Southern Illinois University Carbondale
OpenSIUC

Dissertations | Theses and Dissertations

8-1-2016

Lyapunov-type inequality and eigenvalue estimates for fractional problems

Nimishaben Shailesh Pathak
Southern Illinois University Carbondale, nimishasp@siu.edu

LYAPUNOV-TYPE INEQUALITY AND EIGENVALUE ESTIMATES FOR FRACTIONAL PROBLEMS

by

Nimishaben Pathak

Bachelor of Science in Mathematics, The M. S. University of Baroda, India, 1992

Master of Science in Mathematics, The M. S. University of Baroda, India, 1994

Master of Philosophy in Mathematics, The M. S. University of Baroda, India, 1998

A Dissertation
Submitted in Partial Fulfillment of the Requirements for the
Doctor of Philosophy Degree

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
August, 2016

DISSERTATION APPROVAL

LYAPUNOV-TYPE INEQUALITY AND EIGENVALUE ESTIMATES FOR FRACTIONAL PROBLEMS

By

Nimishaben Pathak

A Thesis Submitted in Partial

Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in the field of Mathematics

Approved by:

Dr. Michael Sullivan, Chair

Dr. Kathleen Pericak-Spector

Dr. Rasit Koc

Dr. Jerzy Kocik

Dr. Randy Hughes

Graduate School
Southern Illinois University Carbondale
May 5^{th}, 2016

AN ABSTRACT OF THE DISSERTATION OF

NIMISHABEN PATHAK, for the Doctor of Philosophy degree in MATHEMATICS, presented on MAY 5^{th}, 2016, at Southern Illinois University Carbondale.

TITLE: LYAPUNOV-TYPE INEQUALITY AND EIGENVALUE ESTIMATES FOR FRACTIONAL PROBLEMS

MAJOR PROFESSOR: Dr. Michael Sullivan

In this work, we establish the Lyapunov-type inequalities for the fractional boundary value problems with Hilfer derivative for different boundary conditions. We apply this inequality to fractional eigenvalue problems and prove one of the important results of real zeros of certain Mittag-Leffler functions and improve the bound of the eigenvalue using the Cauchy-Schwarz inequality and Semi-maximum norm. We extend it for higher order cases.

DEDICATION

I dedicate this dissertation to my parents, Mr. Mahadev and Mrs. Rama Raval, and my parents-in-law, Mr. Arvind and Mrs. Indira Pathak. They have supported me throughout my life, and have helped me find the inspiration and determination to overcome the tough challenges I have faced.

ACKNOWLEDGMENTS

My sincere thanks to Dr. Om P. Agrawal for persevering with me as my advisor, and providing invaluable guidance and insights leading to the writing of this dissertation.

The members of my dissertation committee, Dr. Michael Sullivan, Dr. Kathleen Pericak-Spector, Dr. Rasit Koc, Dr. Jerzy Kocik, Dr. Randy Hughes, have dedicated time and expertise to improve my work. I thank them for their patience and support.

I would also like to thank everyone in the Mathematics department at Southern Illinois University. They have been very supportive of my ambitions, and for this I will be forever grateful. I thank Professor Rasit Koc and everyone in the Mechanical Engineering department for their continued support for my research work.

I am very much thankful to Professor Salah Mohammed for making valuable remarks throughout my research. I must also acknowledge my co-researchers Yufeng and Rajesh for pointing out and discussing interesting results.

I am grateful to the Department of Mathematics at Maharaja Sayajirao University, India, for providing this opportunity to pursue my Ph.D. at Southern Illinois University. I appreciate the continued help and support of my colleagues Nilima and Snehal.

I am especially thankful to my uncle Mr. Mahendra Raval, and my brothers Chandrahas, Samir and Amit for providing motivation and support throughout my pursuit.

Finally, love and thanks to my husband Shailesh, and our children, Shivani and Nishil for their continuous faith in me. They bring out the best in me, and without them this dissertation would not have been a reality.

TABLE OF CONTENTS

Abstract ii

Dedication iii

Acknowledgments iv

List of Tables viii

List of Figures ix

Introduction 1

1 Preliminaries 5

1.1 Spaces of Integrable, Absolutely Continuous, And Continuous Functions . . 5

1.2 Special Functions 7

1.2.1 Gamma Function 7

1.2.2 Beta Function 8

1.2.3 Hypergeometric Function 8

1.2.4 Mittag-Leffler Function 8

1.3 Fractional Integrals, Fractional Derivatives and their properties 10

1.3.1 Fractional Integral 10

1.3.2 Riemann-Liouville Fractional Derivative 12

1.3.3 Caputo Fractional Derivative 14

1.3.4 Hilfer Fractional Derivative 14

1.3.5 Lyapunov Inequality 16

2 Fractional Boundary Value and Eigenvalue Problems 18

2.1 General Fractional Boundary value Problem 18

2.2 General Fractional Eigenvalue Problem 20

3 Lyapunov-type Inequality and Eigenvalue Estimates for Fractional Problems of order α, $\alpha \in (1, 2]$ 23

3.1 Lyapunov-type Inequality For FBVP with the Dirichlet boundary conditions 23

3.1.1 Eigenvalue Problem with the Dirichlet boundary conditions and Eigenvalue Estimates 26
3.2 Lyapunov-type inequality for FBVP with a mixed set of Dirichlet and Neumann boundary conditions 32
3.2.1 Eigenvalue Problem with a mixed set of Dirichlet and Neumann boundary conditions and Eigenvalue Estimates 34
4 Lyapunov-type Inequality and Eigenvalue Estimates for Fractional Problems of order α, $\alpha \in (2, 3]$ 40
4.1 Lyapunov-type Inequality For FBVP With a Mixed Set of Dirichlet and Neumann Boundary Conditions 40
4.1.1 Eigenvalue Estimate For Fractional Eigenvalue Problem With a Mixed Set of Dirichlet and Neumann Boundary Conditions 43
4.2 Lyapunov-type Inequality For FBVP With a Mixed Set of fractional Dirichlet, Neumann and fractional Neumann Boundary Conditions 49
4.2.1 Eigenvalue Estimate For Fractional Eigenvalue Problem With a Mixed Set of fractional Dirichlet, Neumann and fractional Neumann Boundary Conditions 50
5 Lyapunov-type Inequality and Eigenvalue Estimates for Fractional Problems of order α, $\alpha \in (3, 4]$ 55
5.1 Fractional boundary and eigenvalue problems with first boundary conditions of B3 56
5.1.1 Lyapunov-type Inequality For Fractional Boundary Value Problem with first Boundary conditions of B3 56
5.1.2 Eigenvalue Problem with first boundary conditions of B3 and Eigenvalue Estimates 59
5.2 Fractional boundary and eigenvalue problems with second boundary conditions of B3 65

5.2.1 Lyapunov-type Inequality For Fractional Boundary Value Problem with second Boundary conditions of B3 65
5.2.2 Eigenvalue Problem with second Boundary conditions of B3 and Eigenvalue Estimates 67
5.3 Fractional boundary and eigenvalue problems with third boundary conditions of B3 72
5.3.1 Lyapunov-type Inequality For Fractional Boundary Value Problem with third Boundary conditions of B3 73
5.3.2 Eigenvalue Problem with third Boundary conditions of B3 and Eigenvalue Estimates 74
References 79
Appendix 83
A-1 Volterra Integral Equations 84
A-2 Fubini's Theorem 84
A-3 Some Special Cases of Mittag-Leffler Function $E_{\alpha,\beta}(z)$ 87
A-4 Semigroup Property of $\mathcal{K}_P^{\alpha}$-Operator 87
Vita 91

LIST OF TABLES

3.1 Results for $\alpha \in (1, 2]$ and $\beta = 0$ (FBVP P4 and FEP P5 with Riemann-Liouville derivative) 28

3.2 Results for $\alpha \in (1, 2]$ and $\beta = 1$ (FBVP P4 and FEP P5 with Caputo derivative) 29

3.3 Results for $\alpha \in (1, 2]$ and $\beta = 0$ (FBVP P6 and FEP P7 with Riemann-Liouville derivative) 37

3.4 Results for $\alpha \in (1, 2]$ and $\beta = 1$ (FBVP P6 and FEP P7 with Caputo derivative) 39

LIST OF FIGURES

3.1 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 0$, Riemann-Liouville derivative FEP P5) 30

3.2 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 1$, Caputo derivative FEP P5) 31

3.3 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 0$, Riemann-Liouville derivative FEP P7) 38

3.4 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 1$, Caputo derivative FEP P7) 38

4.1 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ)($\beta = 0$, Riemann-Liouville derivative FEP P9) 46

4.2 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($\beta = 0.5$, mixed behavior of Riemann-Liouville and Caputo derivative FEP P9) 47

4.3 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ)($\beta = 1$, Caputo derivative FEP P9) 47
4.4 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ)($\beta = 0$, Riemann-Liouville derivative FEP P11) 53
4.5 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ)($\beta = 0.5$, mixed behavior of Riemann-Liouville and Caputo derivative FEP P11) 53
4.6 Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ)($\beta = 1$, Caputo derivative FEP P11) 54
5.1 Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($\beta = 0$, Riemann-Liouville derivative FEP P14) 63
5.2 Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($\beta = 1$, Caputo derivative FEP P14) 64

5.3 Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($\beta = 1$, Caputo derivative FEP P17) 71
5.4 Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($\beta = 0$, Riemann-Liouville derivative FEP P19) 77
5.5 Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($\beta = 1$, Caputo derivative FEP P19) 78

INTRODUCTION

The theory of fractional derivatives goes back to Leibniz's note in his letter to L'Hôspital, dated 30 September 1695, in which the meaning of a one-half ordered derivative is discussed. Leibniz's note led to the appearance of the theory of derivatives and integrals of arbitrary order, which by the end of nineteenth century took more or less finished form due primarily to Liouville, Grünwald, Letnikov, Riemann and Caputo. Recently, there have been several books on the subject of fractional derivatives and fractional integrals, see [30], [36], [39], [44], [49]. More recently, a remarkably large family of generalized RiemannLiouville fractional derivative of order α $(0 < \alpha < 1)$ and type β $(0 \leq \beta \leq 1)$ was introduced [23], [25]. Which is written in the more general form as the generalized Riemann-Liouville fractional derivative (GRLFD) or Hilfer fractional derivative (HFD) of order α $(n-1 < \alpha \leq n \in \mathbb{N}$ and type β $(0 \leq \beta \leq 1)$ [24], [51], [52].

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in selfsimilar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science.

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear fractional differential equations. A remarkable research is done on the Lyapunov-type inequality (LTI) for integer order boundary value problems (see [7], [10], [21], [22], [41], [47], [50], [54], [55], [56] and the references therein). The Lyapunov inequality [34] has proved to be very useful in the study of spectral properties of ordinary differential equations (see [7], [41]). This inequality can be stated as follows:

Theorem 0.0.1. *(See [34]) A necessary condition for the Boundary Value Problem (BVP) BVP 1*

$$y''(t) + q(t)y(t) = 0, \quad a < t < b,$$
$$y(a) = 0, \quad y(b) = 0,$$

to have nontrivial solutions is that

$$\int_a^b |q(s)|ds > \frac{4}{b-a},$$

where q is a real and continuous function. The constant 4 in the above inequality is sharp so that it cannot be replaced by a larger number.

There are several generalizations and extensions of Theorem 0.0.1. Hartman and Wintner [19] proved that if u is a nontrivial solution to BVP 1, then

$$\int_a^b (b-s)(s-a)q^+(s)ds > b-a,$$

where $q^+(s)$ is the positive part of q defined as

$$q^+(s) = \max\{q(s), 0\}.$$

We call the above inequality as Hartman and Wintner inequality. For other generalizations and extensions of the classical Lyapunov's inequality, we refer to [3], [8], [9], [10], [11], [14], [19], [21], [32], [41], [42], [55] and the references therein. Recently, there are some papers dealing with the Lyapunov-type inequality of Fractional Boundary Value Problems (FBVPs) have appeared. Ferreira in [15] and [16], Jleli and Samet [26], [27], [28], and Rong and Bai [48] have established Lyapunov-type inequalities (LTIs) for FBVPs of order α, $\alpha \in (1, 2]$ and different boundary conditions. In [43], we obtained the LTI for FBVP of order $2 < \alpha \leq 3$. We also improved the lower bound of the smallest eigenvalue of the eigenvalue problem using the semi-maximum norm and Cauchy-Schwarz inequalities. In these work the authors considered the FBVPs with either Riemann-Liouville or Caputo derivatives. Motivated by the above work, in this work we consider FBVPs involving a Hilfer derivative operator.

The aim in writing this paper is to establish the Lyapunov-type inequality for fractional boundary value problems with Hilfer derivative operator $D_{a^+}^{\alpha,\beta}$ of order α, $\alpha \in (1,2]$, $\alpha \in (2,3]$ and $\alpha \in (3,4]$, and type $\beta \in [0,1]$. The advantage of considering the FBVP and fractional eigenvalue problem (FEP) with the Hilfer derivative is that the obtained results allow us to give results for Riemann-Liouville as well as Caputo derivative FBVPs and FEPs as its particular cases. Here possible, basic ideas are studied by using the equivalent integral equation form of the fractional boundary value problems and the properties of corresponding Green's function. We consider both integer and fractional order eigenvalue problems, determine a lower bound for the smallest eigenvalue using a Lyapunov-type inequality, and improve this bound using a semi maximum norm and Cauchy-Schwarz inequality. We use the improved lower bounds to obtain intervals where certain Mittag-Leffler functions have no real zeros. Further, for both the fractional and the integer order eigenvalue problems, we give a comparison between the smallest eigenvalue and its lower bounds obtained from the semi maximum norm and Lyapunov-type and Cauchy-Schwarz inequalities. Results show that the Lyapunov-type inequality gives the worse and the Cauchy-Schwarz inequality gives the best lower bound estimates for the smallest eigenvalues.

It is necessary to note that many authors have been devoted to studying zeros of Mittag-Leffler function (see [38] and references therein), and the basic works in this direction are due to A. Veeman and M.M. Dzhrbashjan (see [4], [13]). In [5] the authors carried out spectral analysis of one class of integral operators associated with fractional order differential equations which arise in mechanics by establishing a connection between the eigenvalues of these operators and the zeros of Mittag-Leffler type functions. This may become the extension of our work in future.

The outline of the thesis is as follows.

Chapter 1 deals with preliminary materials; definitions and lemmas necessary for the derivations in this work.

In chapter 2, we use the basic results from chapter 1 and explain the procedure to establish a Lyapunov-type inequality of general fractional boundary value problem. Also, give three methods to obtain the lower bound estimate of the smallest eigenvalue of the general fractional eigenvalue problem.

Chapter 3 contains the Lyapunov-type inequality and eigenvalue estimate for fractional problems of order α, $\alpha \in (1, 2]$.

Chapter 4 includes the Lyapunov-type inequalities and eigenvalue estimates for fractional problems of order α, $\alpha \in (2, 3]$ with a mixed set of Dirichlet and Neumann, and a mixed set of fractional Dirichlet, Neumann and fractional Neumann boundary conditions.

In chapter 5, we discuss about the Lyapunov-type inequalities and eigenvalue estimates for fractional problems of order α, $\alpha \in (3, 4]$ with different boundary conditions.

CHAPTER 1

PRELIMINARIES

In this chapter we briefly review the definitions of fractional operators, properties from such topics of Analysis as functional spaces, special functions, Laplace transforms and some preliminary materials.

1.1 SPACES OF INTEGRABLE, ABSOLUTELY CONTINUOUS, AND CONTINUOUS FUNCTIONS

Here we present definitions of spaces of p-integrable, Lebesgue integrable, absolutely continuous, and continuous functions. Most of the results stated here are well known and can be found in any standard textbook, for example [30], [31], [49].

Definition. Consider the space $X_c^p(a,b)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $||f||_{X_c^p} < \infty$, where the norm is defined by

$$||f||_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad c \in \mathbb{R}, \quad 1 \leq p < \infty,$$

and for the case $p = \infty$

$$||f||_{X_c^\infty} = ess\ sup_{a \leq x \leq b}[x^c |f(x)|].$$

In particular, when $c = 1/p$, the space X_c^p coincides with the classical $\mathbf{L}^p(a,b)$-space with

$$||f||_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{1.1}$$

$$||f||_\infty = ess\ sup_{a \leq x \leq b}|f(x)|. \tag{1.2}$$

The case $p = 1$ describes the space of Lebesque integrable functions.

Definition. The space $L(a,b)$ of Lebesgue measurable functions $f(t)$ on a finite interval $[a,b](b > a)$ of the real line $\mathbb{R}$ is defined as

$$\mathbf{L}(a,b) = \{f : ||f||_1 = \int_a^b |f(t)|dt < \infty\}.$$

For norm in (1.1) we shall also use the notations

$$||f||_p = ||f||_{L^p} = ||f||_{\mathbf{L}^p(a,b)}. \tag{1.3}$$

Let us give some properties of $\mathbf{L}^p$-spaces:

1. The Minkowsky inequality

$$||f + g||_p \leq ||f||_p + ||g||_p \ , \tag{1.4}$$

   so that $\mathbf{L}^p(a,b)$ is a normed space. It is also known that $\mathbf{L}^p(a,b)$ is a complete space.

2. The Hölder inequality

$$\int_a^b |f(x)g(x)|dx \leq ||f||_p ||g||_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.5}$$

   where $f(x) \in \mathbf{L}^p(a,b)$, $g(x) \in \mathbf{L}^q(a,b)$. In particular, if $p = q = 2$ describes the Cauchy-Schwarz inequality

$$\int_a^b |f(x)g(x)|dx \leq ||f||_2 ||g||_2. \tag{1.6}$$

Definition. Let $[a,b]$ be a finite interval and let $AC[a,b]$ be the space of functions f which are absolutely continuous on $[a,b]$. It is known that (see [31], p. 338) $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \phi(t)dt, \quad \phi(t) \in \mathbf{L}(a,b), \tag{1.7}$$

and therefore an absolutely continuous function $f(x)$ has a summable derivative $f'(x) = \phi(x)$ almost everywhere on $[a,b]$. Thus (1.7) yields

$$\phi(t) = f'(t), \ c = f(a). \tag{1.8}$$

Definition. Let $AC[a,b]$ be the space of real-valued functions $f(t)$ which are absolutely continuous on $[a,b]$. We denote by $AC^n[a,b]$ the space of real-valued functions $f(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(t) \in AC^n[a,b]$:

$$AC^n[a,b] = \left\{ f : [a,b] \to \mathbb{R} : (D^{n-1}f)(t) \in AC[a,b]; \ D \equiv \frac{d}{dt} \right\}, \quad n \in \mathbb{N}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

This space is characterized by the following assertion [49].

Lemma 1.1.1. *The space $AC^n[a,b]$ consists of those and only those functions $f(x)$ which can be represented in the form*

$$f(x) = (I^n_{a+}\phi)(x) + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{1.9}$$

where $\phi(t) \in \mathbf{L}(a,b)$, $c_k(k = 0, 1 \cdots, n-1)$ are arbitrary constants, and

$$(I^n_{a+}\phi)(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\phi(t)dt. \tag{1.10}$$

It follows from (1.9) that

$$\phi(t) = f^n(t),\ c_k = \frac{f^k(a)}{k!}.$$

Proof. The proof can be found at Lemma 2.4 in Samko et al [49]. □

1.2 SPECIAL FUNCTIONS

In this section we give definitions and basic properties of those functions which are relevant in the theory of Fractional Calculus. These include the Gamma function, the Beta function, the Hypergeometric function and the Mittag-Leffler function.

1.2.1 Gamma Function

The Euler's Gamma function $\Gamma(z)$ which generalizes the factorial $n!$ and allows n to take also non-integer and complex values [44].

Definition. The gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \tag{1.11}$$

which converges in the right half of the complex plane $\mathbf{R}(z) > 0$.

The Gamma function has one of the basic properties given by

$$\Gamma(z+1) = z\Gamma(z), \quad \mathbf{R}(z) > 0. \tag{1.12}$$

This recursion relationship can be used to define the factorial operation for integral values of z, because $\Gamma(1) = 1$. Thus, we can write

$$\Gamma(n+1) = n\Gamma(n) = n!\ , \tag{1.13}$$

when n is an integer.

1.2.2 Beta Function

Definition. The beta function is defined by the Euler integral of the first kind: [30]

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \quad \mathbf{R}(z) > 0, \mathbf{R}(w) > 0. \tag{1.14}$$

This function is connected with the gamma functions by the relation

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(G+w)}. \tag{1.15}$$

1.2.3 Hypergeometric Function

Definition. The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined in the unit disk as the sum of the hypergeometric series [49]

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{1.16}$$

where $|z| < 1$; a, b, c and the variable z may be complex ($c \neq 0, -1, -2, \cdots$) and $(a)_k$ is the Pochhammer symbol given by $(a)_k = a(a+1)\cdots(a+k-1),\ k = 1, 2, \cdots, \quad (a)_0 \equiv 1.$

1.2.4 Mittag-Leffler Function

The MittagLeffler function, which is generalization of exponential function, plays an important role in the theory of fractional differential equations and is connected with gamma function.

Definition. The function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z, \alpha \in \mathbb{C}, \mathbf{R}(\alpha) > 0, \tag{1.17}$$

was introduced by Mittag-Leffler [37] and is known as one parameter Mittag-Leffler function. It is an entire function of z. The basic properties of this function are as follows [20], [30]: When $\alpha = 1$ and $\alpha = 2$, we have

$$E_1(z) = e^z, \quad E_2(z^2) = \cosh(z) \text{ and } E_2(-z^2) = \cos(z). \tag{1.18}$$

Definition. A two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta, z \in \mathbf{C}, \quad \mathbf{R}(\alpha), \mathbf{R}(\beta) > 0. \tag{1.19}$$

In particular, when $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function

$$E_{\alpha,1}(z) = E_\alpha(z). \tag{1.20}$$

It follows from (1.19) that

$$E_{1,2}(z) = \frac{e^z - 1}{z} \quad \text{and} \quad E_{2,2}(z^2) = \frac{\sinh(z)}{z}. \tag{1.21}$$

We obtain some other special cases which are discussed in section A-3 (see appendix). The following differentiation formula is satisfied by (1.19) [30], [20]:

$$\frac{d^m}{dz^m}[z^{\beta-1}E_{\alpha,\beta}(\pm\lambda z^\alpha)] = z^{\beta-m-1}E_{\alpha,\beta-m}(\pm\lambda z^\alpha), \quad \lambda \in \mathbb{C}, \mathbf{R}(\beta - m) > 0, m \in \mathbf{N}. \tag{1.22}$$

In particular, when $m = 1$ and $\beta = 1$ the relationship [35]

$$\frac{d}{dz}[E_{\alpha,1}(-\lambda z^\alpha)] = -\lambda z^{\alpha-1}E_{\alpha,\alpha}(-\lambda z^\alpha) \tag{1.23}$$

holds. The function $E_{\alpha,\beta}(z)$ has the integral representation

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi} \int_C \frac{t^{\alpha-\beta}e^t}{t^\alpha - z} dt, \tag{1.24}$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disk $|t| \leq |z|^{1/\alpha}$ in the positive sense: $|\arg(t)| \leq \pi$ on C.

Definition. The Laplace transform of the function $f(z), z \in (0, \infty)$ is defined by

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-sz} f(z) dz, \quad s \in \mathbb{C}.$$

If the above integral is convergent at a point $s_0 \in \mathbb{C}$, then it converges absolutely for $s \in \mathbb{C}$, $\mathbf{R}(s) > \mathbf{R}(s_0)$.

Using the above definition, the Laplace transform of the function $\phi(z) = z^{\beta-1} E_{\alpha,\beta}(\pm\lambda z^\alpha)$ is given as

$$(\mathcal{L}\phi)(s) = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}, \quad \mathbf{R}(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1,$$

and its inverse relationship is given as

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}\right] = z^{\beta-1} E_{\alpha,\beta}(\pm\lambda z^\alpha), \quad \mathbf{R}(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1, \tag{1.25}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform operator.

1.3 FRACTIONAL INTEGRALS, FRACTIONAL DERIVATIVES AND THEIR PROPERTIES

Considerable work has been done on fractional calculus in recent years. Several definitions of the fractional integrals and derivatives have been proposed. These include the Riemann-Liouville, Caputo, Grünwald-Letnikov, Weyl, Marchaud, Miller-Ross, Riesz and Hilfer fractional derivatives (see [23], [30], [36], [39], [44], [49] for details). Much of the tools from fractional calculus necessary for this work could be found, among others, in [30], [36], [39], [44], [49], [51] and [52].

1.3.1 Fractional Integral

For an n-fold integral there is a well known formula

$$\int_a^x dx \int_a^x dx \cdots \int_a^x \phi(x) dx = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \phi(t) dt. \tag{1.26}$$

Since $(n-1)! = \Gamma(n)$, we observe that the right-hand side of (1.26) may have a meaning for non-integer values of n. So, it is natural to define integration of a non-integer order as follows.

Definition. Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order $\alpha > 0$ are defined by

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{1.27}$$

and

$$(I^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \tag{1.28}$$

respectively. They are also called left-sided and right-sided Riemann-Liouville fractional integrals respectively [30], [49]. Fractional integrals (1.27) and (1.28) are defined for functions $f(x) \in \mathbf{L}(a, b)$, existing almost everywhere.

The following result yields the boundedness of the fractional integration operators $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ from the space $\mathbf{L}^p(a, b)(1 \leq p \leq \infty)$ with the norm$||f||_p$ defined in (1.1).

Lemma 1.3.1. *The fractional integration operators I^{α}_{a+} and I^{α}_{b-} with $\alpha > 0$ are bounded in $\mathbf{L}^p(a, b), 1 \leq p \leq \infty$:*

$$||I^{\alpha}_{a+}f||_p \leq K||f||_p, \quad ||I^{\alpha}_{b-}f||_p \leq K||f||_p, \quad \Big(K = \frac{(b-a)^{\alpha}}{\alpha|\Gamma(\alpha)|}\Big). \tag{1.29}$$

Proof. The proof can be found in Samko et al [49]. □

We note a simple relation for the "reflection operator" Q: $(Q\phi)(x) = \phi(a+b-x)$

$$QI^{\alpha}_{a+} = I^{\alpha}_{b-}Q, \quad QI^{\alpha}_{b-} = I^{\alpha}_{a+}Q. \tag{1.30}$$

The fractional integration by parts formula

$$\int_a^b \phi(x)(I^{\alpha}_{a+}\psi)(x)dx = \int_a^b \psi(x)(I^{\alpha}_{b-})(x)dx \tag{1.31}$$

is valid. It can be proved directly by interchanging the order of integration by Dirichlet formula (A-2) in the left-hand side of (1.31). Formula (1.31) is true if

$$\phi(x) \in \mathbf{L}^p,\ \psi(x) \in \mathbf{L}^q, \quad \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha,\ p \geq 1,\ q \geq 1.$$

Fractional integration has the following semi-group property:

If $\alpha > 0$ and $\beta > 0$, then the equations

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} \phi = I^{\alpha+\beta}_{a^+} \phi \quad \text{and} \quad I^{\alpha}_{b^-} I^{\beta}_{b^-} \phi = I^{\alpha+\beta}_{b^-} \phi \tag{1.32}$$

are satisfied in any point for $\phi(t) \in \mathbf{C}[a, b]$ and almost every point for $\phi(t) \in \mathbf{L}(a, b)$. They are true in any point even for $\phi(t) \in \mathbf{L}(a, b)$ if $\alpha + \beta \geq 1$. The proof of (1.32) is direct

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} \phi = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} dt \int_a^t (t-s)^{\beta-1} \phi(s) ds$$

and interchanging the order of integration by Fubini's theorem (A-2.1) and setting $t = s + \tau(x - s)$, we have

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} \phi = \frac{B(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-s)^{\alpha+\beta-1} \phi(s) ds,$$

which gives (1.32).

It can be directly verified that the the fractional integral of the power function $\phi(x) = (x - a)^{\beta-1}$, $\mathbf{R}(\beta) > 0$, yields power function of the same form. It is given by [49]

$$I^{\alpha}_{a^+} \phi = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}, \quad \alpha > 0. \tag{1.33}$$

We obtain the semi-group properties of generalized $\mathcal{K}^{\alpha}_{P}$-operator defined in [1]. These are discussed in section A-4 (see appendix). We note that the work on generalized fractional operators is done in [2], [29] and [43]. For fractional differentiation, it is natural to introduce it as an operation inverse to fractional integration.

1.3.2 Riemann-Liouville Fractional Derivative

We define the Riemann-Liouville fractional derivatives according to Bai [6], Kilbas et al [30], I. Podlubny [44] and Samko et al [49].

Definition. The Riemann-Liouville fractional derivative operators $D^{\alpha}_{a^+}f$ and $D^{\alpha}_{b^-}f$ of order $\alpha > 0$ of a continuous function $f : (a, \infty) \to \mathbb{R}$ are defined by

$$(D^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dx}\Big)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \tag{1.34}$$

and

$$(D^{\alpha}_{b^-}f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\Big(\frac{d}{dx}\Big)^n \int_x^b (t-x)^{n-\alpha-1} f(t)dt, \tag{1.35}$$

where $n = [\alpha] + 1$, provided that the right sides are pointwise defined on (a, ∞).

The composition of the fractional integration operator $I^{\alpha}_{a^+}$ with the fractional differentiation operator $D^{\alpha}_{a^+}$ is given by the following result.

Lemma 1.3.2. *Let $\alpha > 0$ and $n = [\alpha] + 1$.*
(a) If $1 \le p \le \infty$ and $f(x) \in I^{\alpha}_{a^+}(\mathbf{L}^p)$, then

$$(I^{\alpha}_{a^+}D^{\alpha}_{a^+}f)(x) = f(x). \tag{1.36}$$

(b) If $f(x) \in \mathbf{L}(a, b)$ and $(I^{n-\alpha}_{a^+}f)(x) \in AC^k[a, b]$, $0 \le k \le n-1$ then the equality [52]

$$(I^{\alpha}_{a^+}D^{\alpha}_{a^+}f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{d^k}{dx^k}(I^{n-\alpha}_{a^+}f)(a^+)\frac{(x-a)^{\alpha-n+k}}{\Gamma(\alpha-n+k+1)}, \tag{1.37}$$

holds almost everywhere on $[a, b]$.
(c) The equality

$$(D^{\alpha}_{a^+}I^{\alpha}_{a^+}f)(x) = f(x) \tag{1.38}$$

is valid for any summable function $f(x)$.

Proof. The proof is given in Samko et al. [49], (Theorem 2.4). □

It can be directly verified that the the Riemann-Liouville derivative of the power function $\phi(x) = (x-a)^{\beta-1}$, $\mathbf{R}(\beta) > 0$ yields power function of the same form. It is given by [49]

$$D^{\alpha}_{a^+}\phi = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \quad \alpha > 0. \tag{1.39}$$

1.3.3 Caputo Fractional Derivative

Next we present the definitions and some properties of the Caputo fractional derivatives.

Definition. Let $\alpha \geq 0$. If $f(x) \in AC^n[a,b]$ then the left-sided and right-sided Caputo fractional derivatives $({}^C D^\alpha_{a^+} f)(x)$ and $({}^C D^\alpha_{b^-} f)(x)$ exist almost everywhere on $[a,b]$, and are represented by

$$({}^C D^\alpha_{a^+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t)dt, \tag{1.40}$$

and

$$({}^C D^\alpha_{b^-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^n(t)dt, \tag{1.41}$$

respectively. Where $n = [\alpha] + 1$.

The following inverse property for Caputo fractional derivative is valid.

Lemma 1.3.3. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f(x) \in AC^n[a,b]$ then*

$$(I^\alpha_{a^+} {}^C D^\alpha_{a^+} f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^k(a)}{k!}(x-a)^k. \tag{1.42}$$

Proof. This is Lemma 2.22 of Kilbas et al. [30]. □

Similar result exists for right-sided Caputo fractional derivative as well.

1.3.4 Hilfer Fractional Derivative

In [23], [24] an infinite family of fractional Riemann-Liouville derivatives having the same order were introduced as follows.

Definition. The Hilfer Fractional Derivative (HFD) or generalized Riemann-Liouville fractional derivative (GRLFD) of order $0 < \alpha < 1$, and type $0 \leq \beta \leq 1$ with respect to t, is defined as

$$\left(D^{\alpha,\beta}_{a^+} y\right)(t) = \left(I^{\beta(1-\alpha)}_{a^+} \frac{d}{dt}\left(I^{(1-\beta)(1-\alpha)}_{a^+} y\right)\right)(t), \tag{1.43}$$

whenever the right-hand side exists.

In Hilfer et al. [25], this definition for $n-1<\alpha\leq n, n\in\mathbb{N}$ and $0\leq\beta\leq 1$, is rewritten in a more general form:

$$\left(D_{a^+}^{\alpha,\beta}y\right)(t)=\left(I_{a^+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}\left(I_{a^+}^{(1-\beta)(n-\alpha)}y\right)\right)(t)=\left(I_{a^+}^{\beta(n-\alpha)}D_{a^+}^{\beta+\beta n-\alpha\beta}y\right)(t). \tag{1.44}$$

In the above definition, type β allows $D_{a^+}^{\alpha,\beta}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative. As in the case $\beta=0$, equation (1.44) reduces to the classical Riemann-Liouville fractional derivative (1.34) and for $\beta=1$, it gives the Caputo fractional derivative (1.40).

The difference between fractional derivatives of different types becomes apparent from Laplace transformation. The Laplace transform formula of (1.44) is defined as follows [51], [52]:

For $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and $0\leq\beta\leq 1$, the Laplace transform formula

$$\mathcal{L}\{D_{0^+}^{\alpha,\beta}y(t);s\}=s^{\alpha}Y(s)-\sum_{k=0}^{n-1}s^{n-k-1-\beta(n-\alpha)}\frac{d^k}{dt^k}\left(I_{0^+}^{(1-\beta)(n-\alpha)}y\right)(0^+), \tag{1.45}$$

is valid. In [52], the compositional property of Riemann-Liouville fractional integral operator with the HFD operator is obtained.

Lemma 1.3.4. *[52] Let $y\in\mathbf{L}(a,b)$, $n-1<\alpha\leq n, n\in\mathbb{N}$, $0\leq\beta\leq 1$, $I_{a^+}^{(n-\alpha)(1-\beta)}y\in AC^k[a,b]$. Then the Riemann-Liouville fractional integral $I_{a^+}^{\alpha}$ and the HFD operator $D_{a^+}^{\alpha,\beta}$ are connected by the relation*

$$\left(I_{a^+}^{\alpha}D_{a^+}^{\alpha,\beta}y\right)(t)=y(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)}\lim_{t\to a^+}\frac{d^k}{dt^k}\left(I_{a^+}^{(n-\alpha)(1-\beta)}y\right)(t). \tag{1.46}$$

Proof. Using the representation (1.44) and applying the compositional properties (1.32) and (1.37) we get

$$\begin{aligned}\left(I_{a^+}^{\alpha}D_{a^+}^{\alpha,\beta}y\right)(t) &= \left(I_{a^+}^{\alpha}I_{a^+}^{\beta(n-\alpha)}D_{a^+}^{\alpha+\beta n-\alpha\beta}y\right)(t)=\left(I_{a^+}^{\alpha+\beta(n-\alpha)}D_{a^+}^{\alpha+\beta(n-\alpha)}y\right)(t)\\ &= y(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)}\lim_{t\to a^+}\frac{d^k}{dt^k}\left(I_{a^+}^{(n-\alpha)(1-\beta)}y\right)(t).\end{aligned}$$

□

We obtain a result for the Hilfer derivative of the power function, which is given in the following Lemma.

Lemma 1.3.5. *The following result holds true for the fractional derivative operator* $D_{a^+}^{\alpha,\beta}$ *defined by (1.44):*

$$D_{a^+}^{\alpha,\beta}\left[(x-a)^{\mu-1}\right](t) = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}(t-a)^{\mu-\alpha-1}, \quad (t > a; \alpha > 0; 0 \leq \beta \leq 1; \mathbf{R}(\mu) > 0). \tag{1.47}$$

Proof. We observe from equations (1.39) and (1.33) that

$$D_{a^+}^{\alpha+n\beta-\alpha\beta}\left[(x-a)^{\mu-1}\right](t) = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha-n\beta+\alpha\beta)}(t-a)^{\mu-\alpha-n\beta+\alpha\beta-1},$$

and

$$I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\alpha+n\beta-\alpha\beta}\left[(x-a)^{\mu-1}\right](t) = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha-n\beta+\alpha\beta)} \cdot \frac{\Gamma(\mu-\alpha-n\beta+\alpha\beta)}{\Gamma(\mu-\alpha)}(t-a)^{\mu-\alpha-1},$$

which, in light of the definition (1.44) yield

$$\begin{aligned} D_{a^+}^{\alpha,\beta}\left[(x-a)^{\mu-1}\right](t) &= I_{a^+}^{\beta(n-\alpha)} D_{a^+}^{\alpha+n\beta-\alpha\beta}\left[(x-a)^{\mu-1}\right](t) \\ &= \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}(t-a)^{\mu-\alpha-1}. \end{aligned}$$

just as in the assertion (1.47) of the Lemma. □

1.3.5 Lyapunov Inequality

In this work we establish the Lyapunov-type inequality for the fractional boundary value problems. The Lyapunov inequality is established for the integer order problem. We restate the Theorem 0.0.1 discussed in introduction to list in this chapter as one of the necessary preliminaries for this work. It is stated in the following result.

Theorem 1.3.6. *(See [34]) A necessary condition for the Boundary Value Problem (BVP) Problem P1:*

$$\begin{aligned} y''(t) + q(t)y(t) = 0, &\quad a < t < b, \\ y(a) = 0, &\quad y(b) = 0, \end{aligned} \tag{1.48}$$

to have nontrivial solutions is that

$$\int_a^b |q(s)| ds > \frac{4}{b-a}, \tag{1.49}$$

where q is a real and continuous function. The constant 4 in equation (1.49) is sharp so that it cannot be replaced by a larger number.

CHAPTER 2

FRACTIONAL BOUNDARY VALUE AND EIGENVALUE PROBLEMS

In this chapter we consider the general fractional boundary value problems (FBVPs) and fractional eigenvalue problems (FEVPs). We also discuss three methods for eigenvalue estimate.

2.1 GENERAL FRACTIONAL BOUNDARY VALUE PROBLEM

Proposition 2.1.1. *Let $n-1<\alpha\leq n, n\in\mathbb{N}$ and $\beta\in[0,1]$. We consider the FBVP*

Problem P2:

$$\left(D_{a^+}^{\alpha,\beta}y\right)(t)+q(t)y(t)=0,\quad a<t<b, \tag{2.1}$$

where q is a real valued continuous function in interval $[a,b]$ and boundary conditions are:

Boundary conditions B1:

$$\begin{aligned} a_1y(a)+a_2\left(DI_{a^+}^{(2-\alpha)(1-\beta)}y\right)(a) &= 0;\quad D\equiv\frac{d}{dt},\\ b_1y(b)+b_2Dy(b) &= 0, \end{aligned} \tag{2.2}$$

with $a_1^2+a_2^2\neq 0, b_1^2+b_2^2\neq 0$. OR

Boundary conditions B2:

$$\begin{aligned} d_1y(a)+d_2\left(I_{a^+}^{(3-\alpha)(1-\beta)}y\right)(a) &= 0\\ d_3y'(a)+d_4\left(D^2I_{a^+}^{(3-\alpha)(1-\beta)}y\right)(a) &= 0,\\ e_1y(b)+e_2y'(b) &= 0, \end{aligned} \tag{2.3}$$

with $d_1^2+d_2^2\neq 0, d_3^2+d_4^2\neq 0, e_1^2+e_2^2\neq 0$. OR

Boundary conditions B3:

$$y^i(a)=y^i(b)=0,\quad i=0,1 \tag{2.4}$$

or

$$y^i(a)=y^i(b)=0,\quad i=0,2 \tag{2.5}$$

or

$$y^i(a) = y''(b) = 0, \quad i = 0, 1, 2. \tag{2.6}$$

Then the FBVP (2.1) can be written in its equivalent integral form as

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{2.7}$$

where $G(t,s)$ is a Green's function. Green's function depends on the BVPs which will be addressed latter in the chapters.

From (2.7), it follows that if y is a nontrivial continuous solution of the FBVPs (2.1)-(2.2) or (2.1) and (2.3) or (2.1) and (2.4) or (2.1) and (2.5) or (2.1) and (2.6) then

$$|y(t)| \leq \int_a^b |G(t,s)q(s)||y(s)|ds. \tag{2.8}$$

Let $B = \mathbf{C}[a,b]$ be a Banach space endowed a norm

$$||y||_\infty = \max_{a \leq t \leq b} |y(t)|, \quad y \in B. \tag{2.9}$$

Hence, from (2.8) we get

$$||y||_\infty \leq \max_{a \leq t \leq b} \int_a^b |G(t,s)q(s)|ds||y||_\infty,$$

or equivalently,

$$1 \leq \max_{a \leq t \leq b} \int_a^b |G(t,s)q(s)|ds. \tag{2.10}$$

Using the properties of Green's function $G(t,s)$ particularly, $\max_{a \leq t,s \leq b} |G(t,s)| = G_{\max}$ in (2.10) gives the inequality

$$\int_a^b |q(s)|ds \geq \frac{1}{G_{\max}}, \tag{2.11}$$

called the Lyapunov-type inequality for FBVPs (2.1)-(2.2) or (2.1) and (2.3) or (2.1) and (2.4) or (2.1) and (2.5) or (2.1) and (2.6). Additionally from (2.7) and the Cauchy-Schwarz inequality (CSI) we obtain that

$$y^2(t) \leq \left[\int_a^b |G(t,s)q(s)|^2 ds\right] \left[\int_a^b y^2(s)ds\right]. \tag{2.12}$$

Integrating this inequality over $[a,b]$ and then dividing the result by $||y||_2$, we get

$$1 \leq \left[\int_a^b \int_a^b |G(t,s)q(s)|^2 ds dt\right], \tag{2.13}$$

we call (2.13) the CSI for FBVPs (2.1)-(2.2) or (2.1) and (2.3) or (2.1) and (2.4) or (2.1) and (2.5) or (2.1) and (2.6).

2.2 GENERAL FRACTIONAL EIGENVALUE PROBLEM

Now, consider the following linear Fractional Differential Equation (FDE) and the boundary conditions. Let $n - a < \alpha \leq n, n \in \mathbb{N}$ and $\beta \in [0,1]$.

Problem P3:

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) + \lambda y(t) = 0, \quad a < t < b, \tag{2.14}$$

with the boundary conditions B1 OR B2 OR B3 ie.

$$\begin{aligned} a_1 y(a) + a_2 \left(D I_{a^+}^{(2-\alpha)(1-\beta)} y\right)(a) &= 0, \\ b_1 y(b) + b_2 D y(b) &= 0, \end{aligned}$$

OR

$$\begin{aligned} d_1 y(a) + d_2 \left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a) &= 0 \\ e_1 y(b) + e_2 y'(b) &= 0, \end{aligned}$$

or

$$\begin{aligned} d_3 y'(a) + d_4 \left(D^2 I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a) &= 0, \\ e_1 y(b) + e_2 y'(b) &= 0, \end{aligned}$$

OR

$$y^i(a) = y^i(b) = 0, \quad i = 0, 1$$

or

$$y^i(a) = y^i(b) = 0, \quad i = 0, 2$$

or

$$y^i(a) = y''(b) = 0, \quad i = 0, 1, 2.$$

where the function $y(t)$ and the number λ are unknown. A function $y(t)$ that satisfies equations (2.14) and the boundary conditions B1 or B2 or B3 is known as an eigenfunction, the corresponding λ the eigenvalue associated with $y(t)$, and the problem a fractional eigenvalue problem (FEP). Next, we give three methods to estimate the lower bound for the smallest eigenvalue of Problem P3. Note that FBVPs (2.1) with the boundary conditions B1 or B2 or B3, and P2 are the same except that q(t) in equation (2.1) has been replaced with λ to obtain equation (2.14). Thus, the LTI equation (2.11) and the CSI equation (2.13) for FBVPs (2.1) with the boundary conditions B1 or B2 or B3 can be used to find a lower bound for the smallest eigenvalue of Problem P3. These are called two methods; LTI and CSI methods. In the discussion to follow, we will use the following definition for a Lyapunov inequality lower bound.

Definition. A Lyapunov Inequality Lower Bound (LILB) is defined as a lower estimate for the smallest eigenvalue obtained from Lyapunov-type inequality given in equation (2.11).

Setting $q(t) = \lambda$ in (2.11), we obtain LILB of Problem P3 as

$$\lambda \geq \frac{1}{(b-a)G_{\max}}.$$

If we replace $q(t) = \lambda$ in (2.13), then we obtain a lower bound for the smallest eigenvalue of Problem P3

$$\lambda \geq \left[\int_a^b \int_a^b G^2(t,s)dsdt\right]^{-\frac{1}{2}}. \qquad (2.15)$$

In the discussion to follow, we define a Cauchy-Schwarz Inequality Lower Bound as follows:

Definition. A Cauchy-Schwarz Inequality Lower Bound (CSILB) is defined as an estimate of the lower bound for the smallest eigenvalue obtained from the Cauchy-Schwarz inequality of type given in equation (2.15).

To describe the Semi Maximum Norm method, note that a linear FBVP P2 reduces to

$$1 \leq \max_{a \leq t \leq b} \int_a^b |G(t,s)q(s)|ds$$

(see (2.10)), and for a FEP P3, $q(s)$ in the above equation is replaced with λ to obtain

$$\lambda \geq \frac{1}{\max_{a \leq t \leq b} \int_a^b |G(t,s)|ds}. \tag{2.16}$$

The above inequality gives a lower bound estimate for the smallest eigenvalue. In this case, we do not take the maximum norm of $|G(t,s)|$ but only the maximum norm of the integral $\int_a^b |G(t,s)|ds$ over $[a,b]$, and for this reason, we call this method of obtaining a lower bound for λ the Semi Maximum Norm method. Also note that

$$\max_{a \leq t \leq b} \int_a^b |G(t,s)|ds \leq (b-a) \max_{[a,b] \times [a,b]} |G(t,s)|$$

and therefore the Semi Maximum Norm method provides a better estimate for the smallest eigenvalue than that provided by the Lyapunov-type inequalities. In the sequel we define a Semi Maximum Norm Lower Bound as follows.

Definition. A Semi Maximum Norm Lower Bound (SMNLB) is defined as the lower estimate for the smallest eigenvalue obtained from the Semi Maximum Norm inequality of type given in (2.16).

CHAPTER 3

LYAPUNOV-TYPE INEQUALITY AND EIGENVALUE ESTIMATES FOR FRACTIONAL PROBLEMS OF ORDER α, $\alpha \in (1, 2]$

In this section we establish Lyapunov-type inequalities for the FBVPs with the Dirichlet, and a mixed set of Dirichlet and Neumann boundary conditions. We also obtain the eigenvalue estimates for the smallest eigenvalue of FEPs. We apply these estimates to obtain the interval in which certain Mittag-Leffler functions have no real zeros.

3.1 LYAPUNOV-TYPE INEQUALITY FOR FBVP WITH THE DIRICHLET BOUNDARY CONDITIONS

Replacing $a_1 = b_1 = 1$, $a_2 = b_2 = 0$ in equation (2.2) we obtain the FBVP from (2.1) with $n = 2$ as follows.

Problem P4:

$$\begin{aligned} \left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) &= 0, \quad a < t < b,\ 1 < \alpha \leq 2,\ 0 \leq \beta \leq 1, \\ y(a) = y(b) &= 0. \end{aligned} \tag{3.1}$$

Lemma 3.1.1. *Problem P4 can be written as (2.7) where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)} (b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \left(\frac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)} (b-s)^{\alpha-1}, & a \leq t \leq s \leq b, \end{cases} \tag{3.2}$$

is the Green's function for the problem.

Proof. Taking $I_{a^+}^{\alpha}$ on the first equation of P4 and using Lemma 1.3.4, we obtain

$$y(t) = c_1 \frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_2 \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)y(s)ds, \tag{3.3}$$

where c_1 and c_2 are the real constants given by

$$c_1 = \left(I_{a^+}^{(2-\alpha)(1-\beta)} y\right)(a), \quad c_2 = \frac{d}{dt}\left(I_{a^+}^{(2-\alpha)(1-\beta)} y\right)(a).$$

Since $y(a) = 0$, we get $c_1 = 0$. Now $y(b) = 0$ gives

$$c_2 = \frac{\Gamma(2-(2-\alpha)(1-\beta))}{\Gamma(\alpha)(b-a)^{1-(2-\alpha)(1-\beta)}} \int_a^b (b-s)^{\alpha-1} q(s)y(s)ds.$$

Hence, equality (3.3) becomes

$$y(t) = \frac{1}{\Gamma(\alpha)}\Big(\frac{t-a}{b-a}\Big)^{1-(2-\alpha)(1-\beta)} \int_a^b (b-s)^{\alpha-1} q(s)y(s)ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)y(s)ds,$$

which can be written as equation (2.7) with $G(t,s)$ given by (3.2). This concludes the proof. □

Lemma 3.1.2. *The function G defined in Lemma 3.1.1 satisfies the following property:*

$$|G(t,s)| \;\leq\; \frac{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}, \tag{3.4}$$

$(t,s) \in [a,b] \times [a,b]$.

Proof. Let us define two functions

$$G_1(t,s) := (b-s)^{\alpha-1}\Big(\frac{t-a}{b-a}\Big)^{1-(2-\alpha)(1-\beta)} - (t-s)^{\alpha-1}, a \leq s \leq t \leq b,$$

and

$$G_2(t,s) := (b-s)^{\alpha-1}\Big(\frac{t-a}{b-a}\Big)^{1-(2-\alpha)(1-\beta)}, a \leq t \leq s \leq b.$$

Here, G_2 is an increasing function in t. And $0 \leq G_2(t,s) \leq G_2(s,s)$. Using $\Big(\frac{t-a}{t-s}\Big)^{2-\alpha} > \Big(\frac{b-a}{b-s}\Big)^{2-\alpha}$ and since $0 \leq \beta(2-\alpha) < 1$, we get $(\frac{t-a}{b-a})^{\beta(2-\alpha)} < 1$, for $a \leq s < t \leq b$, we get

$$\frac{\partial G_1}{\partial s} = (\alpha-1)(t-a)^{\alpha-2}\left[\Big(\frac{t-a}{t-s}\Big)^{2-\alpha} - \Big(\frac{b-a}{b-s}\Big)^{2-\alpha}\Big(\frac{t-a}{b-a}\Big)^{1+\beta(2-\alpha)}\right] \geq 0. \tag{3.5}$$

Hence, for a given t, $G_1(t,s)$ is an increasing function of $s \in [a,t]$. Hence,

$$\max_{t\in[a,b]} |G(t,s)| = G(t,t).$$

Here,

$$G(t,t) = \frac{1}{\Gamma(\alpha)}\Big(\frac{t-a}{b-a}\Big)^{1-(2-\alpha)(1-\beta)}(b-t)^{\alpha-1}.$$

Let

$$f(t) = (t-a)^{1-(2-\alpha)(1-\beta)}(b-t)^{\alpha-1}, \quad t \in [a,b].$$

Now, we differentiate $f(t)$ on (a, b), and we obtain after simplifications

$$f'(t) = (t-a)^{-(2-\alpha)(1-\beta)}(b-t)^{\alpha-2}\left[(1-(2-\alpha)(1-\beta))(b-t)-(\alpha-1)(t-a)\right].$$

Observe that $f'(t)$ has a unique zero, attained at the point

$$t = t^* = \frac{b(2-\alpha)(1-\beta)-b-a(\alpha-1)}{(2-\alpha)(1-\beta)-\alpha}.$$

Since, $f''(t^*) \leq 0$, we conclude that

$$\max_{t\in[a,b]} f(t) = f(t^*) = \frac{(b-a)^{\alpha-(2-\alpha)(1-\beta)}[1-(2-\alpha)(1-\beta)]^{1-(2-\alpha)(1-\beta)}[\alpha-1]^{\alpha-1}}{[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}.$$

This gives

$$|G(t,t)| \leq \frac{(b-a)^{\alpha-1}[1-(2-\alpha)(1-\beta)]^{1-(2-\alpha)(1-\beta)}[\alpha-1]^{\alpha-1}}{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}.$$

This completes the proof of Lemma. □

Theorem 3.1.3. *If a nontrivial continuous solution of the problem P4 exists, then for P4 the LTI is*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}} \tag{3.6}$$

and in particular, for $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ in P4 gives the standard Lyapunov inequality for BVP (1.48) as (1.49).

Proof. Using (3.4) in LTI equation (2.11) proves the inequality (3.6). Replacing $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ in (3.6) we obtain (1.49). □

3.1.1 Eigenvalue Problem with the Dirichlet boundary conditions and Eigenvalue Estimates

Setting $a_1 = b_1 = 1$, $a_2 = b_2 = 0$ in equation (2.2) and from (2.14) with $n = 2$, we obtain the FEP

Problem P5:

$$\begin{aligned} \left(D_{a^+}^{\alpha,\beta} y\right)(t) + \lambda y(t) &= 0, \quad a < t < b,\ 1 < \alpha \leq 2,\ 0 \leq \beta \leq 1, \\ y(a) = y(b) &= 0. \end{aligned} \tag{3.7}$$

Corollary 3.1.4. *Let λ be the smallest eigenvalue of FEP P5. Then for $\alpha \in (1, 2]$ and $\beta \in [0, 1]$, the smallest eigenvalue estimates of FEP P5 are given by*

1. *the LILB*
$$\lambda \geq \frac{\Gamma(\alpha)[\alpha - (2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha}[\alpha - 1 + \beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}} \tag{3.8}$$
and in particular, for integer order eigenvalue problem (IOEP) P5, i.e. $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ this bound is
$$\lambda \geq \frac{4}{(b-a)^2} \tag{3.9}$$

2. *the SMNLB*
$$\lambda \geq \frac{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{1-\beta(2-\alpha)}}}{(b-a)^{\alpha}[\alpha-1+\beta(2-\alpha)]^{\frac{\alpha-1+\beta(2-\alpha)}{1-\beta(2-\alpha)}}[1-\beta(2-\alpha)]} \tag{3.10}$$
and in particular, for IOEP P5, this bound is
$$\lambda \geq \frac{8}{(b-a)^2} \tag{3.11}$$

3. *and CSILB*
$$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{4\alpha - 1 + 2\beta(2-\alpha)}{2\alpha(2\alpha-1)[2\alpha-1+2\beta(2-\alpha)]} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}, \tag{3.12}$$
where $C_1(\alpha) = \int_0^1 t^{\alpha-(2-\alpha)(1-\beta)+1}{}_2F_1(1-\alpha, 1; \alpha+1; t)dt$ and ${}_2F_1(a, b; c; t)$ is a hypergeometric function and in particular, for IOEP P5, CSILB is
$$\lambda \geq \frac{3\sqrt{10}}{(b-a)^2}. \tag{3.13}$$

Proof. Setting $q(t) = \lambda$ in equations (3.6) and (1.49), the inequalities in the first part follow. Substituting the Green's function from equation (3.2), in (2.16) and (2.15), and simplifying the results, we obtain the inequalities respectively in equations (3.10) and (3.12). Setting $\alpha = 2$ in equations (3.8), (3.10) and (3.12) we get the inequalities (3.9), (3.11) and (3.13). □

We first consider the integer order case, i.e. $\alpha = 2$ and $\beta = 0$ or $\beta = 1$, and $a = 0$ and $b = 1$. For this case, the LILB, SMNLB and CSILB for the smallest λ of FEP P5 are given as 4, 8 and $3\sqrt{10} \simeq 9.48683$, respectively (see equations (3.9), (3.11) and (3.13)) . For $\alpha = 2$, the FEP P5 with $a = 0$ and $b = 1$ can be solved in closed form using the tools from integer order calculus. Results show, that the smallest eigenvalue of FEP P5 for $\alpha = 2$ is the root of $\sin(\sqrt{\lambda}) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 9.86960$. Comparing this λ with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue. The FEP P5 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (1, 2]$ as a root of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$. This is explained in the following theorem and its proof.

Theorem 3.1.5. *For $1 < \alpha \leq 2$, $\beta \in [0, 1]$, $a = 0$ and $b = 1$, the FEP P5 has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$, i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda) = 0. \tag{3.14}$$

Proof. To prove this, we take Laplace transform of the first equation in P5 with $a = 0$ and $b = 1$, using (1.45) for $n = 2$ which after some manipulations leads to

$$Y(s) = \frac{a_0 s^{1-\beta(2-\alpha)}}{s^\alpha + \lambda} + \frac{a_1 s^{-\beta(2-\alpha)}}{s^\alpha + \lambda}, \tag{3.15}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $a_i = D^i \left[I_{0+}^{(1-\beta)(2-\alpha)} y \right] (0^+)$, $i = 0, 1$. Taking inverse Laplace transform of equation (3.15) and using equation (1.25), we obtain

$$y(t) = a_0 t^{(\alpha-1+\beta(2-\alpha))-1} E_{\alpha,\alpha-1+\beta(2-\alpha)}(-\lambda t^\alpha) + a_1 t^{\alpha+\beta(2-\alpha)-1} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda t^\alpha). \tag{3.16}$$

Using the boundary conditions of P5 we obtain (3.14). $\square$

We compute the smallest eigenvalues for FEP P5 from equation (3.14) and its LILB, SMNLB and CSILB for different α, $\alpha \in (1, 2]$ and $\beta = 0, 1$ from equations (3.8), (3.10) and (3.12). Notice that according the definition of Hilfer derivative in (1.44), $\beta = 0, \beta = 1$ and $n = 2$ give respectively the results for classical Riemann-Liouville and Caputo derivative FBVP as well as FEVP. A few results reduce to the the work on LTI for FBVPs in [15], and [16]. Particularly, for $\beta = 0$ and $\beta = 1$ in FBVP P4 and FEP P5, reduce to the results in [15] and [16] respectively. The results are shown in the following tables 3.1 and 3.2.

LTI	LILB	SMNLB	CSILB
$\int_a^b \lvert q(s)\rvert ds \geq$	$\lambda \geq$	$\lambda \geq$	$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\cdot$
$\frac{\Gamma(\alpha)4^{\alpha-1}}{(b-a)^{\alpha-1}}$ [15]	$\frac{\Gamma(\alpha)4^{\alpha-1}}{(b-a)^{\alpha}}$[15]	$\frac{\Gamma(\alpha+1)\alpha^{\alpha}}{(b-a)^{\alpha}(\alpha-1)^{\alpha-1}}$	$\left[\frac{4\alpha-1}{2\alpha(2\alpha-1)^2} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}$;
			$C_1(\alpha) =$
			$\int_0^1 t^{2\alpha-1}{}_2F_1(1-\alpha, 1; \alpha+1; t)dt$

Table 3.1. Results for $\alpha \in (1, 2]$ and $\beta = 0$ (FBVP P4 and FEP P5 with Riemann-Liouville derivative)

For comparison purpose, we compute the smallest eigenvalues for FEP P5 with $a = 0$ and $b = 1$ for particular values of type $\beta = 0$ and $\beta = 1$ and its LILB, SMNLB and CSILB for different α, $\alpha \in (1, 2]$ from Tables 3.1 and 3.2. The results are shown in figures 3.1 and 3.2 respectively. These figures clearly demonstrate that among the three estimates considered here, the LILB provides the worse estimate and the CSILB and SMNLB provide better estimate for the smallest eigenvalues of FEP P5 for $\beta = 0, 1$. We use MATHEMATICA and MATLAB code to find the smallest eigenvalue of the Mittag-Leffler functions. We note that the MATLAB code was contributed by Podlubny [45], and the algorithm is based on

LTI	LILB	SMNLB	CSILB
$\int_a^b \lvert q(s)\rvert ds \geq$ $\frac{\Gamma(\alpha)\alpha^{\alpha}}{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}}$[16]	$\lambda \geq$ $\frac{\Gamma(\alpha)\alpha^{\alpha}}{(b-a)^{\alpha}(\alpha-1)^{\alpha-1}}$[16]	$\lambda \geq$ $\frac{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{\alpha-1}}}{(b-a)^{\alpha}(\alpha-1)}$	$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\cdot$ $\left[\frac{2\alpha+3}{6\alpha(2\alpha-1)} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2};$ $C_1(\alpha) =$ $\int_0^1 t^{\alpha+1} {}_2F_1(1-\alpha, 1; \alpha+1; t)dt$

Table 3.2. Results for $\alpha \in (1, 2]$ and $\beta = 1$ (FBVP P4 and FEP P5 with Caputo derivative)

the paper of Gorenflo et al. [17]. By this code we can calculate the MittagLeffler function with desired accuracy. Throughout this work we calculate the MittagLeffler function with the accuracy 10^{-5}. Setting $\beta = 1$ in equation (3.14), it reduces to $E_{\alpha,2}(-\lambda) = 0$. We analyzed that $E_{\alpha,2}(z)$ has no solution for $\alpha = 1.1$ to $\alpha = 1.5991152$. Furthermore, for $\alpha = 1.5991152$, $E_{\alpha,2}(z)$ has no real zeros and an infinite number of complex zeros. Whereas for $\alpha = 1.5991153$, $E_{\alpha,2}(z)$ has two real zeros and an infinite number of complex zeros. (see [12], [18]). We note that if $\alpha = 1.5991153$ to $\alpha = 2$, the FEP P5 with $a = 0$, $b = 0$ and $\beta = 1$ has zero solutions. For $\alpha = 1.5991153, 1.6, 1.7, 1.8, 1.9, 2$, we calculate the eigenvalues. Which is shown in figure 3.2.

We now consider an application of the lower bounds for the smallest eigenvalues of FEP P5 found in Corollary 3.1.4 and Theorem 3.1.5. In [15], [16], [26], [27], [28] and [48], the authors have applied the LILB to the FEPs for $\alpha \in (1, 2]$ to find the interval in which certain Mittag-Leffler functions have no real zeros. On the other hand, in [43], we applied the improved bounds to obtain these intervals for certain Mittag-Leffler functions for $\alpha \in (2, 3]$. We follow a similar procedure, which is discussed in the following theorem.

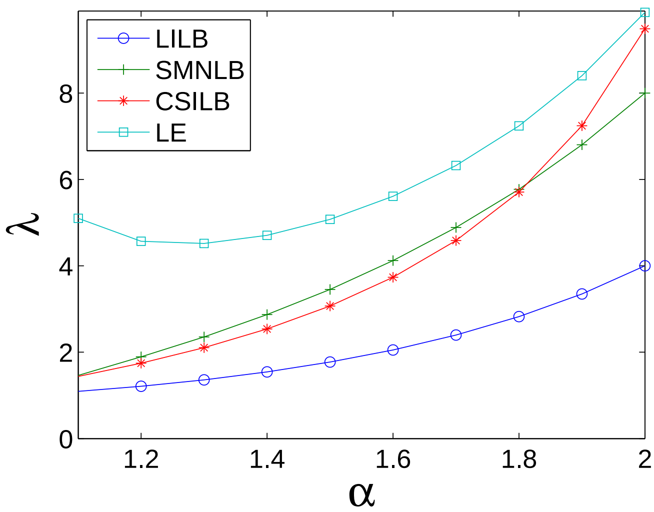

Figure 3.1. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 0$, Riemann-Liouville derivative FEP P5)

Theorem 3.1.6. *Let* $1 < \alpha \leq 2$ *if* $\beta = 0$, *and* $1.5991153 \leq \alpha \leq 2$ *if* $\beta \in (0, 1]$. *Then based on the LILB, SMNLB and CSILB inequalities, the Mittag-Leffler function* $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ *has no real zeros in the following domains:*

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}, 0\right], \tag{3.17}$$

SMNLB inequality:

$$z \in \Big(- \frac{\Gamma(\alpha+1)\alpha^{\frac{\alpha}{1-\beta(2-\alpha)}}}{[\alpha-1+\beta(2-\alpha)]^{\frac{\alpha-1+\beta(2-\alpha)}{1-\beta(2-\alpha)}}[1-\beta(2-\alpha)]}, 0\Big], \tag{3.18}$$

CSILB inequality:

$$z \in \left(-\Gamma(\alpha)\left[\frac{4\alpha-1+2\beta(2-\alpha)}{2\alpha(2\alpha-1)[2\alpha-1+2\beta(2-\alpha)]} - \frac{2}{\alpha}C_1(\alpha)\right]^{-1/2}, 0\right]. \tag{3.19}$$

Proof. Let λ be the smallest eigenvalue of the FEP P5, then $z = \lambda$ is the smallest value of z for which $E_{\alpha,\alpha+\beta(2-\alpha)}(-z) = 0$. If there is another z smaller than λ for which

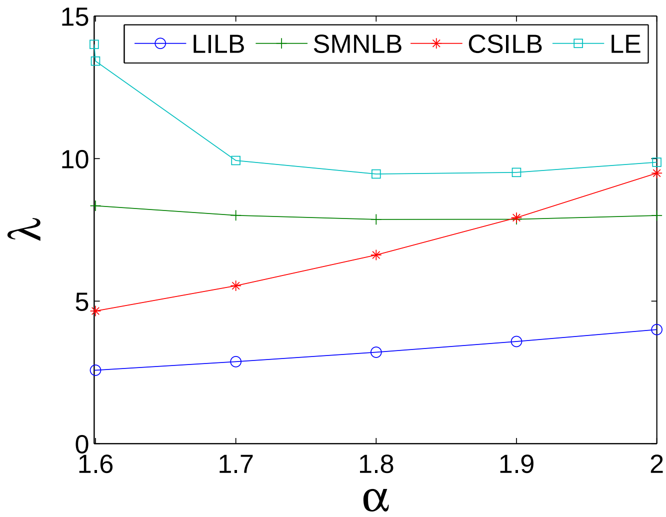

Figure 3.2. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. $(-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a=0, b=1, \beta=1$, Caputo derivative FEP P5)

$E_{\alpha,\alpha+\beta(2-\alpha)}(-z)=0$, then it will contradict that λ is the smallest eigenvalue. Therefore, $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ has no real zero for $z\in(-\lambda,0]$. Now, according to LILB,

$$\lambda \geq \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}$$

(see equation (3.8)). Thus, $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ has no real zero for

$$z\in\left(-\frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}},0\right].$$

This proves equation (3.17). Equations (3.18) and (3.19) are proved in a similar fashion. $\Box$

From figures 3.1 and 3.2, it is clear that among the three inequalities discussed here, LILB provides the smallest interval, and CSILB and SMNLB provide the larger intervals in which the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)}(z)$ has no real zero. Particularly, we discuss two cases, $\beta=0$ and $\beta=1$.

3.2 LYAPUNOV-TYPE INEQUALITY FOR FBVP WITH A MIXED SET OF DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

Setting $a_1 = b_2 = 1, a_2 = b_1 = 0$ in equation (2.2) and from (2.1) with $n = 2$, we obtain the FBVP

Problem P6:

$$\begin{aligned}\left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) &= 0, \quad a < t < b,\ 1 < \alpha \leq 2,\ 0 \leq \beta \leq 1, \\ y(a) = 0, Dy(b) &= 0. \end{aligned} \tag{3.20}$$

We note that to establish the Lyapunov-type inequality for Problem P6, we employ the methods of Jleli, Ragoub and Samet [26], [27] in the following argument. We begin by rewriting the Greens function in terms of $H(t,s)$.

Lemma 3.2.1. *Problem P6 can be written as (2.7) where* $G(t,s) = \frac{H(t,s)}{\Gamma(\alpha)(b-s)^{2-\alpha}}$ *and* $H(t,s)$ *is given by*

$$H(t,s) = \begin{cases} \frac{(\alpha-1)(t-a)^{1-(2-\alpha)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{1-(2-\alpha)(1-\beta)} - (t-s)^{\alpha-1}(b-s)^{2-\alpha}, & a \leq s \leq t \leq b, \\ \frac{(\alpha-1)(t-a)^{1-(2-\alpha)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{1-(2-\alpha)(1-\beta)}, & a \leq t \leq s \leq b. \end{cases} \tag{3.21}$$

Proof. Taking $I_{a^+}^{\alpha}$ on the first equation of P6 and using Lemma 1.3.4, we get equation (3.3) as discussed in Lemma 3.1.1, that is

$$y(t) = c_1 \frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(1-(2-\alpha)(1-\beta))} + c_2 \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)y(s)ds.$$

Since, $y(a) = 0$, we obtain $c_1 = 0$. Thus we get

$$y(t) = c_2 \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)y(s)ds.$$

The time derivative of the above equation gives

$$Dy(t) = c_2[1-(2-\alpha)(1-\beta)] \frac{(t-a)^{-(2-\alpha)(1-\beta)}}{\Gamma(2-(2-\alpha)(1-\beta))} - \frac{\alpha-1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)y(s)ds. \tag{3.22}$$

Now $y'(b) = 0$ gives

$$c_2 = \frac{\Gamma(2-(2-\alpha)(1-\beta))(\alpha-1)(b-a)^{(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2} q(s)y(s)ds.$$

Hence, we get

$$\begin{aligned} y(t) &= \frac{(\alpha-1)(t-a)^{1-(\alpha-2)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{\Gamma(\alpha)[1-(2-\alpha)(1-\beta)]} \int_a^b (b-s)^{\alpha-2} q(s)y(s)ds \\ &\quad -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s)y(s)ds. \end{aligned}$$

This concludes the proof. □

Lemma 3.2.2. *The function H defined in Lemma 3.2.1 satisfies the following property:*

$$|H(t,s)| \leq \frac{b-a}{\alpha-1+\beta(2-\alpha)} \max\Big\{\alpha-1, \beta(2-\alpha)\Big\},$$

$(t,s) \in [a,b] \times [a,b]$.

Proof. Here $H(t,s)$ is an increasing function of t for $a \leq t < s \leq b$. For $a \leq s < t \leq b$ and a fixed $s \in [a,b]$, since, $\left(\frac{b-a}{t-a}\right)^{(2-\alpha)(1-\beta)} < \left(\frac{b-a}{t-a}\right)^{(2-\alpha)} < \left(\frac{b-s}{t-s}\right)^{2-\alpha}$, we get

$$\frac{\partial H}{\partial t} = (\alpha-1)\left[\left(\frac{b-a}{t-a}\right)^{(2-\alpha)(1-\beta)} - \left(\frac{b-s}{t-s}\right)^{2-\alpha}\right] \leq 0.$$

So, in $a \leq s < t \leq b$ for a given s, $H(t,s)$ is a decreasing function of $t \in [s,b]$. Hence,

$$\max_{t\in[a,b]} H(t,s) \leq \max\{|H(s,s)|, |H(b,s)|\}.$$

After some calculations we obtain

$$|H(b,s)| \leq \frac{b-a}{\alpha-1+\beta(2-\alpha)} \max\{\alpha-1, \beta(2-\alpha)\}$$

and

$$\begin{aligned} |H(s,s)| &\leq \frac{(\alpha-1)(s-a)^{1-(2-\alpha)(1-\beta)}}{\alpha-1+\beta(2-\alpha)(b-a)^{(\alpha-2)(1-\beta)}} \\ &\leq \frac{(\alpha-1)(b-a)}{\alpha-1+\beta(2-\alpha)}, \end{aligned}$$

which concludes the proof. □

Theorem 3.2.3. *If a nontrivial continuous solution of the FBVP P6 exists, then the LTI is given by*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geq \frac{\Gamma(\alpha)[\alpha-1+\beta(2-\alpha)]}{(b-a)\max\Big\{\alpha-1, \beta(2-\alpha)\Big\}}. \tag{3.23}$$

Proof. Using Lemma 3.2.1 in equation (2.10) we obtain

$$1 \leq \frac{1}{\Gamma(\alpha)} \max_{a\leq t\leq b} \int_a^b (b-s)^{\alpha-2}|H(t,s)q(s)|ds.$$

Now an application of Lemma 3.2.2 proves the inequality (3.23). □

3.2.1 Eigenvalue Problem with a mixed set of Dirichlet and Neumann boundary conditions and Eigenvalue Estimates

Setting $a_1 = b_2 = 1$, $a_2 = b_1 = 0$ in equation (2.2) and from (2.14) with $n = 2$, we obtain the FEP

Problem P7:

$$\begin{aligned} \left(D^{\alpha,\beta}_{a^+} y\right)(t) + \lambda y(t) &= 0, \quad a < t < b,\ 1 < \alpha \leq 2,\ 0 \leq \beta \leq 1, \\ y(a) = Dy(b) &= 0. \end{aligned} \tag{3.24}$$

The eigenvalue estimates for the smallest eigenvalue of FEP P7 can be obtained in the similar way as we discussed in Corollary 3.1.4.

Corollary 3.2.4. *Let λ be the smallest eigenvalue of FEP P7. For $\alpha \in (1,2]$ and $\beta \in [0,1]$ the eigenvalue estimates for the smallest eigenvalue of FEP P7 are given by*

1. *the LILB*

$$\lambda \geq \frac{\Gamma(\alpha)(\alpha-1)[\alpha-1+\beta(2-\alpha)]}{(b-a)^\alpha \max\Big\{\alpha-1, \beta(2-\alpha)\Big\}} \tag{3.25}$$

and in particular, for IOEP P7, i.e. $\alpha = 2$ and $\beta = 0$ or $\beta = 1$ this bound is

$$\lambda \geq \frac{1}{(b-a)^2} \tag{3.26}$$

2. *the SMNLB*

$$\lambda \geq \frac{\Gamma(\alpha+1)[\alpha-1+\beta(2-\alpha)]^{\alpha}}{(b-a)^{\alpha}\left[2(\alpha-1)^{\alpha-1}-(\alpha-1+\beta(2-\alpha))^{\alpha-1}(1-\beta(2-\alpha))\right]} \tag{3.27}$$

and in particular, for IOEP P7, SMNLB is

$$\lambda \geq \frac{2}{(b-a)^2} \tag{3.28}$$

3. *and CSILB*

$$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{2\alpha(\alpha-1)^2(2\alpha-1)+[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)}{2\alpha(2\alpha-1)[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)}-\frac{2(\alpha-1)C_2(\alpha)}{\alpha[1-(2-\alpha)(1-\beta)]}\right]^{-1/2}, \tag{3.29}$$

where $C_2(\alpha)=\int_0^1 t^{\alpha-(2-\alpha)(1-\beta)+1}{}_2F_1(2-\alpha,1;\alpha+1;t)dt$, $\alpha>\frac{3}{2}$ *and in particular, for IOEP P7, CSILB is*

$$\lambda \geq \frac{\sqrt{6}}{(b-a)^2}. \tag{3.30}$$

Proof. Setting $q(t)=\lambda$ in equation (3.23) and evaluating the resulting integral, the first inequality in the first part follows. Substituting the Green's function from equation (3.21), in (2.15) and simplifying the result, we obtain the inequality in equation (3.29). Substituting $\alpha=2$ and $\beta=0$ or $\beta=1$, in inequalities (3.25) and (3.29), prove the inequalities (3.26) and (3.30) respectively. To prove (2), notice that the maximum of $\int_a^b |G(t,s)|ds$ occurs at $t=b$ for $s\in[a,t]$. From (3.21) we get

$$\frac{(b-a)(\alpha-1)}{1-(2-\alpha)(1-\beta)}-(b-s)=0$$

which is satisfied by

$$s=s^*=\frac{b\beta(2-\alpha)+a(\alpha-1)}{\alpha-1+\beta(2-\alpha)}.$$

Hence,

$$\max_{t\in[a,b]}\int_a^b |G(t,s)|ds=\int_a^{s^*}|G(b,s)|ds+\int_{s^*}^b |G(b,s)|ds. \tag{3.31}$$

Using $G(t,s)$ from (3.21) in (3.31) we obtain

$$\max_{t\in[a,b]}\int_a^b |G(t,s)|ds = \frac{2(\alpha-1)^{\alpha-1}-(\alpha-1+\beta(2-\alpha))^{\alpha-1}(1-\beta(2-\alpha))}{(b-a)^{-\alpha}\Gamma(\alpha+1)[\alpha-1+\beta(2-\alpha)]^{\alpha}}. \tag{3.32}$$

Substituting (3.32) in (2.16) completes the proof. □

We notice that (3.29) becomes unbounded when $\alpha \leq \frac{3}{2}$. Hence the CSILB for FEP P7 holds for $\alpha > \frac{3}{2}$.

For the integer order case, i.e. $\alpha = 2, a = 0$ and $b = 1$, the LILB, SMNLB and CSILB for the smallest λ of FEP P7 are given as 1, 2 and $\sqrt{6} \simeq 2.4495$, respectively (see equations (3.26), (3.28) and (3.30)) . For $\alpha = 2$, the smallest eigenvalue of FEP P7 with $a = 0$ and $b = 1$ is the root of $\cos(\sqrt{\lambda}) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 2.4674011$. Comparing this λ with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue.

The eigenvalues of the FEP P7 for $\alpha \in (1, 2]$ are the roots of the Mittag-Leffler function given in the following theorem.

Theorem 3.2.5. *The FEP P7 for $1 < \alpha \leq 2$, $\beta \in [0, 1]$, $a = 0$ and $b = 1$ has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$, i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(2-\alpha)-1}(-\lambda) = 0 \tag{3.33}$$

Proof. The proof is similar to the proof of Theorem 3.1.5. □

We compute the smallest eigenvalues for FEP P7 from equation (3.33) and its LILB, SMNLB and CSILB for different α, $\alpha \in (1, 2]$ and $\beta = 0$ and $\beta = 1$ from equations (3.25), (3.27) and (3.29). The results are shown in the following tables 3.3 and 3.4.

For comparison purpose, we compute the smallest eigenvalues for FEP P7 with $a = 0$ and $b = 1$ for particular values of type $\beta = 0$ and $\beta = 1$ and its LILB, SMNLB and CSILB for different α, $\alpha \in (1, 2]$ from tables 3.3 and 3.4. The results are shown in figures 3.3 and

LTI	LILB	SMNLB	CSILB
$\int_a^b (b-s)^{\alpha-2}\lvert q(s)\rvert ds$ $\geq \frac{\Gamma(\alpha)}{b-a}$	$\lambda \geq$ $\frac{\Gamma(\alpha)(\alpha-1)}{(b-a)^\alpha}$	$\lambda \geq$ $\frac{\Gamma(\alpha+1)(\alpha-1)^\alpha}{(b-a)^\alpha}$	$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^\alpha}\cdot$ $\left[\frac{4\alpha-3}{2\alpha(2\alpha-1)(2\alpha-3)} - \frac{2C_2(\alpha)}{\alpha}\right]^{-1/2}$, $\alpha > \frac{3}{2}$, $C_2(\alpha) =$ $\int_0^1 t^{2\alpha-1}{}_2F_1(2-\alpha,1;\alpha+1;t)dt$

Table 3.3. Results for $\alpha \in (1, 2]$ and $\beta = 0$ (FBVP P6 and FEP P7 with Riemann-Liouville derivative)

3.4 respectively. We note that a few results for the particular case $\beta = 1$ in FBVP and FEP P7, reduce to the results in [27] (page 447, 449). From the figures it is clear that the CSILB and SMNLB provide better estimate for the smallest eigenvalues than LILB of FEP P7 for $\beta = 0$ and $\beta = 1$. We notice that in figure 3.3, the CSILB is valid for $\alpha \in (1.5, 2]$.

We apply the lower bounds for the smallest eigenvalues of FEP P7 with $a = 0$ and $b = 1$ found in Corollary 3.2.4 and Theorem 3.2.5 for $\alpha \in (1, 2]$ to find the interval in which the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zeros.

Theorem 3.2.6. *Let $1 < \alpha \leq 2$. The Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zeros in the following domains:*

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)(\alpha-1)[\alpha-1+\beta(2-\alpha)]}{\max\left\{\alpha-1, \beta(2-\alpha)\right\}}, 0\right], \tag{3.34}$$

SMNLB inequality:

$$z \in \Big(-\frac{\Gamma(\alpha+1)[\alpha-1+\beta(2-\alpha)]^\alpha}{[2(\alpha-1)^{\alpha-1} - (\alpha-1+\beta(2-\alpha))^{\alpha-1}(1-\beta(2-\alpha))]}, 0\Big], \tag{3.35}$$

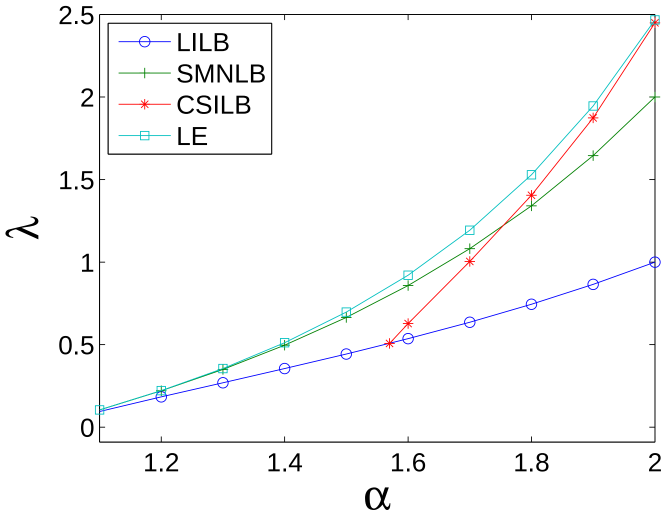

Figure 3.3. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 0$, Riemann-Liouville derivative FEP P7)

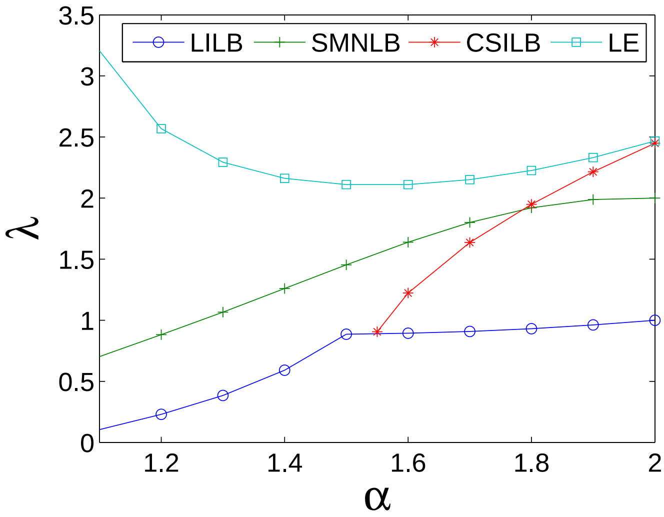

Figure 3.4. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ) ($a = 0, b = 1, \beta = 1$, Caputo derivative FEP P7)

LTI	LILB	SMNLB	CSILB
$\int_a^b (b-s)^{\alpha-2}\lvert q(s)\rvert ds \geq \frac{\Gamma(\alpha)}{(b-a)\max\{\alpha-1,2-\alpha\}}$ [27]	$\lambda \geq \frac{\Gamma(\alpha)(\alpha-1)}{(b-a)^\alpha \max\{\alpha-1,2-\alpha\}}$ [27]	$\lambda \geq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \cdot \frac{(\alpha-1)^{-1}}{2(\alpha-1)^{\alpha-2}-1}$	$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^\alpha} \cdot \left[\frac{2(\alpha-1)^2\alpha(2\alpha-1)+3(2\alpha-3)}{6\alpha(2\alpha-1)(2\alpha-3)} - \frac{2(\alpha-1)C_2(\alpha)}{\alpha}\right]^{\frac{-1}{2}}$; $\alpha > \frac{3}{2}$, $C_2(\alpha) = \int_0^1 t^{\alpha+1} {}_2F_1(2-\alpha, 1; \alpha+1; t)dt$

Table 3.4. Results for $\alpha \in (1, 2]$ and $\beta = 1$ (FBVP P6 and FEP P7 with Caputo derivative)

CSILB inequality:

$$z \in \left(-\Gamma(\alpha)\left[\frac{2\alpha(\alpha-1)^2(2\alpha-1)+[\alpha-1+\beta(2-\alpha)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)}{2\alpha(2\alpha-1)[1-(2-\alpha)(1-\beta)]^2[2\alpha-1+2\beta(2-\alpha)](2\alpha-3)} - \frac{2(\alpha-1)C_2(\alpha)}{\alpha[1-(2-\alpha)(1-\beta)]}\right]^{-1/2}, 0\right]. \tag{3.36}$$

Proof. The proof is similar to the proof of Theorem 3.1.6. □

From figures 3.3 and 3.4, it is clear that among the three inequalities, LILB provides the smallest interval, and CSILB and SMNLB provide the larger intervals in which the Mittag-Leffler functions $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ for $\beta = 0$ and $\beta = 1$, have no real zero.

CHAPTER 4

LYAPUNOV-TYPE INEQUALITY AND EIGENVALUE ESTIMATES FOR FRACTIONAL PROBLEMS OF ORDER α, $\alpha \in (2,3]$

In this chapter we establish Lyapunov-type inequalities for the Hilfer derivative fractional boundary value problem with a mixed set of Dirichlet and Neumann, and a mixed set of fractional Dirichlet, Neumann and fractional Neumann boundary conditions. We also obtain the eigenvalue estimates for the smallest eigenvalue of FEPs. We apply these estimates to obtain the interval in which Mittag-Leffler functions have no real zeros.

4.1 LYAPUNOV-TYPE INEQUALITY FOR FBVP WITH A MIXED SET OF DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

Setting $d_1 = d_3 = 1$, $d_2 = d_4 = 0$, and $e_2 = 1, e_1 = 0$ in equation (2.3) we obtain from equation (2.1) with $n = 3$ the FBVP
Problem P8:

$$\begin{aligned} &\left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad \alpha \in (2,3], \beta \in [0,1],\ a < t < b, \\ &y(a) = 0,\ y'(a) = 0,\ y'(b) = 0, \end{aligned} \tag{4.1}$$

Lemma 4.1.1. *Problem P8 can be written as (2.7) where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{1-(3-\alpha)(1-\beta)}[2-(3-\alpha)(1-\beta)]} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \frac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{1-(3-\alpha)(1-\beta)}[2-(3-\alpha)(1-\beta)]}, & a \le t \le s \le b, \end{cases} \tag{4.2}$$

is the Green's function for the problem.

Proof. Taking $I_{a^+}^{\alpha}$ on the first equation of P8 and using Lemma 1.3.4 with $n = 3$, we obtain

$$\begin{aligned} y(t) &= c_1 \frac{(t-a)^{-(3-\alpha)(1-\beta)}}{\Gamma(1-(3-\alpha)(1-\beta))} + c_2 \frac{(t-a)^{1-(3-\alpha)(1-\beta)}}{\Gamma(2-(3-\alpha)(1-\beta))} + c_3 \frac{(t-a)^{2-(3-\alpha)(1-\beta)}}{\Gamma(3-(3-\alpha)(1-\beta))} \\ &- \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)y(s)ds, \end{aligned} \tag{4.3}$$

where c_1, c_2 and c_3 are the real constants given by

$$c_1 = \left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a), c_2 = \frac{d}{dt}\left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a), c_3 = \frac{d^2}{dt^2}\left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a).$$

Since $y(a) = 0$, we get $c_1 = 0$. Taking time derivative of equation (4.3) we obtain

$$\begin{aligned} y'(t) &= c_2 \frac{[1-(3-\alpha)(1-\beta)](t-a)^{-(3-\alpha)(1-\beta)}}{\Gamma(2-(3-\alpha)(1-\beta))} + c_3 \frac{[2-(3-\alpha)(1-\beta)](t-a)^{1-(3-\alpha)(1-\beta)}}{\Gamma(3-(3-\alpha)(1-\beta))} \\ &- \frac{(\alpha-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} q(s) y(s) ds, \end{aligned} \tag{4.4}$$

and setting $y'(a) = y'(b) = 0$, we get $c_2 = 0$ and

$$c_3 = \frac{\Gamma(3-(3-\alpha)(1-\beta))(\alpha-1)}{\Gamma(\alpha)(b-a)^{1-(3-\alpha)(1-\beta)}[2-(3-\alpha)(1-\beta)]} \int_a^b (b-s)^{\alpha-2} q(s) y(s) ds.$$

Hence, equality (4.3) becomes

$$\begin{aligned} y(t) &= \frac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}}{\Gamma(\alpha)(b-a)^{1-(3-\alpha)(1-\beta)}[2-(3-\alpha)(1-\beta)]} \int_a^b (b-s)^{\alpha-2} q(s) y(s) ds \\ &- \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) y(s) ds, \end{aligned}$$

which can be written as equation (2.7) with $G(t,s)$ given by (4.2). This concludes the proof. □

Lemma 4.1.2. *The Green's function G defined by (4.2) satisfies the following properties:*

1. *$G(t,s)$ is a continuous function on $[a,b] \times [a,b]$.*
2. *$G(t,s) \geq 0$ for all $a \leq t, s \leq b$.*
3. $$|G(t,s)| \leq \frac{2(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}{\Gamma(\alpha)[2-(3-\alpha)(1-\beta)]^{\alpha-1}}, \quad (t,s) \in [a,b] \times [a,b]. \tag{4.5}$$

Proof. Let us define two functions

$$G_1(t,s) := \frac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{1-(3-\alpha)(1-\beta)}(2-(3-\alpha)(1-\beta))} - (t-s)^{\alpha-1}, a \leq s \leq t \leq b,$$

and

$$G_2(t,s) := \frac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{1-(3-\alpha)(1-\beta)}(2-(3-\alpha)(1-\beta))}, a \le t \le s \le b.$$

Property (1) is trivial. Indeed, it is clear that both G_1 and G_2 are continuous on their domains, and that $G_1(s,s) = G_2(s,s)$, whence (1) follows. To prove (2), note that $G_2(t,s) \ge 0$. To prove that $G_1(t,s) \ge 0$, note that $0 < \alpha - 1$, $0 \le \beta(3-\alpha)$, and for $a \le s < t \le b$, we have $(b-s)(t-a) - (t-s) = (b-t)(s-a) \ge 0$, $(\frac{b-s}{b-a})^{\alpha-2} > (\frac{b-s}{b-a})^{\alpha-1} > (\frac{t-s}{t-a})^{\alpha-1}, t-a < b-a$. Since, $0 \le \beta(3-\alpha) < 1$, we get $0 < (\frac{t-a}{b-a})^{\beta(3-\alpha)} < 1$. Hence,

$$\begin{aligned} G_1(t,s) &= (t-a)^{\alpha-1}\left[\frac{\alpha-1}{2-(3-\alpha)(1-\beta)}\Big(\frac{b-s}{b-a}\Big)^{\alpha-2}\Big(\frac{t-a}{b-a}\Big)^{\beta(3-\alpha)} - \Big(\frac{t-s}{t-a}\Big)^{\alpha-1}\right] \\ &> (t-a)^{\alpha-1}\left[\Big(\frac{b-s}{b-a}\Big)^{\alpha-1}\Big(\frac{t-a}{b-a}\Big)^{\beta(3-\alpha)} - \Big(\frac{t-s}{t-a}\Big)^{\alpha-1}\right] \ge 0. \end{aligned}$$

Which concludes that (2) is true. To prove (3), since $(t-a)^{2-(3-\alpha)(1-\beta)}$ is an increasing function in t so, for a given s, $G_2(t,s)$ is an increasing function of t. Similarly, using $(\frac{b-s}{b-a})^{\alpha-2} > (\frac{t-s}{t-a})^{\alpha-2}$ for $a \le s < t \le b$, we get

$$\frac{\partial G_1}{\partial t} = (\alpha-1)(t-a)^{\alpha-2}\left[\Big(\frac{t-a}{b-a}\Big)^{\beta(3-\alpha)}\Big(\frac{b-s}{b-a}\Big)^{\alpha-2} - \Big(\frac{t-s}{t-a}\Big)^{\alpha-2}\right] \ge 0.$$

So, for a given s, $G_1(t,s)$ is an increasing function of $t \in (s,b]$. Hence,

$$\max_{t\in[a,b]}|G(t,s)| = G(b,s) = \frac{(\alpha-1)(b-a)(b-s)^{\alpha-2} - (b-s)^{\alpha-1}[2-(3-\alpha)(1-\beta)]}{\Gamma(\alpha)[2-(3-\alpha)(1-\beta)]}. \tag{4.6}$$

Let

$$f(s) = (\alpha-1)(b-a)(b-s)^{\alpha-2} - (b-s)^{\alpha-1}[2-(3-\alpha)(1-\beta)], t \in [a,b].$$

Now, we differentiate $f(s)$ on (a,b), and we obtain after simplifications

$$f'(s) = (\alpha-1)(b-s)^{\alpha-3}[(2-(3-\alpha)(1-\beta))(b-s) - (\alpha-2)(b-a)].$$

Observe that $f'(s)$ has a unique zero, attained at the point

$$s = s^* = \frac{b(1-\beta(3-\alpha)) + a(\alpha-2)}{2-(3-\alpha)(1-\beta)}.$$

Since, $f''(s^*) \leq 0$, which concludes that

$$\max_{t\in[a,b]} f(s) = f(s^*) = \frac{2(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}{[2-(3-\alpha)(1-\beta)]^{\alpha-2}}.$$

Hence

$$G(b,s) \leq \frac{2(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}{\Gamma(\alpha)\,[2-(3-\alpha)(1-\beta)]^{\alpha-1}}.$$

This proves part (3). This completes the proof. □

Theorem 4.1.3. *If a nontrivial continuous solution of the FBVP P8 exists, then*

$$\begin{aligned} &\int_a^b \left[(\alpha-1)(b-a)(b-s)^{\alpha-2} - (b-s)^{\alpha-1}[2-(3-\alpha)(1-\beta)]\right] |q(s)|ds \\ \geq\ & \Gamma(\alpha)\,[2-(3-\alpha)(1-\beta)]^{\alpha-1} \end{aligned} \tag{4.7}$$

and more specifically,

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)\,[2-(3-\alpha)(1-\beta)]^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}, \tag{4.8}$$

and for integer order case $\alpha = 3$ and $\beta = 0$ or $\beta = 1$,

$$\int_a^b |q(s)|ds \geq \frac{8}{(b-a)^2}, \tag{4.9}$$

Proof. Using equation (4.6) into (2.10), proves the inequality (4.7) and using property (3) of Lemma 4.1.2 in LTI equation (2.11) proves the inequality in equation (4.8). Setting $\alpha = 3$ and $\beta = 0$ or $\beta = 1$ in (4.8), we get (4.9). □

The inequality in equation (4.7) is called a Hartman-Wintner type inequality, and the inequalities in equations (4.8) and (4.9) are called the Lyapunov-type inequalities for problem P8 of fractional and integer orders, respectively.

4.1.1 Eigenvalue Estimate For Fractional Eigenvalue Problem With a Mixed Set of Dirichlet and Neumann Boundary Conditions

In this section we consider the FEP and discuss the smallest eigenvalue estimates using the inequalities discussed in Chapter 2. Setting $d_1 = d_3 = 1$, $d_2 = d_4 = 0$, and

$e_2 = 1, e_1 = 0$ in equation (2.3) and from equation (2.1) with $n = 3$ we get

Problem P9:

$$\left(D^{\alpha,\beta}_{a^+} y\right)(t) + \lambda y(t) = 0, \quad \alpha \in (2,3], \beta \in [0,1], \ a < t < b,$$
$$y(a) = 0, y'(a) = 0, y'(b) = 0, \tag{4.10}$$

Corollary 4.1.4. *Let λ be the smallest eigenvalue of FEP P9. Then for $\alpha \in (2,3]$ and $\beta \in [0,1]$ the smallest eigenvalue estimates of FEP P9 are given by*

1. *the LILB*
$$\lambda \geq \frac{\Gamma(\alpha)\left[2-(3-\alpha)(1-\beta)\right]^{\alpha-1}}{(b-a)^{\alpha}(\alpha-2)^{\alpha-2}} \tag{4.11}$$
and in particular, for IOEP P9, i.e. $\alpha = 3$, and $\beta = 0$ or $\beta = 1$ this bound is
$$\lambda \geq \frac{8}{(b-a)^3} \tag{4.12}$$

2. *the SMNLB*
$$\lambda \geq \frac{\Gamma(\alpha+1)[\alpha-1+\beta(3-\alpha)]^{\alpha+1}}{(b-a)^{\alpha}\left[2(\alpha-1)^{\alpha-1}[\alpha-1+\beta(3-\alpha)]-(1-\beta(3-\alpha))[\alpha-1+\beta(3-\alpha)]^{\alpha}\right]} \tag{4.13}$$
and in particular, for IOEP P9, this bound is
$$\lambda \geq \frac{12}{(b-a)^3} \tag{4.14}$$

3. *and CSILB*
$$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{(\alpha-1)^2}{[2-(3-\alpha)(1-\beta)]^2(2\alpha-3)[5-2(3-\alpha)(1-\beta)]}+\frac{1}{2\alpha(2\alpha-1)}-\frac{2(\alpha-1)C_1(\alpha)}{\alpha[2-(3-\alpha)(1-\beta)]}\right]^{-1/2}, \tag{4.15}$$
where $C_1(\alpha) = \int_0^1 t^{\alpha-(3-\alpha)(1-\beta)+2} {}_2F_1(2-\alpha, 1; \alpha+1; t)dt$ and in particular, for IOEP P9, CSILB is
$$\lambda \geq \frac{\sqrt{315}}{(b-a)^3}. \tag{4.16}$$

Proof. Substituting $q(t) = \lambda$ in equation (4.8), the inequality in the first part follow. Substituting the Green's function from equation (4.2), in (2.16) and (2.15), and simplifying the results, we obtain the inequalities respectively in equations (4.13) and (4.15). Setting $\alpha = 3$ in equations (4.11), (4.13) and (4.15) we get the inequalities (4.12), (4.14) and (4.16). $\square$

We first consider the integer order case, i.e. $\alpha = 3$, and $\beta = 0$ or $\beta = 1$. For this case, the LILB, SMNLB and CSILB for the smallest λ of FEP P9 with $a = 0$ and $b = 1$ are given as 8, 12 and $\sqrt{315} \simeq 17.7482$, respectively (see equations (4.12), (4.14) and (4.16)) . For $\alpha = 3$, the FEP P9 can be solved in closed form using the tools from integer order calculus. Results show, that the smallest eigenvalue of FEP P9 for $\alpha = 3$ is the root of $\exp(-3\lambda^{1/3}/2) - 2\sin(\sqrt{3}\lambda^{1/3}/2 + \pi/6) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 27.4545$. Comparing this λ with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue.

The FEP P9 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (2, 3]$ and $\beta \in [0, 1]$ as a root of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(3-\alpha)-1}(-\lambda)$. This is discussed in the following theorem.

Theorem 4.1.5. *The FEP P9 with $a = 0$ and $b = 1$, for $2 < \alpha \leq 3$ has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(3-\alpha)-1}(z)$, i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(2-\alpha)-1}(-\lambda) = 0. \tag{4.17}$$

Proof. To prove this, we take Laplace transform of the first equation in P9, using (1.45) for $n = 3$ which after some manipulations leads to

$$Y(s) = \frac{a_0 s^{2-\beta(3-\alpha)}}{s^\alpha + \lambda} + \frac{a_1 s^{1-\beta(3-\alpha)}}{s^\alpha + \lambda} + \frac{a_3 s^{-\beta(3-\alpha)}}{s^\alpha + \lambda}, \tag{4.18}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $a_i = D^i \left[I_{0^+}^{(1-\beta)(3-\alpha)} y \right] (0^+)$, $i = 0, 1, 2.$

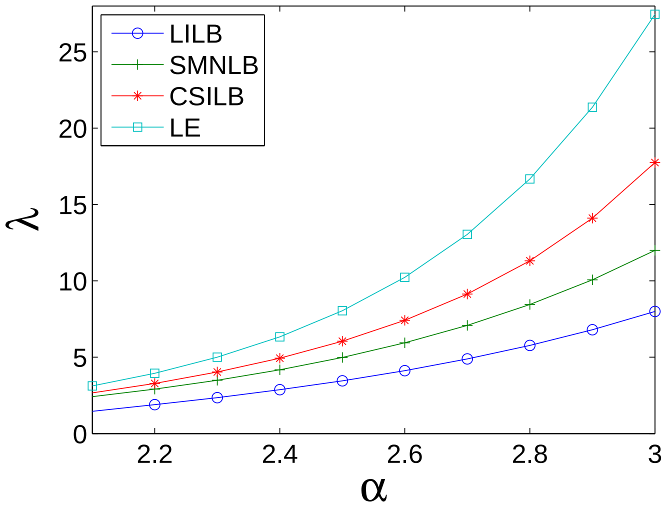

Figure 4.1. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\Box-$: LE - the Lowest Eigenvalue λ)($\beta = 0$, Riemann-Liouville derivative FEP P9)

Taking inverse Laplace transform of equation (4.18) and using equation (1.25), we obtain

$$\begin{aligned} y(t) &= a_0 t^{\alpha+\beta(3-\alpha)-3} E_{\alpha,\alpha+\beta(3-\alpha)-2}(-\lambda t^\alpha) + a_1 t^{\alpha+\beta(3-\alpha)-2} E_{\alpha,\alpha+\beta(3-\alpha)-1}(-\lambda t^\alpha) \\ &+ a_3 t^{\alpha+\beta(3-\alpha)-1} E_{\alpha,\alpha+\beta(3-\alpha)}(-\lambda t^\alpha). \end{aligned} \quad (4.19)$$

Using the boundary conditions of P9 we obtain (4.17). □

For comparison purpose, we compute the smallest eigenvalues for FEP P9 from equation (4.17) and its LILB, SMNLB and CSILB for different α, $\alpha \in (2,3]$ and $\beta = 0, 0.5$ and 1 from equations (4.11), (4.13) and (4.15). The results are shown in figures 4.1, 4.2 and 4.3.

These figures clearly demonstrate that among the three estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP P9. Furthermore, figure 4.2 for $\beta = 0.5$ shows the eigenvalue estimate for mixed behavior of Riemann-Liouville and Caputo derivative FEP. We now

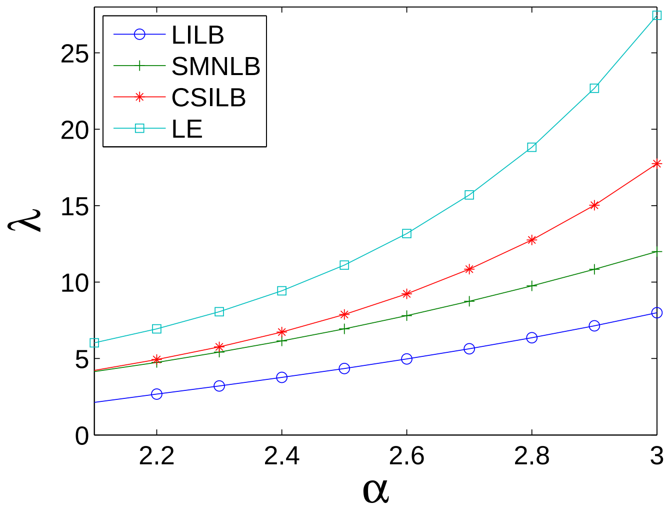

Figure 4.2. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($\beta = 0.5$, mixed behavior of Riemann-Liouville and Caputo derivative FEP P9)

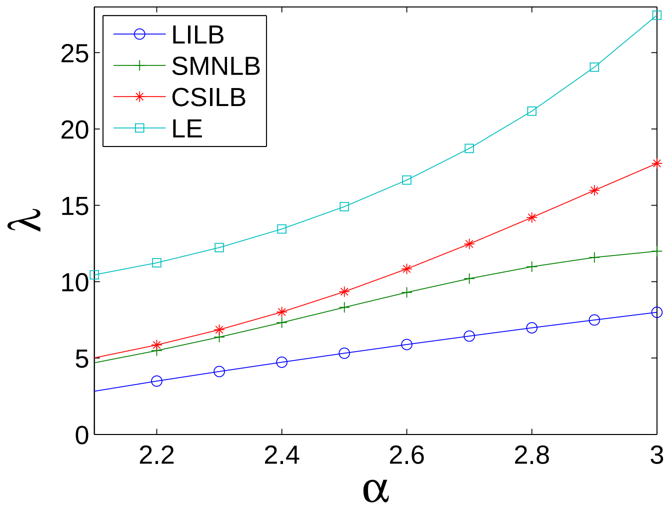

Figure 4.3. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ)($\beta = 1$, Caputo derivative FEP P9)

consider an application of the lower bounds for the smallest eigenvalues of FEP P9 found in Corollary 4.1.4 and Theorem 4.1.5. In [15], [16], [28] and [48], the authors have applied the LILB to the FEPs for $\alpha \in (1, 2]$ to find the interval in which certain Mittag-Leffler functions have no real zeros. On the other hand, we apply the improved lower bounds to to find the interval in which certain Mittag-Leffler functions have no real zeros. The results for $\alpha \in (2, 3]$ are given in the following theorem.

Theorem 4.1.6. *Let $2 < \alpha \leq 3$ and $\beta \in [0, 1]$. Then based on the LILB, SMNLB and CSILB inequalities, the Mittag-Leffler function $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zeros in the following domains:*

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)\,[2-(3-\alpha)(1-\beta)]^{\alpha-1}}{(\alpha-2)^{\alpha-2}}, 0\right], \tag{4.20}$$

SMNLB inequality:

$$z \in \Big(-\frac{\Gamma(\alpha+1)[\alpha-1+\beta(3-\alpha)]^{\alpha+1}}{2(\alpha-1)^{\alpha-1}[\alpha-1+\beta(3-\alpha)]-(1-\beta(3-\alpha))[\alpha-1+\beta(3-\alpha)]^{\alpha}}, 0\Big], \tag{4.21}$$

CSILB inequality:

$$z \in \Big(-\Gamma(\alpha)\left[\tfrac{(\alpha-1)^2}{[2-(3-\alpha)(1-\beta)]^2(2\alpha-3)[5-2(3-\alpha)(1-\beta)]} + \tfrac{1}{2\alpha(2\alpha-1)} - \tfrac{2(\alpha-1)C_1(\alpha)}{\alpha[2-(3-\alpha)(1-\beta)]}\right]^{-1/2}, 0\Big]. \tag{4.22}$$

Proof. Let λ be the smallest eigenvalue of the FEP P9 then $z = \lambda$ is the smallest value of z for which $E_{\alpha,\alpha+\beta(2-\alpha)-1}(-z) = 0$. If there is another z smaller than λ for which $E_{\alpha,\alpha+\beta(2-\alpha)-1}(-z) = 0$, then it will contradict that λ is the smallest eigenvalue. Therefore, $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zero for $z \in (-\lambda, 0]$. Now, according to LILB, $\lambda \geq \frac{\Gamma(\alpha)[2-(3-\alpha)(1-\beta)]^{\alpha-1}}{(\alpha-2)^{\alpha-2}}$ (see equation (4.11)). Thus, $E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zero for $z \in \left(-\frac{\Gamma(\alpha)[2-(3-\alpha)(1-\beta)]^{\alpha-1}}{(\alpha-2)^{\alpha-2}}, 0\right]$. This proves equation (4.20). Equations (4.21) and (4.22) are proved in a similar way. □

From figures 4.1, 4.2 and 4.3, it is clear that among the LILB provides the smallest interval and CSILB provide the largest interval in which the Mittag-Leffler function

$E_{\alpha,\alpha+\beta(2-\alpha)-1}(z)$ has no real zero. We note that for $\beta = 0$ and $\alpha \in (2,3]$ in (4.1) and (4.10) in [43], we obtain the similar results. So, this section is the generalized case of our work in [43].

4.2 LYAPUNOV-TYPE INEQUALITY FOR FBVP WITH A MIXED SET OF FRACTIONAL DIRICHLET, NEUMANN AND FRACTIONAL NEUMANN BOUNDARY CONDITIONS

In this section we obtain Lyapunov-type inequality and eigenvalue estimate for FBVP and FEP with a mixed set of fractional Dirichlet, Neumann and fractional Neumann boundary conditions. Setting $d_2 = d_4 = 1$, $d_1 = d_3 = 0$ and $e_2 = 1, e_1 = 0$ in equation (2.3) we obtain from equation (2.1) with $n = 3$ the FBVP

Problem P10:

$$\begin{aligned}
&\left(D_{a^+}^{\alpha,\beta} y\right)(t) + q(t)y(t) = 0, \quad \alpha \in (2,3], \beta \in [0,1],\ a < t < b,\\
&\left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a) = 0,\ y'(b) = 0,\ \frac{d^2}{dt^2}\left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a) = 0.
\end{aligned} \tag{4.23}$$

Since in (4.23), first, second and third boundary conditions include the fractional Dirichlet, integer order derivative, and composition of derivative and fractional integral respectively, we call the boundary conditions in (4.23) as a mixed set of fractional Dirichlet, Neumann and fractional Neumann boundary conditions. Here we will use procedure similar to the procedure used in the previous section 4.1 to derive the Lyapunov-type inequality for FBVP (4.23). We will do this by finding the Green's function $G(t,s)$.

Lemma 4.2.1. *Problem P10 can be written as (2.7) where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(\alpha-1)(t-a)^{1-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{-(3-\alpha)(1-\beta)}[1-(3-\alpha)(1-\beta)]} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \frac{(\alpha-1)(t-a)^{1-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{-(3-\alpha)(1-\beta)}[1-(3-\alpha)(1-\beta)]}, & a \le t \le s \le b, \end{cases} \tag{4.24}$$

is the Green's function for the problem.

Proof. The proof is similar to Lemma 4.1.1. $\square$

Lemma 4.2.2. *The function G defined in Lemma 4.2.1 satisfies the following property:*

$$|G(t,s)| \leq \frac{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}{\Gamma(\alpha)[1-(3-\alpha)(1-\beta)]^{\alpha-1}}, \quad (t,s) \in [a,b] \times [a,b]. \tag{4.25}$$

Proof. For $a \leq t \leq s \leq b$, $G(t,s)$ is an increasing function of t we get for $a \leq s \leq t \leq b$ and fixed $s \in (a,b)$, $\frac{\partial G}{\partial t} \geq 0$. Which gives

$$\max_{t\in[a,b]} |G(t,s)| = |G(b,s)| = \frac{(\alpha-1)(b-s)^{\alpha-2} - [1-(3-\alpha)(1-\beta)](b-s)^{\alpha-1}}{\Gamma(\alpha)[1-(3-\alpha)(1-\beta)]}.$$

This proves the inequality (4.25) after some calculations. $\square$

Theorem 4.2.3. *If a nontrivial continuous solution of the FBVP P10 exists, then the Lyapunov-type inequality is given by*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)\,[1-(3-\alpha)(1-\beta)]^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}, \tag{4.26}$$

and for integer order case $\alpha = 3$ and $\beta = 0$ or $\beta = 1$ in (4.23) the LTI is given by

$$\int_a^b |q(s)|ds \geq \frac{2}{(b-a)^2}, \tag{4.27}$$

Proof. Using equation (4.25) into (2.11), proves the inequality (4.26) and taking $\alpha = 3$ and $\beta = 0$ or $\beta = 1$ in (4.26), we get the inequality in equation (4.27). $\square$

4.2.1 Eigenvalue Estimate For Fractional Eigenvalue Problem With a Mixed Set of fractional Dirichlet, Neumann and fractional Neumann Boundary Conditions

We consider the FEP by taking $d_2 = d_4 = 1$, $d_1 = d_3 = 0$ and $e_2 = 1, e_1 = 0$ in equation (2.3) and from equation (2.14) with $n = 3$ as follows.

Problem P11:

$$\begin{aligned}
&\left(D_{a^+}^{\alpha,\beta} y\right)(t) + \lambda y(t) = 0, \quad \alpha \in (2,3], \beta \in [0,1],\ a < t < b, \\
&\left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a) = 0,\ y'(b) = 0,\ \frac{d^2}{dt^2}\left(I_{a^+}^{(3-\alpha)(1-\beta)} y\right)(a) = 0.
\end{aligned} \tag{4.28}$$

We use the method similar to section 4.1.1 to obtain the eigenvalue estimate for FEP P11.

Corollary 4.2.4. *Let λ be the smallest eigenvalue of FEP P11. Then the smallest eigenvalue estimates of FEP P11 for $\alpha \in (2,3]$ and $\beta \in [0,1]$ are given by*

1. *the LILB*
$$\lambda \geq \frac{\Gamma(\alpha)\left[1-(3-\alpha)(1-\beta)\right]^{\alpha-1}}{(b-a)^{\alpha}(\alpha-2)^{\alpha-2}}. \tag{4.29}$$
and in particular, for IOEP P11, i.e. $\alpha = 3$ and $\beta = 0$ or $\beta = 1$ this bound is
$$\lambda \geq \frac{2}{(b-a)^3} \tag{4.30}$$

2. *the SMNLB*
$$\lambda \geq \frac{\Gamma(\alpha+1)[\alpha-2+\beta(3-\alpha)]}{2(b-a)^{\alpha}(\alpha-1)^{2-\beta(3-\alpha)}}. \tag{4.31}$$
and in particular, for IOEP P11, this bound is
$$\lambda \geq \frac{3}{(b-a)^3} \tag{4.32}$$

3. *and CSILB*
$$\lambda \geq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{(\alpha-1)^2}{[1-(3-\alpha)(1-\beta)]^2(2\alpha-3)[3-2(3-\alpha)(1-\beta)]}+\frac{1}{2\alpha(2\alpha-1)}-\frac{2(\alpha-1)C_1(\alpha)}{\alpha[1-(3-\alpha)(1-\beta)]}\right]^{-1/2}, \tag{4.33}$$
where $C_1(\alpha) = \int_0^1 t^{\alpha-(3-\alpha)(1-\beta)+1}{}_2F_1(2-\alpha, 1; \alpha+1; t)dt$ and in particular, for IOEP P11, CSILB is
$$\lambda \geq \frac{\sqrt{15}}{(b-a)^3}. \tag{4.34}$$

Proof. Setting $q(t) = \lambda$ in equation (4.26), the first inequality in the first part follow. Taking $\alpha = 3$ and $\beta = 0$ or $\beta = 1$ in (4.29), proves (4.30). Substituting the Green's function from equation (4.24), in (2.16) and (2.15), and simplifying the results, we obtain the inequalities respectively in equations (4.31) and (4.33). Setting $\alpha = 3$ in equations (4.29), (4.31) and (4.33) we get the inequalities (4.30), (4.32) and (4.34). □

For $\alpha = 3$, $a = 0$ and $b = 1$, the FEP P11 can be solved in closed form. Results show, that the smallest eigenvalue of FEP P11 for $\alpha = 3$ is the root of $\exp(-\lambda^{1/3}) + 2\exp(\lambda^{1/3}/2)\cos(\sqrt{3}/2\lambda^{1/3}) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 6.3297$. Comparing this λ with its estimate above, it is clear that among LILB, SMNLB and CSILB for integer α, the CSILB provides the best estimate for the smallest eigenvalue.

The FEP P11 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (2, 3]$ and $\beta \in [0, 1]$ as a root of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(3-\alpha)-2}(-\lambda)$.

Theorem 4.2.5. *The FEP P11 for* $2 < \alpha \leq 3$, $a = 0$ *and* $b = 1$ *has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function* $E_{\alpha,\alpha+\beta(3-\alpha)-2}(z)$, *i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(2-\alpha)-2}(-\lambda) = 0. \tag{4.35}$$

Proof. Taking the Laplace transform of the first equation of FEP P11 and using its boundary conditions proves (4.35). $\square$

We compute the smallest eigenvalues for FEP P11 from equation (4.35) and compare it with its LILB, SMNLB and CSILB for different α, $\alpha \in (2, 3]$ and $\beta = 0, 0.5$ and 1 from equations (4.29), (4.31) and (4.33). The results are shown in figures 4.3, 4.4 and 4.5. These figures clearly demonstrate that among the three estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP P11. Moreover, figure 4.5 for $\beta = 0.5$ shows the eigenvalue estimate for mixed behavior of Riemann-Liouville and Caputo derivative FEP. We now consider an application of the lower bounds for the smallest eigenvalues of FEP P11 found in Corollary 4.2.4 and Theorem 4.2.5.

Theorem 4.2.6. *Let* $2 < \alpha \leq 3$, *and* $\beta \in [0, 1]$. *Then based on the LILB, SMNLB and CSILB inequalities, the Mittag-Leffler function* $E_{\alpha,\alpha+\beta(2-\alpha)-2}(z)$ *has no real zeros in the*

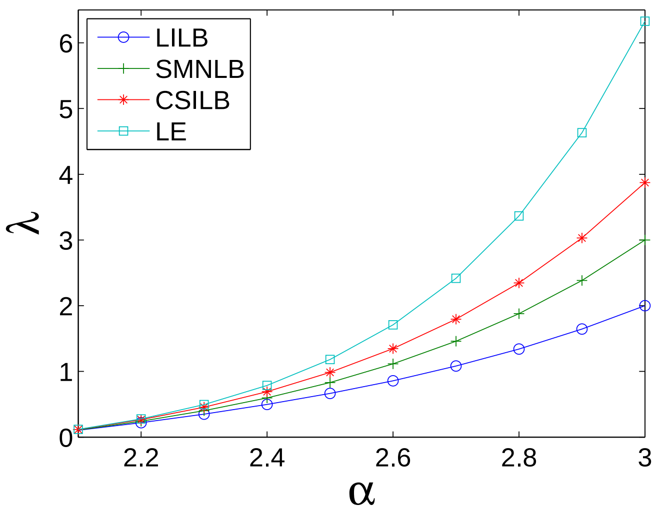

Figure 4.4. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ)($\beta = 0$, Riemann-Liouville derivative FEP P11)

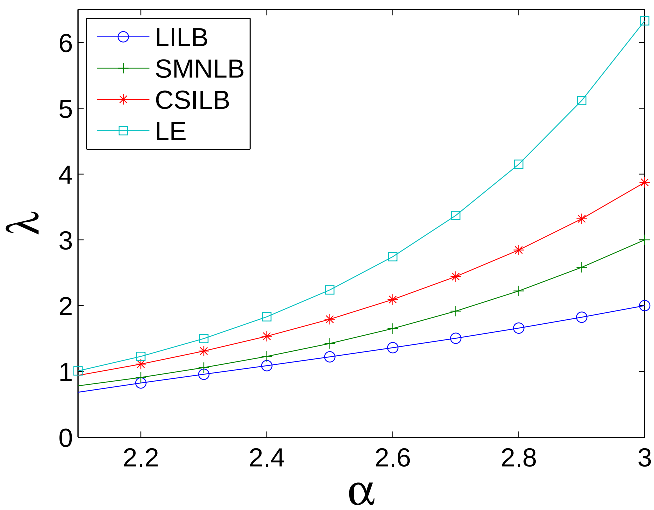

Figure 4.5. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ)($\beta = 0.5$, mixed behavior of Riemann-Liouville and Caputo derivative FEP P11)

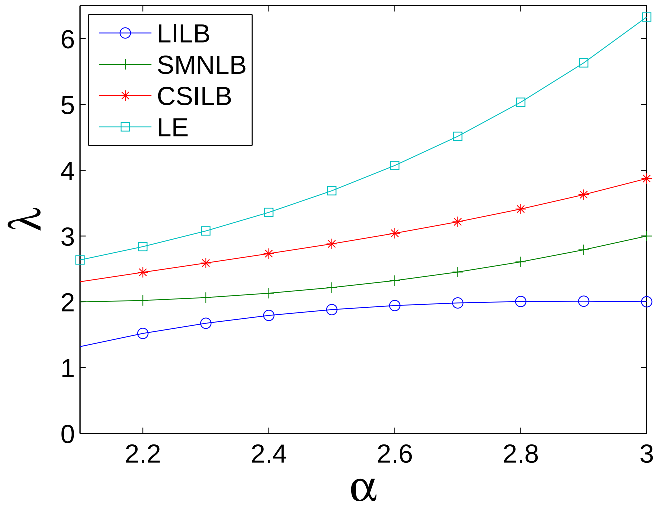

Figure 4.6. Comparison of the lower bounds for λ obtained from maximum norm, Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-+-$: SMNLB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ)($\beta=1$, Caputo derivative FEP P11)

following domains:

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)\left[1-(3-\alpha)(1-\beta)\right]^{\alpha-1}}{(\alpha-2)^{\alpha-2}}, 0\right], \tag{4.36}$$

SMNLB inequality:

$$z \in \Big(-\frac{\Gamma(\alpha+1)[\alpha-2+\beta(3-\alpha)]}{2(\alpha-1)^{2-\beta(3-\alpha)}}, 0\Big], \tag{4.37}$$

CSILB inequality:

$$z \in \Big(-\Gamma(\alpha)\left[\tfrac{(\alpha-1)^2}{[1-(3-\alpha)(1-\beta)]^2(2\alpha-3)[3-2(3-\alpha)(1-\beta)]}+\tfrac{1}{2\alpha(2\alpha-1)}-\tfrac{2(\alpha-1)C_1(\alpha)}{\alpha[1-(3-\alpha)(1-\beta)]}\right]^{-1/2}, 0\Big]. \tag{4.38}$$

Proof. The proof is similar to the proof of Theorem 4.1.6. $\square$

From figures 4.4, 4.5 and 4.6, it is clear that among the three inequalities, CSILB provide the largest interval in which the Mittag-Leffler functions $E_{\alpha,\alpha+\beta(2-\alpha)-2}(z)$, $\beta=0$, $\beta=0.5$ and $\beta=1$ have no real zero.

CHAPTER 5

LYAPUNOV-TYPE INEQUALITY AND EIGENVALUE ESTIMATES FOR FRACTIONAL PROBLEMS OF ORDER α, $\alpha \in (3,4]$

In this chapter we consider the FBVPs and FEPs of order α, $\alpha \in (3,4]$. We consider the FBVP by replacing $q(t)$ by $-q(t)$, $a = 0, b = 1$ and $n = 4$ in (2.1) and λ by $-\lambda$ in (2.14) with boundary conditions B3. i.e we consider the general FBVP as

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) - q(t)y(t) = 0, \quad 0 < t < 1,\ 3 < \alpha \leq 4,\ 0 \leq \beta \leq 1, \tag{5.1}$$

with the boundary conditions B3:

$$y^i(0) = y^i(1) = 0, \quad i = 0, 1 \tag{5.2}$$

or

$$y^i(0) = y^i(1) = 0, \quad i = 0, 2 \tag{5.3}$$

or

$$y^i(0) = y''(1) = 0, \quad i = 0, 1, 2. \tag{5.4}$$

and FEP as

$$\left(D_{a^+}^{\alpha,\beta} y\right)(t) - \lambda y(t) = 0, \quad 0 < t < 1,\ 3 < \alpha \leq 4,\ 0 \leq \beta \leq 1, \tag{5.5}$$

with the boundary conditions (5.2) or (5.3) or (5.4). We establish Lyapunov-type inequalities with different integer order boundary conditions. We also obtain the eigenvalue estimates for the smallest eigenvalue of FEPs using the LILB and CSILB methods discussed in Chapter 2. We apply these estimates to obtain the intervals in which certain Mittag-Leffler functions have no real zeros.

5.1 FRACTIONAL BOUNDARY AND EIGENVALUE PROBLEMS WITH FIRST BOUNDARY CONDITIONS OF B3

In this section we consider the FBVP (5.1) and FEP (5.5) with boundary conditions (5.2).

5.1.1 Lyapunov-type Inequality For Fractional Boundary Value Problem with first Boundary conditions of B3

We first consider FBVP (5.1)-(5.2).

Problem P12:

$$\begin{aligned} &\left(D_{a^+}^{\alpha,\beta} y\right)(t) - q(t)y(t) = 0, \quad 0 < t < 1, \\ &y(0) = 0, y'(0) = 0,\ y(1) = 0, y'(1) = 0. \end{aligned} \tag{5.6}$$

In particular for P12, if $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ the integer order boundary value problem (IOBVP) is

Problem P13:

$$\begin{aligned} &y''''(t) - q(t)y(t) = 0, \quad 0 < t < 1, \\ &y(0) = 0, y'(0) = 0,\ y(1) = 0, y'(1) = 0, \end{aligned} \tag{5.7}$$

which represents the differential equation with clamped beam conditions.

Lemma 5.1.1. *Problem P12 can be written as (2.7) with $a = 0$ and $b = 1$ where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-2}\left\{t^{2-(4-\alpha)(1-\beta)}\left[(\alpha-1) - (3-(4-\alpha)(1-\beta))(1-s)\right]\right. \\ \left. +t^{3-(4-\alpha)(1-\beta)}\left[(2-(4-\alpha)(1-\beta))(1-s) - (\alpha-1)\right]\right\} \\ +(t-s)^{\alpha-1}, 0 \le s \le t \le 1, \\ (1-s)^{\alpha-2}\left\{t^{2-(4-\alpha)(1-\beta)}\left[(\alpha-1) - (3-(4-\alpha)(1-\beta))(1-s)\right]\right. \\ \left. +t^{3-(4-\alpha)(1-\beta)}\left[(2-(4-\alpha)(1-\beta))(1-s) - (\alpha-1)\right]\right\} \\ , 0 \le t \le s \le 1, \end{cases} \tag{5.8}$$

is the Green's function for the problem.

Proof. Taking $I_{0^+}^{\alpha}$ on the first equation of P12 and using Lemma 1.3.4 with $a = 0$ and for $n = 4$, we obtain

$$\begin{aligned} y(t) &= c_0 \frac{t^{-(4-\alpha)(1-\beta)}}{\Gamma(1-(4-\alpha)(1-\beta))} + c_1 \frac{t^{1-(4-\alpha)(1-\beta)}}{\Gamma(2-(4-\alpha)(1-\beta))} + c_2 \frac{t^{2-(4-\alpha)(1-\beta)}}{\Gamma(3-(4-\alpha)(1-\beta))} \\ &+ c_3 \frac{t^{3-(4-\alpha)(1-\beta)}}{\Gamma(4-(4-\alpha)(1-\beta))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)y(s)ds, \end{aligned} \tag{5.9}$$

where $c_i = \frac{d^i}{dt^i}\left(I_{0^+}^{(4-\alpha)(1-\beta)} y\right)(0^+)$, $i = 0, 1, 2, 3$ are the real constants. Applying the first two boundary conditions of P12, we get $c_0 = c_1 = 0$ and using the last two boundary conditions of P12, we obtain

$$c_2 = \frac{\Gamma(3-(4-\alpha)(1-\beta))}{\Gamma(\alpha)} \int_0^1 \left[(\alpha-1) - (3-(4-\alpha)(1-\beta))(1-s)\right](1-s)^{\alpha-2} q(s)y(s)ds.$$

and

$$c_3 = \frac{\Gamma(4-(4-\alpha)(1-\beta))}{\Gamma(\alpha)} \int_0^1 \left[(2-(4-\alpha)(1-\beta))(1-s) - (\alpha-1)\right](1-s)^{\alpha-2} q(s)y(s)ds.$$

Hence, equality (5.9) becomes

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \left\{ t^{2-(4-\alpha)(1-\beta)} \int_0^1 \left[(\alpha-1) - (3-(4-\alpha)(1-\beta))(1-s)\right](1-s)^{\alpha-2} q(s)y(s)ds \right. \\ &+ t^{3-(4-\alpha)(1-\beta)} \int_0^1 \left[(2-(4-\alpha)(1-\beta))(1-s) - (\alpha-1)\right](1-s)^{\alpha-2} q(s)y(s)ds \\ &+ \left. \int_0^t (t-s)^{\alpha-1} q(s)y(s)ds \right\}, \end{aligned}$$

which can be written as equation (2.7) with $G(t,s)$ given by (5.8). This concludes the proof. $\square$

Lemma 5.1.2. *The function G defined in equation (5.8) satisfies the following property:*

$$|G(t,s)| \leq \frac{(\alpha-2)(\alpha-1)^{\alpha-1}[3-(4-\alpha)(1-\beta)]^{3-(4-\alpha)(1-\beta)}}{\Gamma(\alpha)[2\alpha-2+\beta(4-\alpha)]^{2\alpha-2+\beta(4-\alpha)}}, \tag{5.10}$$

$(t,s) \in [0,1] \times [0,1]$.

Proof. It follows from (5.8) that for $0 \le t < s \le 1$, G is an increasing function of t and

$$\begin{aligned} |G(t,s)| \le |G(s,s)| &\le \frac{s^{2-(4-\alpha)(1-\beta)}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[(\alpha-2)s-\beta(4-\alpha)(1-s)\right] \\ &\le \frac{(\alpha-2)s^{3-(4-\alpha)(1-\beta)}(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned} \tag{5.11}$$

For $0 < s < t \le 1$, we have $(1-s)^{\alpha-2} > (1-s/t)^{\alpha-2}$, $2-3t < 1$ and $1-s > 1-t$. Now, for a fixed $s \in (0,1)$ let us define

$$\begin{aligned} \Gamma(\alpha)\psi_s(t) &= (t-s)^{\alpha-1} + (1-s)^{\alpha-2}\left\{t^{2-(4-\alpha)(1-\beta)}\left[(\alpha-1)-(3-(4-\alpha)(1-\beta))(1-s)\right]\right. \\ &\left. +t^{3-(4-\alpha)(1-\beta)}\left[(2-(4-\alpha)(1-\beta))(1-s)-(\alpha-1)\right]\right\}. \end{aligned}$$

Taking the time derivative of this equation and after some calculations gives

$$\begin{aligned} \Gamma(\alpha)\psi_s'(t) &= t^{\alpha-2}\left\{(\alpha-1)(1-s/t)^{\alpha-2} + t^{\beta(4-\alpha)-1}(1-s)^{\alpha-2}\left\{(\alpha-1)\left[(2-3t)-(1-t)\right.\right.\right. \\ &\left.\left.\left.(4-\alpha)(1-\beta)\right]-(1-s)(1-t)(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))\right\}\right\} \\ &< t^{\alpha-2}(1-s)^{\alpha-2}\left\{(\alpha-1)+t^{\beta(4-\alpha)-1}\left[(\alpha-1)-(\alpha-1)(4-\alpha)(1-\beta)(1-t)\right.\right. \\ &- \left.\left.(1-t)^2(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))\right]\right\} < 0, t \in (s,1). \end{aligned}$$

On the other hand, we have

$$\lim_{t \to s^+} \psi_s(t) = |G(s,s)|.$$

Hence, for $0 < s < t \le 1$, we have

$$|G(t,s)| \le \max\left\{|\psi_s(s)|, |\psi_s(1)|\right\}. \tag{5.12}$$

However, $\psi_s(1) = 0$. Hence we get

$$\max_{t \in [0,1]} |G(t,s)| \le |G(s,s)|. \tag{5.13}$$

To prove equation (5.10), let us take in equation (5.11)

$$\phi(s) = s^{3-(4-\alpha)(1-\beta)}(1-s)^{\alpha-1}, s \in [0,1].$$

Differentiating $\phi(s)$ with respect to s, and setting $\phi'(s)$ to 0, we obtain that $\phi(s)$ has an extremum at $s = s^* = (3-(4-\alpha)(1-\beta))/(2\alpha-2+\beta(4-\alpha))$, $s^* \in (0,1)$. We notice that $\phi''(s) < 0$ at $s = s^*$. which indicates that $\phi(s)$ is maximum at $s = s^*$, and

$$\max_{0\leq s\leq 1} \phi(s) = \phi(s^*) = \frac{(3-(4-\alpha)(1-\beta))^{3-(4-\alpha)(1-\beta)}(\alpha-1)^{\alpha-1}}{[2\alpha-2+\beta(4-\alpha)]^{2\alpha-2+\beta(4-\alpha)}},$$

which together with equations (5.13) and (5.11) proves (5.10). □

Theorem 5.1.3. *If a nontrivial continuous solution of the FBVP P12 exists, then the LTI is given by*

$$\int_0^1 |q(s)|ds \geq \frac{\Gamma(\alpha)[2\alpha-2+\beta(4-\alpha)]^{2\alpha-2+\beta(4-\alpha)}}{(\alpha-2)(\alpha-1)^{\alpha-1}[3-(4-\alpha)(1-\beta)]^{3-(4-\alpha)(1-\beta)}}, \tag{5.14}$$

and in particular, for $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in (5.14) gives the Lyapunov-type inequality for IOBVP P13 as

$$\int_0^1 |q(s)|ds \geq 192. \tag{5.15}$$

Proof. Using (5.10) in ((2.12) proves the inequality (5.14). Replacing $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in (5.14), we obtain (5.15). □

We note that the inequality in (5.15) is the Lyapunov-type inequality for P13, which is obtained by Yang in [54].

5.1.2 Eigenvalue Problem with first boundary conditions of B3 and Eigenvalue Estimates

We consider FEP (5.5) with boundary conditions (5.2) to obtain the FEP Problem P14:

$$\begin{aligned} \left(D_{a^+}^{\alpha,\beta} y\right)(t) - \lambda y(t) = 0, \quad 0 < t < 1 \\ y(0) = 0, y'(0) = 0, \ y(1) = 0, y'(1) = 0. \end{aligned} \tag{5.16}$$

Corollary 5.1.4. *Let λ be the smallest eigenvalue of FEP P14 for $\alpha \in (3,4]$ and $\beta \in [0,1]$. Then the smallest eigenvalue estimates of FEP P14 are given by*

1. *the LILB*

$$\lambda \geq \frac{\Gamma(\alpha)[2\alpha - 2 + \beta(4-\alpha)]^{2\alpha-2+\beta(4-\alpha)}}{(\alpha-2)(\alpha-1)^{\alpha-1}[3-(4-\alpha)(1-\beta)]^{3-(4-\alpha)(1-\beta)}} \tag{5.17}$$

and in particular, for integer order eigenvalue problem P14, i.e. $\alpha = 4$ *and* $\beta = 0$ *or* $\beta = 1$ *this bound is*

$$\lambda \geq 192 \tag{5.18}$$

2. *and CSILB*

$$\begin{aligned}
\lambda \;\; &\geq \;\; \Gamma(\alpha)\left\{\frac{2}{\alpha}\left[(\alpha-1)C_1(\alpha) - (3-(4-\alpha)(1-\beta))C_2(\alpha) - (\alpha-1)C_4(\alpha)\right.\right.\\
&+ \;\; \left.(2-(4-\alpha)(1-\beta))C_3(\alpha)\right] + \frac{1}{7-2(4-\alpha)(1-\beta)}\left[\frac{(\alpha-1)^2}{2\alpha-3}\right.\\
&- \;\; \left.\frac{(\alpha+1-\beta(4-\alpha))(2-(4-\alpha)(1-\beta))}{2\alpha-1}\right] + \frac{2-(4-\alpha)(1-\beta)}{2(3-(4-\alpha)(1-\beta))}\\
&- \;\; \frac{(\alpha-1)^2}{(2\alpha-3)(3-(4-\alpha)(1-\beta))} - \frac{2-(4-\alpha)(1-\beta)}{2\alpha-1} + \frac{2\alpha^2-\alpha+1}{2\alpha(2\alpha-1)}\\
&+ \;\; \left.\frac{1}{5-2(4-\alpha)(1-\beta)}\left[\frac{(\alpha-1)^2}{2\alpha-3} - \frac{(\alpha-\beta(4-\alpha))(3-(4-\alpha)(1-\beta))}{2\alpha-1}\right]\right\}^{-1/2},
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
C_1(\alpha) &= \int_0^1 t^{\alpha+2-(4-\alpha)(1-\beta)}{}_2F_1(2-\alpha,1;\alpha+1;t)dt,\\
C_2(\alpha) &= \int_0^1 t^{\alpha+2-(4-\alpha)(1-\beta)}{}_2F_1(1-\alpha,1;\alpha+1;t)dt,\\
C_3(\alpha) &= \int_0^1 t^{\alpha+3-(4-\alpha)(1-\beta)}{}_2F_1(1-\alpha,1;\alpha+1;t)dt,\\
C_4(\alpha) &= \int_0^1 t^{\alpha+3-(4-\alpha)(1-\beta)}{}_2F_1(2-\alpha,1;\alpha+1;t)dt.
\end{aligned}$$

And in particular, for IOEP P14, CSILB is

$$\lambda \geq 1260\sqrt{\frac{11}{71}}. \tag{5.20}$$

Proof. Setting $q(t) = \lambda$ in equations (5.14) and (5.15), the inequalities in the first part follow. Substituting the Green's function from equation (5.8), in (2.13) and simplifying the result, we obtain the inequality in equation (5.19). Setting $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in equation (5.19), we get the inequality (5.20). $\square$

We first consider the integer order case, i.e. $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in equation (5.16) (IOEP P14). For this case, the LILB and CSILB for the smallest λ of IOEP P14 are given as 192 and $1260\sqrt{\frac{11}{71}} \simeq 495.95$, respectively (see equations (5.18) and (5.20)). The IOEP P14 can be solved in closed form. Result shows, that the smallest eigenvalue of IOEP P14 is the root of $\cosh(\lambda^{1/4})\cos(\lambda^{1/4}) = 1$, which gives the smallest eigenvalue as $\lambda \simeq 500.564$. Comparing this λ with its estimate above, it is clear that among LILB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue. The FEP P14 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (3, 4]$ as a root of a combination of Mittag-Leffler functions. This is explained in the following theorem.

Theorem 5.1.5. *For $3 < \alpha \leq 4$ and $\beta \in [0, 1]$, the FEP P14 has an infinite number of eigenvalues, and they are the roots of combination of the Mittag-Leffler functions*

$$E_{\alpha,\alpha+\beta(4-\alpha)-2}(z)E_{\alpha,\alpha+\beta(4-\alpha)}(z) - (E_{\alpha,\alpha+\beta(4-\alpha)-1}(z))^2,$$

i.e. the eigenvalues satisfy

$$E_{\alpha,\alpha+\beta(4-\alpha)-2}(\lambda)E_{\alpha,\alpha+\beta(4-\alpha)}(\lambda) - (E_{\alpha,\alpha+\beta(4-\alpha)-1}(\lambda))^2 = 0. \tag{5.21}$$

Proof. To prove this, we take Laplace transform of the first equation in P14 using (1.45) for $n = 4$, we obtain

$$Y(s) = \frac{a_0 s^{3-\beta(4-\alpha)}}{s^\alpha - \lambda} + \frac{a_1 s^{2-\beta(4-\alpha)}}{s^\alpha - \lambda} + \frac{a_2 s^{1-\beta(4-\alpha)}}{s^\alpha - \lambda} + \frac{a_3 s^{-\beta(4-\alpha)}}{s^\alpha - \lambda}, \tag{5.22}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $a_i = D^i\left[I_{0^+}^{(1-\beta)(4-\alpha)}y\right](0^+)$, $i = 0, 1, 2, 3$.

Taking inverse Laplace transform of equation (5.22) using equation (1.25), we get

$$\begin{aligned} y(t) &= a_0 t^{\alpha+\beta(4-\alpha)-4} E_{\alpha,\alpha+\beta(4-\alpha)-3}(\lambda t^\alpha) + a_1 t^{\alpha+\beta(4-\alpha)-3} E_{\alpha,\alpha+\beta(4-\alpha)-2}(\lambda t^\alpha) \\ &+ a_2 t^{\alpha+\beta(4-\alpha)-2} E_{\alpha,\alpha+\beta(4-\alpha)-1}(\lambda t^\alpha) + a_3 t^{\alpha+\beta(4-\alpha)-1} E_{\alpha,\alpha+\beta(4-\alpha)}(\lambda t^\alpha). \end{aligned} \tag{5.23}$$

Using the boundary conditions of P14 in (5.23), we obtain (5.21). □

For comparison purpose, we compute the smallest eigenvalues for FEP P14 from equation (5.21) and its LILB and CSILB for $\beta = 0$ and $\beta = 1$ for different α, $\alpha \in (3, 4]$ from equations (5.17) and (5.19). The results are shown in figures 5.1 and 5.2 respectively. These figures clearly demonstrate that among the two estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP P14 for $\beta = 0$ and $\beta = 1$. We use MATHEMATICA to find the smallest eigenvalues of combinations of the Mittag-Leffler functions. For solving equation (5.21) taking $\beta = 1$ and $\alpha \in (3, 4]$ for λ, we examine using MATHEMATICA that

$$E_{\alpha,2}(\lambda)E_{\alpha,4}(\lambda) - (E_{\alpha,3}(\lambda))^2 = 0 \tag{5.24}$$

has no solution for $\alpha = 3.1$ to $\alpha = 3.469391976$. Whereas for $\alpha = 3.469391977$ to $\alpha = 4$, equation (5.24) has solutions. For $\alpha = 4$ the smallest eigenvalue of (5.24) is 500.56390. Hence, for the combination of Mittag-Leffler functions (5.24), we calculate the eigenvalues for $\alpha = 3.469391977, 3.5, 3.6, 3.7, 3.8, 3.9$ and 4. Which is shown in figure 5.2.

We apply the improved bounds to obtain the interval in which the combination of Mittag-Leffler functions have no real zeros in the following theorem.

Theorem 5.1.6. *Let* $3 < \alpha \leq 4$ *if* $\beta \in [0, 1)$, *and* $3.469391977 \leq \alpha \leq 4$ *if* $\beta = 1$. *Then based on the LILB and CSILB inequalities, the combination of Mittag-Leffler functions*

$$E_{\alpha,\alpha+\beta(4-\alpha)-2}(z)E_{\alpha,\alpha+\beta(4-\alpha)}(z) - (E_{\alpha,\alpha+\beta(4-\alpha)-1}(z))^2 \tag{5.25}$$

have no real zeros in the following domains:

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)[2\alpha - 2 + \beta(4-\alpha)]^{2\alpha-2+\beta(4-\alpha)}}{(\alpha-2)(\alpha-1)^{\alpha-1}[3-(4-\alpha)(1-\beta)]^{3-(4-\alpha)(1-\beta)}}, 0 \right], \tag{5.26}$$

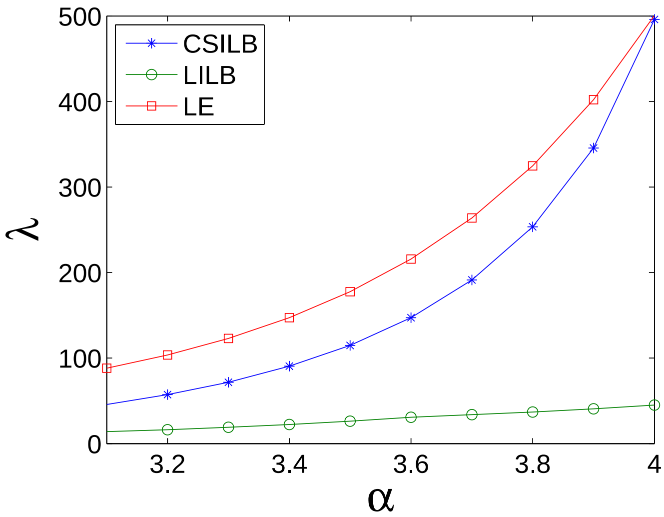

Figure 5.1. Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($\beta = 0$, Riemann-Liouville derivative FEP P14)

CSILB inequality:

$$\begin{aligned}
z \;&\in\; \Bigg(-\Gamma(\alpha)\Bigg\{\frac{2}{\alpha}\left[(\alpha-1)C_1(\alpha)-(3-(4-\alpha)(1-\beta))C_2(\alpha)-(\alpha-1)C_4(\alpha)\right.\\
&+\; \left.(2-(4-\alpha)(1-\beta))C_3(\alpha)\right]+\frac{1}{7-2(4-\alpha)(1-\beta)}\left[\frac{(\alpha-1)^2}{2\alpha-3}\right.\\
&-\; \left.\frac{(\alpha+1-\beta(4-\alpha))(2-(4-\alpha)(1-\beta))}{2\alpha-1}\right]+\frac{2-(4-\alpha)(1-\beta)}{2(3-(4-\alpha)(1-\beta))}\\
&-\; \frac{(\alpha-1)^2}{(2\alpha-3)(3-(4-\alpha)(1-\beta))}-\frac{2-(4-\alpha)(1-\beta)}{2\alpha-1}+\frac{2\alpha^2-\alpha+1}{2\alpha(2\alpha-1)}\\
&+\; \frac{1}{5-2(4-\alpha)(1-\beta)}\left[\frac{(\alpha-1)^2}{2\alpha-3}-\frac{(\alpha-\beta(4-\alpha))(3-(4-\alpha)(1-\beta))}{2\alpha-1}\right]\Bigg\}^{-1/2},0\Bigg].
\end{aligned} \tag{5.27}$$

Proof. Let λ be the smallest eigenvalue of the FEP P14, then $z = \lambda$ is the smallest value of z for which

$$E_{\alpha,\alpha+\beta(4-\alpha)-2}(z)E_{\alpha,\alpha+\beta(4-\alpha)}(z)-(E_{\alpha,\alpha+\beta(4-\alpha)-1}(z))^2=0. \tag{5.28}$$

If there is another z smaller than λ for which equation (5.28) is satisfied by z, then it

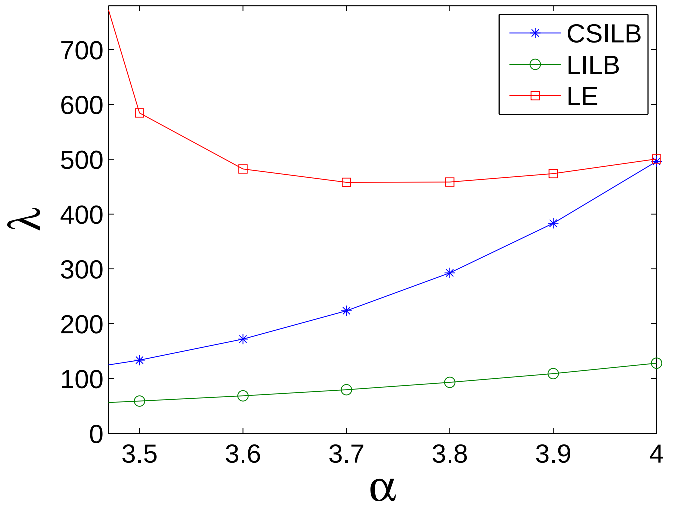

Figure 5.2. Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($\beta = 1$, Caputo derivative FEP P14)

will contradict that λ is the smallest eigenvalue. Therefore, (5.25) has no real zero for $z \in (-\lambda, 0]$. Now, according to LILB,

$$\lambda \geq \frac{\Gamma(\alpha)[2\alpha - 2 + \beta(4-\alpha)]^{2\alpha-2+\beta(4-\alpha)}}{(\alpha-2)(\alpha-1)^{\alpha-1}[3-(4-\alpha)(1-\beta)]^{3-(4-\alpha)(1-\beta)}}$$

(see equation (5.17)). Thus, (5.28) has no real zero for

$$z \in \left(-\frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}, 0\right].$$

This proves equation (5.26). Equation (5.27) can be proved in a similar way. $\square$

From figures 5.1 and 5.2, it is clear that among the two inequalities discussed in this chapter, LILB provides the smallest interval, and CSILB provides the largest interval in which the combination of Mittag-Leffler functions have no real zero. Particularly, we discuss two cases, $\beta = 0$ and $\beta = 1$.

In the succeeding sections 5.2-5.3, we follow the same procedure as we discussed in section 5.1 to obtain the Lyapunov-type inequalities for the FBVPs and eigenvalue estimates for the FEPs with other integer order boundary conditions of B3. We omit the proof of some results in the following sections 5.2 and 5.3.

5.2 FRACTIONAL BOUNDARY AND EIGENVALUE PROBLEMS WITH SECOND BOUNDARY CONDITIONS OF B3

In this section we consider the FBVP (5.2) and FEP (5.5) with the boundary conditions (5.3).

5.2.1 Lyapunov-type Inequality For Fractional Boundary Value Problem with second Boundary conditions of B3

We first consider FBVP (5.2) and (5.3).

Problem P15:

$$\begin{aligned} \left(D_{a^+}^{\alpha,\beta} y\right)(t) - q(t)y(t) = 0, \quad 0 < t < 1, \\ y(0) = 0, y''(0) = 0,\ y(1) = 0, y''(1) = 0. \end{aligned} \tag{5.29}$$

In particular for P15, if $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ then the IOBVP is

Problem P16:

$$\begin{aligned} y''''(t) - q(t)y(t) = 0, \quad 0 < t < 1, \\ y(0) = 0, y''(0) = 0,\ y(1) = 0, y''(1) = 0, \end{aligned} \tag{5.30}$$

which represents the differential equation with a simply-supported beam boundary conditions.

Lemma 5.2.1. *Problem P15 can be written as (2.7) where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(1-s)^{\alpha-3}}{2[3-2(4-\alpha)(1-\beta)]} \left\{t^{1-(4-\alpha)(1-\beta)} \left[(\alpha-1)(\alpha-2) - (3-(4-\alpha)(1-\beta))\right.\right. \\ \left.(2-(4-\alpha)(1-\beta))(1-s)^2\right] - t^{3-(4-\alpha)(1-\beta)} \left[(4-\alpha)(1-\beta)\right. \\ \left.\left.(1-(4-\alpha)(1-\beta))(1-s)^2 + (\alpha-1)(\alpha-2)\right]\right\} \\ +(t-s)^{\alpha-1}, 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-3}}{2[3-2(4-\alpha)(1-\beta)]} \left\{t^{1-(4-\alpha)(1-\beta)} \left[(\alpha-1)(\alpha-2) - (3-(4-\alpha)(1-\beta))\right.\right. \\ \left.(2-(4-\alpha)(1-\beta))(1-s)^2\right] - t^{3-(4-\alpha)(1-\beta)} \left[(4-\alpha)(1-\beta)\right. \\ \left.\left.(1-(4-\alpha)(1-\beta))(1-s)^2 + (\alpha-1)(\alpha-2)\right]\right\}, 0 \le t \le s \le 1, \end{cases} \tag{5.31}$$

is the Green's function for the problem.

Proof. The proof is similar to Lemma 5.1.1. □

Lemma 5.2.2. *The Green's function defined in equation (5.31) satisfies the following property:*

$$|G(t,s)| \le \frac{(\alpha-1)(\alpha-2)(\alpha-3)^{\alpha-3}[\alpha-3+\beta(4-\alpha)]^{\alpha-3+\beta(4-\alpha)}}{2\Gamma(\alpha)[2\alpha-6+\beta(4-\alpha)]^{2\alpha-6+\beta(4-\alpha)}[3-2(4-\alpha)(1-\beta)]}, \tag{5.32}$$

$(t,s) \in [0,1] \times [0,1]$.

Proof. The proof is similar to Lemma 5.1.2. □

Theorem 5.2.3. *If a nontrivial continuous solution of the FBVP P15 exists, then the LTI is given by*

$$\int_0^1 |q(s)|ds \ge \frac{2\Gamma(\alpha)[2\alpha-6+\beta(4-\alpha)]^{2\alpha-6+\beta(4-\alpha)}[3-2(4-\alpha)(1-\beta)]}{(\alpha-1)(\alpha-2)(\alpha-3)^{\alpha-3}[\alpha-3+\beta(4-\alpha)]^{\alpha-3+\beta(4-\alpha)}}, \tag{5.33}$$

and in particular, for $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in (5.29) gives the Lyapunov-type inequality for IOBVP P16 as

$$\int_0^1 |q(s)|ds \ge 24. \tag{5.34}$$

Proof. Using Lemma 5.2.2 in LTI equation (2.11), we obtain the inequality (5.33). Setting $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in (5.33), proves (5.34). $\square$

5.2.2 Eigenvalue Problem with second Boundary conditions of B3 and Eigenvalue Estimates

Let us consider FEP (5.5) with the boundary conditions (5.3) given as follows.

Problem P17:

$$\begin{gathered}\left(D_{a^+}^{\alpha;\beta} y\right)(t) - \lambda y(t) = 0, \quad 0 < t < 1 \\ y(0) = 0, y''(0) = 0,\; y(1) = 0, y''(1) = 0.\end{gathered} \tag{5.35}$$

Corollary 5.2.4. *Let λ be the smallest eigenvalue of FEP P17 for $\alpha \in (3, 4]$ and $\beta \in [0, 1]$. Then the smallest eigenvalue estimates of FEP P17 are given by*

1. *the LILB*

$$\lambda \geq \frac{2\Gamma(\alpha)[2\alpha - 6 + \beta(4-\alpha)]^{2\alpha-6+\beta(4-\alpha)}[3 - 2(4-\alpha)(1-\beta)]}{(\alpha-1)(\alpha-2)(\alpha-3)^{\alpha-3}[\alpha - 3 + \beta(4-\alpha)]^{\alpha-3+\beta(4-\alpha)}} \tag{5.36}$$

and in particular, for integer order eigenvalue problem (IOEP) P18, i.e. $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ this bound is

$$\lambda \geq 24, \tag{5.37}$$

2. and CSILB

$$
\begin{aligned}
\lambda \;\geq\; & \Gamma(\alpha)\left\{\frac{1}{\alpha(3-2(4-\alpha)(1-\beta))}\left[(\alpha-1)(\alpha-2)C_1(\alpha)-(\alpha-1)(\alpha-2)\right.\right.\\
& C_4(\alpha)-(3-(4-\alpha)(1-\beta))(2-(4-\alpha)(1-\beta))C_2(\alpha)-(4-\alpha)\\
& \left.(1-\beta)(1-(4-\alpha)(1-\beta))C_3(\alpha)\right]+\left[\frac{(\alpha-1)^2(\alpha-2)^2}{2\alpha-5}\right.\\
+ \;& \frac{(3-(4-\alpha)(1-\beta))^2(2-(4-\alpha)(1-\beta))^2}{2\alpha-1}-(3-(4-\alpha)(1-\beta))\cdot\\
& \left.(2-(4-\alpha)(1-\beta))\frac{2(\alpha-1)(\alpha-2)}{2\alpha-3}\right]\frac{1}{4[3-2(4-\alpha)(1-\beta)]^3}\\
& +\frac{1}{2\alpha(2\alpha-1)}+\left[\frac{(\alpha-1)^2(\alpha-2)^2}{2\alpha-5}+\frac{(4-\alpha)^2(1-\beta)^2(1-(4-\alpha)(1-\beta))^2}{2\alpha-1}\right.\\
+ \;& \left.\frac{2(\alpha-1)(\alpha-2)(4-\alpha)(1-\beta)(1-(4-\alpha)(1-\beta))}{2\alpha-3}\right]\cdot\\
& \frac{1}{4[3-2(4-\alpha)(1-\beta)]^2(7-2(4-\alpha)(1-\beta))}\\
- \;& \frac{1}{2[3-2(4-\alpha)(1-\beta)]^2(5-2(4-\alpha)(1-\beta))}\left[\frac{(\alpha-1)^2(\alpha-2)^2}{2\alpha-5}+\frac{(\alpha-1)}{2\alpha-3}\right.\\
& (\alpha-2)\left[(4-\alpha)(1-\beta)(1-(4-\alpha)(1-\beta))-(3-(4-\alpha)(1-\beta))\right.\\
& \left.(2-(4-\alpha)(1-\beta))\right]-(1-(4-\alpha)(1-\beta))(4-\alpha)(1-\beta)\\
& \left.\left.\frac{(3-(4-\alpha)(1-\beta))(2-(4-\alpha)(1-\beta))}{2\alpha-1}\right]\right\}^{-1/2}, \qquad (5.38)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(\alpha) &= \int_0^1 t^{\alpha+1-(4-\alpha)(1-\beta)}{}_2F_1(3-\alpha,1;\alpha+1;t)dt,\\
C_2(\alpha) &= \int_0^1 t^{\alpha+1-(4-\alpha)(1-\beta)}{}_2F_1(1-\alpha,1;\alpha+1;t)dt,\\
C_3(\alpha) &= \int_0^1 t^{\alpha+3-(4-\alpha)(1-\beta)}{}_2F_1(1-\alpha,1;\alpha+1;t)dt,\\
C_4(\alpha) &= \int_0^1 t^{\alpha+3-(4-\alpha)(1-\beta)}{}_2F_1(3-\alpha,1;\alpha+1;t)dt.
\end{aligned}
$$

And in particular, for IOEP P17, CSILB is

$$\lambda \geq 15\sqrt{42}. \qquad (5.39)$$

Proof. Setting $q(t) = \lambda$ in equations (5.33) and (5.34), the inequalities in the first part follow. Substituting the Green's function from equation (5.31), in (2.15) with $a = 0$ and $b = 1$, and simplifying the result, we obtain the inequality in equation (5.38). Setting $\alpha = 4$, and $\beta = 0$ or $\beta = 1$, in equation (5.38), we get the inequality (5.39). $\square$

We first consider the integer order case, i.e. $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in equation (5.35) (IOEP P17). For this case, the LILB and CSILB for the smallest λ of IOEP P17 are given as 24 and $15\sqrt{42} \simeq 97.211$, respectively (see equations (5.37), and (5.39). For $\alpha = 4$, the IOEP P17 can be solved in closed form. Result shows, that the smallest eigenvalue of P18 is the root of $\sinh(\lambda^{1/4})\sin(\lambda^{1/4}) = 0$, which gives the smallest eigenvalue as $\lambda = \pi^4 \simeq 97.4091$. Comparing this λ with its estimate above, it is clear that among LILB, and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue. The FEP P17 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (3, 4]$ as a root of the combination of Mittag-Leffler functions. This is explained in the following theorem.

Theorem 5.2.5. *For $3 < \alpha \leq 4$ and $\beta \in [0, 1]$, the FEP P17 has an infinite number of eigenvalues, and they are the roots of combination of the Mittag-Leffler functions*

$$E_{\alpha,\alpha+\beta(4-\alpha)}(z)E_{\alpha,\alpha+\beta(4-\alpha)-4}(z) - (E_{\alpha,\alpha+\beta(4-\alpha)-2}(z))^2(z),$$

i.e. the eigenvalues satisfy

$$E_{\alpha,\alpha+\beta(4-\alpha)}(\lambda)E_{\alpha,\alpha+\beta(4-\alpha)-4}(\lambda) - (E_{\alpha,\alpha+\beta(4-\alpha)-2}(\lambda))^2 = 0. \tag{5.40}$$

Proof. To prove this, we take Laplace transform of the first equation in P19 and we obtain equation (5.22). Taking inverse Laplace transform of equation (5.22) and using equation

(1.25), we get (5.23) as discussed in Theorem 5.1.5. Using the first boundary condition of P17 in (5.23) we get $a_0 = 0$ and taking the time derivative of (5.23) we get

$$\begin{aligned} y'(t) &= a_1 t^{\alpha+\beta(4-\alpha)-4} E_{\alpha,\alpha+\beta(4-\alpha)-3}(\lambda t^\alpha) + a_2 t^{\alpha+\beta(4-\alpha)-3} E_{\alpha,\alpha+\beta(4-\alpha)-2}(\lambda t^\alpha) \\ &+ a_3 t^{\alpha+\beta(4-\alpha)-2} E_{\alpha,\alpha+\beta(4-\alpha)-1}(\lambda t^\alpha). \end{aligned} \tag{5.41}$$

Using the second boundary condition of P17 in (5.41) we get $a_1 = 0$. Taking the time derivative of (5.41) and using the last two boundary conditions of P17 we obtain (5.40). $\square$

Remark. We notice that for $\beta = 1$ in FEP P17 then from (5.41) we have

$$y'(t) = a_1 E_{\alpha,1}(\lambda t^\alpha) + a_2 t E_{\alpha,2}(\lambda t^\alpha) + a_3 t^2 E_{\alpha,3}(\lambda t^\alpha). \tag{5.42}$$

Taking the time derivative of (5.42) using (1.23) we obtain

$$y''(t) = a_1 \lambda t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) + a_2 E_{\alpha,1}(\lambda t^\alpha) + a_3 t E_{\alpha,2}(\lambda t^\alpha),$$

which after using the last three boundary conditions gives

$$\lambda E_{\alpha,\alpha}(\lambda) E_{\alpha,4}(\lambda) - (E_{\alpha,2}(\lambda))^2 = 0. \tag{5.43}$$

Using MATHEMATICA we examine that for $\alpha = 3.1$ to $\alpha = 3.32$, equation (5.43) has no solutions. Whereas, for $\alpha = 3.33$, equation (5.43) has solution. We compute the eigenvalues for $\alpha = 3.33, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9$ and 4 from equation (5.43). Which is shown in figure 5.3.

Remark. If $\beta \in [0, 1]$ in FEP P17 then in equation (5.40) for $\alpha \in (3, 4]$, $\alpha+\beta(4-\alpha)-4 \leq 0$ which does not satisfy the condition of the definition of Mittag-Leffler function (1.19). Hence, equation(5.40) can not be solved for $\beta \in [0, 1]$.

For comparison purpose, we compute the smallest eigenvalues of FEP P17 with $\beta = 1$ from equation (5.43), and its LILB and CSILB for different α, $\alpha \in [3.33, 4]$ and $\beta = 1$ from equations (5.36) and (5.38). The results are shown in figure 5.3. This figure clearly

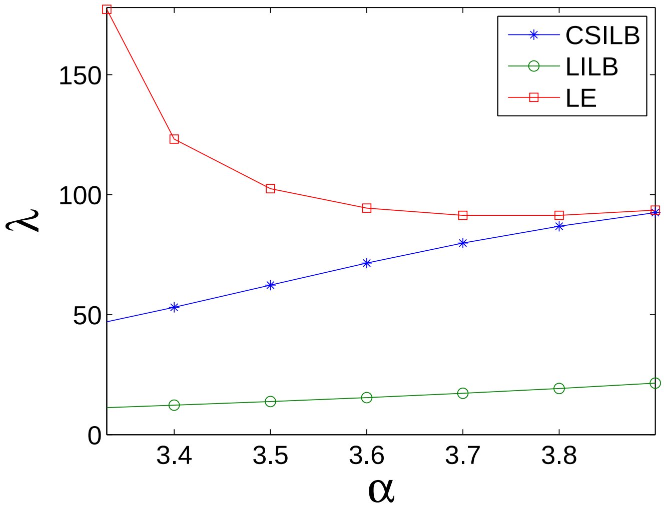

Figure 5.3. Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($\beta = 1$, Caputo derivative FEP P17)

demonstrates that among the two estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP P17 for $\beta = 1$.

We apply the improved bounds to obtain the interval in which the combination of Mittag-Leffler functions have no real zeros in the following theorem.

Theorem 5.2.6. *Let* $3.33 \leq \alpha \leq 4$. *Then based on the LILB and CSILB inequalities, the combination of Mittag-Leffler functions*

$$zE_{\alpha,\alpha}(z)E_{\alpha,4}(z) - (E_{\alpha,2}(z))^2 \tag{5.44}$$

have no real zeros in the following domains:

LILB inequality:

$$z \in \left(-\frac{6\Gamma(\alpha)(\alpha-2)^{\alpha-3}}{(\alpha-1)(\alpha-3)^{\alpha-3}}, 0\right], \tag{5.45}$$

CSILB inequality:

$$z \in \left(-\Gamma(\alpha)\left\{\frac{1}{3\alpha}\left[(\alpha-1)(\alpha-2)C_1(\alpha)-(\alpha-1)(\alpha-2)C_4(\alpha)-6C_2(\alpha)\right]+\frac{1}{2\alpha(2\alpha-1)}\right.\right.$$
$$\left.\left.-\frac{2(\alpha-1)(\alpha-2)}{45(2\alpha-3)}+\frac{2(\alpha-1)^2(\alpha-2)^2}{945(2\alpha-5)}+\frac{1}{3(2\alpha-1)}\right\}^{-1/2},0\right]. \tag{5.46}$$

Proof. Let λ be the smallest eigenvalue of the FEP P17 with $\beta = 1$, then $z = \lambda$ is the smallest value of z for which

$$zE_{\alpha,\alpha}(z)E_{\alpha,4}(z) - (E_{\alpha,2}(z))^2 = 0.$$

If there is another z smaller than λ for which above equation is satisfied by z, then it will contradict that λ is the smallest eigenvalue. Therefore, (5.44) has no real zero for $z \in (-\lambda, 0]$. Now, according to LILB with $\beta = 1$, we get

$$\lambda \geq \frac{6\Gamma(\alpha)(\alpha-2)^{\alpha-3}}{(\alpha-1)(\alpha-3)^{\alpha-3}}$$

(see equation (5.36)). Thus, (5.44) has no real zero for

$$z \in \left(-\frac{6\Gamma(\alpha)(\alpha-2)^{\alpha-3}}{(\alpha-1)(\alpha-3)^{\alpha-3}}, 0\right].$$

This proves equation (5.45). Equation (5.46) can be proved in a similar way by setting $\beta = 1$ in equation (5.38). $\square$

5.3 FRACTIONAL BOUNDARY AND EIGENVALUE PROBLEMS WITH THIRD BOUNDARY CONDITIONS OF B3

In this section we consider by replacing $-q(t)$ by $q(t)$ in FBVP (5.1), and $-\lambda$ by λ in FEP (5.5) with boundary conditions (5.4).

5.3.1 Lyapunov-type Inequality For Fractional Boundary Value Problem with third Boundary conditions of B3

We first consider FBVP (5.1) with the boundary conditions (5.4).

Problem P18:

$$\left(D_{a^+}^{\alpha,\beta}y\right)(t)+q(t)y(t)=0,\quad 0<t<1,$$
$$y(0)=0,y'(0)=0,\ y''(0)=0,y''(1)=0. \tag{5.47}$$

Lemma 5.3.1. *Problem P18 can be written as (2.7) where*

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{(\alpha-1)(\alpha-2)}{[2-(4-\alpha)(1-\beta)][3-(4-\alpha)(1-\beta)]}t^{3-(4-\alpha)(1-\beta)}(1-s)^{\alpha-3}-(t-s)^{\alpha-1},\\ 0\le s\le t\le 1,\\ \frac{(\alpha-1)(\alpha-2)}{[2-(4-\alpha)(1-\beta)][3-(4-\alpha)(1-\beta)]}t^{3-(4-\alpha)(1-\beta)}(1-s)^{\alpha-3},\\ 0\le t\le s\le 1,\end{cases} \tag{5.48}$$

is the Green's function for the problem P18.

Proof. Taking $I_{0^+}^{\alpha}$ on the first equation of P18 and using Lemma 1.3.4 with $a=0$ and for $n=4$, we obtain

$$\begin{aligned}y(t)&=c_0\frac{t^{-(4-\alpha)(1-\beta)}}{\Gamma(1-(4-\alpha)(1-\beta))}+c_1\frac{t^{1-(4-\alpha)(1-\beta)}}{\Gamma(2-(4-\alpha)(1-\beta))}+c_2\frac{t^{2-(4-\alpha)(1-\beta)}}{\Gamma(3-(4-\alpha)(1-\beta))}\\&+c_3\frac{t^{3-(4-\alpha)(1-\beta)}}{\Gamma(4-(4-\alpha)(1-\beta))}-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)y(s)ds,\end{aligned}$$

where $c_i=\frac{d^i}{dt^i}\left(I_{0^+}^{(4-\alpha)(1-\beta)}y\right)(0^+),\ i=0,1,2,3$ are the real constants. Applying the first three boundary conditions of P18, we get $c_0=c_1=c_2=0$ and using the last boundary condition of P18, we obtain

$$c_3=\frac{\Gamma(4-(4-\alpha)(1-\beta))(\alpha-1)(\alpha-2)}{\Gamma(\alpha)(3-(4-\alpha)(1-\beta))(2-(4-\alpha)(1-\beta))}\int_0^1(1-s)^{\alpha-3}q(s)y(s)ds.$$

Hence, we get

$$\begin{aligned}y(t)&=\frac{(\alpha-1)(\alpha-2)t^{3-(4-\alpha)(1-\beta)}}{\Gamma(\alpha)(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))}\int_0^1(1-s)^{\alpha-3}q(s)y(s)ds\\&-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)y(s)ds,\end{aligned}$$

which can be written as equation (2.7) with $G(t,s)$ given by (5.48). This concludes the proof. $\square$

Lemma 5.3.2. *The Green's function defined in equation (5.48) satisfies the following property:*

$$|G(t,s)| \leq |G(1,s)| \leq \frac{2(\alpha-2)^{\frac{\alpha-1}{2}}(\alpha-3)^{\frac{\alpha-3}{2}}}{\Gamma(\alpha)\left[(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))\right]^{\frac{\alpha-1}{2}}}, \tag{5.49}$$

$(t,s) \in [0,1] \times [0,1]$.

Proof. The proof is similar to Lemma 5.1.2. $\square$

Theorem 5.3.3. *If a nontrivial continuous solution of the FBVP P18 exists, then the LTI is given by*

$$\int_0^1 |q(s)|ds \geq \frac{\Gamma(\alpha)\left[(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))\right]^{\frac{\alpha-1}{2}}}{2(\alpha-2)^{\frac{\alpha-1}{2}}(\alpha-3)^{\frac{\alpha-3}{2}}}, \tag{5.50}$$

and in particular, for $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in (5.47) this bound is

$$\int_0^1 |q(s)|ds \geq 9\sqrt{3}. \tag{5.51}$$

Proof. Using Lemma 5.3.2 in equation (2.11) we obtain the inequality (5.50). Setting $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in (5.50), proves (5.51). $\square$

We notice that the inequalities in (5.50) and (5.51) give better estimates than the inequalities given in [40].

5.3.2 Eigenvalue Problem with third Boundary conditions of B3 and Eigenvalue Estimates

We now consider FEP (5.5) with boundary conditions (5.4) as

Problem P19:

$$\left(D_{a^+}^{\alpha,\beta}y\right)(t) + \lambda y(t) = 0, \quad 0 < t < 1$$
$$y(0) = 0, y'(0) = 0, \ y''(0) = 0, y''(1) = 0. \tag{5.52}$$

Corollary 5.3.4. *Let λ be the smallest eigenvalue of FEP P19. Then for $\alpha \in (3,4]$ and $\beta \in [0,1]$, the smallest eigenvalue estimates of FEP P19 are given by*

1. *the LILB*
$$\lambda \geq \frac{\Gamma(\alpha)\left[(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))\right]^{\frac{\alpha-1}{2}}}{2(\alpha-2)^{\frac{\alpha-1}{2}}(\alpha-3)^{\frac{\alpha-3}{2}}} \tag{5.53}$$
and in particular, for IOEP P19, i.e. $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in P19, this bound is
$$\lambda \geq 9\sqrt{3} \tag{5.54}$$

2. *and CSILB*
$$\begin{aligned}\lambda \;\geq\; & \Gamma(\alpha)\left\{\frac{1}{2\alpha(2\alpha-1)} - \frac{2(\alpha-1)(\alpha-2)C_1(\alpha)}{\alpha(3-(4-\alpha)(1-\beta))(2-(4-\alpha)(1-\beta))} + \frac{(\alpha-1)^2}{(2\alpha-5)}\right. \\ & \cdot \left.\frac{(\alpha-2)^2}{(3-(4-\alpha)(1-\beta))^2(2-(4-\alpha)(1-\beta))^2(7-2(4-\alpha)(1-\beta))}\right\}^{-1/2},\end{aligned} \tag{5.55}$$
where $C_1(\alpha) = \int_0^1 t^{\alpha+3-(4-\alpha)(1-\beta)}{}_2F_1(3-\alpha, 1; \alpha+1; t)dt$ and in particular, for IOEP P19, CSILB is
$$\lambda \geq 72\sqrt{\frac{35}{71}}. \tag{5.56}$$

Proof. Setting $q(t) = \lambda$ in equations (5.50) and (5.51), the inequalities in the first part follow. Substituting the Green's function from equation (5.48), in (2.15) with $a = 0$ and $b = 1$, and simplifying the result, we obtain the inequality in equation (5.55). Setting $\alpha = 4$, and $\beta = 0$ or $\beta = 1$, in equation (5.55), we get the inequality (5.56). □

We first consider the integer order case, i.e. $\alpha = 4$ and $\beta = 0$ or $\beta = 1$ in equation (5.52). For this case, the LILB and CSILB for the smallest λ of FEP P19 are given as $9\sqrt{3} \simeq 15.588$ and $72\sqrt{\frac{35}{71}} \simeq 50.5519$, respectively (see equations (5.54) and (5.56)). We notice that equations (5.53) and (5.54) give a better lower bound estimate for the smallest λ than that is given in [40]. For $\alpha = 4$ and $\beta = 0$ or $\beta = 1$, the FEP P19 can be solved in closed form. Result shows, that the smallest eigenvalue of FEP P19 for $\alpha = 4$ and $\beta = 0$ or

$\beta = 1$ is the root of $\sin(\lambda^{1/4}/\sqrt{2})\cosh(\lambda^{1/4}/\sqrt{2}) + \cos(\lambda^{1/4}/\sqrt{2})\sinh(\lambda^{1/4}/\sqrt{2}) = 0$, which gives the smallest eigenvalue as $\lambda \simeq 125.140$. Comparing this λ with its estimate above, it is clear that among LILB and CSILB for integer α the CSILB provides the best estimate for the smallest eigenvalue. The FEP P19 can also be solved and its eigenvalues can be determined for arbitrary α, $\alpha \in (3, 4]$ as a root of the certain Mittag-Leffler function. This is explained in the following theorem.

Theorem 5.3.5. *For $3 < \alpha \leq 4$ and $0 \leq \beta \leq 1$, the FEP P19 has an infinite number of eigenvalues, and they are the roots of the Mittag-Leffler function $E_{\alpha,\alpha+\beta(4-\alpha)-2}(z)$ i.e. the eigenvalues satisfy*

$$E_{\alpha,\alpha+\beta(4-\alpha)-2} = 0. \tag{5.57}$$

Proof. The proof is similar to Theorem 5.1.5. □

For comparison purpose, we compute the smallest eigenvalues of FEP P19 from equation (5.57) and its LILB and CSILB for different α, $\alpha \in (3, 4]$, $\beta = 0$ and $\beta = 1$ from equations (5.53) and (5.55). The results are shown in figures 5.4 and 5.5. These figures clearly demonstrate that among the two estimates considered here, the LILB provides the worse estimate and the CSILB provides the best estimate for the smallest eigenvalues of FEP P19 for $\beta = 0$ and $\beta = 1$.

In [40], the authors have applied the LILB to the FEPs with Riemann-Liouville derivative for $\alpha \in (3, 4]$ to find the interval in which certain Mittag-Leffler functions have no real zeros. On the other hand, we apply the improved bounds to obtain these intervals for certain Mittag-Leffler functions. Which is given in the following theorem.

Theorem 5.3.6. *Let $3 \leq \alpha \leq 4$ and $\beta \in [0, 1]$. Then based on the LILB and CSILB inequalities, the Mittag-Leffler function $E_{\alpha,\alpha+\beta(4-\alpha)-2}(z)$ has no real zeros in the following domains:*

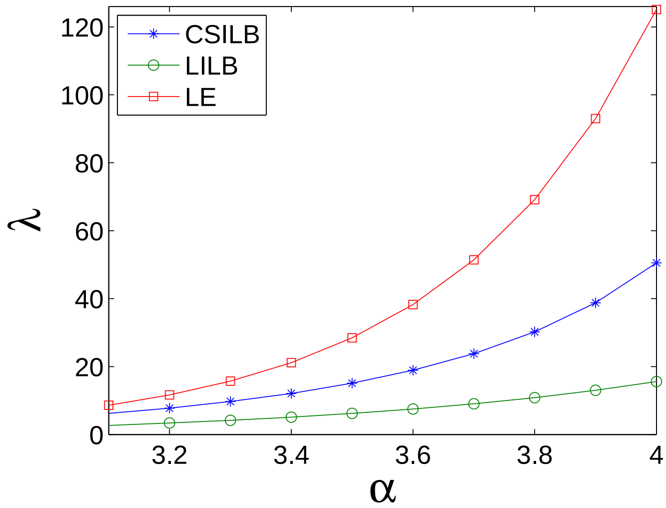

Figure 5.4. Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($\beta = 0$, Riemann-Liouville derivative FEP P19)

LILB inequality:

$$z \in \left(-\frac{\Gamma(\alpha)\left[(2-(4-\alpha)(1-\beta))(3-(4-\alpha)(1-\beta))\right]^{\frac{\alpha-1}{2}}}{2(\alpha-2)^{\frac{\alpha-1}{2}}(\alpha-3)^{\frac{\alpha-3}{2}}}, 0 \right], \tag{5.58}$$

CSILB inequality:

$$\begin{aligned} z \in & \left(-\Gamma(\alpha) \left\{ \frac{1}{2\alpha(2\alpha-1)} - \frac{2(\alpha-1)(\alpha-2)C_1(\alpha)}{\alpha(3-(4-\alpha)(1-\beta))(2-(4-\alpha)(1-\beta))} + \frac{(\alpha-1)^2}{(2\alpha-5)} \right.\right. \\ & \cdot \left.\left. \frac{(\alpha-2)^2}{(3-(4-\alpha)(1-\beta))^2(2-(4-\alpha)(1-\beta))^2(7-2(4-\alpha)(1-\beta))} \right\}^{-1/2}, 0 \right]. \end{aligned} \tag{5.59}$$

Proof. The proof is similar to Theorem 5.1.6. $\square$

From figures 5.4 and 5.5, it is clear that among the two inequalities discussed in this chapter, LILB provides the smallest interval, and CSILB provides the largest interval in which the Mittag-Leffler function $E_{\alpha,\alpha+\beta(4-\alpha)-2}(z)$ has no real zero. Particularly, we discuss two cases, $\beta = 0$ and $\beta = 1$.

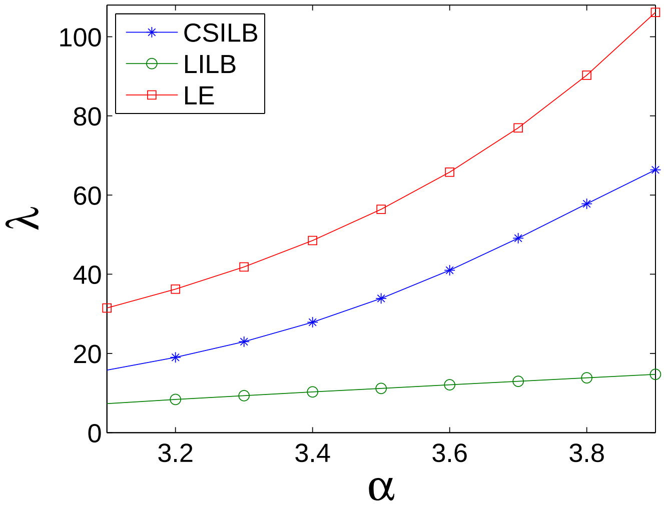

Figure 5.5. Comparison of the lower bounds for λ obtained from Lyapunov-type and Cauchy-Schwarz inequalities with the lowest eigenvalue. ($-\circ-$: LILB; $-*-$: CSILB; $-\square-$: LE - the Lowest Eigenvalue λ) ($\beta = 1$, Caputo derivative FEP P19)

REFERENCES

[1] Agrawal, O.P. *Generalized variational problems and Euler-Lagrange equations*, Computers & Mathematics with Applications, **59** (2010), 1852-1864.

[2] Agrawal, O.P. *Some generalized fractional calculus operators and their applications in integral equations*, Fract. Calc. Appl. Anal. **15**(2012), 700-711.

[3] Aktas, M. F. *Lyapunov-type inequalities for a certain class of n-dimensional quasilinear systems*, Electron. J. Differ. Equ. **67** (2013).

[4] Aleroev, T.S. *The boundary problems for differential equations with fractional derivatives*, Dissertation, doctor of Physical and Mathematical Sciences, Moscow State University, 2000.

[5] Aleroev, T. S. *On the problem of zeros of the Mittag-Leffler function and the spectrum of a differential fractional-order operator*, Differ. Uravn., 36, **9** (2000), 1414-1415 .

[6] Bai, Z and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. of Math. Anal. Appl. **311** (2005), 495-505.

[7] Brown, R. C. and Hinton, D. B. *Lyapunov inequalities and their applications*, in Survey on classical inequalities, 1-25, Math. Appl.**517** Kluwer Acad. Publ., Dordrecht.

[8] Çakmak, D. *Lyapunov-type integral inequalities for certain higher order differential equations*, Appl. Math. Comput. **216** (2010), 368-373.

[9] Çakmak, D. *On Lyapunov-type inequality for a class of nonlinear systems*, Math. Inequal. Appl. **16** (2013), 101-108.

[10] Cheng, S. S. *A discrete analogue of the inequality of Lyapunov*, Hokkaido Math. J. **12** (1983), 105-112.

[11] Das, A.M. and Vatsala, A.S. *Green function for nn boundary value problem and an analogue of Hartman's result*, J. Math. Anal. Appl., **51** (1975), pp. 670-677.

[12] Duan, J. S., Wangc, Z., Liu, Y. L. and Qiu,X. *Eigenvalue problems for fractional ordinary differential equations*, Chaos Solutions Fract., **46** (2013), 46-53.

[13] Dzhrbashyan, M.M. *A boundary value problem for a Sturm-Liouville type differential operator of fractional order*, (in Russian), Izv. Akad. Nauk Armyan. SSR, Ser. Mat. 5, **2** (1970), 71-96.

[14] Eliason, S. B. *A Lyapunov inequality for a certain nonlinear differential equation*, J. Lond. Math. Soc. **2** (1970), 461-466.

[15] Ferreira, R. A. C. *A Lyapunov-type inequality for a fractional boundary value problem*, Fract. Calc. Appl. Anal., 4, **16**, (2013), 978-984.

[16] Ferreira, R. A. C. *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, J. Math. Anal. Appl., 2, **412** (2014), 1058-1063.

[17] Gorenflo, R., Loutchko, J. and Luchko Y. *Computation of the MittagLeffler function* $E_{\alpha,\beta}(z)$ *and its derivative*, Fract Calc Appl Anal.**5** (2002), 491-518.

[18] Hanneken, J. W., Narahari Achar, B. N. and Vaught, D. M. *An Alpha-Beta Phase Diagram Representation of the Zeros and Properties of the Mittag-Leffler Function*, Hindavi Publl. corporation, (2013).

[19] Hartman, P., Wintner, A. *On an oscillation criterion of Liapunov*, Am. J. Math. **73** (1951), 885-890.

[20] Haubold, H.J., Mathai, A.M. and Saxena, R.K. *Mittag-Leffler functions and their applications*, J. Appl. Math. **2011** (2011) 298-628.

[21] He, X, and Tang, X. H. *Lyapunov-type inequalities for even order differential equations*, Commun. Pure Appl. Anal. **11** (2012), 465-473.

[22] He, X. and Zhang, Q. *Lyapunov-type inequalities for a class of even-order differential equations*, J. Inequal. Appl., **5** (2012).

[23] Hilfer, R. *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[24] Hilfer, R. *Experimental evidence for fractional time evolution in glass forming materials*, Chem. Phys. **284** (2002) 399-408.

[25] Hilfer, R., Luchko, Y. and Tomovski, Ž. *Operational method for the solution of frac-*

tional differential equations with generalized RiemannLiouville fractional derivatives, Fract. Calc. Appl. Anal.,3, **12** (2009) 299-318.

[26] Jleli, M., Ragoub, L. and Samet, B. *A Lyapunov-type inequality for a fractional differential equation under a Robin boundary condition*, Journal of Function Spaces. **501** (2014), 468-536.

[27] Jleli, M. and Samet, B. *Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions*, Mathematical Inequalities and applications, 2, **18** (2015), 443-451.

[28] Jleli, M. and Samet, B. *Lyapunov-type inequalities for fractional boundary value problems*, Ele. Journal of Differential Equations, **88** (2015), pp. 111.

[29] Katugampola, U.N. *On Generalized Fractional Integrals and Derivatives*, Ph.D. Dissertation, Southern Illinois University, Carbondale. 2011.

[30] Kilbas, A. A., Srivastava, H. M. and Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[31] Kolmogorov, A. N. and Fomin, S. V. *Elements of Theory of Functions and Functional Analysis*, Nauka, Moscow, 1968.

[32] Kwong, M. K. *On Lyapunovs inequality for disfocality*, J. Math. Anal. Appl. **83** (1981), 486-494.

[33] Linz P., *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

[34] Lyapunov, A. M. *A. M., Probleme général de la stabilité du mouvement*, Ann. Fac. Sci. Univ. Toulouse. **2** (1907), 27–247.

[35] Magin R L, *Fractional Calculus in Bioengineering*, (Redding, CT: Begell House Publisher), 2006.

[36] Miller, K.S. and Ross, B., *An introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[37] Mittag-Leffler, G. M., *Sur la nouvelle function E_α*, C.R. Acad. Sci., Paris. **137**.

[38] Nachushev, A.M. *The fractional calculating and its application*, M. Fiz- matlit. **272** (2003).

[39] Oldham, K.B. and Spanier, J., *The Fractional Calculus*, Academic Press, New York, 1974.

[40] ORegan, D. and Samet, B., *Lyapunov-type inequalities for a class of fractional differential equations*, J. Inequal. Appl. **247** (2015).

[41] Pachpatte, B. G. *Mathematical Inequalities*, North Holland Mathematical Library, 2005.

[42] Parhi, N. and Panigrahi, S. *On Lyapunov-type inequality for third-order differential equations*, J. Math. Anal. Appl. **233** (1999), 445-460.

[43] Pathak, N. and Agrawal, O. P. *Lyapunov-type inequality and eigenvalue estimate for fractional problems of order* $\alpha, \alpha \in (2, 3]$, Mathematical inequalities and applications (to be appear), (2016).

[44] Podlubny, I. *Fractional differential equations*, Academic Press, San Diego, CA, 1999.

[45] Podlubny, I. *MittagLeffler function; Calculates the MittagLeffler function with desired accuracy (File ID: No.8738, mlf.m)*, MATLAB Central/File Exchange, 2005.

[46] Prabhakar, T. R. *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama. Math. J. **19** (1971), 7-15.

[47] Reid, W.T. *A generalized Liapunov inequality*, J. Differential Equations **13** (1973), 182-196.

[48] Rong, J. and Bai, C. *Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions*, Advances in Difference equations, (2015).

[49] Samko, S. G., Kilbas, A. A. and Marichev, O. I. *Fractional integrals and derivatives*, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[50] Tiryaki, A. *A Recent developments of Lyapunov-type inequalities*, Adv. Dyn. Syst. Appl. **5** (2010), 231-248.

[51] Tomovski, $\check{Z}$. and Hilfer, R. *Fractional and operational calculus with generalized frac-*

tional derivative operators and MittagLeffler type functions, Integral Transforms and Special Functions, 21, **11**, (2010), 797-814.

[52] Tomovski, *Ž. Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator*, Nonlinear Analysis, (2012), DOI: 10.1016/j.na.2011.12.034.

[53] Xu, Y., Agrawal, O.P. and Pathak, N. *Solution of New Generalized Diffusion-Wave Equation Defined in a Bounded Domain*, Journal of Applied Nonlinear Dynamics, **3** (2014), 159-171.

[54] Yang, X. *On inequalities of Lyapunov type*, Appl. Math. Comput., **134** (2003), pp. 293-300.

[55] Yang, X. *On Lyapunov-type inequality for certain higher-order differential equations*, Appl. Math. Comput. **134** (2003), 307-317.

[56] Yang, X. and Lo, K *Lyapunov-type inequality for a class of even-order differential equations*, Appl. Math. Comput. **215** (2010),3884-3890.

APPENDICES

APPENDIX I

DEFINITIONS AND THEOREMS

Important well-known general theorems and definitions that are mentioned in the main body of text are collected below for easy reference.

A-1 VOLTERRA INTEGRAL EQUATIONS

Definition. A Volterra integral equation of the first kind is an integral equation of the form

$$f(x) = \int_a^x K(x,t)\phi(t)dt,$$

where $K(x,t)$ is a known integral kernel.

Definition. A Volterra integral equation of the second kind is an integral equation of the form

$$\phi(x) = f(x) + \int_a^x K(x,t)\phi(t)dt,$$

where $K(x,t)$ is a known integral kernel and $f(x)$ is a given function [33]

A-2 FUBINI'S THEOREM

This allows us to interchange the order of integration in repeated integrals:

Theorem A-2.1. *Let* $\Omega_1 = [a,b]$, $\Omega_2 = [c,d]$, $-\infty \leq a < b \leq \infty$ *and let* $f(x,y)$ *be a measurable function defined on* $\Omega_1 \times \Omega_2$*. If at least one of the integrals*

$$\int_{\Omega_1} dx \int_{\Omega_2} f(x,y)dy, \quad \int_{\Omega_2} dy \int_{\Omega_1} f(x,y)dx, \quad \iint\limits_{\Omega_1 \times \Omega_2} f(x,y)dxdy \tag{A-1}$$

is absolutely convergent, then they coincide (see [49], p. 9).

DIRICHLET FORMULA

The foll0owing particular case of the Fubini's theorem A-2.1 holds, namely

$$\int_a^b dx \int_a^x f(x,y)dy = \int_a^b dy \int_y^b f(x,y)dx \tag{A-2}$$

assuming that one of these integrals is absolutely convergent. This is called the Dirichlet formula [49].

ABEL'S EQUATION

Here we give the definition and a method to solve the Abels equation adopted by Samko, et al, [49]. The integral equation

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)dt}{(x-t)^{1-\alpha}} = f(x), \quad x > 0, \tag{A-3}$$

where $0 < \alpha < 1$, is called Abel's equation. Equation (A-3) may be solved in the following way. Changing x to t and t to s respectively in (A-3), multiplying both sides of the equation by $(x-t)^\alpha$ and integrating we have

$$\int_a^x \frac{dt}{(x-t)^\alpha} \int_a^t \frac{\phi(s)}{(t-s)^{1-\alpha}} = \Gamma(\alpha) \int_a^x \frac{f(t)dt}{(x-t)^\alpha}. \tag{A-4}$$

Interchanging the order of integration in the left-hand side by Dirichlet formula A-2) we arrive at

$$\int_a^x \phi(s)ds \int_a^x \frac{dt}{(x-t)^\alpha (t-s)^{1-\alpha}} = \Gamma(\alpha) \int_a^x \frac{f(t)dt}{(x-t)^\alpha}. \tag{A-5}$$

The inner integral is evaluated using the change of variable $t = s + \tau(x-s)$ and application of the formulae (1.14) and (1.12):

$$\begin{aligned} \int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1} dt &= \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha} d\tau \\ &= B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha). \end{aligned}$$

Therefore

$$\int_a^x \phi(s)ds = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^\alpha}.$$

After differentiation we have:

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)dt}{(x-t)^\alpha}. \tag{A-6}$$

So, if (A-3) has a solution, this solution is necessarily given by (A-6) and therefore it is unique. The case $\alpha = 1$ is clear, while the case $\alpha > 1$ is reduced to the case $0 < \alpha < 1$, by differentiating (A-6). Analogously, the Abel equation of the form

$$\frac{1}{\Gamma(\alpha)} \int_x^b \frac{\phi(t)dt}{(t-x)^{1-\alpha}} = f(x), \quad x \le b \tag{A-7}$$

is considered and instead of (A-3), one obtains for $0 < \alpha < 1$ the following inversion formula

$$\phi(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(t)dt}{(t-x)^\alpha}. \tag{A-8}$$

[49].

APPENDIX II

A-3 SOME SPECIAL CASES OF MITTAG-LEFFLER FUNCTION $E_{\alpha,\beta}(Z)$

We obtain some special cases of the Mittag-Leffler function $E_{\alpha,\beta}(z)$.

Lemma A-3.1. *Let $\alpha, \beta, \lambda \in \mathbb{C}$ then*

$$t^2 E_{3,3}(-\lambda^3 t^3) = \frac{1}{3\lambda^2}\left[e^{-\lambda t} - e^{\lambda t/2}\cos\left(\frac{\sqrt{3}}{2}\lambda t\right) + \sqrt{3}e^{\lambda t/2}\sin\left(\frac{\sqrt{3}}{2}\lambda t\right)\right] \tag{A-9}$$

$$t^3 E_{4,4}(-\lambda^4 t^4) = \frac{1}{2\sqrt{2}\lambda^3}\left[-2\cos\left(\frac{\lambda}{\sqrt{2}}t\right)\sinh\left(\frac{\lambda}{\sqrt{2}}t\right) + 2\sin\left(\frac{\lambda}{\sqrt{2}}t\right)\cosh\left(\frac{\lambda}{\sqrt{2}}t\right)\right] \tag{A-10}$$

Proof. Taking the Laplace transform on left hand side of equations (A-9) and (A-10) we get,

$$\begin{aligned}\mathcal{L}\left[t^2 E_{3,3}(-\lambda^3 t^3)\right] &= \frac{1}{s^3+\lambda^3} \\ &= \frac{1}{3\lambda^2}\left[\frac{3\lambda^2}{s^3+\lambda^3}\right].\end{aligned} \tag{A-11}$$

and

$$\begin{aligned}\mathcal{L}\left[t^3 E_{4,4}(-\lambda^4 t^4)\right] &= \frac{1}{s^4+\lambda^4} = \frac{1}{2\sqrt{2}\lambda^3}\left[\frac{2\sqrt{2}\lambda^3}{s^4+\lambda^4}\right] \\ &= \frac{1}{2\sqrt{2}\lambda^3}\left[\frac{\sqrt{2}\lambda(\lambda^2-s^2)}{(\lambda^2-\sqrt{2}\lambda s+s^2)(\lambda^2+\sqrt{2}\lambda s+s^2)}\right. \\ &+ \left.\frac{\sqrt{2}\lambda(\lambda^2+s^2)}{(\lambda^2-\sqrt{2}\lambda s+s^2)(\lambda^2+\sqrt{2}\lambda s+s^2)}\right]\end{aligned} \tag{A-12}$$

respectively. Using (1.25), we take the inverse Laplace transform on the last parts of (A-11) and (A-12) give respectively the right hand sides of equations (A-9) and (A-10). This completes the proof. □

A-4 SEMIGROUP PROPERTY OF $\mathcal{K}_P^{\alpha}$-OPERATOR

The $\mathcal{K}_P^{\alpha}$-operator was introduced in [1].

Definition. Let $f(x) \in \mathbf{L}(a,b)$. The operator $\mathcal{K}_P^\alpha$ is defined as

$$\mathcal{K}_P^\alpha f(x) = r\int_a^x k_\alpha(x,t)f(t)dt + q\int_x^b k_\alpha(t,x)f(t)dt, \tag{A-13}$$

where r, q and α $(n-1<\alpha<n)$, a, b $(a<b)$ are some real parameters, n is a positive integer, $P =< a,x,b,r,q >$ is a parameter set, and $k_\alpha(x,t)$ is a kernel which may depend on α.

We note that in [53], we have taken $k_\alpha(x,t) = \exp(-\alpha t)$. If we take

$$k_\alpha(x,t) = \frac{(x-t)^{\alpha-1}e^{-(x-t)}}{\Gamma(\alpha)} \tag{A-14}$$

in (A-13) then the following results hold.

Proposition A-4.1. *Let $\alpha, \beta > 0$ and if $f(x) \in \mathbf{L}(a,b)$, then the semi-group properties for operator in (A-13) with kernel (A-14) given by*

$$\mathcal{K}_{P_1}^\alpha \mathcal{K}_{P_1}^\beta f(x) = \mathcal{K}_{P_1}^{\alpha+\beta} f(x) \tag{A-15}$$

$$\mathcal{K}_{P_2}^\alpha \mathcal{K}_{P_2}^\beta f(x) = \mathcal{K}_{P_2}^{\alpha+\beta} f(x) \tag{A-16}$$

hold at almost every point $x \in [a,b]$. Where $P_1 =< a,x,b,1,0 >$ and $P_2 =< a,x,b,0,1 >$. If $\alpha+\beta > 1$, then the above relations hold at any point of $[a,b]$ [30],[49].

Proof. For $P = P_1 =< a,x,b,1,0 >$ in equation (A-13) gives

$$\mathcal{K}_{P_1}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}e^{-(x-t)}f(t)dt, x \in (a,b].$$

Now

$$\begin{aligned}
\mathcal{K}_{P_1}^\alpha \mathcal{K}_{P_1}^\beta f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-s)^{\alpha-1}e^{-(x-s)}ds\int_a^s (s-t)^{\beta-1}e^{-(s-t)}f(t)dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-s)^{\alpha-1}ds\int_a^s (s-t)^{\beta-1}e^{-(x-t)}f(t)dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x e^{-(x-t)}f(t)dt\int_t^x (x-s)^{\alpha-1}(s-t)^{\beta-1}ds \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_a^x e^{-(x-t)}(x-t)^{\alpha+\beta-1}f(t)dt \\
&= K_{P_1}^{\alpha+\beta} f(x).
\end{aligned}$$

Which proves (A-15). Similarly, to prove equation (A-16) using $P = P_2 =< a, x, b, 0, 1 >$ in equation (A-13) we get,

$$\mathcal{K}^{\alpha}_{P_2} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-(t-x)} f(t)dt, x \in (a, b].$$

Now

$$\begin{aligned}
\mathcal{K}^{\alpha}_{P_2}\mathcal{K}^{\beta}_{P_2} f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b (s-x)^{\alpha-1} e^{-(s-x)} ds \int_s^b (t-s)^{\beta-1} e^{-(t-s)} f(t)dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b (s-x)^{\alpha-1} ds \int_s^b (t-s)^{\beta-1} e^{-(t-x)} f(t)dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b e^{-(t-x)} f(t)dt \int_x^t (s-x)^{\alpha-1}(t-s)^{\beta-1} ds \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_x^b e^{-(t-x)} (t-x)^{\alpha+\beta-1} f(t)dt \\
&= K^{\alpha+\beta}_{P_2} f(x).
\end{aligned}$$

This completes the proof. $\square$

If we take

$$k(x,t) = (x-t)^{\beta-1} E^{\sigma}_{\alpha,\beta}\left[(x-t)^{\alpha}\right], \tag{A-17}$$

in (A-13), where

$$E^{\sigma}_{\alpha,\beta}[z] = \sum_{k=0}^{\infty} \frac{(\sigma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \alpha, \beta, \sigma \in \mathbb{C}\ (\mathbf{R}(\alpha), \mathbf{R}(\beta), \mathbf{R}(\sigma) > 0),$$

the generalized Mittag-Leffler function. It was introduced by Prabhakar [46]. The following results hold.

Proposition A-4.2. *Let $\alpha, \beta, \nu > 0$ and if $f(x) \in \mathbf{L}(a, b)$, then the semi-group properties for operator in (A-13) with kernel (A-17) given by*

$$\mathcal{K}^{\beta}_{P_1}\mathcal{K}^{\nu}_{P_1} f(x) = \mathcal{K}^{\beta+\nu}_{P_1} f(x) \tag{A-18}$$

$$\mathcal{K}^{\beta}_{P_2}\mathcal{K}^{\nu}_{P_2} f(x) = \mathcal{K}^{\beta+\nu}_{P_2} f(x) \tag{A-19}$$

hold at almost every point $x \in [a, b]$. If $\beta + \nu > 1$, then the above relations hold at any point of $[a, b]$ [30],[49].

Proof. For $P = P_1 =< a, x, b, 1, 0 >$ in equation (A-13)

$$\mathcal{K}_{P_1}^{\beta} f(x) = \int_a^x (x-u)^{\beta-1} E_{\alpha,\beta}^{\sigma}\left[(x-u)^{\alpha}\right] f(u)du, x \in (a,b]$$

$$\begin{aligned}
\mathcal{K}_{P_1}^{\beta}\mathcal{K}_{P_1}^{\nu} f(x) &= \int_a^x (x-u)^{\beta-1} E_{\alpha,\beta}^{\sigma}\left[(x-u)^{\alpha}\right] du \int_a^u (u-t)^{\nu-1} E_{\alpha,\nu}^{\mu}\left[(u-t)^{\alpha}\right] f(t)dt \\
&= \int_a^x \left[\int_t^x (x-u)^{\beta-1}(u-t)^{\nu-1} E_{\alpha,\beta}^{\sigma}\left[(x-u)^{\alpha}\right] E_{\alpha,\nu}^{\mu}\left[(u-t)^{\alpha}\right] du\right] f(t)dt \\
&= \int_a^x \left[\int_0^{x-t} (x-t-\tau)^{\beta-1}\tau^{\nu-1} E_{\alpha,\beta}^{\sigma}\left[(x-t-\tau)^{\alpha}\right] E_{\alpha,\nu}^{\mu}\left[\tau^{\alpha}\right] du\right] f(t)dt.
\end{aligned} \tag{A-20}$$

We have taken $u - t = \tau \Rightarrow du = d\tau, u \to t \Rightarrow \tau \to 0, u \to x \Rightarrow \tau \to x - t$ in the above derivation. We first prove the following:

$$\int_0^{x-t} (x-t-\tau)^{\beta-1} E_{\alpha,\beta}^{\sigma}\left[(x-t-\tau)^{\alpha}\right] \tau^{\nu-1} E_{\alpha,\nu}^{\mu}\left[\tau^{\alpha}\right] du = x^{\beta+\nu-1} E_{\alpha,\beta+\nu}^{\sigma+\mu}\left[x^{\alpha}\right]. \tag{A-21}$$

Using

$$\mathcal{L}\left[\int_0^{x-t} k(x-t-\tau)\phi(t)dt\right](s) = \mathcal{L}\left[k(x)\right](s)\mathcal{L}\left[\phi(x)\right](s),$$

we take the Laplace transform on equation (A-21) we obtain

$$\begin{aligned}
&\mathcal{L}\left[\int_0^{x-t} (x-t-\tau)^{\beta-1} E_{\alpha,\beta}^{\sigma}\left[(x-t-\tau)^{\alpha}\right] \tau^{\nu-1} E_{\alpha,\nu}^{\mu}\left[\tau^{\alpha}\right] du\right] \\
= \ & \frac{s^{-\beta}}{(1-s^{-\alpha})^{\sigma}} \frac{s^{-\nu}}{(1-s^{-\alpha})^{\mu}} = \frac{s^{-(\beta+\nu)}}{(1-s^{-\alpha})^{\sigma+\mu}}.
\end{aligned} \tag{A-22}$$

Taking inverse Laplace transform on (A-22) proves (A-21). Hence using (A-21) in (A-20) we obtain

$$\mathcal{K}_{P_1}^{\beta}\mathcal{K}_{P_1}^{\nu} f(x) = \int_a^x (x-u)^{\beta+\nu-1} E_{\alpha,\beta+\nu}^{\sigma+\mu}\left[(x-u)^{\alpha}\right] f(u)du = \mathcal{K}_{P_1}^{\beta+\nu} f(x).$$

□

This proves (A-18). In the similar way (A-19) can be proved.

VITA

Graduate School
Southern Illinois University

Nimishaben Pathak

email address nimishasp@hotmail.com

The Maharaja Sayajirao University of Baroda
Bachelor of Science, Mathematics, May 1992

The Maharaja Sayajirao University of Baroda
Master of Science, Mathematics, May 1994

The Maharaja Sayajirao University of Baroda
Master of Philosophy, Mathematics, May 1998

Special Honors and Awards:

1. Chaired one of the contributed talk sessions at the Joint Mathematics Meetings, Seattle, Washington, January, 2016.

2. Winner of the MathSciNet challenge award at the Joint Mathematics Meetings, Seattle, Washington, January, 2016.

3. Received dissertation research assistantship award at Southern Illinois University, Fall 2015.

4. Reviewer of Bulletin of the Iranian Mathematical Society.

5. Member, American Mathematical Society, since 2013 - Present.

6. Member, Society for Special Functions and its Applications, since 1998 - Present.

Dissertation Title:
Lyapunov-type Inequality And Eigenvalue Estimates For Fractional Problems

MAJOR PROFESSOR: Dr. Michael Sullivan

Publications:

1. "Solution of New Generalized Diffusion-Wave Equation Defined in a Bounded Domain" in Journal of Applied Nonlinear Dynamics, **3**, pp. 159-171.

2. "Lyapunov-type inequality and eigenvalue estimates for fractional problems of order α, $\alpha \in (2, 3]$" in Journal of Mathematical Inequalities and Applications is accepted for publication.

Presentations:

1. "Lyapunov-type inequalities for fractional order differential Equations with Caputo and Riemann-Liouville derivatives of order α, $3 < \alpha \leq 4$" at the Joint Mathematics Meetings, Seattle, Washington, USA, January, 2016.

2. "Lyapunov-type inequality for fractional differential equation under mixed and fractional integral boundary conditions" at the University of North Carolina, Greensboro, USA, October, 2015.

3. "Variational method for the regular generalized fractional Sturm-Liouville problem" a colloquium talk at the Department of Mathematics, Southern Illinois University, Carbondale, USA, January, 2015.

4. "Regular Generalized Sturm-Liouville Problem with Discrete Spectrum" in 34th Southeastern-Atlantic Regional Conference on Differential Equations at the University of Memphis, Tennessee, USA, October, 2014.

5. "Solution of a generalized diffusion-wave equation defined in a bounded domain" in 33rd Southeastern-Atlantic Regional Conference on Differential Equations at the University of Tennessee, Knoxville, USA, September, 2013.