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Abstract—We analyze the problem of zero-error communication through timing channels that can be interpreted as discrete-time queues with bounded waiting times. The channel model includes the following assumptions: 1) time is slotted; 2) at most N particles are sent in each time slot; 3) every particle is delayed in the channel for a number of slots chosen randomly from the set $\{0, 1, \ldots, K\}$; and 4) the particles are identical. It is shown that the zero-error capacity of this channel is $\log r$, where r is the unique positive real root of the polynomial $x^{K+1} - x^K - N$. Capacity-achieving codes are explicitly constructed, and a linear-time decoding algorithm for these codes devised. In the particular case $N = 1$, $K = 1$, the capacity is equal to $\log \phi$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio, and constructed codes give another interpretation of the Fibonacci sequence.

Index Terms—Zero-error capacity, zero-error code, timing channel, timing capacity, molecular communications, discrete-time queue, Fibonacci sequence.

I. Preliminaries

THE study of timing channels, i.e., channels that arise when the information is being encoded in the transmission times of messages, has resulted in many interesting and relevant models. Two important and relatively recent examples are the models adopted from queuing theory [3], [4], [15] and those that arise in molecular communications [5]. We analyze here the problem of *zero-error* communication over certain channels of this type. The study is partly motivated by settings in which the communication is done with rather unconventional physical carriers, such as particles, molecules, items, etc. These channels can also be viewed as discrete-time queues with bounded waiting times, and the results can thus be seen as supplementing in a sense the work carried out in [4] and [15] (see also [10], [12]); however, due to the combinatorial nature of zero-error information theory [8], [14], the methods used are quite different from those in [4] and [15].

A. The Channel Model

We assume that multiple transmissions can occur at the same time instant without interfering with each other. In this
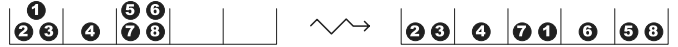
Fig. 1. Illustration of the DTPC(4, 2). The particles are numbered only for the purpose of illustration, they are assumed identical.

regard, we will use the term *particle* (instead of *symbol* or *packet*) for the unit of transmission. We believe that this convention will make the discussion clearer.

Let $\mathbb{N}$ denote the set of nonnegative integers $\{0, 1, \ldots\}$.

Definition 1: The Discrete-Time Particle Channel with parameters $N, K \in \mathbb{N}$, denoted DTPC(N, K), is the communication channel described by the following assumptions:

1) Time is slotted, meaning that the particles are sent and received in integer time instants;
2) At most N particles are sent in each time slot;
3) Every particle is delayed in the channel for a number of slots chosen randomly from the set $\{0, 1, \ldots, K\}$;
4) The particles are indistinguishable, and hence the information is conveyed via timing only, or equivalently, via the number of particles in each slot. ▲

We elaborate briefly on the definition of the DTPC. If the duration of the transmission is n slots, then the assumption 4) implies that the sequence of particles can be identified with an n-tuple of integers $(x_1, \ldots, x_n) \in \{0, 1, \ldots, N\}^n$, where x_i represents the number of particles in the i'th slot. For example, Figure 1 illustrates a situation where the transmitted sequence is $(3, 1, 4, 0, 0)$ and the received sequence is $(2, 1, 2, 1, 2)$. Hence, the DTPC can be defined purely in terms of sequences of nonnegative integers, and in the rest of the paper we will rely entirely on this representation.

As for the assumption 3), observe that if the delays of the particles were unbounded (as is the case, e.g., in queues having service times with geometric distribution [4]), the zero-error capacity would be zero. Therefore, in order to obtain interesting models, some restrictions on the delays have to be imposed. Similarly, if there is no restriction on the number of particles sent in each slot, then the zero-error capacity is infinite for any $K \in \mathbb{N}$, which justifies the assumption 2).

Note that we have not imposed a restriction on the number of particles at the output of the DTPC(N, K) in a single slot (though it is obviously bounded by $(K + 1)N$). It is not hard to argue that this does not affect the zero-error capacity of the above channel, i.e., it would be the same if this number were also bounded by N. This is proven in Appendix A.

Let us also give several more concrete interpretations of the DTPC. Namely, the "particles" referred to in the definition of this channel can be interpreted in various ways depending on

the context, e.g., as:

- "Molecules" in the so-called molecular communications, where the transmission of information via the number of molecules and their emission times is considered. The molecules are usually assumed identical, and their arrival times are random due to their interaction with the fluid medium. The codes described in the present paper are relevant precisely for the channels of this type, at least in discrete-time models [5].
- "Customers" in queuing systems, an important example of which are queues of "packets" formed in network routers (see the discussion in Remark 1 below).
- "Packets" in channels introducing random delays (caused by effects different than queuing). Note that the packets referred to in this and the previous paragraph are not identical in practice and usually carry information via their contents. In this paper we will be interested in the transmission of information via *timing only*, similarly as in [4]. Alternatively, one can imagine a receiver that is not processing the packets (e.g., a low power node in a wireless sensor network), but only infers their arrivals through energy detection.
- "Energy quanta" in a simultaneous transmission of energy and information [11].

Remark 1 (DTPC vs. Discrete-Time Queues): We have pointed out already that the results of the paper apply also to queuing systems of certain type. We introduce them here in a bit more detail. Denote by DTQR(N, K) the Discrete-Time Queue[1] with N servers/processors (meaning that N particles can be processed simultaneously), with at most N arrivals per slot, and with Residence times bounded by K slots (the residence time of the particle is the total time that it spends in the queue, either waiting to be processed or being processed). It is not difficult to argue that the DTPC(N, K) and the DTQR(N, K) have identical zero-error codes and zero-error capacities. The key difference between these channels is that the delays of the particles in the DTPC are independent, while in the DTQR they are not as they are affected also by the service procedure (for example, in FIFO queues the particles cannot be reordered). The assumption that the particles are identical, however, makes this difference irrelevant in the zero-error case. ▲

B. Notation and Definitions

By a "sequence" of length n over a nonempty alphabet $\mathbb{A}$ we mean an n-tuple from $\mathbb{A}^n$. When there is no risk of confusion, a sequence $(x_1, \ldots, x_n)$ will also be written as $x_1 \cdots x_n$. If, for a given channel, the sequence $\mathbf{x}$ at its input can produce the sequence $\mathbf{y}$ at its output with nonzero probability, then we write $\mathbf{x} \rightsquigarrow \mathbf{y}$. For any two sequences $\mathbf{x}$ and $\mathbf{y}$, their concatenation is denoted by $\mathbf{x} \circ \mathbf{y}$, or sometimes simply by $\mathbf{x} \mathbf{y}$. Also, if Z is a set of sequences, we let $\mathbf{x} \circ Z = \{\mathbf{x} \circ \mathbf{z} : \mathbf{z} \in Z\}$ and $Z \circ \mathbf{x} = \{\mathbf{z} \circ \mathbf{x} : \mathbf{z} \in Z\}$. We assume that $\mathbf{x} \circ \emptyset = \emptyset \circ \mathbf{x} = \emptyset$, and $\mathbf{x} \circ \varnothing = \varnothing \circ \mathbf{x} = \mathbf{x}$, where $\emptyset$ denotes an empty set and $\varnothing$ an empty sequence. For a sequence $\mathbf{x}$ and a number $k \in \mathbb{N}$,

$\mathbf{x}^k$ will denote the concatenation of k copies of $\mathbf{x}$, where it is assumed that $\mathbf{x}^0 = \varnothing$. The weight of a sequence $\mathbf{x} = x_1 \ldots x_n$, $x_i \in \mathbb{N}$, is defined as $wt(\mathbf{x}) = \sum_{i=1}^{n} x_i$.

A code of length n for the DTPC(N, K) is a subset of $\{0, 1, \ldots, N\}^n$. Codes will be denoted by calligraphic letters $\mathcal{C}, \mathcal{D}$, etc., or $\mathcal{C}(n), \mathcal{D}(n)$, if their length needs to be emphasized. The set of codewords of $\mathcal{C}$ having prefix $\mathbf{u}$ is denoted by $\mathcal{C}^{\mathbf{u}}$, and the code obtained by removing this prefix by $\mathcal{C}^{\underline{\mathbf{u}}} = \{\mathbf{v} : \mathbf{u} \circ \mathbf{v} \in \mathcal{C}\}$. Clearly, $\mathcal{C}^{\mathbf{u}} = \mathbf{u} \circ \mathcal{C}^{\underline{\mathbf{u}}}$.

Definition 2: $\mathcal{C}$ is said to be a zero-error code for the DTPC if for any $m \geq 1$ and any two distinct sequences $\mathbf{x} = \mathbf{x}_1 \ldots \mathbf{x}_m$ and $\mathbf{y} = \mathbf{y}_1 \ldots \mathbf{y}_m$, where $\mathbf{x}_i, \mathbf{y}_i \in \mathcal{C}$, there exists no sequence $\mathbf{z}$ such that both $\mathbf{x} \rightsquigarrow \mathbf{z}$ and $\mathbf{y} \rightsquigarrow \mathbf{z}$. ▲

In words, no two sequences of codewords of $\mathcal{C}$ can produce the same channel output, and hence there is no confusion about which sequence was sent. Note that we demand the distinguishability of *sequences of codewords*, rather that just of *codewords*. This is necessary in the delay channels. To illustrate this, let $\mathcal{C} = \{000, 100, 001\}$ be a code of length three for the DTPC(1, 1), introducing delays of at most one slot. Then it is easy to check that no two codewords can produce the same channel output, but on the other hand $001000 \rightsquigarrow 000100$, and hence the sequences of codewords $001, 000$ and $000, 100$ are confusable. $\mathcal{C}$ is therefore not a zero-error code.

This problem can easily be circumvented by simply padding each codeword with K zeros (empty slots, in the original terminology). Empty slots at the end of each codeword serve to "catch" the particles that are (potentially) sent in the preceding slots and are (potentially) delayed in the channel. In this way these particles do not interfere with the following codeword.

Definition 3: A code $\mathcal{C}(n)$ for the DTPC(N, K) is said to be *zero-padded* if all of its codewords end with $\min\{n, K\}$ zeros. ▲

Clearly, a zero-padded code is zero-error if and only if for every two distinct *codewords* $\mathbf{x}, \mathbf{y}$, there exists no sequence $\mathbf{z}$ with $\mathbf{x} \rightsquigarrow \mathbf{z}$ and $\mathbf{y} \rightsquigarrow \mathbf{z}$.

Definition 4: The rate of a code $\mathcal{C}(n)$ for the DTPC(N, K) is defined as $\frac{1}{n} \log |\mathcal{C}(n)|$. The zero-error capacity of the DTPC is the supremum of the rates of all zero-error codes for this channel. The base of log is assumed to be 2 and hence the rates and capacities are expressed in bits per time slot. ▲

It is easy to show that this supremum is equal to the lim sup of the rates of the largest zero-error codes for the DTPC. Furthermore, when considering the capacity of the DTPC(N, K), there is no loss in generality to restrict oneself to zero-padded codes, because padding with a constant number of zeros does not affect the code rate in the asymptotic sense.

II. OPTIMAL ZERO-ERROR CODES FOR THE DTPC

In this section we give two constructions of optimal zero-padded zero-error codes for the DTPC. The results have similar flavor to those obtained for some other types of combinatorial channels, see [1], [2], [7], [18].

A. Recursive Construction

The claim that follows establishes a general property of zero-padded zero-error codes for the DTPC(N, K), from which the construction of optimal codes will follow in a

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[1] Service procedure, service time distribution, and interarrival distribution are irrelevant in this context and hence are not specified.

straightforward way. It states that such codes can, without loss of generality, be assumed to contain only codewords with prefixes N and $i \circ 0^K$, $i \in \{0, 1, \ldots, N-1\}$.

Proposition 1: Let $\mathcal{C}(n)$, $n > K$, be a zero-padded zero-error code for the DTPC(N, K). Then there exists a zero-padded zero-error code $\mathcal{D}(n)$ of the same size, and such that:

$$\mathcal{D} = \mathcal{D}^N \cup \bigcup_{i=0}^{N-1} \mathcal{D}^{i \circ 0^K}. \qquad (1)$$

Proof: Let $\mathcal{C}(n)$ be a zero-padded zero-error code for the DTPC(N, K). We will construct $\mathcal{D}(n)$ by removing the codewords of $\mathcal{C}(n)$ that do not satisfy the desired form, and add the corresponding codewords that do. For any codeword of $\mathcal{C}(n)$ of the form $\mathbf{x} = \mathbf{u} \circ \mathbf{v}$, where $\mathbf{u} = u_1 \cdots u_{K+1}$ is of length $K+1$ and weight $wt(\mathbf{u}) = q$, we let the corresponding codeword $\tilde{\mathbf{x}}$ of $\mathcal{D}(n)$ be specified as follows: If $q < N$, then $\tilde{\mathbf{x}} = q \circ 0^K \circ \mathbf{v}$, while if $q \geq N$, then $\tilde{\mathbf{x}} = N \circ \tilde{\mathbf{u}} \circ \mathbf{v}$, where $\tilde{\mathbf{u}} = \tilde{u}_2 \cdots \tilde{u}_{K+1}$ is some sequence of length K and weight $q - N$ satisfying $\tilde{u}_i \leq u_i$, $i = 2, \ldots, K+1$ (in other words, the prefix of $\tilde{\mathbf{x}}$ is in the latter case constructed from $\mathbf{u}$ by removing $N - u_1$ of its particles from slots $2, \ldots, K+1$, and placing them in the first slot, together with the u_1 particles that are already there). Thus, we can write $\mathcal{D}(n) = \{\tilde{\mathbf{x}} : \mathbf{x} \in \mathcal{C}(n)\}$. It is now not difficult to argue that $|\mathcal{D}(n)| = |\mathcal{C}(n)|$ and that the fact that $\mathcal{C}(n)$ is a zero-padded zero-error code implies that $\mathcal{D}(n)$ is such a code too. The key observation is that $\mathcal{C}(n)$ cannot contain two distinct codewords of the form $\mathbf{x}_1 = \mathbf{u}_1 \circ \mathbf{v}$ and $\mathbf{x}_2 = \mathbf{u}_2 \circ \mathbf{v}$, where the prefixes $\mathbf{u}_1, \mathbf{u}_2$ are of length $K+1$ and have the same weight, that is $wt(\mathbf{u}_1) = wt(\mathbf{u}_2) = q$. This is because $\mathcal{C}(n)$ is zero-error, and clearly $\mathbf{x}_1 \rightsquigarrow 0^K \circ q \circ \mathbf{v}$ and $\mathbf{x}_2 \rightsquigarrow 0^K \circ q \circ \mathbf{v}$. ∎

Lemma 1: Let $\mathcal{D}(n)$, $n > K$, be a zero-padded code of the form (1) for the DTPC(N, K). Then $\mathcal{D}(n)$ is zero-error if and only if $\mathcal{D}^{\underline{N}}(n-1)$ and $\mathcal{D}^{i \circ 0^K}(n-K-1)$, $0 \leq i < N$, are all zero-error.

Proof: If $\mathcal{D}^{\underline{N}}$ and $\mathcal{D}^{i \circ 0^K}$ are zero-error, then so are $\mathcal{D}^N$ and $\mathcal{D}^{i \circ 0^K}$. Observe also that no two sequences $\mathbf{x}$, $\mathbf{y}$, such that $\mathbf{x}$ has prefix $i \circ 0^K$ and $\mathbf{y}$ either $j \circ 0^K$ or N (where $i, j \in \{0, 1, \ldots, N-1\}$, $i \neq j$), can produce the same channel output, implying that $\mathcal{D}$ is zero-error. The opposite direction is also easy. ∎

The above claims imply that an optimal zero-padded zero-error code of length n for the DTPC(N, K) can be constructed recursively from the codes of length $n-1$ and $n - K - 1$. To start the recursion, optimal zero-padded zero-error codes of length $j \in \{0, \ldots, K\}$ are needed, which are trivially $\{0^j\}$.

Theorem 1: The largest zero-padded zero-error code for the DTPC(N, K), denoted $\mathcal{C}_{N,K}$, is given by:

$$\mathcal{C}_{N,K}(n) = \left(N \circ \mathcal{C}_{N,K}(n-1)\right) \cup$$
$$\bigcup_{i=0}^{N-1} \left(i \circ 0^K \circ \mathcal{C}_{N,K}(n-K-1)\right), \qquad (2)$$

for $n > K$, and $\mathcal{C}_{N,K}(n) = \{0^n\}$ for $0 \leq n \leq K$. ∎

In the following subsection we will describe a different, perhaps more intuitive construction of the codes $\mathcal{C}_{N,K}$.

Theorem 1 implies that the cardinalities of the codes $\mathcal{C}_{N,K}$ satisfy the recurrence relation:

$$\left|\mathcal{C}_{N,K}(n)\right| = \left|\mathcal{C}_{N,K}(n-1)\right| + N \left|\mathcal{C}_{N,K}(n-K-1)\right|, \qquad (3)$$

with initial conditions $\left|\mathcal{C}_{N,K}(n)\right| = 1$, $0 \leq n \leq K$, which further implies that:

$$\left|\mathcal{C}_{N,K}(n)\right| = \sum_{k=1}^{K+1} a_k r_k^n, \qquad (4)$$

where r_k are the (complex) roots of the polynomial $x^{K+1} - x^K - N$, and a_k are (complex) constants.

Remark 2: In the particular case $N = 1$, $K = 1$, the analysis of the channel amounts to analyzing binary sequences whose 1's are being shifted by at most one position to the right (hence, the DTPC$(1, 1)$ can also be seen as a type of a "bit-shift" channel [9], [13]). In this case, the codes $\mathcal{C}_{1,1}$ satisfy the relation[2]:

$$\mathcal{C}_{1,1}(n) = \left(1 \circ \mathcal{C}_{1,1}(n-1)\right) \cup \left(00 \circ \mathcal{C}_{1,1}(n-2)\right), \qquad (5)$$

with $\mathcal{C}_{1,1}(0) = \{\varnothing\}$, $\mathcal{C}_{1,1}(1) = \{0\}$, which implies that $|\mathcal{C}_{1,1}(n)| = |\mathcal{C}_{1,1}(n-1)| + |\mathcal{C}_{1,1}(n-2)|$, with $|\mathcal{C}_{1,1}(0)| = |\mathcal{C}_{1,1}(1)| = 1$. In other words, $(|\mathcal{C}_{1,1}(n)|)$ is the Fibonacci sequence[3] (F_n). ▲

B. Direct Construction

Let $\mathcal{D}_{N,K}(n)$ be the code defined by the following procedure. First enumerate in the inverse lexicographic order all sequences of length n over $\{0, 1, \ldots, N\}$ ending with $\min\{n, K\}$ zeros (so that, for $n > K$, the first sequence on the list is $N^{n-K} \circ 0^K$, the second one is $N^{n-K-1} \circ (N-1) \circ 0^K$, etc.; see Table I). Then repeat the following step until there are no more sequences to process: Select the first sequence on the list that has not been processed, call it $\mathbf{x}$, to be a codeword, and then exclude all sequences $\mathbf{y}$ such that $\mathbf{x} \rightsquigarrow \mathbf{y}$. Table I illustrates the construction for $N = 2$, $K = 1$ (only the codewords are listed to save space).

Proposition 2: $\mathcal{D}_{N,K}(n) = \mathcal{C}_{N,K}(n)$ for every $n \in \mathbb{N}$.

Proof: Since $\mathcal{D}_{N,K}(n) = \mathcal{C}_{N,K}(n) = \{0^n\}$ for $0 \leq n \leq K$, it is enough to show that $\mathcal{D}_{N,K}(n)$ satisfy the relation (2). Observe that $\mathcal{D}_{N,K}^N(n) = N \circ \mathcal{D}_{N,K}(n-1)$ because adding a fixed prefix to a set of sequences does not affect the process of construction and, moreover, the prefix N puts the sequences on the top of the list. It is left to prove that $\mathcal{D}_{N,K}^i(n) = i \circ 0^K \circ \mathcal{D}_{N,K}(n-K-1)$, for $0 \leq i < N$. First consider the case $i = N - 1$. Let $\mathbf{x}$ be a sequence with prefix $(N-1) \circ \mathbf{u}$, where $\mathbf{u}$ is of length K and strictly positive weight, and construct $\tilde{\mathbf{x}}$ as in the proof of Proposition 1. Now, if $\tilde{\mathbf{x}}$ is a codeword, then $\mathbf{x}$ is not because $\tilde{\mathbf{x}} \rightsquigarrow \mathbf{x}$ and so $\mathbf{x}$ would have been excluded in the process of construction. On the other hand, if $\tilde{\mathbf{x}}$ is not a codeword, then it has itself been excluded by some sequence $\mathbf{y}$ that precedes it in the inverse

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[2] As one of the referees pointed out, this resembles a well known characterization of F_n as the number of binary sequences of length $n - 1$ with no consecutive ones. Such a set of sequences, $\mathcal{S}(n)$, obeys the recursion $\mathcal{S}(n) = \left(0 \circ \mathcal{S}(n-1)\right) \cup \left(10 \circ \mathcal{S}(n-2)\right)$, with $\mathcal{S}(0) = \{\varnothing\}$, $\mathcal{S}(1) = \{0, 1\}$.

[3] The name Fibonacci code would thus be appropriate here, but it has already been used in some other contexts [6], [18].

TABLE I
ZERO-ERROR CODES OF LENGTH UP TO 5 FOR THE DTPC(2, 1).
CARDINALITY OF THE CODES IS SHOWN IN
THE RIGHTMOST COLUMN

2	2	2	2	0	1
2	2	2	1	0	
2	2	2	0	0	3
2	2	1	0	0	
2	2	0	0	0	5
2	1	0	2	0	
2	1	0	1	0	
2	1	0	0	0	
2	0	0	2	0	
2	0	0	1	0	
2	0	0	0	0	11
1	0	2	2	0	
1	0	2	1	0	
1	0	2	0	0	
1	0	1	0	0	
1	0	0	0	0	
0	0	2	2	0	
0	0	2	1	0	
0	0	2	0	0	
0	0	1	0	0	
0	0	0	0	0	21

lexicographic order ($\mathbf{y} \rightsquigarrow \tilde{\mathbf{x}}$). But then we also have $\mathbf{y} \rightsquigarrow \mathbf{x}$, and therefore $\mathbf{x}$ is not a codeword either. We have shown that $\mathcal{D}_{N,K}^{N-1}(n)$ does not contain codewords having prefix $(N-1) \circ \mathbf{u}$, where $wt(\mathbf{u}) > 0$, and hence it can only contain codewords starting with $(N-1) \circ 0^K$. Since none of the sequences with this prefix could have been excluded in the process of construction by a codeword from $\mathcal{D}_{N,K}^{N}(n)$, it follows that they have been processed independently of the rest of the list and so $\mathcal{D}_{N,K}^{N-1}(n) = (N-1) \circ 0^K \circ \mathcal{D}_{N,K}(n-K-1)$. One can now prove by induction that $\mathcal{D}_{N,K}^{i}(n) = i \circ 0^K \circ \mathcal{D}_{N,K}(n-K-1)$ for $i = N-1, N-2, \ldots, 1, 0$; the argument is very similar to the above and is omitted. ∎

C. Decoding Algorithm

The structure of the codes $\mathcal{C}_{N,K}$, captured by the relation (2), suggests a very simple algorithm for recovering the transmitted sequence $\mathbf{x} = x_1 \cdots x_n \in \mathcal{C}_{N,K}(n)$ from the received sequence $\mathbf{y} = y_1 \cdots y_n$. The procedure is as follows:

Set $\mathbf{y}^{(1)} = \mathbf{y}$, and observe the prefix of $\mathbf{y}^{(1)}$ of length $K+1$, namely $y_1 \cdots y_{K+1}$, and its weight q.

1. If $q < N$, conclude that $x_1 \cdots x_{K+1} = q \circ 0^K$ (see (2)), and set $\mathbf{y}^{(2)} = y_{K+2} \cdots y_n$. Note that $\mathbf{y}^{(2)}$ is the (possible) output of the DTPC(N, K) when the input is the codeword $x_{K+2} \cdots x_n$ from $\mathcal{C}_{N,K}(n-K-1)$.

2. If $q \geq N$, conclude that $x_1 = N$. If also $y_1 < N$, this means that some of the particles from the first slot have been delayed in the channel. In that case remove $N - y_1$ of these particles from slots $2, \ldots, K+1$ (first taking particles from slot 2, then slot 3, etc., until $N - y_1$ of them are collected) and put them in the first slot. Then set $\mathbf{y}^{(2)} = y_2' \cdots y_{K+1}' \circ y_{K+2} \cdots y_n$, where $y_2' \cdots y_{K+1}'$ is obtained from $y_2 \cdots y_{K+1}$ by removing the particles in the above-described way, i.e., for some $k \in \{2, \ldots, K+1\}$ we have $y_i' = 0$ for $i \in \{2, \ldots, k-1\}$, $y_k' = \sum_{i=1}^{k} y_i - N \geq 0$, and $y_i' = y_i$ for $i \in \{k+1, \ldots, K+1\}$. Note that $\mathbf{y}^{(2)}$ is the (possible) output of the DTPC(N, K) when the input is the codeword $x_2 \cdots x_n \in \mathcal{C}_{N,K}(n-1)$.

The procedure is repeated with $\mathbf{y}^{(2)}$ by considering its prefix of length $K+1$, and so on.

Since at least one symbol of $\mathbf{x}$ is determined in every iteration, the algorithm will terminate in at most n iterations (in fact, at most $n - K$ due to the trailing zeros).

III. ZERO-ERROR CAPACITY OF THE DTPC

The results of Section II-A imply that the capacity of the DTPC(N, K) can be simply found as $\lim_{n \to \infty} \frac{1}{n} |\mathcal{C}_{N,K}(n)|$, and by using the fact that the asymptotic behavior of $|\mathcal{C}_{N,K}(n)|$ is determined by the largest (in modulus) root of the polynomial $x^{K+1} - x^K - N$ (see (4)).

Lemma 2: The largest (in modulus) root r of the polynomial $x^{K+1} - x^K - N$ is real and greater than 1. Moreover, if $K \to \infty$, then $r \to 1$.

Proof: The following theorem is proven in [16, Ch. 3, Th. 2] (see also [17]): If $p(x) = c_m x^m + c_{m-1} x^{m-1} + \cdots + c_1 x + c_0$ is an arbitrary polynomial with complex coefficients, and $c_0 \cdot c_m \neq 0$, then all roots of $p(x)$ lie in the (complex) circle $|x| \leq r$, where r is the *unique positive real* root of $\tilde{p}(x) = |c_m| x^m - |c_{m-1}| x^{m-1} - \cdots - |c_1| x - |c_0|$. Since our polynomial is precisely of the form $\tilde{p}(x)$, we conclude that it has a unique positive real root r, and that all other roots are smaller in modulus than r. This root can be found as the point of intersection of the curves x^K and $N(x-1)^{-1}$ (viewed as real functions). By analyzing these curves it follows easily that $r > 1$ and that $r \to 1$ when $K \to \infty$. ∎

Theorem 2: The zero-error capacity of the DTPC(N, K) is equal to $\log r$, where r is the unique positive real root of the polynomial $x^{K+1} - x^K - N$. ∎

The zero-error capacity of the DTPC$(1, 1)$ is therefore $\log \phi$, where $\phi = (1+\sqrt{5})/2$. More generally, the zero-error capacity of the DTPC$(N, 1)$ equals $\log\left(\frac{1}{2}(1 + \sqrt{1+4N})\right)$. Explicit expressions can also be obtained in the following two cases, which are intuitively clear: The zero-error capacity of the DTPC$(N, 0)$ is $\log(N+1)$, while that of the DTPC(N, ∞) (which allows arbitrarily large delays) is zero.

The following proposition states some basic properties of the capacity, regarded as a function of the channel parameters N and K. This function is also illustrated in Figure 2.

Proposition 3: Both r and $\log r$ are monotonically increasing concave functions of N, for fixed K, and monotonically decreasing convex functions of K, for fixed N.

Proof: The function r is defined implicitly by $r^{K+1} - r^K - N = 0$, $r > 1$, and the function $c = \log r$ by $2^{c(K+1)} - 2^{cK} - N = 0$, $c > 0$. Note that r and c are well-defined for all $N, K \in \mathbb{R}_+$, not necessarily integers. One can therefore differentiate them with respect to N and K and verify that $\dot{r}_N > 0$, $\ddot{r}_N < 0$, $\dot{c}_K < 0$, $\ddot{c}_K > 0$. The remaining claims follow from the properties of the logarithm and the exponential function. ∎

APPENDIX A
RESTRICTING THE CHANNEL OUTPUT

In this section we demonstrate that bounding the number of particles that can be received in a slot by N (or by $N' \geq N$)
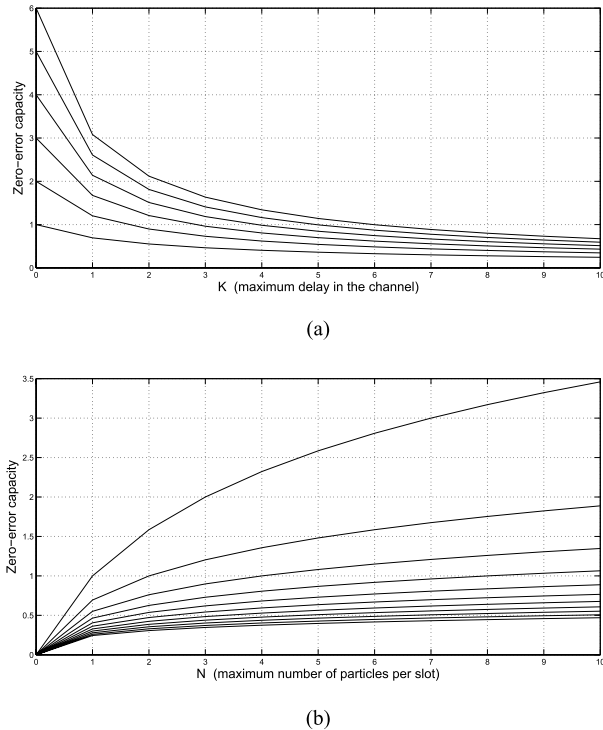
(a)

(b)

Fig. 2. The behavior of the zero-error capacity of the DTPC(N, K). (a) Dependence on K, for $N = 1, 3, 7, 15, 31, 63$. (b) Dependence on N, for $K = 0, 1, \ldots, 10$.

does not change the zero-error capacity of the DTPC. For the purpose of this argument we will refer to the channel with this additional restriction as the DTPC$(N, K; N)$. To clarify what is meant by the DTPC$(N, K; N)$, we emphasize that there is no "limiter" in the channel that drops some of the particles if their number in a slot exceeds N. As in the DTPC(N, K), all particles must arrive at the destination, only now their delays, in addition to being $\leq K$, have to be such that the number of particles at the channel output in every slot is $\leq N$. One can perhaps imagine a "membrane" at the channel output allowing at most N particles per slot to pass through.

Proposition 4: Any zero-error code for the DTPC(N, K) is a zero-error code for the DTPC$(N, K; N)$, and vice versa.

Proof: Let **x** and **y** be two sequences such that they can both produce $\mathbf{z} = z_1 \cdots z_l$ at the output of the DTPC(N, K). Then there exists $\mathbf{w} = w_1 \cdots w_l$ such that $\mathbf{x} \rightsquigarrow \mathbf{w}$ and $\mathbf{y} \rightsquigarrow \mathbf{w}$ in the DTPC$(N, K; N)$, i.e., such that $w_i \leq N$. To see this, observe that if $z_i > N$ for some $i \in \{1, \ldots, l\}$, then some of these z_i particles have not been delayed for a maximal number of slots (K) and could be further delayed. We can therefore find the desired **w** by going through slots $1, \ldots, l$, respectively, and whenever we find that $z_i' > N$, we move $z_i' - N$ of these particles to slot $i+1$, where z_i' is the sum of z_i and the number of particles that were moved from slots $1, \ldots, i-1$ to slot i during this procedure. We conclude that if a code is *not* a zero-error code for the DTPC(N, K), then it is *not* a zero-error code for the DTPC$(N, K; N)$ either. The opposite direction is obvious. ∎

We note, however, that bounding the number of received particles in a slot by $N' < N$ reduces the zero-error capacity because it excludes some sequences as valid inputs.

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