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SQUARED BESSEL PROCESS WITH DELAY

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Abstract. We discuss a generalization of the well known squared Bessel process with real nonnegative parameter δ by introducing a predictable almost everywhere positive process $\gamma(t, \omega)$ into the drift and diffusion terms. The resulting generalized process is nonnegative with instantaneous reflection at zero when δ is positive. When δ is a positive integer, the process can be constructed from δ-dimensional Brownian motion. In particular, we consider $\gamma_t = X_{t-\tau}$ which makes the process a solution of a stochastic delay differential equation with a discrete delay. The solutions of these equations are constructed in successive steps on time intervals of length τ. We prove that if $0 < \delta < 2$, zero is an accessible boundary and the process is instantaneously reflecting at zero. If $\delta \leq 2$, $\liminf_{t\to\infty} X_t = 0$. Zero is inaccessible if $\delta \geq 2$.

1. Introduction

Bessel processes have been extensively studied and used in modeling many real world phenomena such as population dynamics [2] and interest rates in finance [1]. We begin by introducing the squared Bessel process which is constructed as the squared distance between the origin and the position of the δ-dimensional Brownian motion at time t [12, p. 439]. The process was first constructed for integer values of δ and subsequently defined and analyzed for real $\delta \geq 0$.

Definition 1.1 (Squared Bessel Process)**.** Let B_t be $\delta-$dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ be the natural filtration. The process

$$X = B_1^2 + B_2^2 + \cdots + B_\delta^2$$

is called a squared Bessel process of dimension δ (BESQ$^\delta$).

Construction of this process for positive integer δ is straightforward. Using the Itô formula,

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dW_t, \qquad X_0 = x, \tag{1.1}$$

where W_t is a one-dimensional Brownian motion constructed from B_t. The process $BESQ^\delta$ for general real $\delta \geq 0$ is defined as the solution of (1.1). This process is well suited for modeling population dynamics and financial asset pricing. Because the coefficients satisfy a linear growth condition, the equation has a global solution (no explosion in finite time) [4, p. 177]. Note that the diffusion term is non-Lipschitz and hence the uniqueness follows from the Yamada-Watanabe theorem [12, p. 390]:

Theorem 1.2 (Yamada-Watanabe Theorem)**.** *Suppose that for a one-dimensional SDE,*

$$dX_t = a(t, X_t)\, dt + \sigma(t, X_t)\, dW_t; \qquad X_0 = x_0, \tag{1.2}$$

the function a is Lipschitz continuous and there exists a strictly increasing function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\int_0^{0+} h^{-2}(x)\, dx = +\infty$ *such that,*

$$|\sigma(t, x) - \sigma(t, y)| \leq h\,(|x - y|)$$

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for all t and $x, y \in \mathbb{R}$. Then pathwise uniqueness holds.

The construction of the squared Bessel process and other properties are discussed in Revuz and Yor and we state the main theorem here [12, p. 442]:

Theorem 1.3. *The stochastic differential equation*

$$dX_t = \delta\, dt + 2\sqrt{|X_t|}\, dW_t, \qquad X_0 = x_0,$$

has a unique strong solution and, if δ, $x_0 \geq 0$, then $X_t \geq 0$. Furthermore,

(1) *for $\delta > 3$ the process is transient, and for $\delta \leq 2$, it is recurrent,*
(2) *for $\delta \geq 2$, zero is polar and, for $\delta \leq 1$, zero is reached a.s.,*
(3) *for $\delta = 0$, zero is an absorbing boundary,*
(4) *for $\delta > 0$, the process is instantaneously reflecting at zero.*

The squared Bessel process was analyzed by Feller [2] in a generalized form and used in the Cox model [1]. It was further generalized by Göing-Jaeschke and Yor to allow $\delta < 0$ dimensions [5,7]. Parameter estimation of the squared Bessel process was discussed by Göing-Jaeschke in her Ph.D. dissertation [6]. Squared Bessel processes in non-colliding particle systems were analyzed in [3], [8], and [9].

In this paper, we construct a generalization of the Bessel process by introducing a predictable almost everywhere positive process $\gamma(t, \omega)$ to the drift and diffusion terms. The motivation for this construction is to define a process X_t where the dynamics depend on $\gamma_t = X_{t-\tau}$, the state of the process at a fixed time τ units in the past. For instance, in a hypothetical population, the growth in the number of adults at time t depends on the number of adults able to reproduce at time $t - \tau$, where τ is the time required for an egg to mature to adulthood. In this model, the adult population could rebound even after reaching zero, due to the delay in the hatching of eggs and the maturation of individuals. The behavior of such a process at boundary zero is of particular interest. This analysis may be applied to other fields where the past needs to be considered in the present dynamics of the model.

In [10], Mohammed gives a general formulation of a stochastic delay differential equation (SDDE). We consider the one-dimensional case. For each solution path $X(t)$, define the segment $x_t\colon [-\tau, 0] \to \mathbb{R}$ by $x_t(s) = X(t+s)$. The initial segment x_0 is given by $\theta\colon [-\tau, 0] \to \mathbb{R}$ and the coefficients of the stochastic differential equation are functionals h and g of the segments, x_t:

$$dX(t) = h(t, x_t)\, dt + g(t, x_t)\, dW(t), \quad X(t) = \theta(t) \text{ for } t \in [-\tau, 0]. \tag{1.3}$$

As it is noted in [10], in the discrete delay case where h and g only depend on the values of $X(t - \tau)$ and $X(t)$, such a system can be solved in successive steps as stochastic ordinary differential equations (SODE) on time intervals of length τ. For $t \in [0, \tau]$, the coefficients of the SODE are determined by $\theta(t)$. The coefficients of the SODE for $t \in [n\tau, (n+1)\tau]$ depend on the already determined solution for $t \in [(n-1)\tau, n\tau]$.

In the following sections, we will present the construction of an SDDE that generalizes the squared Bessel process:

$$dX_t = \delta X_{t-\tau}\, dt + 2\sqrt{X_t X_{t-\tau}}\, dW_t, \quad X_t = \theta_t \text{ for } t \in [-\tau, 0]\,. \tag{1.4}$$

This equation can also be solved in successive steps. Because the diffusion coefficient does not satisfy the Lipschitz condition, the uniqueness results of [10] do not apply. We turn to the Yamada-Watanabe Theorem to show uniqueness.

2. Construction of the Generalized Squared Bessel Process

Let δ be a positive integer, B_t be δ-dimensional Brownian motion, $\mathcal{F}_t$ be the natural filtration and $\gamma : [0,\infty] \times \Omega \to \mathbb{R}$ be an almost everywhere positive, predictable, locally integrable process.
Let time-changed Brownian motions Y_i be defined by

$$Y_i = \int_0^t \sqrt{\gamma_s}\, dB_s^i.$$

Define

$$X = Y_1^2 + Y_2^2 + \cdots + Y_\delta^2.$$

Then, as for the squared Bessel process, using the Itô formula, we get

$$dX_t = \delta\gamma_t\, dt + 2\sqrt{\gamma_t X_t}\, dW_t, \quad X_0 = x_0,$$

where W_t is a one-dimensional Brownian motion.

We extend this construction for a general real $\delta \geq 0$ below. To study the boundary at zero, we follow Revuz and Yor and use the following theorems.

Theorem 2.1 (Local time theorem for Semi-martingales [12, p. 225])**.** *For any continuous semimartingale X, there exists a modification of the local time process $\{L_t^a; a \in \mathbb{R}, t \in \mathbb{R}_+\}$ such that the map $(a,t) \to L_t^a$ is a.s. continuous in t and cadlag in a. Moreover, if $X = M + V$, then*

$$L_t^a - L_t^{a-} = 2\int_0^t 1_{\{X_s = a\}} dX_s = 2\int_0^t 1_{\{X_s = a\}} dV_s.$$

Let $\langle X, X\rangle_t$ denote the quadratic variation process for semi-martingale X.

Theorem 2.2 (Occupation Time Formula [12, p. 224])**.** *If X is a continuous semimartingale, there is a P-negligible set outside of which*

$$\int_0^t \Pi(X_s)\, d\langle X, X\rangle_s = \int_{-\infty}^{\infty} \Pi(a)\, L_t^a\, da$$

for every t and every positive Borel function Π.

We introduce the following theorem for non-negative real δ.

Theorem 2.3. *Let $\delta, x_0 \geq 0$. Suppose that outside of a set of probability zero, γ_t is an almost everywhere positive predictable process, locally integrable with respect to t. Then the SDE*

$$dX_t = \delta\gamma_t\, dt + 2\sqrt{\gamma_t X_t}\, dW_t, \quad X_0 = x_0, \tag{2.1}$$

has a unique strong solution. If $\delta > 0$, the process almost surely instantaneously reflects at zero and is a.e. positive. If $\delta = 0$, the boundary at zero is absorbing.

Proof. The existence and uniqueness follow as in the proof of Theorem 1.3 [12]. Equation 2.1 may also be solved by means of a random time change of the $BESQ^\delta$ process with time change rate γ_t [11, p. 153]. Pathwise uniqueness for solutions follows with the application of the Yamada-Watanabe Theorem (Theorem 1.2), noting that $|\sqrt{x} - \sqrt{y}| < \sqrt{x-y}$ for $x > y \geq 0$.

We modify the arguments of Revuz and Yor [12, p. 442] to analyze the boundary behavior of the process at zero. Note that $d\langle X, X\rangle_t = 4\gamma_t X_t dt$ and use Theorem 2.1 to get

$$L_t^0 = 2\delta \int_0^t 1_{\{X_s=0\}}\gamma_s ds.$$

Since $\gamma_s > 0$ a.e., then almost surely for fixed t,

$$\begin{aligned}\int_0^t \gamma_s ds &\geq \int_0^t 1_{\{X_s>0\}}\gamma_s ds\\ &= \int_0^t 1_{\{X_s>0\}}\gamma_s \left(4\gamma_s X_s\right)^{-1} d\langle X, X\rangle_s\\ &= \int_0^\infty 1 \cdot (4a)^{-1} L_t^a da,\end{aligned}$$

where the last equality follows from the Occupation Time Formula. Since $\int_0^t \gamma_s ds < \infty$, this implies that $L_t^0 = 2\delta \int_0^t 1_{\{X_s=0\}}\gamma_s\, ds = 0$. Hence

$$|s > 0 : X_s = 0| = 0$$

when $\delta > 0$. When $\delta = 0$, $X_t \equiv 0$, $t \geq s$, is the unique solution with $X_s = 0$ and thus zero is an absorbing boundary. □

3. The Stochastic Delay Differential Equation

For the rest of the results we choose $\gamma_t = X_{t-\tau}$ where $\tau > 0$ is the fixed delay time. We now present the main theorem of this paper.

Theorem 3.1. *Let $\tau > 0$ be fixed and let θ_t be positive, integrable, and independent of $\mathcal{F}_0$ for $t \in [-\tau, 0]$. Then the SDDE*

$$dX_t = \delta X_{t-\tau}\, dt + 2\sqrt{X_t X_{t-\tau}}\, dW_t, \quad X_t = \theta_t \text{ for } t \in [-\tau, 0] \tag{3.1}$$

has a unique strong solution and this solution is nonnegative. In addition:

1. *If $\delta = 0$, zero is an absorbing boundary.*
2. *If $\delta > 0$, zero is a reflecting boundary.*
3. *If $\delta < 2$, the process reaches zero with positive probability and $\liminf_{t\to\infty} X_t = 0$ almost surely.*
4. *If $\delta = 2$, X_t reaches arbitrarily small values ϵ, $0 < \epsilon < \theta_0$, in finite time almost surely.*
5. *If $\delta \geq 2$, zero is inaccessible and X_t reaches any $m > \theta_0$ in a finite time almost surely.*

Proof. The global existence and uniqueness of solutions follow by successive applications of Theorem 2.3 to time intervals of length τ. Note that for $t \in [0, \tau]$, $\gamma_t = X_{t-\tau} = \theta_{t-\tau}$. Once a continuous nonnegative solution is obtained up to time $n\tau$, $\gamma_t = X_{t-\tau}$ is determined for $t \in [n\tau, (n+1)\tau]$ [10].

If $\delta = 0$, it follows that zero is an absorbing boundary and therefore X_t is nonnegative. If $\delta > 0$, we claim that $X_t \geq 0$ and X_t instantaneously reflects at 0. Arguing inductively, let Y_t be the solution of the stochastic differential equation,

$$dY_t = 2\sqrt{Y_t X_{t-\tau}}\, dW_t, \quad Y_{n\tau} = X_{n\tau}, \tag{3.2}$$

for $t \in [n\tau, (n+1)\tau]$. If X_t is nonnegative in the interval $[(n-1)\tau, n\tau]$, then by applying a comparison theorem to the processes defined in (3.1) and (3.2) [12, p. 394], it follows that $X_t \geq Y_t \geq 0$ for $t \in [n\tau, (n+1)\tau]$. Furthermore, by Theorem 2.3, X_t instantaneously reflects at zero and is positive a.e. on $[n\tau, (n+1)\tau]$ if it is positive a.e. on $[(n-1)\tau, n\tau]$. The result follows because, on the initial segment, $\theta_t > 0$ by hypothesis.

Now we consider the boundary behavior for different values of δ. Suppose $0 < \delta < 2$. Define stopping time

$$\lambda_n = \inf\{t > 0 : X_t = n \text{ or } X_t = 0\}.$$

First we claim that for almost all ω, there exists an n such that $\lambda_n = \infty$ or $X_{\lambda_n} = 0$. Let $X_0 = \theta_0$ and $U_n : [0, \infty) \to \mathbb{R}$ be twice continuously differentiable on $(0, \infty)$. Fix $T > 0$. By the Itô formula [11, p. 44], we have

$$\int_0^{\lambda_n \wedge T} dU_n(X_s) = \int_0^{\lambda_n \wedge T} \left[\delta X_{s-\tau} U_n'(X_s) + 2X_s X_{s-\tau} U_n''(X_s)\right] ds + \int_0^{\lambda_n \wedge T} \left[2\sqrt{X_s X_{s-\tau}}\, U_n'(X_s)\right] dW_s. \tag{3.3}$$

Solving the boundary problem

$$\delta U_n'(x) + 2xU_n''(x) = 0, \quad U_n(0) = 0, \quad U_n(n) = 1,$$

we obtain

$$U_n(x) = \frac{x^{(1-\delta/2)}}{n^{(1-\delta/2)}}.$$

Substituting U_n in (3.3) and applying the Optional Stopping Theorem [12, p. 69], we have

$$\mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\right] = U_n(\theta_0) = \frac{\theta_0^{(1-\delta/2)}}{n^{(1-\delta/2)}},$$

where $\mathbb{E}^\theta$ is expectation conditioned on the initial segment θ. Since $U_n(x) \geq 0$ for $x \in [0, n]$,

$$\begin{aligned} U_n(\theta_0) &= \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\right] \\ &= \mathbb{E}^\theta\left[U_n(X_{\lambda_n})\,;\, \lambda_n < \infty\right] + \lim_{T\to\infty} \mathbb{E}^\theta\left[U_n(X_{\lambda_n \wedge T})\,;\, \lambda_n = \infty\right] \\ &\geq 1 \cdot P^\theta(X_{\lambda_n} = n) \end{aligned}$$

where P^θ is the conditional law of X_t with initial segment θ. Taking the limit $n \to \infty$,

$$\lim_{n\to\infty} P^\theta(X_{\lambda_n} = n) = \lim_{n\to\infty} U_n(\theta_0) = 0.$$

Therefore, since

$$P^\theta(X_{\lambda_n} = n) + P^\theta(\lambda_n = \infty \text{ or } X_{\lambda_n} = 0) = 1,$$

we have that

$$P^\theta\left(\cup_{n=1}^\infty \{\lambda_n = \infty \text{ or } X_{\lambda_n} = 0\}\right) = \lim_{n\to\infty} P^\theta(\lambda_n = \infty \text{ or } X_{\lambda_n} = 0) = 1,$$

which proves the claim.

Now we show that $\liminf_{t\to\infty} X_t(\omega) = 0$ almost surely. Let $V_n(x) = (n-x)/\delta$. Then V_n satisfies

$$\delta V_n'(x) + 2xV_n''(x) = -1$$

with $V_n(n) = 0$ and $V_n(x) \geq 0$ on $(0, n)$. Substituting V_n for U_n in (3.3), we have

$$\lim_{T\to\infty} \mathbb{E}^\theta \left[V_n(X_{\lambda_n \wedge T})\right] = V_n(\theta_0) - \lim_{T\to\infty} \mathbb{E}^\theta \left[\int_0^{\lambda_n \wedge T} X_{t-\tau}\, dt\right].$$

It follows then that

$$\mathbb{E}^\theta \left[\int_0^{\lambda_n} X_{t-\tau}\, dt\,;\, \lambda_n < \infty\right] + \mathbb{E}^\theta \left[\int_0^{\infty} X_{t-\tau}\, dt\,;\, \lambda_n = \infty\right]$$
$$= V_n(\theta_0) - \lim_{T\to\infty} \mathbb{E}^\theta \left[V_n(X_{\lambda_n \wedge T})\right] \leq V_n(\theta_0) < \infty$$

and thus

$$\mathbb{E}^\theta \left[\int_0^{\infty} X_{t-\tau}\, dt\,;\, \lambda_n = \infty\right] < \infty.$$

If there is an n such that $P^\theta\left(\lambda_n = \infty\right) > 0$, then on that event, $\liminf_{t\to\infty} X_t = 0$. Otherwise X_t reaches zero and reflects instantaneously. Since the same is true if the process is considered for $t \in [S, \infty)$, for arbitrarily large time S, $\liminf_{t\to\infty} X_t = 0$ a.s.

Now consider the case $\delta = 2$. Suppose that $0 < \epsilon < \theta_0 < n$. Define

$$\lambda_{\epsilon n} = \inf\left\{t > 0 : X_t = \epsilon \text{ or } X_t = n\right\}.$$

Again, $V_n(x) = (n - x)/2$ is a solution of $2V_n'(x) + 2xV_n(x) = -1$, with boundary condition $V_n(n) = 0$. Hence,

$$\begin{aligned}\mathbb{E}^\theta \left[\int_0^{T} X_{t-\tau}\, dt\,;\, \lambda_{\epsilon n} = \infty\right] &= V_n(\theta_0) - \mathbb{E}^\theta \left[V_n(X_{\lambda_{\epsilon n} \wedge T})\right] \\ &\quad - \mathbb{E}^\theta \left[\int_0^{\lambda_{\epsilon n} \wedge T} X_{t-\tau}\, dt\,;\, \lambda_{\epsilon n} < \infty\right] \\ &\leq V_n(\theta_0) < \infty.\end{aligned}$$

Since $X_t > \epsilon$ for all t when $\lambda_{\epsilon n} = \infty$, letting $T \to \infty$, it follows that $P^\theta(\lambda_{\epsilon n} = \infty) = 0$.

Now let $U_n(x) = 1 - (\log x)/(\log n)$. Then $U_n'(x) + xU_n''(x) = 0$ and $U_n(n) = 0$. Thus

$$\begin{aligned}U_n(\theta_0) &= \lim_{T\to\infty} \mathbb{E}^\theta \left[U_n(X_{\lambda_{\epsilon n} \wedge T})\right] \\ &= \left(1 - \frac{\log \epsilon}{\log n}\right) P^\theta(X_{\lambda_{\epsilon n}} = \epsilon).\end{aligned}$$

Therefore $\lim_{n\to\infty} P^\theta(X_{\lambda_{\epsilon n}} = \epsilon) = 1$, which proves that the process almost surely reaches every positive $\epsilon < \theta_0$ in finite time.

Arguing in a similar way, let $U_n(x) = 1 + (\log x)/(\log \epsilon)$. Thus

$$\begin{aligned}U_n(\theta_0) &= \lim_{T\to\infty} \mathbb{E}^\theta \left[U_n(X_{\lambda_{\epsilon n} \wedge T})\right] \\ &= \left(1 + \frac{\log n}{\log \epsilon}\right) P^\theta(X_{\lambda_{\epsilon n}} = n).\end{aligned}$$

In this case, $\lim_{\epsilon\to 0} P^\theta(X_{\lambda_{\epsilon n}} = n) = 1$ for every n. Since this continuous process almost surely reaches every positive integer before it hits zero, it almost surely cannot reach zero in finite time.

Now suppose $\delta > 2$. We redefine $\lambda_n = \inf\{t > 0 : X_t = 1/n\}$ and solve $\delta U_n'(x) + 2xU_n''(x) = 0$, with boundary condition $U_n(1/n) = 1$, to get

$$U_n(x) = \frac{1}{x^{(\delta/2-1)} n^{(\delta/2-1)}}.$$

With an argument similar to the $\delta < 2$ case, we get,

$$P^\theta(\cup_{n=1}^\infty \{\lambda_n = \infty\}) = 1. \tag{3.4}$$

Now we prove that X_t reaches any $m > \theta_0$ a.s. when $\delta > 2$. Define $\mu_m = \inf\{t > 0 : X_t = m\}$. Then $V_m = (m - x)/\delta$ is a solution of $\delta V_m'(x) + 2xV_m''(x) = -1$, with boundary condition $V_m(m) = 0$. Therefore, by an argument similar to before,

$$\mathbb{E}^\theta \left[\int_0^{\mu_m} X_{t-\tau}\, dt \right] < \infty,$$

and thus,

$$\mathbb{E}^\theta \left[\int_0^{\mu_m} X_{t-\tau}\, dt\,;\, \mu_m < \infty \right] + \mathbb{E}^\theta \left[\int_0^{\mu_m} X_{t-\tau}\, dt\,;\, \mu_m = \infty \right] < \infty.$$

Arguing by contradiction, suppose that $P(\mu_m = \infty) > 0$. Then using (3.4), there exists a positive integer n_0 such that $P(\lambda_{n_0} = \infty, \mu_m = \infty) > 0$ and thus

$$\mathbb{E}^\theta \left[\int_0^{\infty} X_{t-\tau}\, dt\,;\, \mu_m = \infty \right] = \infty,$$

which leads to a contradiction. Therefore, $P(\mu_m = \infty) = 0$. □

Remark 3.2. Because the coefficient functions in the stochastic delay differential equations depend on the history of the process, these processes are not in general Markov [10]. Nevertheless, we identify properties analogous to the properties of recurrence and transience as defined for Markov processes. When $\delta \le 2$, $\liminf_{t\to\infty} X_t = 0$ almost surely and thus it satisfies a recurrence property at zero. On the other hand, the preceding proof shows that when $\delta > 2$, $\liminf_{t\to\infty} X_t > 0$ almost surely, and when $\delta \ge 2$, $\limsup_{t\to\infty} X_t = \infty$ almost surely.

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References

[1] J. C. Cox, J. E. Ingersoll, and S. A. Ross, *A theory of the term structure of interest rates*, Econometrica **53** (March 1985), no. 2, 385–408.
[2] W. Feller, *Diffusion processes in genetics*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1951, pp. 227–246.
[3] P. Graczyk and J. Małecki, *Multidimensional Yamada-Watanabe theorem and its applications to particle systems*, J. Math. Phys. **54** (2013), no. 021503.
[4] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North Holland, Amsterdam, 1989.
[5] A. Göing-Jaeschke and M. Yor, *A survey and some generalizations of Bessel processes*, Bernoulli **9** (2003), no. 2, 313–349.
[6] A. Göing-Jaeschke, *Parameter Estimation and Bessel Processes in Financial Models and Numerical Analysis in Hamiltonian Dynamics*, Ph.D. Dissertation, ETH Zürich, 1998.
[7] A. Göing-Jaeschke, *Some generalizations of Bessel processes*, RiskLab, ETH Zürich, 1997.
[8] M. Katori and H. Tanemura, *Noncolliding squared Bessel processes*, J. Stat. Phys. **142** (2011), 592–615.

[9] W. König and N. O'Connell, *Eigenvalues of the Laguerre process as non-colliding squared Bessel processes*, Electronic Journal of Probability **6** (2001), no. 11, 107–114.

[10] Salah-Eldin A. Mohammed, *Stochastic differential systems with memory: theory, examples and applications*, Proceedings of the Sixth Oslo-Silivri Workshop Geilo 1996, 1997, pp. 412–489.

[11] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 6th ed., Springer, Berlin, 2003.

[12] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd ed., Comprehensive Studies in Mathematics, vol. 293, Springer, New York, 1999.

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