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A Scaling Method and its Applications to Problems in Fractional Dimensional Space

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A scaling method and its applications to problems in fractional dimensional space

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A scaling method is proposed to find (1) the volume and the surface area of a generalized hypersphere in a fractional dimensional space and (2) the solid angle at a point for the same space. It is demonstrated that the total dimension of the fractional space can be obtained by summing the dimension of the fractional line element along each axis. The regularization condition is defined for functions depending on more than one variable. This condition is applied (1) to find a closed form expression for the fractional Gaussian integral, (2) to establish a relationship between a fractional dimensional space and a fractional integral, (3) to develop the Bochner theorem, and (4) to obtain an expression for the fractional integral of the Mittag–Leffler function. Some possible extensions of this work are also discussed. © *2009 American Institute of Physics.* [doi:10.1063/1.3263940]

I. INTRODUCTION

Mandelbrot[1] introduced the concept of "fractal" to describe irregular geometries of complex objects such as clouds, mountains, cost lines, and travel path of a lightning. The first fractional physical phenomenon, which was observed sometime ago and is still a subject of many investigations, is the Brownian motion.[1] Feynman and Hibbs[2] were the first to formulate the nonrelativistic quantum mechanics as an integral over Brownian paths, which is now known as the Feynman path integral. The concept of fractals has been applied in many other areas of physics ranging from the dynamics of fluids in porous media to the resistivity networks in electronics.[3–5]

One of the key issues in fractal geometry is measuring the dimension of the fractional space. Several methods have been proposed related to this issue. For example, the fractional mass dimension obeys the scaling rule $M(r)=kr^{D_M}$,[1] where M is the mass of the fractal medium, r is the radius of the sphere, D_M is the mass fractal dimension, and k is a constant that depends on the scaling rule. This definition could be used to compute the dimension of the fractional space. An experimental measurement suggests that the fractional mass dimension D_M of our real world is (3 ± 10^{-6}).[6] The same technique can be used to find the fractional charge dimension via the relation $Q(r)=kr^{D_Q}$, where the terms Q and D_Q are defined similarly. This scaling rule has one more implication; it suggests that a charge may be understood as a distribution instead of a monopole, a dipole, or a quadrupole, as it is traditionally done.[7]

Several applications of fractional dimensional space could be cited. In the 1970s, Stillinger[6] described a procedure for integration on a fractional space of dimension D (where D is a noninteger number) and generalized the Laplace operator $\Delta_D\psi(\mathbf{r})$ in this space as

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$$\Delta_D = \frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin^{D-2}\theta}\frac{\partial}{\partial \theta}\sin^{D-2}\theta\frac{\partial}{\partial \theta}.$$

Many investigations into low-dimensional semiconductors[8,9] have used this Laplacian to solve the Schrödinger equation for hydrogenlike atoms for anisotropic solids to obtain energy bounds and the optical spectra as a function of fractional spatial dimension D.

As a next application, consider the N-quarks statistical mechanics with a potential $U(\mathbf{r_1},\mathbf{r_2},\ldots,\mathbf{r_N})=\Sigma_{1\le i<j\le N}q_iq_j|\mathbf{r_i}-\mathbf{r_j}|^{\beta}$, where β is a parameter. From the theory of statistical mechanics of many body problem, it is shown in Ref. 10 that the potential energy term defined above will be thermodynamically stable if $0<\beta\le 2$.

In quantum field theory, the dimensional regularization technique,[11] i.e.,

$$\int f(x)d(V)_D = \frac{2\pi^{(D)/2}}{\Gamma\left(\frac{D}{2}\right)}\int_0^{\infty} f(x)x^{D-1}dx,$$

suggests that D (the parameter representing noninteger dimension of the space) could be of relevance as a regularization parameter in the Feynman integral. The technique is used as a method of removing the divergent term from the Feynman integral. Clearly, the fractional dimensional space could be important in many applications.

Related to fractal geometry and fractional dimensional space is the area of fractional derivatives and integrals which have recently been applied in many applications including particle physics,[12] fractional Hamiltonian systems,[13] chaotic dynamics,[5] astrophysics,[14] physics of fractals and complex media,[4] and recent studies of scaling phenomena.[15–17] However, as demonstrated in Refs. 4 and 18, the areas of fractals, fractional dimensional space, and fractional derivatives are not completely independent.

In this paper, we examine applications of fractional dimensional space and its relationship with fractional derivatives and integrals. To accomplish this, we first present a generalization of a differential volume element in a fractional space of dimension $D>0$ (also denoted as the D-dimensional space). We use this definition (1) to obtain the volume and the surface area of a hypersphere in the D-dimensional space and (2) to define the solid angle about a point in the same space. This hypersphere is also denoted as a D-hypersphere or simply a D-sphere (and an N-hypersphere or simply an N-sphere when $D=N$ is an integer). The proposed extension allows us to define a scaling rule for a fractional line element. It is shown that this definition leads to the same expressions for the volume and the surface area of the hypersphere, and the solid angle about a point in the fractional space. Furthermore, the overall dimension of the space could be obtained by summing the dimensions of the fractional line elements. This paper also presents a regularization condition for two-parameter functions defined in a D-dimensional space. This definition differs from that presented in Ref. 19, where Tarasov used the formula of fractional regularization only for fractional space of dimension D, $0<D\le 3$. The regularization condition defined here is used (1) to find a closed form expression for the fractional Gaussian integral, (2) to establish a relationship between a fractional dimensional space and a fractional integral, (3) to develop the Bochner theorem, and (4) to obtain an expression for the fractional integral of a Mittag–Leffler function. Some possible extensions of this work are also discussed.

II. THE VOLUME AND THE SURFACE AREA OF A *D*-HYPERSPHERE

In this section we define the volume and the surface area of a generalized D-hypersphere for $D>0$ and the total solid angle around a point for the same space. The equations give the starting point for the concept of a fractional line element and the regularization condition for a fractional space. To accomplish this, first consider the volume and the surface area of an N-hypersphere in an N-dimensional space, where N is an integer. Note that the experts in geometry and topology define an N-hypersphere differently.[20–23] Here, we shall follow the definition used by the experts in geometry.[20,21] We assume that the radius of this N-hypersphere is R, and its center coincides with the origin of the coordinate system in the space. Using the Cartesian coordinates $x=(x_1,\ldots,x_N)$, the differential volume element and the volume of the N-sphere in the N-dimensional space are given by

$$(dV)_N = dx_1 \cdots dx_N = \prod_{j=1}^{N} dx_j \tag{1}$$

and

$$V_N = \int_\Omega \prod_{j=1}^{N} dx_j, \tag{2}$$

where Π is the product symbol and $\Omega=\{x \in R^N | x_1^2+\cdots+x_N^2 \le R^2\}$.

To simplify the derivation, let us define the transformation equations between the Cartesian coordinates and a set of polar coordinates $(r,\theta_1,\ldots,\theta_{N-1})$ as (Ref. 24, Vol. II, Chap. IX)

$$x_j = r\cos(\theta_j)\prod_{k=0}^{j-1}\sin(\theta_k), \quad j=1,\ldots,N-1,$$

$$x_N = r\prod_{k=1}^{N-1}\sin(\theta_k), \tag{3}$$

where $\theta_0=\pi/2$, $0\le\theta_j\le\pi$ $(j=1,2,\ldots,N-2)$, $0\le\theta_{N-1}\le 2\pi$, and r is the radial coordinate defined as $r=(\Sigma_{j=1}^N x_j^2)^{1/2}$. Using Eq. (3), the volume element in the new coordinates can be written as

$$(dV)_N = r^{N-1} dr d\Theta_N, \tag{4}$$

where

$$d\Theta_N = \prod_{k=1}^{N-1} (\sin\,\theta_k)^{N-1-k} d\theta_k \tag{5}$$

is the hypersolid angle at the origin defined by the hypersurface element

$$(dS)_N = r^{N-1} d\Theta_N. \tag{6}$$

Substituting Eqs. (4) and (5) into Eq. (2), integrating over the complete range of angles $\theta_1,\ldots,\theta_{N-1}$, and using the identity

$$\int_0^\pi \sin^j\,\theta d\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+2}{2}\right)}, \tag{7}$$

we obtain

$$V_N = \int_\Omega (dV)_N = \int_0^R \frac{2\pi^{N/2}}{\Gamma(N/2)} r^{N-1} dr, \tag{8}$$

where Γ is the gamma function. Equation (8) suggests the following transformation:

$$\int_{\Omega} \rightarrow \int_0^R (dV)_N \rightarrow \frac{2\pi^{N/2}}{\Gamma(N/2)} r^{N-1} dr, \tag{9}$$

where "$a \rightarrow b$" indicates that "a could be replaced with b." Note that the transformation indicated in Eq. (9) must take place simultaneously. Furthermore, Eq. (9) assumes that it is possible to integrate over all angles separately (as is the case here). If this condition fails, then the transformation given in (9) will not be valid. Therefore, transformation (9) must be applied carefully. Note that the value of R could extend to ∞.

Using Eq. (8), we obtain the volume of the N-sphere as

$$V_N = \frac{2\pi^{N/2} R^N}{N\Gamma(N/2)}. \tag{10}$$

Differentiating Eq. (10) with respect to R, the hypersurface area of the N-sphere is given as

$$S_N = \frac{2\pi^{N/2}}{\Gamma(N/2)} R^{N-1}. \tag{11}$$

For $N > 3$, this hypersurface can be of dimension of more than 2. Note that S_N is proportional to R^{N-1}. Dividing S_N by R^{N-1}, we obtain the expression for the hypersolid angle about a point in the N-dimensional space as

$$\Theta_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{12}$$

Note that for $N=2$ and $N=3$, the hypersolid angle about a point, the hypersurface area, and the hypervolume of a sphere of radius R are given as $\Theta_2 = 2\pi$, $S_2 = 2\pi R$, $V_2 = \pi R^2$, $\Theta_3 = 4\pi$, $S_3 = 4\pi R^2$, and $V_3 = (4/3)\pi R^3$. For $N=2$, the hypersphere is denoted as the circle, the solid angle, the surface area, and the volume are simply called the angle, the circumference, and the area of the circle. For $N=1$, the hypersurface area and the hypervolume is reduced to 2 points and a line, and we obtain $\Theta_1 = 2$, $S_1 = 2$, and $V_1 = 2R$, and thus for $N=1$, the hypervolume is essentially the length of the line. For $N=1$, it is not common to call Θ_1, S_1, and V_1 as the hypersolid angle, the hypersurface area, and the hypervolume.

Equation (9) could be thought of as a transformation from an N-dimensional space to a one-dimensional space. Therefore, it is our conjecture that one could write the following transformation:

$$\int_{\Omega_D} \rightarrow \int_0^R (dV)_D \rightarrow \frac{2\pi^{D/2}}{\Gamma(D/2)} r^{D-1} dr, \tag{13}$$

where $D > 0$ is the dimension of the space (which could be a noninteger), Ω_D is the domain of the D-sphere, and $(dV)_D$ is the fractional volume element in the D-dimensional space. Like before, this transformation is valid only when the angles could be integrated separately. Furthermore, both transformations in (13) must be applied at the same time. Otherwise, this transformation may not be valid. Our interpretation of the meaning of D will be discussed in Sec. III.

Using Eq. (13), the volume of the D-sphere of radius R is given as

$$V_D = \int_{\Omega_D} (dV)_D = \int_0^R \frac{2\pi^{D/2}}{\Gamma(D/2)} r^{D-1} dr = \frac{2\pi^{D/2} R^D}{D\Gamma(D/2)}. \tag{14}$$

Differentiating Eq. (14) with respect to R, the hypersurface area of the D-sphere is given as

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} R^{D-1}. \tag{15}$$

Note that this area is proportional to R^{D-1}, and therefore, dividing S_D by R^{D-1}, we get the hypersolid angle about a point in the D-dimensional space as

$$\Theta_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{16}$$

It could be verified that these definitions of the volume and the surface area of a hypersphere and the solid angle at a point in the D-dimensional space are also applicable when D is an integer. As discussed later, fractionalization of Cartesian coordinates also leads to the same results. One of the major significance of Eq. (13) is that it allows us to propose a scaling rule.

III. THE SCALING METHOD AND THE DIMENSION OF THE SPACE

In Secs. I and II, we used "dimension D of a fractional space" and similar phrases. One may ask "What is the dimension D of a space for a positive noninteger D?" Here we shall try to answer this question.

Let us say that N coordinates $x_1, \ldots, x_N$ are needed to locate a point in a space. In the case where a space is filled with regular geometric objects, and the curves and the surfaces are smooth, it is common to call this number N as the dimension of the space. Thus, a straight line, a plane surface, and a cube are of dimensions 1, 2, and 3. This is also true if these spaces have curvatures. For example, motions along the circumference of a circle and on the surface of a sphere can be considered as motions in one- and two-dimensional spaces even though our true motions may be in a three-dimensional space. In such cases, infinitesimal line, area, and volume elements in the Cartesian coordinates are defined as dx_1, dx_1dx_2, and $dx_1dx_2dx_3$, respectively, and even in the case of a space with curvature, the distance between two points sufficiently closed to each other is given by a quadratic expression. However, this is not the situation in the case of fractal lines, surfaces, volumes, and hypervolumes. Thus, in these cases, there is a clear distinction between the number of coordinates used to locate a point and the dimension of the space.

As discussed in Ref. 1, the dimensions of the fractals can be defined in various ways. Here, we shall use the scaling method $d^{\alpha}x = f(\alpha)|x|^{\alpha-1}dx$ to relate the differential fractional line element $d^{\alpha}x$ with the differential straight line element dx, where α is a parameter called the fractional dimension of the line and $f(\alpha)$ is a function of α. Here, we take the absolute value of x in $|x|^{\alpha-1}$ to indicate that our scaling is symmetric about the origin. A space filled with such lines is called a fractional space. Theoretically, one can speculate α to have any positive value. However, we shall restrict the value of α such that $0 < \alpha \leq 1$. The case of $\alpha > 1$ is left for future considerations. For $\alpha = 1$, $f(\alpha)$ also must be 1 so that the scaling is a unit scaling (or an identity transformation). Many functions would satisfy this requirement. However, based on Sec. II, we take $f(\alpha) = \pi^{\alpha/2}/\Gamma(\alpha/2)$ [see Eq. (13)]. The presence of a factor of 2 in Eq. (13) is due to the fact that for $D=1$, r is integrated from $-R$ to R, and when the limits are taken as 0 and R, one gets a factor of 2. Thus, if a point in a space is located using N points $x_1, \ldots, x_N$, then the scaling between $d^{\alpha_i}x_i$ and dx_i is defined as

$$d^{\alpha_i}x_i = \frac{\pi^{\alpha_i/2}|x_i|^{\alpha_i-1}}{\Gamma(\alpha_i/2)} dx_i, \quad i = 1, \ldots, N. \tag{17}$$

We call this space as a fractional space, and

$$D = \alpha_1 + \cdots + \alpha_N \tag{18}$$

as the dimension of the space. When $\alpha_1 = \cdots = \alpha_N = 1$, we call the scaling as the unit or the identity scaling, and in this case $D=N$, i.e., the fractional dimension agrees with the normal dimension of the space. If $\alpha_1 = \cdots = \alpha_N = \alpha$, where $0 < \alpha \leq 1$, we call the scaling as the isotropic scaling, otherwise an anisotropic scaling. Regardless of the isotropic or the anisotropic scaling, the dimension of the space is always given by Eq. (18). Furthermore, we call the space a fractional dimensional space as long as α_i for at least one i is not equal to 1. When α_j is not equal to 1 for some j, then we also say that the coordinate x_j has been fractionalized.

Following the above discussion, the volume element in fractional dimensional space D is defined as

$$(dV)_D = d^{\alpha_1}x_1 \cdots d^{\alpha_N}x_N, \tag{19}$$

and the volume of the D-sphere is given as

$$V_D = \int_{\Omega_D} d(V)_D = \int_{\Omega_D} d^{\alpha_1}x_1 \cdots d^{\alpha_N}x_N. \tag{20}$$

Let us examine the consequences of this definition of a fractional volume element. For this purpose, we compute the volume of the D-sphere once again. First note that for $\alpha_1 = \cdots = \alpha_N = 1$, the definition of the volume element given by Eq. (19) coincides with that given in Eq. (1).

Using Eqs. (17) and (18), the differential fractional volume element [Eq. (19)] is given as

$$(dV)_D = \pi^{D/2}\left\{\prod_{i=1}^{N}\frac{|x_i|^{\alpha_i-1}}{\Gamma(\alpha_i/2)}\right\}\left\{\prod_{i=1}^{N} dx_i\right\}. \tag{21}$$

Using Eqs. (1), (3)–(5), (18), and (21), the volume of the D-sphere [Eq. (20)] is given as

$$V_D = 2^N\pi^{D/2}\int_0^R\int_{\theta_1=0}^{\pi/2}\cdots\int_{\theta_{N-1}=0}^{\pi/2}\left\{\prod_{i=1}^{N-1}(\cos\,\theta_i)^{\alpha_i-1}\right\}\left\{\prod_{i=1}^{N}\frac{1}{\Gamma(\alpha_i/2)}\left\{\prod_{k=0}^{i-1}\sin\,\theta_k\right\}^{\alpha_i-1}\right\}$$
$$\times\left\{r^{D-1}dr\prod_{k=1}^{N-1}(\sin\,\theta_k)^{N-1-k}d\theta_k\right\}. \tag{22}$$

Finally, using the identity[14]

$$2\int_0^{\pi/2}\cos^{\nu-1}\,\theta\,\sin^{\mu-1}\,\theta d\theta = \frac{\Gamma(\nu/2)\Gamma(\mu/2)}{\Gamma((\nu+\mu)/2)}, \quad \nu,\mu > 0, \tag{23}$$

in Eq. (22), we obtain the volume of the D-sphere as

$$V_D = \frac{2\pi^{D/2}R^D}{D\Gamma(D/2)}, \tag{24}$$

which is the same as that given by Eq. (14). Accordingly, the scaling method gives the same results for the surface area of the D-sphere and the hypersolid angle around a point in the D-dimensional space as those given by Eqs. (11) and (12). Thus, we have the following.

Theorem 1: If the scaling is defined by Eq. (17), then for arbitrary values of $\alpha_1, \ldots, \alpha_N$, the volume V_D and the surface area S_D of a D-sphere and the hypersolid angle Θ_D around a point in a D-dimensional space are given by Eqs. (14)–(16), respectively, where D is defined by Eq. (18). Furthermore, V_D, S_D, and Θ_D depend on the total dimension of the space and not explicitly on the individual dimensions of the fractional line elements.

IV. DIMENSIONAL REGULARIZATION TECHNIQUE AND ITS APPLICATIONS

The scaling rule presented in Sec. III can be used to develop a regularization technique that will allow us to compute the integral of a function over a fractional domain. Thus, let us assume that we have a function $F(x_1, \ldots, x_N)$ defined over a fractional domain Ω, and we want to compute the integral $\int_\Omega F(x_1, \ldots, x_N)d^{\alpha_1}x_1, \ldots, d^{\alpha_N}x_N$. Using the scaling Eq. (17), this integral is given as

$$\int_{\Omega_D} F(x_1, \ldots ,x_N) d^{\alpha_1}x_1, \ldots ,d^{\alpha_N}x_N = \int_{\Omega} F(x_1, \ldots ,x_N) \pi^{D/2} \left\{ \prod_{i=1}^{N} \frac{|x_i|^{\alpha_i - 1}}{\Gamma(\alpha_i/2)} \right\} \left\{ \prod_{i=1}^{N} dx_i \right\}. \qquad (25)$$

The domains Ω_D and Ω used here may be different from those defined above. This conversion of an integral of a function from a fractional dimensional space to a regular dimensional space is called the regularization technique. This regularization method is used to remove the divergent term in the evaluation of Feynman diagram term. For example, in the space-time dimension of four, *i.e.* $D=4$, the Feynman diagram loop integral

$$\int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2}$$

diverges. To avoid this divergence and to change this integral into convergent loop integral, we introduce integration on the fractional dimensional space $D=4-\epsilon$, where ϵ is small parameter closed to zero. In this case the convergent loop integral is obtained as

$$\lim_{\epsilon \to 0^+} \int \frac{d^{4-\epsilon} p}{(2\pi)^{4-\epsilon}} \frac{1}{(p^2 + m^2)^2} = \lim_{\epsilon \to 0^+} \int \frac{dp}{(2\pi)^{(4-\epsilon)}} \frac{2\pi^{(4-\epsilon)/2}}{\Gamma\left(\frac{4-\epsilon}{2}\right)} \frac{p^{3-\epsilon}}{(p^2 + m^2)^2}.$$

The regularization condition given in Eq. (25) can be specialized for special functions. For radial function, this specialization is expressed as follows.

Theorem 2: Let $F(r)$ be a radial function. The fractional dimensional regularization condition for this function is given as

$$\int_{\Omega_D} F(r)(dV)_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^R F(r) r^{D-1} dr, \qquad (26)$$

where D is the dimension of the space, Ω_D is the domain of the hypersphere, which in regular space is defined as $\Omega = \{x | x_1^2 + \cdots + x_N^2 \le R^2\}$, and R is the radius of the sphere which can be extended to infinity.

Proof: The fractional volume element $(dV)_D$ is defined by Eq. (19). Since $F(r)$ is a radial function, the integration of angle terms in $(dV)_D$ separately is allowed. Thus, the right-hand side of Eq. (26) follows by applying the transformation given by Eq. (13) to the left-hand side of Eq. (26).

We now consider some applications of this dimensional regularization technique.

A. Gaussian function in *D*-dimensional space

The Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad (27)$$

plays a fundamental role in probability and statistics. One can prove the equality in Eq. (27) using the Poisson (or the Liouville) approach.[25,26] Its N-dimensional equivalent is the following identity:

$$\int_{\mathcal{R}^N} e^{-r^2} dV = \pi^{N/2}, \qquad (28)$$

and it follows by writing Eq. (27) for each coordinates and multiplying them together. For fractional dimensional space, this identity is defined as follows.

Theorem 3: The *fractional Gaussian integral* in the D-dimensional space satisfies the following identity:

$$\int_{\mathcal{R}^N} e^{-r^\beta} d(V)_D = \frac{2\pi^{D/2}\Gamma(D/\beta)}{\Gamma(D/2)\beta} \tag{29}$$

and, in particular, for $\beta=2$, we have

$$\int_{\mathcal{R}^N} e^{-r^2} d(V)_D = \pi^{D/2}. \tag{30}$$

Proof: Note that e^{-r^β} is a radial function. Thus, the identity in Eq. (29) follows by using Theorem 2 and the definition the gamma function, and Eq. (30) follows by taking $\beta=2$. Note that both Eqs. (29) and (30) agree with Eq. (28).

B. The relationship between an integral over a fractional dimensional space and a fractional integral

The right Riemann–Liouville fractional integral of order α is defined as[27]

$$ {}_zI_b^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (x-z)^{\alpha-1} f(x) dx. \tag{31}$$

Setting $z=0$ in Eq. (31) and using the scaling defined in Eq. (17), it follows that

$$\int_\Omega f(x) d^D x = \frac{\pi^{D/2}\Gamma(D)}{\Gamma(D/2)} {}_zI_b^\alpha f(0), \tag{32}$$

where $\Omega=[0,b]$. Here, b can be extended to ∞. Thus, one interpretation of a fractional integral of a function computed at the initial point (or at the origin) would be that it is a measure of the integral of the function over a fractional dimension space. Analogously, one could also define an integral of a function over a fractional dimensional space in terms of a fractional integral.

Let us take $b=\infty$. In this case, z could take any value. Using a simple translation of the axis, Eq. (31) could be written as

$$ {}_zI_\infty^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} f(x+z) dx. \tag{33}$$

Thus, a fractional integral could be thought of as an integral of a translated function over a fractional dimensional space.

C. Bochner theorem in a fractional dimensional space

A natural way for the investigation of Riesz integrodifferentiation[27] is the computation of the Fourier transform of the kernel $|x|^{\alpha-N}$, where N is the dimension of the space, $x\in\mathcal{R}^N$, and α is a real positive number.[27] This transform can be computed using the Bochner theorem, which, after some changes in notations, can be stated as follows.[27]

The Bochner theorem: The Fourier transform of a radial function is a radial function, and the following relation is valid:

$$\int_\Omega e^{iz\cdot x}\phi(r)(dV)_N = \frac{(2\pi)^{N/2}}{|z|^{(N-2)/2}} \int_0^R \phi(r) r^{N/2} J_{N/2-1}(|z|r) dr \tag{34}$$

provided $\phi(r)$ is summable in $x\in\Omega$, where $\Omega=\{x\in\mathcal{R}^N \,|\, |x|=r\le R\}$, $z\in\mathcal{R}^N$, "$\cdot$" represents the scalar product, and J_ν is the Bessel function of the first kind of order ν. Furthermore,

$$\int_{\mathcal{R}^{\mathcal{N}}} e^{iz\cdot x}\phi(r)(dV)_N = \frac{(2\pi)^{N/2}}{|z|^{(N-2)/2}}\int_0^\infty \phi(r)r^{N/2}J_{N/2-1}(|z|r)dr \tag{35}$$

for any function $\phi(r)$ provided that

$$\int_0^\infty r^{N-1}(1+r)^{(1-N)/2}|\phi(r)|dr < \infty. \tag{36}$$

The integral on the left-hand side of (36) is interpreted as conventionally convergent. It converges absolutely if $\int_0^\infty r^{N-1}|\phi(r)|dr<\infty$.

Proof: See Ref. 27.

The Bochner theorem presented above can be stated in a fractional dimensional space as follows.

Theorem 4: Assume that z and x are two arbitrary vectors in $\mathcal{R}^N$, $|x|=r\le R$, D is the dimension of the fractional space, and the space normal to z is fractionalized. Then, the following identity holds:

$$\int_\Omega e^{ix\cdot z}\phi(r)(dV)_D = \frac{(2\pi)^{D/2}}{|z|^{D/2-1}}\int_0^R r^{D/2}\phi(r)J_{D/2-1}(r|z|)dr \tag{37}$$

provided that the integration of $\phi(r)$ over the domain Ω exists. Moreover,

$$\int_{\mathcal{R}^{\mathcal{N}}} e^{ix\cdot z}\phi(r)(dV)_D = \frac{(2\pi)^{D/2}}{|z|^{D/2-1}}\int_0^\infty r^{D/2}\phi(r)J_{D/2-1}(r|z|)dr \tag{38}$$

for any $\phi(r)$ such that

$$\int_0^\infty r^{(D-1)/2}\phi(r)dr < \infty. \tag{39}$$

Note that this theorem requires only the dimension of the space and not the individual dimensions of each axis, except that the dimension of the axis x_1 is 1. Proof of this theorem requires the knowledge of the following two lemmas.

Lemma 1: Assume that x is an arbitrary vector in $\mathcal{R}^N$, $\Omega=\{x\in\mathcal{R}^{\mathcal{N}}\,|\,|x|=r\le R\}$, D is the dimension of the fractional space, θ_1 is the angle between the radial line r and the axis x_1, and all coordinates except x_1 are fractionalized. Then, the following relationship holds:

$$\int_\Omega f(r,\cos\,\theta_1)(dV)_D = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}\int_0^R r^{D-1}dr\int_0^\pi f(r,\cos\,\theta_1)\sin^{D-2}\,\theta_1 d\theta_1. \tag{40}$$

Proof: This follows by using Eqs. (17) and (19), setting $\alpha_1=1$ (because x_1 is not fractionalized), and performing integration for variables θ_2 to θ_{N-1}. Equation (40) suggests that if the function being integrated does not depend on θ_2 to θ_{N-1}, then one can take the following transformations:

$$\int_\Omega \to \int_{r=0}^R\int_{\theta_1=0}^\pi,(dV)_D \to \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}r^{D-1}dr\,\sin^{D-2}\,\theta_1 d\theta_1. \tag{41}$$

Lemma 2: The following relationship is valid:

$$\int_0^\pi e^{i|z|\cos\,\theta}\sin^{D-2}\,\theta d\theta = \sqrt{\pi}\Gamma((D-1)/2)\left(\frac{2}{|z|}\right)^{(D/2-1)}J_{D/2-1}(|z|). \tag{42}$$

Proof: This follows by defining $t=\cos(\theta)$ and then using a Poisson's formula.[27]

We now prove Theorem 4. To accomplish this, we assume, without loss in generality, that z coincides with the axis x_1. Thus, x_1 is not fractionalized. Further, we can write $x \cdot z = r|z|\cos(\theta_1)$. We then apply Lemma 1 to obtain

$$\int_\Omega e^{ix\cdot z}\phi(r)(dV)_D = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}\int_0^R r^{D/2}\phi(r)dr\int_{\theta_1=0}^{\pi} e^{ir|z|\cos(\theta_1)}\sin(\theta_1)d\theta_1. \tag{43}$$

Finally, we apply Lemma 2 to obtain Eq. (37). Of course, the integration on the left-hand side of Eq. (37) is valid provided that the integration of $\phi(r)$ over Ω exists. Since $|J_\nu(r)| \le c\sqrt{r}$ as $r \to \infty$, the limit on the right-hand side of Eq. (37) exists provided that Eq. (39) is satisfied. This yields Eq. (38). Note that for integer dimensional space (i.e., for $D=N$), Eqs. (37)–(39) reduce to Eqs. (34)–(36), respectively.

D. Integration of the Mittag–Leffler function over a fractional dimensional space and its relationship with the Fox *H*-function

The Mittag–Leffler function[28,29] is the generalization of the exponential function. The original Mittag–Leffler function and its generalized form are given by

$$E_{\alpha,1}(z) = E_\alpha(z), \quad E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\beta+\alpha j)}. \tag{44}$$

Using the dimensional regularization technique, the integral of the Mittag–Leffler function over the fractional space of dimension $s>0$ is given by

$$\int_0^\infty E_\alpha(x)d^sx = \frac{\pi^{(s)/2}}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty E_\alpha(x)x^{s-1}dx. \tag{45}$$

Note that the Mittag–Leffler function is a special case of the Fox H-function,[29] and the relationship between these two functions is given by

$$E_{\alpha,\beta}(x) = H_{1,2}^{1,1}\left[x\left|\begin{matrix}(0,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right]. \tag{46}$$

The Mellin transform of a single H-function is given as[30]

$$\int_0^\infty x^{s-1}H_{p,q}^{m,n}\left[ax\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]dx = a^{-s}\frac{\Pi_{j=1}^{m}\Gamma(b_j+B_js)\Pi_{j=1}^{n}\Gamma(1-a_j+A_js)}{\Pi_{j=m+1}^{q}\Gamma(1-b_j+B_js)\Pi_{j=n+1}^{p}\Gamma(a_j+A_js)}, \tag{47}$$

where $-\min_{1\le j\le m} R(bj/B_j) < R(s) < 1/A_j - \max_{1\le j\le n} R(aj/A_j), |\arg a| < \frac{1}{2}\pi\lambda$, $\lambda = \Sigma_{j=1}^{m}A_j - \Sigma_{j=n+1}^{p}A_j\Sigma_{j=1}^{m}B_j - \Sigma_{j=m+1}^{q}B_j > 0$. Setting $m=n=p=a_1=A_1=B_1=1$, $b_1=b_2=0$, $q=2$, $s=D$, and $B_2=\alpha$, we obtain

$$\int_0^\infty x^{s-1}H_{1,2}^{1,1}\left[x\left|\begin{matrix}(0,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right]dx = \Gamma(s), \tag{48}$$

which leads to

$$\int_0^\infty E_\alpha(x)d^sx = \frac{\pi^{(s)/2}\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)}. \tag{49}$$

Equation (49) is useful in finding the Mellin transform of the Feynman propagator of a free particle system, which allows us to obtain the Feynman quantum mechanical kernel for the systems.[31] It is also useful in evaluating the Green's function for fractional diffusion processes[32] and in obtaining the probability density for fractional Brownian motion.[33]

V. CONCLUSIONS

In this paper, we considered the generalization of a volume element $(dV)_N$ to a fractional volume element $(dV)_D$ for a fractional space of dimension D, $0<D\leq N$. We presented a generalization of the regularization condition for a fractional space of dimension D. We also developed Bochner theorem in the fractional space. This will enable us to introduce a new technique to solve nonhomogeneous fractional differential equations, which will involve taking Fourier transform of the differential equations in the fractional space. This topic will be discussed in a later paper.

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