Partial dynamical symmetry in quantum Hamiltonians with higher-order terms

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(Dated: January 30, 2012)

A generic procedure is proposed to construct many-body quantum Hamiltonians with partial dynamical symmetry. It is based on a tensor decomposition of the Hamiltonian and allows the construction of a hierarchy of interactions that have selected classes of solvable states. The method is illustrated in the SO(6) limit of the interacting boson model of atomic nuclei and applied to the nucleus ^{196}Pt.

PACS numbers: 21.60.Fw, 21.10.Re, 21.60.Ev, 27.80.+w

The concept of dynamical symmetry (DS) is now widely accepted to be of central importance in our understanding of many-body systems. In particular, it had a major impact on developments in nuclear [1] and molecular [2] physics and made significant contributions to virtually all areas of many-body physics [3]. Its basic paradigm is to write the Hamiltonian of the system under consideration in terms of Casimir operators of a set of nested algebras. Its hallmarks are (i) solvability of the complete spectrum, (ii) existence of exact quantum numbers for all eigenstates, and (iii) pre-determined symmetry-based structure of the eigenfunctions, independent of the Hamiltonian's parameters.

The merits of a DS are self-evident. However, in most applications to realistic systems, the predictions of an exact DS are rarely fulfilled and one is compelled to break it. More often one finds that the assumed symmetry is not obeyed uniformly, *i.e.*, is fulfilled by only some states but not by others. The need to address such situations has led to the introduction of partial dynamical symmetries (PDSs). The essential idea is to relax the stringent conditions of *complete* solvability so that the properties (i)–(iii) are only partially satisfied. Partiality comes in three different guises: (a) part of the eigenspectrum retains all the DS quantum numbers [4, 5], (b) the entire eigenspectrum retains part of the DS quantum numbers [6, 7], and (c) part of the eigenspectrum retains part of the DS quantum numbers [8]. PDS of various types have been shown to be relevant to nuclear [5, 6, 7, 8, 9] and molecular [10] spectroscopy, to systems with mixed chaotic and regular dynamics [11] and to quantum phase transitions [12].

We emphasize that DS as well as its generalization PDS are notions that are not restricted to a specific model but can be applied to any quantal system of interacting particles, bosons and fermions. In this Letter we propose a generic method to construct quantum Hamiltonians with PDS of type (a) and show that its existence is closely related to the order of the interaction among the particles. The procedure is discussed in general terms and subsequently illustrated with a nuclear-physics example with an application to the nucleus ^{196}Pt.

The analysis starts from the chain of nested algebras

$$\begin{array}{ccccccc} G_{\rm dyn} & \supset & G & \supset & \cdots & \supset & G_{\rm sym} \\ \downarrow & & \downarrow & & & & \downarrow \\ [h] & & \langle\Sigma\rangle & & & & \Lambda \end{array} \qquad (1)$$

where, below each algebra, its associated labels of irreducible representations (irreps) are given. Eq. (1) implies that $G_{\rm dyn}$ is the dynamical algebra of the system such that operators of all physical observables can be written in terms of its generators [13]; a single irrep of $G_{\rm dyn}$ contains all states of relevance in the problem. In contrast, $G_{\rm sym}$ is the symmetry algebra and a single of its irreps contains states that are degenerate in energy. A frequently encountered example is $G_{\rm sym} = {\rm SO}(3)$, the algebra of rotations in 3 dimensions, with its associated quantum number of total angular momentum L. Other examples of conserved quantum numbers can be the total spin S in atoms or total isospin T in atomic nuclei.

The classification (1) is generally valid and does not require conservation of particle number. Although the extension from DS to PDS can be formulated under such general conditions, let us for simplicity of notation assume in the following that particle number is conserved. All states, and hence the representation $[h]$, can then be assigned a definite particle number N. For N identical particles the representation $[h]$ of the dynamical algebra $G_{\rm dyn}$ is either symmetric $[N]$ (bosons) or antisymmetric $[1^N]$ (fermions) and will be denoted, in both cases, as $[h_N]$. For particles that are non-identical under a given dynamical algebra $G_{\rm dyn}$, a larger algebra can be chosen such that they become identical under this larger algebra (generalized Pauli principle) [14]. The occurrence of a DS of the type (1) signifies that eigenstates can be labeled as $|[h_N]\langle\Sigma\rangle\ldots\Lambda\rangle$; additional labels (indicated by $\ldots$) are suppressed in the following. Likewise, operators can be classified according to their tensor character under (1) as $\hat{T}_{[h_n]\langle\sigma\rangle\lambda}$. Of specific interest in the construction of a PDS associated with the reduction (1), are the n-particle annihilation operators $\hat{T}$ which satisfy the property

$$\hat{T}_{[h_n]\langle\sigma\rangle\lambda}|[h_N]\langle\Sigma_0\rangle\Lambda\rangle = 0, \qquad (2)$$

for all possible values of Λ contained in a given irrep $\langle\Sigma_0\rangle$. Any number-conserving normal-ordered interaction writ-

ten in terms of these annihilation operators (and their Hermitian conjugates which transform as the corresponding conjugate irreps) can be added to the Hamiltonian with a DS (1), while still preserving the solvability of states with $\langle\Sigma\rangle = \langle\Sigma_0\rangle$. The annihilation condition (2) is satisfied if none of the G irreps $\langle\Sigma\rangle$ contained in the $G_{\rm dyn}$ irrep $[h_{N-n}]$ belongs to the G Kronecker product $\langle\Sigma_0\rangle \times \langle\sigma\rangle$. So the problem of finding interactions that preserve solvability for part of the states (1) is reduced to carrying out a Kronecker product.

Let us now illustrate this procedure with an example taken from nuclear physics where the concept of DS has been applied successfully in the context of the interacting boson model (IBM) [1]. In this model, low-energy collective states are described in terms of N interacting monopole (s) and quadrupole (d) bosons representing valence nucleon pairs. The dynamical algebra is $G_{\rm dyn} = {\rm U}(6)$ and the symmetry algebra is $G_{\rm sym} = {\rm SO}(3)$. Three DS limits occur in the model with leading subalgebras U(5), SU(3), and SO(6), corresponding to typical collective spectra observed in nuclei, vibrational, rotational, and γ-unstable, respectively. Here we focus on the SO(6) limit with predictions that were found to correspond closely to the empirical structure of some platinum nuclei [15] as well as of other nuclei, notably, around mass number $A = 130$ [16].

The classification of states in the SO(6) limit of the IBM is

$$\begin{array}{ccccccccc} {\rm U}(6) & \supset & {\rm SO}(6) & \supset & {\rm SO}(5) & \supset & {\rm SO}(3) & \supset & {\rm SO}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & \langle\Sigma\rangle & & (\tau) & \nu_\Delta & L & & M \end{array} . \quad (3)$$

The multiplicity label ν_Δ in the ${\rm SO}(5) \supset {\rm SO}(3)$ reduction will be omitted in the following when it is not needed. The eigenstates $|[N]\langle\Sigma\rangle(\tau)\nu_\Delta LM\rangle$ are obtained with a Hamiltonian with SO(6) DS which, for one- and two-body interactions, can be transcribed in the form

$$\hat{H}_{\rm DS} \;=\; A\,\hat{P}_+\hat{P}_- + B\,\hat{C}_{{\rm SO}(5)} + C\,\hat{C}_{{\rm SO}(3)}. \quad (4)$$

Here $\hat{C}_G$ denotes the quadratic Casimir operator of G, $\hat{P}_+ \equiv \frac{1}{2}(s^\dagger s^\dagger - d^\dagger\cdot d^\dagger)$, $4\hat{P}_+\hat{P}_- = \hat{N}(\hat{N}+4) - \hat{C}_{{\rm SO}(6)}$ and $\hat{P}_- = \hat{P}_+^\dagger$. The total boson number operator, $\hat{N} = \hat{n}_s + \hat{n}_d$, is the linear Casimir of U(6) and is a constant for all N-boson states. The spectrum of $\hat{H}_{\rm DS}$ is completely solvable with eigenenergies

$$\begin{aligned} E_{\rm DS} \;=\; & \tfrac{1}{4}A\,(N-\Sigma)(N+\Sigma+4) + B\,\tau(\tau+3) \\ & + C\,L(L+1). \end{aligned} \quad (5)$$

In contrast, Hamiltonians with SO(6) PDS preserve the analyticity of only a *subset* of the states (3). The construction of interactions with this property requires n-boson creation and annihilation operators with definite tensor character in the basis (3):

$$\hat{B}^\dagger_{[n]\langle\sigma\rangle(\tau)lm}, \quad \tilde{B}_{[n^5]\langle\sigma\rangle(\tau)lm} \equiv (-1)^{l-m}\left(\hat{B}^\dagger_{[n]\langle\sigma\rangle(\tau)l,-m}\right)^\dagger. \quad (6)$$

TABLE I: Normalized two- and three-boson SO(6) tensors.

n	σ	τ	l	$\hat{B}^\dagger_{[n]\langle\sigma\rangle(\tau)lm}$
2	2	2	4	$\sqrt{\frac{1}{2}}(d^\dagger\times d^\dagger)^{(4)}_m$
2	2	2	2	$\sqrt{\frac{1}{2}}(d^\dagger\times d^\dagger)^{(2)}_m$
2	2	1	2	$(s^\dagger\times d^\dagger)^{(2)}_m$
2	2	0	0	$\sqrt{\frac{5}{12}}(s^\dagger\times s^\dagger)^{(0)}_0 + \sqrt{\frac{1}{12}}(d^\dagger\times d^\dagger)^{(0)}_0$
2	0	0	0	$\sqrt{\frac{1}{12}}(s^\dagger\times s^\dagger)^{(0)}_0 - \sqrt{\frac{5}{12}}(d^\dagger\times d^\dagger)^{(0)}_0$
3	3	3	6	$\sqrt{\frac{1}{6}}((d^\dagger\times d^\dagger)^{(4)}\times d^\dagger)^{(6)}_m$
3	3	3	4	$\sqrt{\frac{7}{22}}((d^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(4)}_m$
3	3	3	3	$\sqrt{\frac{7}{30}}((d^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(3)}_m$
3	3	3	0	$\sqrt{\frac{1}{6}}((d^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(0)}_0$
3	3	2	4	$\sqrt{\frac{1}{2}}((s^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(4)}_m$
3	3	2	2	$\sqrt{\frac{1}{2}}((s^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(2)}_m$
3	3	1	2	$\sqrt{\frac{7}{16}}((s^\dagger\times s^\dagger)^{(0)}\times d^\dagger)^{(2)}_m + \sqrt{\frac{5}{112}}((d^\dagger\times d^\dagger)^{(0)}\times d^\dagger)^{(2)}_m$
3	3	0	0	$\sqrt{\frac{5}{48}}((s^\dagger\times s^\dagger)^{(0)}\times s^\dagger)^{(0)}_0 + \sqrt{\frac{3}{16}}((s^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(0)}_0$
3	1	1	2	$\sqrt{\frac{1}{16}}((s^\dagger\times s^\dagger)^{(0)}\times d^\dagger)^{(2)}_m - \sqrt{\frac{5}{16}}((d^\dagger\times d^\dagger)^{(0)}\times d^\dagger)^{(2)}_m$
3	1	0	0	$\sqrt{\frac{1}{16}}((s^\dagger\times s^\dagger)^{(0)}\times s^\dagger)^{(0)}_0 - \sqrt{\frac{5}{16}}((s^\dagger\times d^\dagger)^{(2)}\times d^\dagger)^{(0)}_0$

Of particular interest are tensor operators with $\sigma < n$. They have the property

$$\tilde{B}_{[n^5]\langle\sigma\rangle(\tau)lm}|[N]\langle N\rangle(\tau)\nu_\Delta LM\rangle = 0, \qquad \sigma < n, \quad (7)$$

for all possible values of τ, L contained in the SO(6) irrep $\langle N\rangle$. This is so because the action of $\tilde{B}_{[n^5]\langle\sigma\rangle(\tau)lm}$ leads to an $(N-n)$-boson state that contains the SO(6) irreps $\langle\Sigma\rangle = \langle N-n-2i\rangle$, $i = 0, 1, \ldots$ which cannot be coupled with $\langle\sigma\rangle$ to yield $\langle\Sigma\rangle = \langle N\rangle$, since $\sigma < n$. Number-conserving normal-ordered interactions that are constructed out of such tensors with $\sigma < n$ (and their Hermitian conjugates) thus have $|[N]\langle N\rangle(\tau)\nu_\Delta LM\rangle$ as eigenstates with zero eigenvalue.

A systematic enumeration of all interactions with this property is a simple matter of SO(6) coupling. For *one-body* operators one has

$$\hat{B}^\dagger_{[1]\langle1\rangle(0)00} = s^\dagger \equiv b^\dagger_0, \quad \hat{B}^\dagger_{[1]\langle1\rangle(1)2m} = d^\dagger_m \equiv b^\dagger_{2m}, \quad (8)$$

and no annihilation operator has the property (7).

Coupled *two-body* operators are of the form

$$\hat{B}^\dagger_{[2]\langle\sigma\rangle(\tau)lm} \propto \sum_{\tau_k\tau_{k'}}\sum_{kk'} C^{\langle\sigma\rangle(\tau)l}_{\langle1\rangle(\tau_k)k,\langle1\rangle(\tau_{k'})k'}(b^\dagger_k\times b^\dagger_{k'})^{(l)}_m, \quad (9)$$

where $(\cdot\times\cdot)^{(l)}_m$ denotes coupling to angular momentum l and the C-coefficients are known ${\rm SO}(6) \supset {\rm SO}(5) \supset {\rm SO}(3)$ isoscalar factors [13]. This leads to the normalized two-boson SO(6) tensors shown in Table I. There is one operator with $\sigma < n = 2$ and it gives rise to the following SO(6)-invariant interaction

$$\hat{B}^\dagger_{[2]\langle0\rangle(0)00}\tilde{B}_{[2^5]\langle0\rangle(0)00} = \tfrac{1}{3}\hat{P}_+\hat{P}_-, \quad (10)$$

which is simply the SO(6) term in $\hat{H}_{\rm DS}$, Eq. (4). This proves that a two-body interaction which is diagonal in $|[N]\langle N\rangle(\tau)\nu_\Delta LM\rangle$ is diagonal in all states $|[N]\langle\Sigma\rangle(\tau)\nu_\Delta LM\rangle$. This result is valid in the SO(6) limit of the IBM, but not in general. For example, from a tensor decomposition of two-boson operators in SU(3) one concludes that the SU(3) limit of the IBM *does* allow a PDS with two-body interactions [4, 5].

Three-body operators with good SO(6) labels can be obtained from an expansion similar to (9) and this leads to the normalized three-boson SO(6) tensors shown in Table I. In terms of the $\hat{P}_+$ operator introduced above, the two operators with $\sigma < n = 3$ are

$$\hat{B}^\dagger_{[3]\langle 1\rangle(1)2m} = \tfrac{1}{2}\hat{P}_+ d^\dagger_m, \quad \hat{B}^\dagger_{[3]\langle 1\rangle(0)00} = \tfrac{1}{2}\hat{P}_+ s^\dagger, \qquad (11)$$

and from these one can construct the interactions with an SO(6) PDS. The only three-body interactions that are partially solvable in SO(6) are thus $\hat{P}_+\hat{n}_s\hat{P}_-$ and $\hat{P}_+\hat{n}_d\hat{P}_-$, involving the s- and d-boson number operators. Since the combination $\hat{P}_+(\hat{n}_s+\hat{n}_d)\hat{P}_- = (\hat{N}-2)\hat{P}_+\hat{P}_-$ is completely solvable in SO(6), there is only one genuine partially solvable three-body interaction which can be chosen as $\hat{P}_+\hat{n}_s\hat{P}_-$, with tensorial components $\sigma = 0, 2$.

The generalization to higher orders now suggests itself. For example, *four-body* interactions with SO(6) PDS are written in terms of $\hat{B}^\dagger_{[4]\langle 2\rangle(\tau)lm}$ and $\hat{B}^\dagger_{[4]\langle 0\rangle(0)00}$, and Hermitian conjugate operators. Without loss of generality, these operators can be written as

$$\hat{B}^\dagger_{[4]\langle 2\rangle(\tau)lm} \propto \hat{P}_+\hat{B}^\dagger_{[2]\langle 2\rangle(\tau)lm}, \quad \hat{B}^\dagger_{[4]\langle 0\rangle(0)00} \propto \hat{P}^2_+. \qquad (12)$$

A four-body interaction with SO(6) PDS is thus of the form $\hat{P}_+\hat{V}_2\hat{P}_-$ where $\hat{V}_2$ is an arbitrary two-body interaction. This interaction leaves solvable the class of states with $\Sigma = N$ but, in general, admixes those with $\Sigma < N$. The conclusion is that we can construct a hierarchy of interactions $\hat{P}^k_+\hat{n}_s\hat{P}^k_-$, $\hat{P}^k_+\hat{V}_2\hat{P}^k_-, \ldots, \hat{P}^k_+\hat{V}_p\hat{P}^k_-$ of order $2k+1, 2k+2, \ldots, 2k+p$, respectively, that retain the SO(6)-DS in selected states. If $\hat{V}_p$ is SO(5)-invariant, τ is a good quantum number for all states.

The SO(6)-DS spectrum of Eq. (5) resembles that of a γ-unstable deformed rotor, where states are arranged in bands with SO(6) quantum number $\Sigma = N - 2v$, $(v = 0, 1, 2, \ldots)$. The in-band rotational splitting is governed by the SO(5) and SO(3) terms in $\hat{H}_{\rm DS}$ (4). A comparison with the experimental spectrum and E2 rates of ^{196}Pt is shown in Fig. 1 and Table II. The SO(6)-DS limit is seen to provide a good description for properties of states in the ground band ($\Sigma = N$). This observation was the basis of the claim [15] that the SO(6)-DS is manifested empirically in ^{196}Pt. However, the resulting fit to energies of excited bands is quite poor. The 0^+_1, 0^+_3, and 0^+_4 levels of ^{196}Pt at excitation energies 0, 1403, 1823 keV, respectively, are identified as the bandhead states of the ground ($v = 0$), first- ($v = 1$) and second- ($v = 2$) excited vibrational bands [15]. Their empirical anharmonicity, defined by the ratio $R = E(v = 2)/E(v = 1) - 2$, is found to be $R = -0.70$. In the SO(6)-DS limit these bandhead states have $\tau = L = 0$ and $\Sigma = N, N-2, N-4$, respectively. The anharmonicity $R = -2/(N+1)$, as calculated from Eq. (5), is fixed by N. For $N = 6$, which is the appropriate boson number for ^{196}Pt, the SO(6)-DS value is $R = -0.29$, which is in marked disagreement with the empirical value. A detailed study of double-phonon excitations within the IBM, has concluded that large anharmonicities can be incorporated only by the inclusion of at least cubic terms in the Hamiltonian [17]. In the IBM there are 17 possible three-body interactions. One is thus confronted with the need to select suitable higher-order terms that can break the DS in excited bands but preserve it in the ground band. This can be accomplished by the PDS construction presented in this work. On the basis of the preceding discussion we propose to use the following Hamiltonian with SO(6)-PDS

$$\hat{H}_{\rm PDS} = \hat{H}_{\rm DS} + \eta\hat{P}_+\hat{n}_s\hat{P}_-, \qquad (13)$$

where the terms are defined in Eqs. (4) and (11). The spectrum of $\hat{H}_{\rm PDS}$ is shown in Fig. 1. The states belonging to the $\Sigma = N = 6$ multiplet remain solvable with energies given by the same DS expression, Eq. (5). States with $\Sigma < 6$ are generally admixed but agree better with the data than in the DS calculation. For example, the bandhead states of the first- (second-) excited bands have the SO(6) decomposition $\Sigma = 4$: 76.5% (19.6%), $\Sigma = 2$: 16.1% (18.4%), and $\Sigma = 0$: 7.4% (62.0%). Thus, although the ground band is pure, the excited bands exhibit strong SO(6) breaking. The calculated SO(6)-PDS anharmonicity for these bands is $R = -0.63$, much closer

TABLE II: Observed [18] and calculated B(E2) values (in e^2b^2) for ^{196}Pt. For both the exact (DS) and partial (PDS) SO(6) dynamical symmetry calculations, the E2 operator is $e_{\rm b}[(s^\dagger\times\tilde{d} + d^\dagger\times\tilde{s})^{(2)} + \chi(d^\dagger\times\tilde{d})^{(2)}]$ with $e_{\rm b} = 0.151$ eb and $\chi = 0.29$.

Transition	Experiment	DS	PDS
$2^+_1 \to 0^+_1$	0.274 (1)	0.274	0.274
$2^+_2 \to 2^+_1$	0.368 (9)	0.358	0.358
$2^+_2 \to 0^+_1$	3.10^{-8}(3)	0.0018	0.0018
$4^+_1 \to 2^+_1$	0.405 (6)	0.358	0.358
$0^+_2 \to 2^+_2$	0.121 (67)	0.365	0.365
$0^+_2 \to 2^+_1$	0.019 (10)	0.003	0.003
$4^+_2 \to 4^+_1$	0.115 (40)	0.174	0.174
$4^+_2 \to 2^+_2$	0.196 (42)	0.191	0.191
$4^+_2 \to 2^+_1$	0.004 (1)	0.001	0.001
$6^+_1 \to 4^+_1$	0.493 (32)	0.365	0.365
$2^+_3 \to 0^+_2$	0.034 (34)	0.119	0.119
$2^+_3 \to 4^+_1$	0.0009 (8)	0.0004	0.0004
$2^+_3 \to 2^+_2$	0.0018 (16)	0.0013	0.0013
$2^+_3 \to 0^+_1$	0.00002 (2)	0	0
$6^+_2 \to 6^+_1$	0.108 (34)	0.103	0.103
$6^+_2 \to 4^+_2$	0.331 (88)	0.221	0.221
$6^+_2 \to 4^+_1$	0.0032 (9)	0.0008	0.0008
$0^+_3 \to 2^+_2$	< 0.0028	0.0037	0.0028
$0^+_3 \to 2^+_1$	< 0.034	0	0

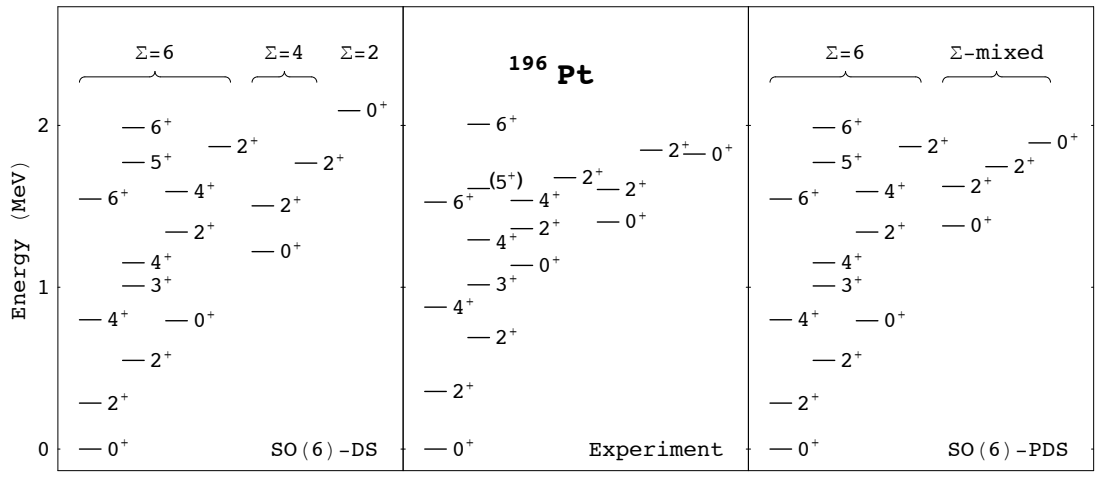

FIG. 1: Observed spectrum of ^{196}Pt [18] compared with the calculated spectra of $\hat{H}_{\rm DS}$ (4), with SO(6) dynamical symmetry (DS), and of $\hat{H}_{\rm PDS}$ (13) with partial dynamical symmetry (PDS). The parameters in $\hat{H}_{\rm DS}$ ($\hat{H}_{\rm PDS}$) are $A = 174.2\,(122.9)$, $B = 44.0\,(44.0)$, $C = 17.9\,(17.9)$, and $\eta = 0\,(34.9)$ keV. The boson number is $N = 6$ and Σ is an SO(6) label.

to the empirical value, $R = -0.70$. We emphasize that not only the energies but also the wave functions of the $\Sigma = N$ states remain unchanged when the Hamiltonian is generalized from DS to PDS. Consequently, the E2 rates for transitions among this class of states are the same in the DS and PDS calculations. This is evident in Table II where most of the E2 data concern transitions between $\Sigma = N = 6$ states. Only transitions involving states from excited bands (*e.g.*, the 0_3^+ state in Table II) can distinguish between DS and PDS. Unfortunately, such interband E2 rates are presently poorly known experimentally. Their measurement is highly desirable for further testing the SO(6)-PDS wave functions.

In conclusion, we have presented a systematic procedure for identifying and selecting interactions, of a given order, with partial dynamical symmetry (PDS). This allows the construction of Hamiltonians that break the dynamical symmetry (DS), but retain selected subsets of solvable eigenstates with good symmetry. As demonstrated in this work, the advantage of using higher-order interactions with PDS is that they can be introduced without destroying results previously obtained with a DS for a segment of the spectrum. These virtues generate an efficient tool which can greatly enhance the scope of algebraic modeling of quantum many-body systems.

This work was supported in part (A.L.) by the Israel Science Foundation and in part (J.E.G.R.) by the Spanish Ministerio de Educación y Ciencia, by the European regional development fund (FEDER) under projects FPA2006-13807-C02-02, FPA2007-63074 and by Junta de Andalucía under projects FQM318, P07-FQM-02962.

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