equal to 0,

$$E_I(N,\beta) = N\left[x\frac{\beta^2}{1+\beta^2} + \frac{1-x}{4}\left(\frac{1-\beta^2}{1+\beta^2}\right)^2\right] . \tag{17}$$

The condition to localize the critical point, Eq. (13), gives in this case $x_c^I = 0.5$. In Fig. 1 we represent as an example the energy surfaces for the hamiltonian (16) (left panel) with three selections for the order parameter x: one at the critical point, one above that value and one below it. For $x > x_c$ an equilibrium spherical shape is obtained, while for $x < x_c$ the equilibrium shape is deformed. The value x_c gives a flat β^4 surface close to $\beta = 0$.

The second schematic hamiltonian we propose is

$$\hat{H}_{II} = x\hat{n}_d - \frac{1-x}{N}\hat{Q}^{\chi=0}\cdot\hat{Q}^{\chi=0} \ , \tag{18}$$

The corresponding energy surface is obtained from Eq. (11) with $\varepsilon_d = x$, $\kappa_2 = -\frac{1-x}{N}$ and all the rest of the parameters equal to 0,

$$E_{II}(N,\beta) = -(5+\beta^2)\,\frac{1-x}{1+\beta^2} + N\,x\,\frac{\beta^2}{1+\beta^2} - 4(N-1)(1-x)\,\frac{\beta^2}{(1+\beta^2)^2} \ . \tag{19}$$

Condition (13) gives in this case the critical point $x_c^{II} = \frac{4N-8}{5N-8}$ that in the large N limit gives 4/5.

In Fig. 1 the corresponding energy surfaces are plotted in the right panel. Same comments as in the preceding case are in order. Thus, we conclude that, in the transition from spherical systems to γ-unstable deformed ones, the critical point in IBM should be associated to a β^4 potential rather that to an infinite square well. The question is then how different are the E(5) predictions from those obtained with a β^4 potential? In order to investigate this point we have solved numerically the Bohr hamiltonian for a potential β^4. The results for energies are presented in Table I and in Fig. 2. Here we keep the label ξ used in the E(5) case. It is related to the label $n_\beta = \frac{n_d - \tau}{2}$, sometimes used in the U(5) classification, by $n_\beta = \xi - 1$, where n_d is the U(5) label and τ is the O(5) label. Particularly interesting are the energy ratios given in Table II which have been used in recent works to identify possible nuclei as critical. In this table the E(5) and β^4 values are shown for comparison. The labeling of the states is $L_{\xi,\tau}$.

Besides the excitation energies, B(E2) transition probabilities can be calculated using the quadrupole operator

$$T^{(E2)}_{\mu} = t\ \beta\ \left[\mathcal{D}^{(2)}_{\mu 0}(\theta_i)\cos\gamma + \frac{1}{\sqrt{2}}\left(\mathcal{D}^{(2)}_{\mu 2}(\theta_i) + \mathcal{D}^{(2)}_{\mu -2}(\theta_i)\right)\sin\gamma\right] \quad , \tag{20}$$

where t is a scale factor. In Table II two important B(E2) ratios are given for E(5) and β^4 cases. In Fig. 2 the B(E2) values for a β^4 potential are shown besides the arrows. They are given normalized to the $B(E2; 2_{1,1} \to 0_{1,0})$ value which is taken as 100.

Comparing Figs. 1 and Table I in Ref. [1] with the present Fig. 2 and Table I we can observe important differences between E(5) and β^4 potentials. In order to see which is the actual large N limit of IBM we have performed calculations with the IBM codes for hamiltonians H_I (Eq. 16) and H_{II} (Eq. 18) at the critical point for different number of bosons. These codes allow to manage a small number of bosons, typically 20. In Fig. 3 the results of these calculations are shown with a full line for Eq. (16) and with a dashed line for Eq. (18). The values for E(5) and β^4 potentials are shown as dotted lines as references. The last two panels labeled with R_1 and R_2 refer to the B(E2) ratios presented in Table II.

From Fig. 3 it is clear that the finite N effects are important and depend on the precise form of the hamiltonian used. However, it is difficult to conclude whether E(5) or β^4 is the large N limit of the corresponding IBM hamiltonian. It is necessary to perform calculations with larger values of N. Fortunately, Dukelsky et al. [11] have recovered an exactly solvable model for pairing proposed by Richardson in the 60's [12]. Following Ref. [11] we have solved the Richardson's equations and obtained the exact eigenvalues for the hamiltonians (16) and (18) up to $N = 1000$, so approaching the large N limit of the corresponding IBM hamiltonians. Details of this method will be given in a longer publication. In Fig. 4 we present the results of these calculations for energy ratios up to $N = 1000$ and B(E2) ratios up to $N = 40$ together with the corresponding values for the E(5) symmetry and the β^4 potential. From this figure it clearly emerges that the large N limit for the studied IBM hamiltonians corresponds to the β^4 potential. Both hamiltonians Eq. (16) and Eq. (18) converge to the same results in the large N limit, although the corresponding corrections for finite N are quite different (see Fig. 3).

We conclude that the large N limit of the IBM hamiltonian at the critical point in the transition from U(5) (spherical) to O(6) (deformed γ-unstable) is represented in the geometrical model by a β^4 potential. The results are similar but not close to those of an infinite square well as in the E(5) critical point symmetry. The analysis of the IBM energy surface followed by an IBM calculation, as presented in Ref. [13], can provide the appropriate finite N corrections and thus lead to the identification of nuclei at the critical points. In that work a systematic study of the properties of the Ru isotopes allowed to select the appropriate form of the hamiltonian. Once it is fixed the construction of the energy surfaces identify the critical nucleus (^{104}Ru in that case). The corresponding IBM calculation for the critical nucleus then provides the correct finite N corrections. We believe that this is a fundamental step if we wish to robustly identify the spectroscopic properties that signal the presence of criticality in the atomic nucleus.

ACKNOWLEDGEMENTS

This work was supported in part by the Spanish DGICYT under project number BFM2002-03315, by CONACyT (México) and by an INFN-DGICYT agreement.

REFERENCES

[1] F. Iachello, *Phys. Rev. Lett.* **85**, 3580 (2000).

[2] A. Bohr and B. Mottelson, *Nuclear Structure, vol II*, Benjamin, Reading, Mass. 1975.

[3] F. Iachello and A. Arima, *The Interacting Boson Model*, Cambridge University Press, Cambridge, 1987.

[4] R. F. Casten and N. V. Zamfir, *Phys. Rev. Lett.* **85**, 3584 (2000).

[5] L. Wilets and M. Jean, *Phys. Rev.* **102**, 788 (1956).

[6] D. Bès, *Nucl. Phys.* **10**, 373 (1959).

[7] J. N. Ginocchio and M. W. Kirson, *Nucl. Phys.* **A350**, 31 (1980).

[8] A. E. L. Dieperink, O. Scholten and F. Iachello, *Phys. Rev. Lett.* **44**, 1747 (1980).

[9] A. Bohr and B. Mottelson, *Phys. Scripta* **22**, 468 (1980).

[10] A. Frank, *Phys. Rev.* **C39**, 652 (1989).

[11] J. Dukelsky, C. Esebbag and P. Schuck, *Phys. Rev. Lett.* **87**, 066403 (2001).

[12] R.W. Richardson, *Phys. Lett.* **3**, 277 (1963). R.W. Richardson and N. Sherman, *Nucl. Phys.* **52**, 221 (1964). R.W. Richardson, *J. Math. Phys. (N.Y.)* **9**, 1327 (1968).

[13] A. Frank, C.E. Alonso, and J.M. Arias, *Phys. Rev.* **C65**, 014301 (2001).

TABLES

TABLE I. Excitation energies for a β^4 potential relative to the energy of the first excited state.

	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$
$\tau = 0$	0.00	2.39	5.15	8.20
$\tau = 1$	1.00	3.63	6.56	9.75
$\tau = 2$	2.09	4.92	8.01	11.34
$\tau = 3$	3.27	6.26	9.50	12.95

TABLE II. Energy and B(E2) transition rate ratios in the E(5) symmetry and for the β^4 potential.

	$E_{4_{1,2}}/E_{2_{1,1}}$	$E_{0_{2,0}}/E_{2_{1,1}}$	$E_{0_{1,3}}/E_{2_{1,1}}$	$E_{0_{2,0}}/E_{0_{1,3}}$	$R_1 = \frac{B(E2;4_{1,2}\to 2_{1,1})}{B(E2;2_{1,1}\to 0_{1,0})}$	$R_2 = \frac{B(E2;0_{2,0}\to 2_{1,1})}{B(E2;2_{1,1}\to 0_{1,0})}$
E(5)	2.20	3.03	3.59	0.84	1.68	0.86
β^4	2.09	2.39	3.27	0.73	1.82	1.41

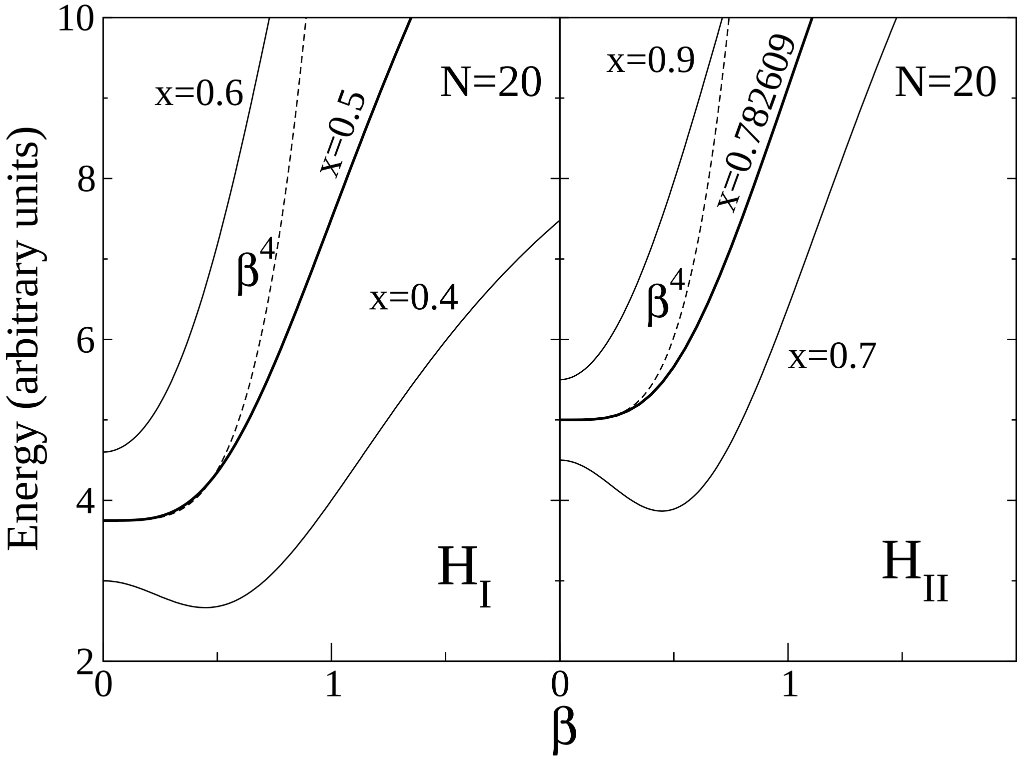

FIG. 1. Representation of the energy surfaces for $N = 20$ as functions of the shape parameter β obtained for two schematic hamiltonians, Eq. (16) (left panel) and Eq. (18) (right panel). In each case three values of the order parameter are presented, one at the critical value, one above and one below that value. The curves have been arbitrarily displaced in energy so as to show clearly the behavior.

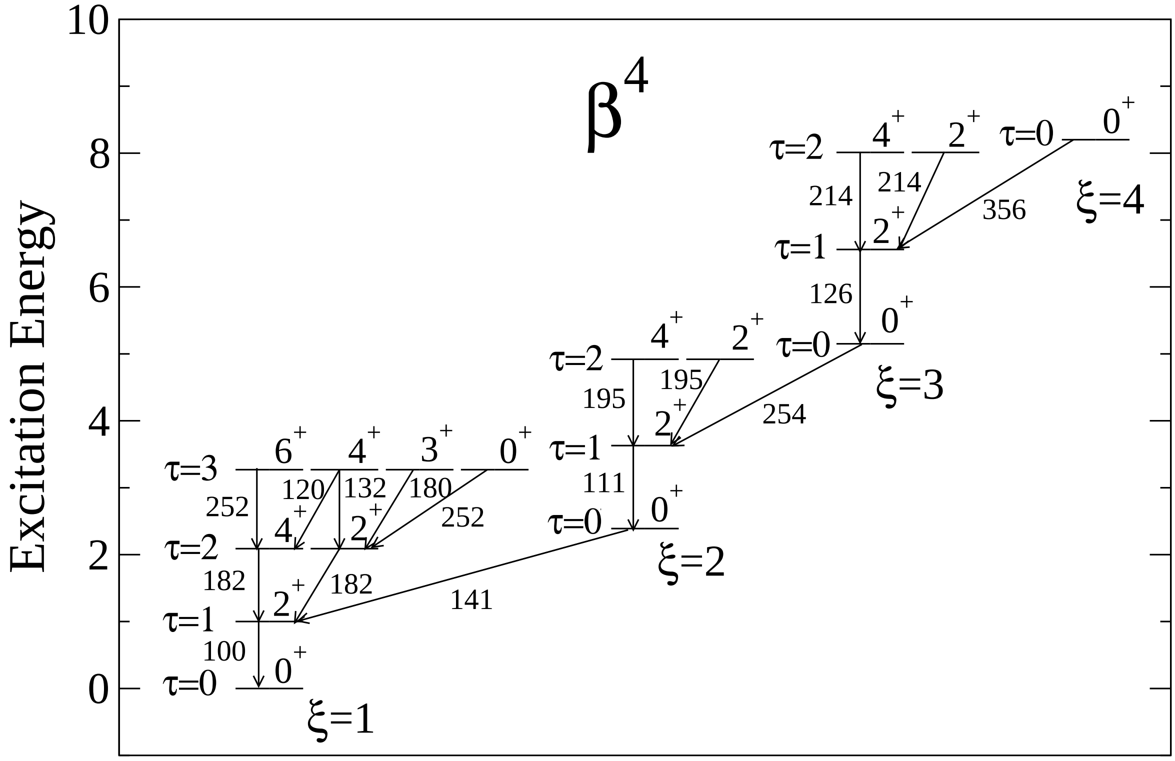

FIG. 2. Schematic spectrum for a β^4 potential. Numbers close to the arrows are B(E2) values. These are relative to the transition $2_{1,1} \rightarrow 0_{1,0}$ whose B(E2) value is taken as 100.

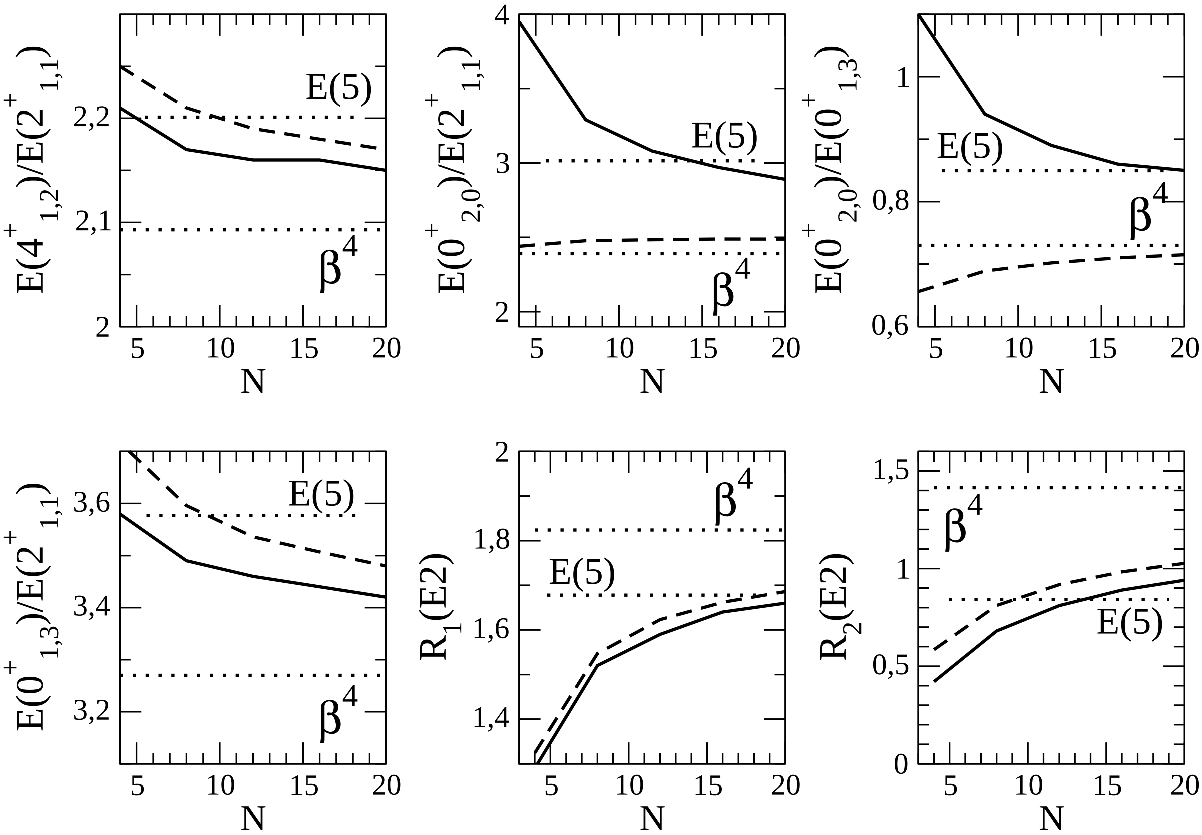

FIG. 3. Variation with the number of bosons (up to $N = 20$) of selected energy and B(E2) ratios for IBM calculations performed at the critical points of hamiltonian (16) (full line) and (18) (dashed line). The corresponding E(5) and β^4 values are marked with dotted lines.

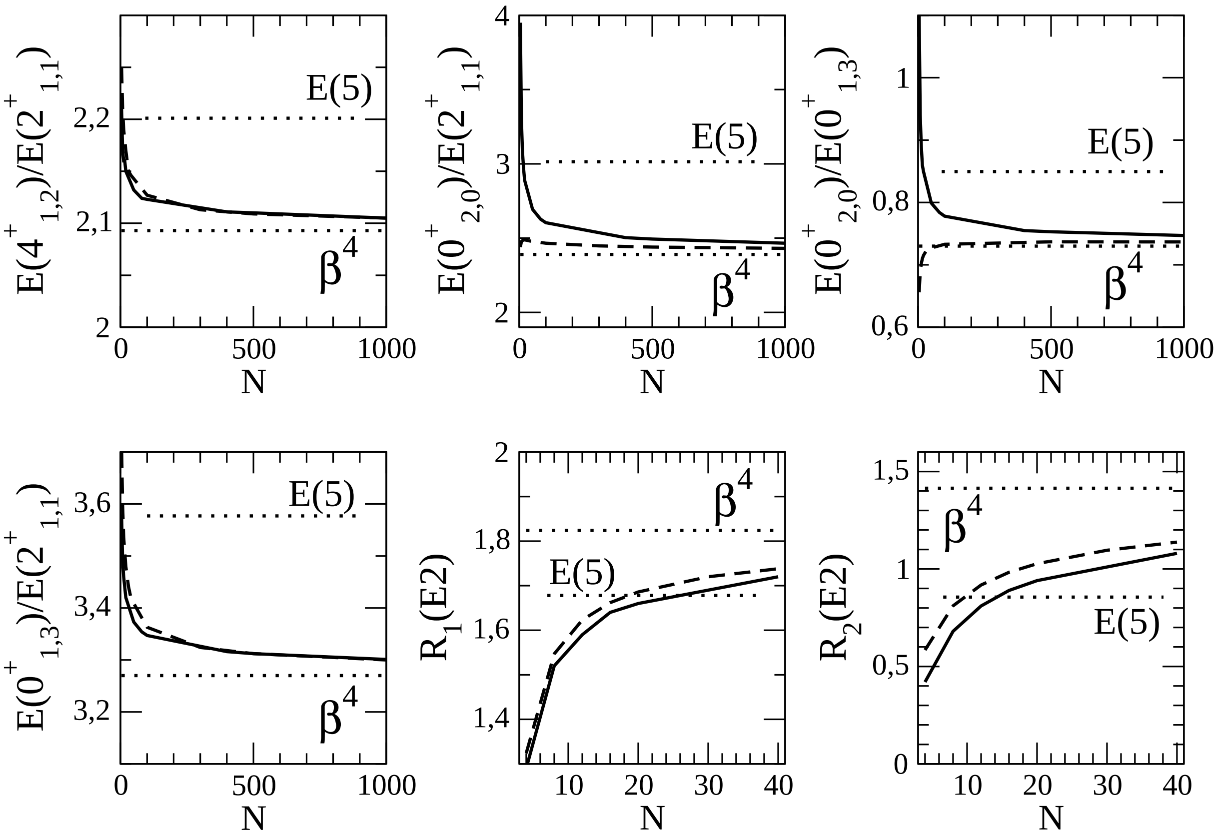

FIG. 4. Same as Fig 3 but here the number of bosons runs up to 1000 in the energy ratios and up to 40 in the B(E2) ratios.