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Digraphs of small defect or excess

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This thesis consists of two parts, the first part consists of unpublished work related to the directed/degree problem. The second part consists of the following papers:

[A] Anita Abildgaard Sillasen, "On k-geodetic digraphs with excess one," submitted to *Electronic Journal of Graph Theory and Applications*, in peer review.

[B] Anita Abildgaard Sillasen, "Subdigraphs of almost Moore digraphs induced by fixpoints of an automorphism," submitted to *Electronic Journal of Graph Theory and Applications*, in peer review.

[C] Leif Kjær Jørgensen and Anita Abildgaard Sillasen, "On the Existence of Friendship Hypergraphs," recommended for publication in *Journal of Combinatorial Designs*.

This thesis has been submitted for assessment in partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis. As part of the assessment, co-author statements have been made available to the assessment committee and are also available at the Faculty. The thesis is not in its present form acceptable for open publication but only in limited and closed circulation as copyright may not be ensured.

This page replaces page iii in the thesis submitted and dated December 13, 2013.

Preface

Already during my time as a graduate student I was intrigued by the graph theoretical research area which is the degree/diameter problem, thus the study of graphs with small defect. At that time I mostly studied the undirected version, and when the time came for me to choose a problem for my PhD, it seemed natural to choose the directed version of the problem. I set out with ambitions to do a lot of research in almost Moore digraphs (defect 1), but along the way I was inspired to also go in other directions, although still concentrating on problems related to the directed degree/diameter problem.

This has broadened not only my knowledge, but also my interests and abilities to see different problems from more than one perspective.

Acknowledments

More than three years has passed since I started as a PhD-student. The work I have been doing has partially been supported by Prof. Carsten Thomassen's Grants 0602-01488B from The Danish Council for Independent Research | Natural Sciences, which I thank him for.

During my studies I have met many wonderful people, some of whom I would like to say a special thanks to.

First of all my supervisor Assoc. Prof. Leif K. Jørgensen. Thank you for taking the leap with me, doing your best to answer all my questions and opening my eyes to possibilities I could not see myself.

A special thanks to Prof. Jozef Širáň for inviting me to Bratislava and inspiring me to work on the vertex-transitive version of the directed degree/diameter problem.

No words can express my gratitude to Prof. Mirka Miller and Dr. Joe Ryan, for taking me under their wings and taking good care of me on several occasions. First during my research stay in Newcastle, Australia and later on my visits to Pilsen and Prague, Czech Republic, where also Prof. Zdeněk Ryjáček made sure I had everything I needed. Special thanks to Prof. Mirka Miller for always encouraging me in my work, and along with Prof. Zdeněk Ryjáček, inspiring me to work on digraphs of degree 2 and defect 3. And thanks to Dr. Joe Ryan for inspiration and our discussions on the oriented version of the MaxDDBS.

Whilst in Australia other people besides Mirka and Joe crossed paths with me. Of these I would like to send my thanks to Dr. Yuqing Lin and Assoc. Prof. Ljiljana Brankovic for making sure I was entertained and saw some of the most amazing places in the area near Newcastle. To Prof. Zdeněk Ryjáček and RNDr. Přemysl Holub for their acquaintance which resulted in meeting them again in Pilsen. To RNDr. Andrea Feňovčíková, her husband Marian Feňovčik and their son Dominik for becoming a part of my family in all senses and later on adding Dorotka to our international family. To Carol and Prof. Don Kreher for invitations to several events and good company during these. And finally to the group at Mayfield Wests, for showing me that Australia is not all about maths.

I would also like to thank my colleagues at Department of Mathematical Sciences, both administrative and academic personnel, for always having an open door policy and helping each other when possible. A special thanks to my fellow PhD student, Sabrina Munch Hansen for providing me with healthy snacks of various kinds and for lending me an ear at all times and occasionally sharing my frustrations.

Thanks also to my family and friends, for being there and taking an interest in me and my work.

And to my husband Kim, the love of my life. Thank you for your support, for believing in me when I do not believe in myself and for your patience with me.

Anita Abildgaard Sillasen
Aalborg University, December 13, 2013

Summary

This thesis is mainly focused on contributing to solving a problem in graph theory, the so called directed degree/diameter problem, where one is given two positive integers d and k and then wishes to determine the largest possible order in a digraph with maximum out-degree at most d and diameter at most k. An upper theoretical bound on the order is given by $M(d, k)$, the so called Moore bound. This bound is known to be obtained only if $d = 1$ or $k = 1$, and thus we wish to establish how close the order can get to this bound. Our contribution is based on proving structural properties of digraphs with order close to the Moore bound, assuming they exist and if possible using these properties to show the non-existence of such digraphs for certain values of d or k.

Part I of the thesis contains unpublished and unsubmitted work, whereas Part II contains three papers, all submitted and in review or in the process of being published.

Chapter 1 contains an introduction to the notation and terminology used throughout the thesis and also used in several of the references. We also introduce some of the known results for the directed degree/diameter problem, especially those results which we use throughout the thesis. Here we also state, that if a digraph has maximum out-degree d, diameter k and $M(d, k) - \delta$ vertices, then it is said to have defect δ. For defect 1 (almost Moore digraphs) we give a structural characterization of the vertices in such a digraph with respect to an automorphism r.

In Chapter 2 we introduce the extra constraint on the digraphs, that they should also be vertex-transitive or arc-transitive, and we prove the non-existence of an infinite number of vertex- and arc-transitive digraphs with order close to the Moore bound.

In Chapter 3 we will focus on digraphs of maximum out-degree 2 and defect 3. We prove, that if they are not diregular, they are almost diregular, and we prove that a diregular digraph of degree $d = 2$ and defect 3 has girth at least k.

In Chapter 4 we introduce a new problem, the oriented maximum degree/diameter-bounded subgraph problem, where given an undirected graph and positive integers d and k, we wish to determine the largest possible order of a subgraph, such that this subgraph has an orientation of maximum out-degree d and diameter k. For sufficiently large d we solve this problem for the planar triangular grid and the planar square grid.

Paper [A] presents a new problem of determining the smallest possible order of a k-geodetic digraph with minimum out-degree d, and a theoretical bound is in fact given by the Moore bound, $M(d, k)$. A k-geodetic digraph with minimum out-degree d and order $M(d, k) + \epsilon$ is then denoted as a digraph of excess ϵ. For $\epsilon = 1$ we prove that if the digraph is not diregular, then it is almost diregular. Furthermore we prove the non-existence of diregular k-geodetic digraphs with $d = 2$ and excess 1 for all $k \geq 3$.

In Paper [B] we prove that given a non-trivial automorphism which fixes at least three vertices in an almost Moore digraph where not all vertices have the same order with respect to the automorphism, this digraph must contain a smaller almost Moore digraph or a diregular digraph of excess 1 as an induced subdigraph. This can be used to characterize the orders of the vertices with respect to the repeat automorphism r and we do this for $d = 4$ and $d = 5$ where we also use the results from Paper [A].

Paper [C] has no obvious relations to the directed degree/diameter problem, as it mainly concerns 3-uniform friendship hypergraphs, which are 3-uniform hypergraphs in which for every triple x, y, z of vertices, there exists a unique vertex w such that xyw, xzw and yzw are edges in the hypergraph. We construct an infinite family of 3-uniform friendship hypergraphs on 2^k vertices and 3-uniform friendship hypergraphs on 20 and 28 vertices along with a 4-uniform friendship hypergraph on 9 vertices.

Danish summary (dansk resumé)

Denne afhandling er hovedsageligt fokuseret på at bidrage til at løse et problem indenfor grafteori, det såkaldte orienterede grad/diameter-problem, hvor man, givet to positive heltal d og k, ønsker at bestemme den stØrt mulige orden i en orienteret graf med maximum udgrad d og diameter k. En øvre teoretisk grænse er givet ved $M(d,k)$, den såkaldte Moore grænse. Det vides at denne grænse kun kan opnås hvis $d = 1$ eller $k = 1$, og dermed er vi interesseret i at bestemme hvor tæt ordenen kan komme på denne grænse. Vores bidrag er baseret på at bevise strukturelle egenskaber for orienterede grafer med orden tæt på Moore grænsen, under antagelse af at de eksisterer, og hvis muligt anvende disse egenskaber til at bevise ikke-eksistensen af sådanne orienterede grafer for visse værdier af d og k.

Del I af afhandlingen indeholder upubliceret og ikke-indsendt arbejde, hvorimod del II indeholder tre artikler, alle indsendt og under vurdering eller i processen med at blive publiceret.

Kapitel 1 indeholder en introduktion til notationen og terminologien som anvendes i afhandlingen og også er anvendt i flere af referencerne. Vi introducerer også nogle af de kendte resultater indenfor det orienterede grad/diameter problem, specielt de resultater som vi anvender i afhandlingen. Vi gør også opmærksom på, at en orienteret graf med maximum udgrad d, diameter k og $M(d,k)-\delta$ knuder, siges at have defekt δ. For defekt 1 angiver vi en strukturel karakterisering af knuderne i en sådan orienteret graf med hensyn til automorfien r.

I kapitel 2 introducerer vi nogle ekstra restriktioner på de orienterede grafer, de skal enten være knudetransitive eller kanttransitive, og vi beviser ikke-eksistensen af et uendeligt antal af orienterede knude- og kanttransitive grafer med orden tæt på Moore grænsen.

I kapitel 3 vil vi fokusere på orienterede grafer med maximum udgrad 2 og defekt 3. Vi beviser, at hvis de ikke er diregulære, så er de næsten diregulære, og vi beviser at en orienteret diregulær graf af grad 2 og defekt 3 ikke indeholder orienterede kredse med længde kortere end k.

I kapitel 4 introducerer vi et nyt problem, det orienterede maximum grad/diameter-afgrænsede delgraf problem, hvori vi, givet en ikke-orienteret graf og positive heltal d og k, ønsker at bestemme den størst mulige orden af en delgraf, således at denne delgraf har en orientering med maximum udgrad d og diameter k. Vi løser dette problem for tilstrækkeligt store værdier af d i de plane triangulære og kvadratiske net.

Artikel [A] præsenterer et nyt problem, hvor man ønsker at bestemme den mindst mulige orden af en orienteret k-geodætisk graf med minimum udgrad d og en teoretisk grænse er givet ved Moore grænsen, $M(d,k)$. En orienteret k-geodætisk graf med minimum udgrad d og orden $M(d,k)+\epsilon$ kaldes så for en orienteret graf med overskud ϵ. For $\epsilon = 1$ beviser vi at hvis den orienterede graf ikke er diregulær, så er den næsten diregulær. Desuden beviser vi ikke-eksistensen af orienterede diregulære k-geodætiske

grafer med $d = 2$ og overskud 1 for alle $k \geq 3$.

I artikel [B] beviser vi, givet en ikke-triviel automorfi, der fastholder mindst tre knuder i en orienteret graf med defekt 1, hvor ikke alle knuder har den samme orden mht. automorfien, at denne orienterede graf enten indeholder en mindre orienteret graf af defekt 1 eller en orienteret diregulær graf med overskud 1. Dette kan anvendes til at karakterisere ordenen af knuderne mht. automorfien r i en orienteret graf med defekt 1 og dette gør vi for $d = 4$ og $d = 5$, hvor vi også anvender resultaterne fra artikel [A].

Artikel [C] har ikke nogen åbenlys forbindelse til det orienterede grad/diameter problem, da den hovedsageligt handler om 3-uniforme venskabshypergrafer, som er 3-uniforme hypergrafer hvori der, for hver 3-mængde x, y, z af knuder, eksisterer en entydig knude w således at xyw, xzw og yzw er kanter i hypergrafen. Vi konstruerer en uendelig familie af 3-uniforme venskabshypergrafer med 2^k knuder og 3-uniforme venskabshypergrafer med 20 og 28 knuder, samt en 4-uniform venskabshypergraf med 9 knuder.

Contents

Part I

Problems related to the directed degree/diameter problem

Chapter 1

Introduction

We start by introducing some of the terminology and notation concerning digraphs, before introducing the directed degree/diameter problem, and some of the results and notation known and used in this research area, as the main part of the results in this part of the thesis are highly related to these.

1 Terminology and notation for digraphs

We will mainly consider directed graphs without loops and multiple arcs, which we denote as digraphs. Such a *digraph* is a pair $G = (V(G), A(G))$ where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $A(G) \subseteq \{(u,v)|u, v \in V(G), u \neq v\}$. If $u \in V(G)$ we say u is a *vertex* in G and if the ordered pair $(u,v) \in A(G)$ we say (u,v) is an *arc* in G. The *order* of G is the number of vertices n. If (u,v) is an arc in G, we say v is an *out-neighbor* of u and u is an *in-neighbor* of v. We let $N^+(u) = \{v \in V(G)|(u,v) \in A(G)\}$ ($N^-(u) = \{v \in V(G)|(v,u) \in A(G)\}$) be the set of out-neighbors (in-neighbors) of u and let $d^+(u) = |N^+(u)|$ be the *out-degree* of u and $d^-(u) = |N^-(u)|$ be the *in-degree* of u. If all vertices in G have the same out-degree (in-degree) d, we say G is *out-regular* (*in-regular*). If G is both out-regular and in-regular, we say G is *diregular*.

A *walk* W of length l in G is an sequence $(u_0, u_1, \ldots, u_l)$ of vertices, where $a_i = (u_{i-1}, u_i) \in A(G)$ for $i = 1, 2, \ldots, l$. The vertices $u_1, u_2, \ldots, u_{l-1}$ are called *internal vertices* of W. A walk is *closed* if $u_0 = u_l$. If the arcs $a_1, a_2, \ldots, a_l$ are all distinct we say that W is a *trail*, and if in addition all the vertices $u_0, u_1, \ldots, u_l$ are distinct we say W is a *path*. A closed trail of length $l > 0$ with all except the first and last vertex distinct is said to be a *cycle*. A walk of length l or respectively at most l will be denoted as a l-*walk* or respectively a $\leq l$-*walk*.

For all integers $i \geq 1$ we define the multisets $N_i^+(u) = N^+(N_{i-1}^+(u))$ where $N_0^+(u) = \{u\}$ and $N_1^+(u) = N^+(u)$. Thus $N_i^+(u)$ represents all the vertices which are reachable from u with an i-walk. Similarly we define $N_i^-(u) = N^-(N_{i-1}^-(u))$ where $N_0^-(u) = \{u\}$ and $N_1^-(u) = N^-(u)$, thus all vertices which can reach u with an i-walk. We also define the multisets $T_i^+(u) = \cup_{j=0}^i N_j^+(u)$, thus $T_i^+(u)$ are all the vertices which are reachable from u with a $\leq i$-walk and similarly we have $T_i^-(u) = \cup_{j=0}^i N_j^-(u)$ as all the vertices which can reach u with a $\leq i$-walk. Notice that for each $\leq i$-walk from u to a vertex v, there is an appearance of v in $T_i^+(u)$.

We will distinguish between counting all the vertices, thus the cardinality, in a multiset S using the notation $|S|$ and only counting the different vertices in a multiset using the notation $||S||$. For a set U we see that $|U| = ||U||$.

The *distance* from a vertex u to v is denoted as $dist(u, v)$ and is defined as the length of the shortest path from u to v. We let $dist(u, u) = 0$ and one should notice that in general we do not have $dist(u, v) = dist(v, u)$ for digraphs. The *diameter* of G is the maximum distance among any two vertices of G. If the diameter k is finite, then any vertex u is reachable in $\leq k$-walks from all other vertices of G.

Furthermore, if we have some automorphism $\varphi : V(G) \mapsto V(G)$, then by $\varphi(W)$ where $W = (u_0, u_1, \ldots, u_l)$, we understand the walk $(\varphi(u_0), \varphi(u_1), \ldots, \varphi(u_l))$. If $\varphi(v) = u$ we use the notation $v = \varphi^-(u)$. Also, for all positive integers $p > 1$ we let $\varphi^p(u) = \varphi^{p-1}(\varphi(u))$ and $\varphi^{-p}(u) = \varphi^{-p+1}(\varphi^-(u))$ where $\varphi^1(u) = \varphi(u)$ and $\varphi^{-1}(u) = \varphi^-(u)$. By the *order* of u with respect to φ, $\omega(u)$, we understand the smallest positive integer such that $\varphi^{\omega(u)}(u) = u$.

A digraph H is a *subdigraph* of G if $V(H) \subseteq V(G)$ and $A(H) \subseteq A(G)$ and an *induced subdigraph* of G if it also that for $u, v \in V(H)$ and $(u, v) \in A(G)$ we have $(u, v) \in A(H)$. When we are dealing with more than one digraph, and it is not clear from the context which digraph a certain parameter refers to, we will use a subscript, for instance $d_G^+(u)$ is the out-degree of u in the digraph G.

2 The directed degree/diameter problem

Let d and k be positive integers, then the directed degree/diameter problem is the problem of finding the maximum number of vertices $n(d, k)$ in a digraph of maximum out-degree at most d and diameter at most k. As the maximum out-degree is d, one easily sees that an upper bound for $n(d, k)$ is given by letting all vertices have out-degree d, thus

$$n(d, k) \leq 1 + d + d^2 + \ldots + d^k = M(d, k). \tag{1.1}$$

The right-hand side of (1.1) is also referred to as *the directed Moore bound $M(d, k)$*, named after E. G. Moore who gave the upper bound for the (undirected) degree/diameter problem, see [1]. A digraph which has maximum out-degree at most d, diameter k and order $M(d, k)$ is denoted as a *Moore digraph.*

Plesník and Znám proved in 1974 [2] that so called *strongly geodetic digraphs* do only exist as $(k+1)$-cycles or complete digraphs. A strongly geodetic digraph is a digraph in which for every ordered pair of vertices u, v there exists exactly one directed $\leq k$-walk from u to v, where k is the diameter, thus one sees that a digraph is strongly geodetic if and only if it is a Moore digraph. Thus Plesník and Znám had proven that Moore digraphs only exist as $(k+1)$-cycles $(d=1)$ and as the complete digraphs K_{d+1} $(k=1)$.

In 1980 Bridges and Toueg, [3], gave a very short and concise proof using the eigenvalues of the adjacency matrix of a directed Moore digraph, proving that Moore digraphs only exist for $d=1$ or $k=1$, thus an alternative proof than that of Plesník and Znám. This proof is probably the most famous proof of the two, as it truly is very short and concise.

Thus, for $k>1, d>1$ we have $n(d,k) \leq M(d,k)-1$, and the question is whether or not it is possible to obtain equality. In fact we are looking for the smallest positive integer δ, such that $n(d,k) = M(d,k) - \delta$. A digraph with maximum out-degree d, diameter k and order $M(d,k)-\delta$ is denoted as a *(d,k)-digraph with defect δ* or a *$(d,k,-\delta)$-digraph*, and (d,k)-digraphs of defect 1 are also referred to as *almost Moore digraphs*.

For small defects, $\delta < M(d,k-1)$, we have out-regularity, as if we assume not, then there would be a vertex with out-degree at most $d-1$ and hence the order n of such a digraph would be

$$\begin{aligned} n &\leq 1 + (d-1) + (d-1)d + (d-1)d^2 + \dots (d-1)d^{k-1} \\ &= M(d,k) - M(d,k-1), \end{aligned}$$

a contradiction to the size of the defect. In-regularity however is more difficult to prove, so far we only know that (d,k)-digraphs with defect 0 and 1 are diregular see [4], whereas it is still an open question for higher defects.

Now, almost Moore digraphs do exist for $k=2$ with $d \geq 2$, line digraphs of complete digraphs are examples hereof, thus $n(d,2) = M(d,2)-1$, see [5]. Using algebraic number theory combined with spectral techniques, it has been proven that no almost Moore digraphs of diameter 3 and 4 exist for $d \geq 2$, see [6] and [7]. These proofs rely on the irreducibility of certain polynomials, and in fact it is possible to prove non-existence of almost Moore digraphs for $k \geq 5$ if the corresponding polynomials are irreducible, see [8]. However, this is not a trivial task.

Using a more graph theoretical approach, thus showing special structural properties, Miller and Fris [9], and Baskoro, Miller, Siran and Sutton [10], proved that no almost Moore digraphs of degree 2 and 3 exist for any $k \geq 3$. In the next section we will look at some of the special structures which was used in the proofs of [9], [10] and various other papers dealing with the directed degree/diameter problem, see [11].

3 Repeats in $(d,k,-\delta)$-digraphs

Let D be a $(d,k,-\delta)$-digraph. Then we define the repeat multiset of $u \in V(D)$ as $R(u) = T_k^+(u) \backslash V(D)$, thus it contains all vertices which occurs more than once in $T_k^+(u)$, the so called *repeats*, and the number of times a repeat occurs in $R(u)$ is one less than the number of times it occurs in $T_k^+(u)$. In fact we have $|R(u)| = \delta$ and we see that $||T_k^-(u)|| = ||T_k^+(u)|| = M(d,k) - \delta$ for all $u \in V(D)$ as k is the diameter. If a vertex $u \in R(u)$, then we say u is a selfrepeat.

We will also be using which vertices a vertex is the repeat of, so let the multiset $R^-(u)$ be given by $R^-(u) = \{w \in V(G) | u \in R(w)\} = T_k^-(u) \backslash V(D)$. Notice that

$$\begin{aligned} 0 &\leq |R^-(u)| \\ &= |T_k^-(u)| - (M(d,k) - \delta) \\ &\leq \delta \cdot (M(d,k) - \delta) \end{aligned}$$

and

$$\sum_{u \in V(G)} |R(u)| = \sum_{u \in V(G)} |R^-(u)| = \delta \cdot (M(d,k) - \delta). \tag{1.2}$$

4 Almost Moore digraphs

For an almost Moore digraph D with $d \geq 2$ and $k \geq 3$ we can obtain the following properties by simple counting arguments, see [12],

- for each pair of vertices u, v in D, there is at most one $\leq (k-1)$-walk from u to v,
- for each vertex u there is a unique vertex v such that there are two $\leq k$-walks from u to v.

As there is only one vertex in $R(u)$, we will denote this vertex by $r(u)$, and this vertex is the unique vertex satisfying the second property. Due to the diregularity of almost Moore digraphs, the mapping $u \mapsto r(u)$ is in fact an automorphism, see [12]. It is worth noticing, that reversing all arcs of D results in yet an $(d,k,-1)$-digraph, thus all results regarding arcs, neighborhoods, distances and the automorphism r, have dual results for almost Moore digraphs.

We will use a graphical representation of $T_k^+(u)$ for a vertex u in a digraph of maximum out-degree $d \geq 2$, diameter $k \geq 3$ and defect δ, consisting of the vertex u, the arcs from u to its out-neighbors and a triangle representing each of the multisets $T_{k-1}^+(v)$ for $v \in N^+(u)$, where the bottom edge in the triangle represents $N_{k-1}^+(v)$. This graphical representation will often also contain the placement of $R(u)$ if possible and sometimes other vertices. Similar graphical representations of $T_k^-(u)$ will also be used.

Where further detail is needed or relevant, these will be provided in the figures, but as it would consume a lot of space, we will omit drawing figures every time we look at some $T_k^+(u)$ for an u, and recommend that the reader draws pictures of the relevant situation, to support the text if needed.

In the following we will show graphical representations of $T_k^+(u)$ where u is a vertex in an almost Moore digraph with $d \geq 2$ and $k \geq 3$, according to how u can be classified as one of four different types of vertices in the almost Moore digraph.

If a vertex u in an almost Moore digraph is a selfrepeat, we say that u is of type 0 and depict thus $T_k^+(u)$ as in Figure 1.1.

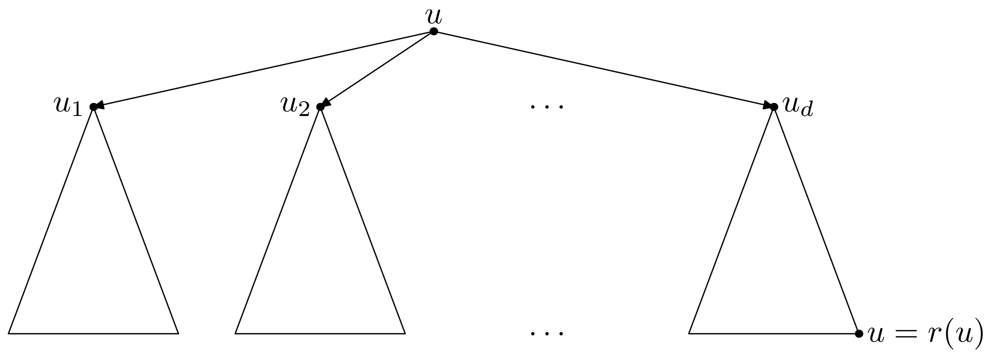


Figure 1.1: The vertex u is a type 0 vertex.

According to [13], we have the following result.

Theorem 1.1. *[13] If an almost Moore digraph with $k \geq 3$ has a selfrepeat, then it has exactly k selfrepeats forming a k-cycle.*

If u is not a selfrepeat and $r(u) \notin N^-(u)$, then we say u is of type 1, see Figure 1.2. Notice that we have no knowledge of the distance from u to its repeat, but we do know that one of the occurrences of $r(u)$ in $T_k^+(u)$ must be in distance k from u due to the fact that there is at most one $< k$-walk from u to $r(u)$.

If u is not a selfrepeat and $r(u) \in N^-(u)$ we say u is a vertex of type 2. We must then have $dist(u, r(u)) = k$ and by the pigeon hole principle and the diregularity of the digraph, we have that one of the out-neighbors of u must have u as a repeat, hence $r^{-1}(u) \in N^+(u)$. With this in mind, we distinguish between the type 2 vertices according to whether the distance from $r^{-1}(u)$ to $r(u)$ is $k-1$, see Figure 1.3, or k, see Figure 1.4. We then say they are of type $2a$ and $2b$ respectively.

Lemma 1.2. *Any type $2a$ (resp. $2b$) vertex has precisely one out-neighbor of type $2a$ (resp. $2b$) and no other out-neighbors of type 2.*

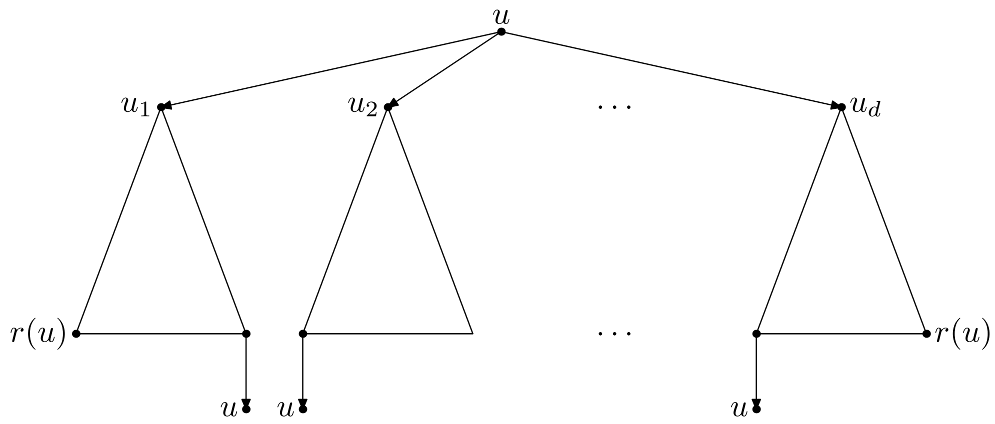


Figure 1.2: The vertex u is a type 1 vertex. Notice that one of the occurences of $r(u)$ could be closer to u.

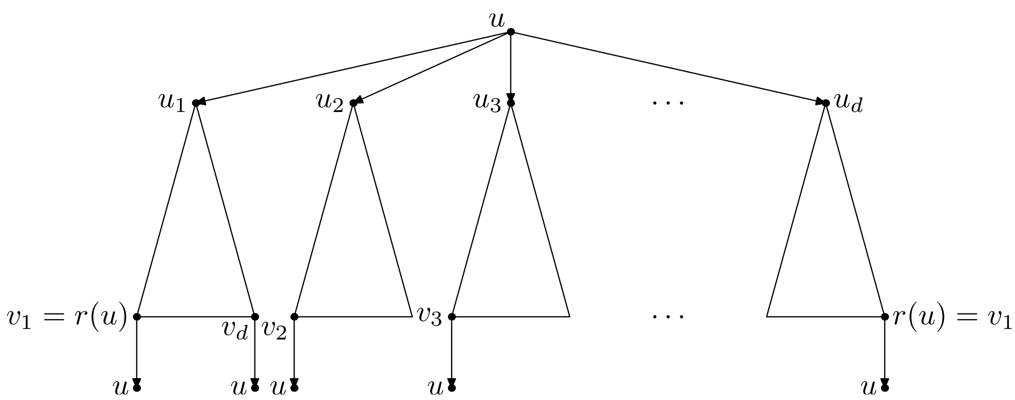


Figure 1.3: The vertex u is a type $2a$ vertex.

Proof. Let u be a type $2a$ vertex and let $u_1, u_2, \dots, u_d$ be the out-neighbors of u as in Figure 1.3.

Then clearly u_1 is of type $2a$, as $u = r(u_1)$ and r is an automorphism. Now assume u_i is of type 2 for an $i = 2, 3, \dots, d$. Then there exists two walks of length two from $r(u)$ to u_i, namely $(r(u), r(u_1) = u, u_i)$ and $(r(u), r(u_i), u_i)$, a contradiction.

The same argument can be applied when u is a type $2b$ vertex as in Figure 1.4 with u_2 in the place of u_1 in the above. $\square$

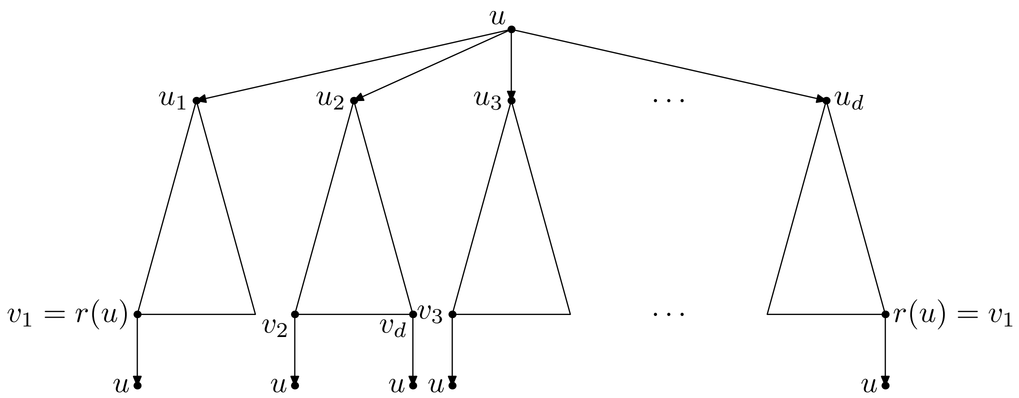


Figure 1.4: The vertex u is a type $2b$ vertex.

Chapter 2

Vertex- and arc-transitive digraphs with small defect

A popular restricted version of the degree/diameter problem for undirected graphs, is that of vertex-transitive graphs, thus we are looking for the largest order of a vertex-transitive graph with maximum degree d and diameter k. Some results regarding upper and lower theoretical bounds for the maximum order have been proven, but nothing more general than that of the general case, see [11]. We do know that all the known Moore graphs are also vertex-transitive. A great deal of work has been put into constructing large vertex-transitive graphs, along with the special case of Cayley-graphs, which are also vertex-transitive.

A digraph D is *vertex-transitive* if for each pair of vertices u, v there exists an automorphism $\varphi : V(D) \mapsto V(D)$ such that $u = \varphi(v)$. Similarly, a digraph D is *arc-transitive* if for each pair of arcs (u, v) and (x, y) there exists an automorphism $\varphi : V(D) \mapsto V(D)$ such that $\varphi(x) = u$ and $\varphi(y) = v$. Notice, that if D is a strongly connected arc-transitive digraph, then it is also vertex-transitive. A nice property of vertex-transitive digraphs is that they are diregular.

For digraphs there has been constructed a great deal of large vertex-transitive digraphs with degree d and diameter k, but no research in finding a theoretical upper bound of the order has been identified, except for that of the general digraphs. We do however know that the Moore digraphs, as complete digraphs and cycles are all vertex-transitive, and that for diameter 2 there exists vertex-transitive almost Moore digraphs for $d \geq 2$ as the Kautz digraphs are all vertex-transitive as they are line digraphs of complete digraphs. In fact, for all values of $d \geq 3$, all the almost Moore digraphs with diameter $k = 2$ are vertex-transitive, see [14].

In this chapter we prove the impossibility of some vertex-transitive digraphs with given degree, diameter and defect and thus also lower some of the upper theoretical bounds of the order of vertex-transitive digraphs. We will assume all digraphs to be of degree $d \geq 2$ and diameter $k \geq 3$. Before stating and proving the results, we will prove the following lemma which concerns vertex-transitive digraphs in general.

Lemma 2.1. *Let G be a vertex-transitive digraph with order n and let $C(l)$ be the number of cycles of length l through any given vertex v. Then $l|n \cdot C(l)$.*

Proof. We count the number of pairs (v, C), where C is a cycle of length l and $v \in C$, in two different ways. First by looking at the vertices, which we know are each contained in exactly $C(l)$ cycles, we get that the number of pairs is $n \cdot C(l)$. Then by looking at the cycles of length l, let the number of these be denoted by r, the number of pairs is $r \cdot l$. Hence

$$n \cdot C(l) = r \cdot l,$$

and thus the result follows. □

1 Vertex-transitive almost Moore digraphs

In this section, we will only prove results concerning vertex-transitive almost Moore digraphs.

Lemma 2.2. *Let G be a vertex-transitive almost Moore digraph. Then all vertices in G are of type* 1.

Proof. Clearly, as G is vertex-transitive, all vertices must be of the same type. At most k vertices can be of type 0, see Theorem 1.1, so none is of type 0. Due to Lemma 1.2, they cannot all be of type 2, so the only remaining is type 1. □

The following theorem follows directly from Lemmas 2.1 and 2.2 and thus $C(k+1) = d$.

Theorem 2.3. *Let d and k be positive integers such that $(k+1) \nmid (M(d,k)-1)d$, then there exists no vertex-transitive almost Moore digraph of degree d and diameter k.*

When looking at a quadratic $m \times m$ table where the (i,j)th entry represents the smallest possible defect of a vertex-transitive digraph of degree i and diameter j, one sees that when $m = 14$ we have more than half of the entries in the table to be larger than 1, and we conjecture that when $m \to \infty$ this ratio will approach 1.

Corollary 2.4. *Let d and k be odd positive integers, then there exist no vertex transitive almost Moore digraph of degree d and diameter k.*

Proof. Let d and k be odd, then

$$n \cdot d = d^2 + d^3 + \ldots + d^{k+1} \equiv 1 \mod 2.$$

But this contradicts that we due to Lemma 2.1 have that $(k+1)|n \cdot d$. □

2 Vertex-transitive digraphs with small defect

Theorem 2.5. *Let δ, d and k be positive integers such that $\delta < d$ and $k \geq 3$. Let p, q be odd primes, such that $p > \delta$, $q > \delta + 1$, $p|k$, $q|(k+1)$ and $d \equiv 1 \mod (p \cdot q)$. Then there is no vertex-transitive digraph with degree d, diameter k and defect δ.*

Proof. Assume G is a vertex-transitive digraph with order n, degree d, diameter k and defect $\delta < d$ and p, q as stated in the theorem. We will look at two cases:

- G **has selfrepeats:** This corresponds to saying the girth of G is k and that $\delta > 1$ due to Lemma 2.2, see also Theorem 1.1. As we assume there is at least one selfrepeat and we know the defect is $\delta < d$, we see that $1 \leq C(k) \leq \delta$. According to Lemma 2.1 we have $k|n \cdot C(k)$ and hence, as $p|k$, $p|n \cdot C(k)$. But $p > \delta$ and therefore $p \nmid C(k)$, so we must have $p|n$. On the other hand, we have $d \equiv 1 \mod (pq)$ and hence

$$\begin{aligned} n &\equiv d^k + d^{k-1} + \ldots + d + 1 - \delta \mod p \\ &\equiv k + 1 - \delta \mod p \\ &\equiv 1 - \delta \mod p, \end{aligned}$$

  thus a contradiction, as $1 < \delta < d$ and $p > \delta$.

- G **has no selfrepeats:** Now we know the girth is $k+1$ and $d \leq C(k+1) \leq d+\delta$. According to Lemma 2.1 and the fact that $q|(k+1)$, we get that $q|n \cdot C(k+1)$. As $q > \delta + 1$ and $d \equiv 1 \mod q$, we have that $q \nmid C(k+1)$ and hence $q|n$. Also

$$\begin{aligned} n &\equiv d^k + d^{k-1} + \ldots + d + 1 - \delta \mod q \\ &\equiv k + 1 - \delta \mod q \\ &\equiv -\delta \mod q, \end{aligned}$$

  thus a contradiction as $\delta + 1 < q$.

□

Corollary 2.6. *Let d and k be positive integers such that $d > 1$ and $k \geq 3$. Let q be an odd prime, $q|(k+1)$ and $d \equiv 1 \mod q$. Then there is no vertex-transitive almost Moore digraph with degree d and diameter k.*

3 Arc-transitive digraphs with small defect

Notice, that as mentioned in the beginning of this chapter, an arc-transitive digraph with finite diameter is also vertex-transitive, so the results in the previous section does also apply in this section.

Lemma 2.7. *Let D be a digraph. If $\varphi : V(D) \mapsto V(D)$ is an automorphism such that the vertex $u \in V(D)$ is a fixpoint, then $\varphi(r(u)) \in R(u)$ for all $r(u) \in R(u)$.*

Proof. Let $r(u) \in R(u)$ and P and Q be two paths from u to $r(u)$. Then $\varphi(P)$ and $\varphi(Q)$ are two paths from u to $\varphi(r(u))$ and hence $\varphi(r(u)) \in R(u)$. $\square$

If D is arc-transitive and $u \in V(D)$ with $N^+(u) = \{u_1, u_2, \ldots, u_d\}$, then there is at least d automorphisms which has the vertex $u \in V(D)$ as fixpoint, namely the automorphisms which maps the arc (u, u_1) to (u, u_i) for $i = 1, 2, \ldots, d$.

Theorem 2.8. *Let δ, d and k be positive integers such that $d \geq 3$, $k \geq 2$ and $\delta \leq \frac{d-1}{2}$. Then there exist no arc-transitive digraph of degree d, diameter k and defect $\delta \geq 1$.*

Proof. Notice that due to the definition of a repeat and the defect δ, there is at most 2δ paths from a vertex u to its set of repeats $R(u)$.

Assume D is a digraph as in the statement and let u be a vertex in $V(D)$ with out-neighbors $\{u_1, u_2, \ldots, u_d\}$.

Now, without loss of generality we assume that the arc (u, u_1) is an arc in a path P from u to one of its repeats. Thus, according to Lemma 2.7 there are at least d paths from u to $R(u)$ as there are at least d automorphisms which has u as a fixpoint. But this contradicts the fact that there are at most $2\delta \leq 2 \cdot \frac{d-1}{2} = d - 1$ paths from u to $R(u)$. $\square$

Theorem 2.9. *Let δ, d and k be positive integers and p be an (odd) prime such that $p|(k+1)$, $p > \delta + 1$ and $d \equiv 1 \mod p$. Then there exist no arc-transitive digraph of degree d, diameter k and defect $\delta < d$.*

Proof. Let D be an arc-transitive digraph as in the statement. Then D cannot contain any selfrepeats, as this would mean all arcs would be in a k-cycle, thus there would be at least d k-cycles, and thus $\delta \geq d$, a contradiction.

The rest of the statement follows from the proof of Theorem 2.5. $\square$

Corollary 2.10. *There exist no arc-transitive digraphs with selfrepeats, degree d, diameter k and defect $\delta < d$.*

Chapter 3

Digraphs of out-degree at most 2, diameter k and defect 3

Throughout this chapter we assume G to be a digraph of maximum out-degree d, diameter $k \geq 3$ and defect 3. We know that there exist a (diregular) $(2, 3, -3)$-digraph, namely the Kautz digraph on 12 vertices and with girth 2, but for all other values of $d \geq 2$ and $k \geq 3$ we do not know any $(d, k, -3)$-digraphs, see [11]. We start by proving that if $d = 2$ and G is not diregular, then there are two, three or four vertices of in-degree 1 and one or two of in-degree 3 and/or 4, whereas the rest of the vertices have in-degree 2, and we know from the Chapter 1, that G must be out-regular as $\delta = 3 < M(2, k-1) - 3$. In the second section we state some general results for a diregular digraph G of degree d, diameter $k \geq 3$ and defect 3 and use these results in the third section which is concerned with diregular digraphs of degree 2, diameter $k \geq 3$ and defect 3, especially we show that for $k \geq 4$ the girth must be at least k.

We start by introducing some general notation concerning digraphs of defect 3. We consider the following possibilities for the multiset $R(v)$ and corresponding notations:

If $R(v) = \{v, v, v\}$ we say v is a triple selfrepeat (TSR), if $R(v) = \{v, v, r(v)\}$ we say that v is a double selfrepeat (DSR), if $R(v) = \{v, r(v), r(v)\}$ we say v is a selfrepeat (SR) and has a double repeat (DR), if $R(v) = \{v, r_1(v), r_2(v)\}$ we say v is a selfrepeat (SR), if $R(v) = \{r(v), r(v), r(v)\}$ we say v has a triple repeat (TR), if $R(v) = \{r_1(v), r_1(v), r_2(v)\}$ we say v has a double repeat (DR), and the last case is that $R(v) = \{r_1(v), r_2(v), r_3(v)\}$.

Notice that triple and double refers to the number of times the repeat occurs in $R(v)$, and not the number of times it occurs in $T_k^+(v)$ (a triple (self) repeat would occur four times in $T_k^+(v)$ and so on).

1 Diregularity of digraphs of maximum out-degree 2 and defect 3

In this section we let G be a digraph of maximum out-degree 2, diameter $k \geq 3$ and defect 3. Also let S be the set of vertices which have in-degree less than 2 in G, hence $S = \{v \in V(G) | d^-(v) = 1\}$. Similarly let $S' = \{x \in V(G) | d^-(x) > 2\}$. Then we have the following lemma.

Lemma 3.1. *$S \subseteq N^+(R(u))$ for all $u \in V(G)$.*

Proof. Let $v \in S$ and let w be the in-neighbor of v. Pick an arbitrary vertex $u \in V(G)$ with out-neighbors u_1 and u_2. Now, for both u_1 and u_2 to be able to reach v, we must have $v \in T_k^+(u_i)$ for $i = 1, 2$. This means that $w \in T_{k-1}^+(u_i) \cup \{u\}$ for $i = 1, 2$ and for one, let's say without loss of generality $i = 1$, we must have $w \in T_{k-1}^+(u_1)$. This implies $\{w, w\} \subseteq T_k^+(u)$ and hence $w \in R(u)$ for all $u \in V(G)$. □

Corollary 3.2. *In G we have*

- $2 \leq |S| \leq 6$
- $|S'| \leq 6$
- *$v_i v_j$ is not an arc for $v_i, v_j \in S$*

Proof. $|S| \leq 6$ and $|S'| \leq 6$ follows directly from Lemma 3.1 and the proof thereof. Assume then $|S| = 1$ and $v \in S$, then $|S'| = 1$ and $d^-(s) = 3$ for $s \in S'$. So

$$\begin{aligned} ||T_k^-(v)|| &\leq 1 + 1 + 3 + 6 + \ldots + 3 \cdot 2^{k-2} \\ &= 2^k + 2^{k-1} - 1 < 2^{k+1} - 4 \\ &= M(2, k) - 3, \end{aligned}$$

a contradiction.

Assume $v_i v_j$ is an arc for $v_i, v_j \in S$. Then

$$\begin{aligned} ||T_k^-(v_j)|| &\leq 1 + 1 + 1 + 8 + 16 + \ldots + 2^k \\ &\leq M(2, k) - 4, \end{aligned}$$

a contradiction.

□

Lemma 3.3. *$|S| \leq 4$.*

Proof. Assume $|S| \geq 5$, so $\{v_1, v_2, v_3, v_4, v_5\} \subseteq S$. Then, due to the proof of Lemma 3.1, we have the fixed repeat-set $R(u) = \{r_1, r_2, r_3\}$ for all $u \in V(G)$ where $r_i \neq r_j$ for $i \neq j$, and without loss of generality we can assume $\{v_1, v_2\} = N^+(r_1)$, $\{v_3, v_4\} = N^+(r_2)$ and $v_5 \in N^+(r_3)$. Now, for v_i to reach v_j in a $\leq k$-path for $i \neq j$, we must have $v_i \in T^-_{k-1}(r_l)$ for $i = 1, 2, 3, 4, 5$ and $l = 1, 2$, thus $v_i \in R(v_i)$ for $i = 1, 2, 3, 4$, which implies there are four different repeats, a contradiction. □

The following corollary follows immediately from Lemma 3.3.

Corollary 3.4. *In G we have $d^-(u) \leq 6$ for all $u \in V(G)$.*

Lemma 3.5. *$N^-(S) \subseteq S'$.*

Proof. Assume the statement is false, hence that there exists a $v_1 \in S$ such that $r_1 = N^-(v_1) \notin S'$, where $r_1 \in R(u)$ for all $u \in V(G)$. Then

$$\begin{aligned} ||T^-_k(v_1)|| &\leq 1 + 1 + 2 + 8 + 16 + \ldots + 2^k \\ &= M(2, k) - 3 \end{aligned}$$

and as we need to have equality we must have $||T^-_k(v_1)|| = |T^-_k(v_1)|$, hence v_1 is not a repeat of any vertex and we must have $|S| = 4$, $|S'| \leq 2$ and $x \in N^-_2(v_1)$ for all $x \in S'$ for equality to hold. Now, $v_i \notin N^+(r_1)$ for $i = 2, 3, 4$, as $v_1 \in T^-_k(v_i)$ would then imply that v_1 is a selfrepeat, a contradiction. So there exist repeats r_2 and r_3 ($R(u) = \{r_1, r_2, r_3\}$ for all $u \in V(G)$) such that $v_2 \in N^+(r_2)$ and $\{v_3, v_4\} = N^+(r_3)$. But this implies that v_3 and v_4 are selfrepeats, hence $v_3, v_4 \in R(u)$ for all $u \in V(G)$ a contradiction to v_iv_j not being an arc, see Corollary 3.2. □

Lemma 3.6. *If $|S| = 2$, then $S' = \{x\}$ and $d^-(x) = 4$. Furthermore we have $x \in R(u)$ for all vertices $u \in V(G)$, $R(x) = \{x, x, w\}$ where $d^-(w) = 2$, $R(v_i) = \{v_i, x, w_i\}$ where $w_i \in N^+(w)$ and $R(z) = \{x, v_1, v_2\}$ where $d^-(z) = 2$, $\{x, x\} \in R(z_i)$ where $S = \{v_1, v_2\}$ and $z_i \in N^-(z)$ for $i = 1, 2$ and the remaining vertices all appear as a repeat exactly twice.*

Proof. Let $S = \{v_1, v_2\}$ and assume $S' = \{x_1, x_2\}$, thus $d^-(x_1) = d^-(x_2) = 3$. Then for $k > 3$ we must have

$$\begin{aligned} ||T^-_k(v_i)|| &\leq 1 + 1 + 3 + 7 + 14 + \ldots + 2^k - 2^{k-3} \\ &< M(2, k) - 3, \end{aligned}$$

a contradiction. If $k = 3$, we can assume $v_1 \in N^+(x_1)$, and $x_2 \in N^-_2(v_1)$ as we must have $||T^-_3(v_1)|| = 1 + 1 + 3 + 7 = 12$. This implies that v_1 isn't a repeat for any vertex in G, thus we especially have $v_1 \notin T^-_2(x_1)$ and hence $v_2 \notin N^+(x_1)$. Thus according to Lemma 3.5 we must have $x_2 \in N^-(v_2)$, and it must be satisfied that $N^+(x_1) = \{v_1, x_2\}$

and $N^+(x_2) = \{v_2, x_1\}$. But then (x_1, x_2, x_1) is a 2-cycle, and thus by looking at $T_2^+(x_1)$ we see that $x_1 \in R^-(v_1)$, a contradiction to the fact that $v_1 \notin R(x_1)$.

So for all $k \geq 3$ we must have $x \in S'$ with $d^-(x) = 4$ and $N^+(x) = \{v_1, v_2\}$. For all vertices $u \in V(G)$ with $N^+(u) = \{u_1, u_2\}$ we must have $x \in T_{k-1}^+(u_i)$ for $i = 1, 2$ for u_i to reach both v_1 and v_2 and thus $x \in R(u)$ for all $u \in V(G)$, see Figure 3.1. Notice, if a vertex $u \neq v_i$ has v_i as a repeat for an $i \in \{1, 2\}$, then it must also have v_j as a repeat for $j \in \{1, 2\} \backslash \{i\}$, as u can only reach v_i through x.

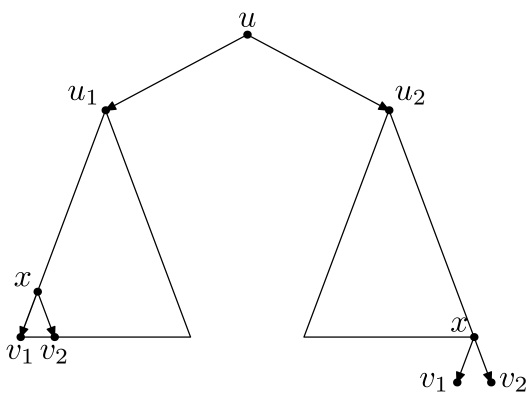


Figure 3.1: The vertex u has x as a repeat (the occurence of x in $T_{k-1}^+(u_1)$ could be placed higher).

We see that x is a DSR, as v_1 and v_2 needs to reach each other, and that $v_i \notin R(x)$ for $i = 1, 2$ and thus $dist(v_i, x) = k - 1$ for $i = 1, 2$, as otherwise $\{x, x, v_1, v_2\} \subseteq R(x)$, a contradiction, see Figure 3.2. On the other hand, x cannot be a TSR, as otherwise for an $i \in \{1, 2\}$ we would have $\{x, x, v_1, v_2, v_i\} \subseteq R(v_i)$, a contradiction. So there must exist some vertex $w \in V(G)$ with $d^-(w) = 2$ such that $R(x) = \{x, x, w\}$.

Now for v_1 and v_2 we see that they must be selfrepeats, as they reach each other through x which has an arc to both of them, see Figure 3.2. Also v_j cannot be a repeat of v_i as then x would be a TSR. We see that for $v \in S$ we have

$$\begin{aligned} |T_k^-(v)| &\geq 1 + 1 + 4 + 8 + \ldots + 2^k \\ &= M(2, k) - 3 + 2 \end{aligned}$$

as $v_i \notin R(x)$, hence $|R^-(v_i)| \geq 2$. Now we already know $v_i \in R^-(v_i)$ so there must exist a vertex $z \in V(G)$ with $d^-(z) = 2$ and $R(z) = \{x, v_1, v_2\}$.

Let $N^+(w) = \{w_1, w_2\}$. We know that $\{w, w\} \in T_k^+(x)$ and as the other in-neighbors of w_1 and w_2 must be in $T_k^+(x)$, there must be at least 3 occurrences of w_i in $T_k^+(v_1) \cup T_k^+(v_2)$ for $i = 1, 2$, so without loss of generality we can assume that $w_i \in R(v_i)$ for $i = 1, 2$. Notice, that if $x = w_i$ for an $i \in \{1, 2\}$, then x will be a double repeat of v_1 or v_2, as there will be 5 occurrences of x in $T_k^+(v_1) \cup T_k^+(v_2)$.

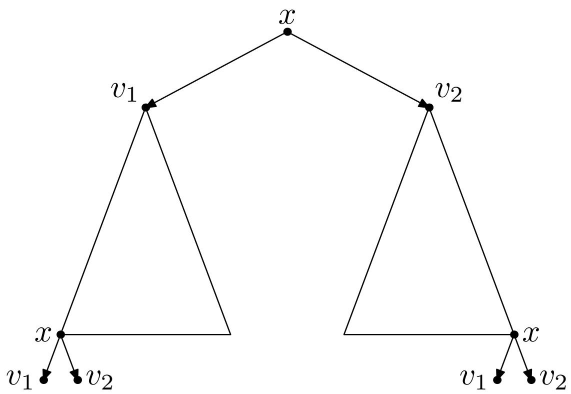


Figure 3.2: The vertex x is a DSR.

For x we see that

$$\begin{aligned}|T_k^-(x)| &\geq 1 + 4 + 8 + \ldots + 2^k + 2^{k+1} - 2\\ &= 2(M(2,k) - 3) + 3,\end{aligned}$$

so $|R^-(x)| \geq M(2,k) - 3 + 3$, and in fact the in-neighbors z_1 and z_2 of z must have x as a DR, as $\{x, x\} \in T_{k-1}^+(z)$ and both out-neighbors of z_i for $i = 1, 2$ have to be able to reach v_1 and v_2.

Now, for $u \in V(G) \backslash \{S, S'\}$ we see that

$$\begin{aligned}|T_k^-(u)| &\geq 1 + 2 + 4 + \ldots + 2^k - 2\\ &= M(2,k) - 3 + 1,\end{aligned}$$

so $|R^-(u)| \geq 1$, with equality satisfied if and only if $u = w$, so for $u \in V(G) \backslash \{S, S', w\}$ we see $|R^-(u)| \geq 2$. Thus according to (1.2) on page 6 we get

$$\begin{aligned}3(M(2,k) - 3) &\geq \sum_{u \in V(G)} |R^-(u)|\\ &= \sum_{u \in V(G) \backslash \{S, S', w\}} |R^-(u)| + \sum_{u \in S} |R^-(u)| + |R^-(x)| + |R^-(w)|\\ &\geq 2(M(2,k) - 3 - 4) + 2 \cdot 2 + (M(2,k) - 3 + 3) + 1\\ &= 3(M(2,k) - 3),\end{aligned}$$

and as we then must have equality throughout, the result follows. □

Lemma 3.7. *If $|S| = 3$, then $S' = \{x_1, x_2\}$ with $d^-(x_1) = 4$, $d^-(x_2) = 3$ and $N^+(x_1) = \{v_1, x_2\}$ and $N^+(x_2) = \{v_2, v_3\}$ where $v_i \in S$ for $i = 1, 2, 3$. Furthermore $R(v) = \{x_1, x_2, v\}$ for $v \in S$, $R(x_1) = \{x_1, x_1, x_2\}$, $R(x_2) = \{x_1, x_2, x_2\}$ and for $u \in V(G)\backslash\{S, S'\}$ we have $R(u) = \{x_1, x_2, w_u\}$ where $w_u \in V(G)\backslash\{S, S'\}$, each u being a repeat exactly once.*

Proof. Let $S = \{v_1, v_2, v_3\}$, then due to Lemma 3.5 we must have $|S'| \geq 2$.

So assume $S' = \{x_1, x_2, x_3\}$, where $d^-(x_1) = d^-(x_2) = d^-(x_3) = 3$. Without loss of generality we can assume that $v_1 \in N^+(x_1)$ and $v_2 \in N^+(x_2)$, due to Lemma 3.5. Now if $v_3 \in N^+(x_i)$ for let's say $i = 2$, then

$$\begin{aligned} ||T_k^-(v_3)|| &\leq 1 + 1 + 3 + 8 + \ldots + 2^k \\ &= M(2, k) - 2, \end{aligned}$$

and v_3 and v_2 are selfrepeats. Also we must have $N^+(x_1) = \{v_1, x_2\}$. Then for x_2 to reach v_1 we have

$$\begin{aligned} ||T_k^-(v_1)|| &\leq 1 + 1 + 3 + 7 + 14 + \ldots + 2^k - 2^{k-3} - 1 \\ &< M(2, k) - 3 \end{aligned}$$

for $k \geq 3$, a contradiction, so we must have $v_3 \in N^+(x_3)$.

As $v_i \in N^+(x_i)$ and $d^+(x_i) = 2$ for $i = 1, 2, 3$ we see that if an x_i has at most one in-neighbor from S', we get

$$\begin{aligned} ||T_k^-(v_i)|| &\leq 1 + 1 + 3 + 7 + 15 + \ldots + 2^k - 2^{k-4} \\ &< M(2, k) - 3, \end{aligned}$$

a contradiction for $k > 3$. If $k = 3$ we can assume without loss of generality that (x_1, x_3, x_2, x_1) is a 3-cycle, as we must have $||T_3^-(v_i)|| = 12 = M(2, 3) - 3$ for $i = 1, 2, 3$. This also implies that $|T_3^-(x_i)| = 24$ and hence $|R^-(x_i)| = 12$ for $i = 1, 2, 3$. So for any vertex $u \in V(G)\backslash S'$ we see that u isn't a repeat of any vertex, and thus $|T_3^-(u)| = 12$. This is only possible for every $u \in V(G)\backslash S'$ if all vertices of in-degree 2 has exactly one in-neighbor of in-degree 2 and one of in-degree 1. Labeling the vertices as in Figure 3.3, we will look more closely at the vertices in $T_3^-(u_2)$, which we can see in Figure 3.4. We need the vertices u_1, u_3, u_4, u_5 to reach u_2 and we see that we cannot have u_3 or u_4 as an in-neighbor of x_3, as otherwise u_3 or u_4 would have v_2 as a repeat, a contradiction. But then either (u_3, u_4) or (u_4, u_3) will be an arc, and thus u_3 or u_4 would have v_2 as a repeat.

Thus for $k \geq 3$ we can safely assume that $S' = \{x_1, x_2\}$, where $d^-(x_1) = 4$, $d^-(x_2) = 3$ and $\{x_1, x_2\} \in R(u)$ for all $u \in V(G)$. As before we assume without loss of generality that $v_1 \in N^+(x_1)$ and $v_2 \in N^+(x_2)$. Now, we must have $x_2 \in N^+(x_1)$, and hence $v_3 \in N^+(x_2)$, as otherwise

$$\begin{aligned} ||T^-(v_2)|| &\leq 1 + 1 + 3 + 6 + 14 + \ldots 2^k - 2^{k-3} \\ &< M(2, k) - 3. \end{aligned}$$

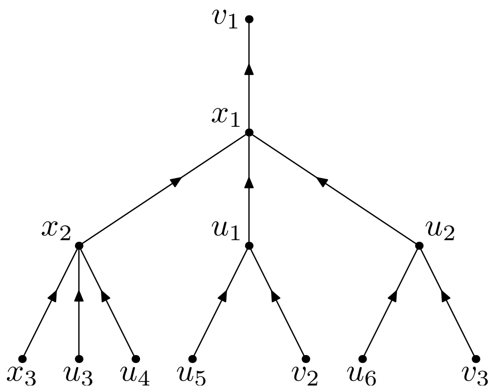


Figure 3.3: Labeling of the vertices in $T_3^-(v_1)$.

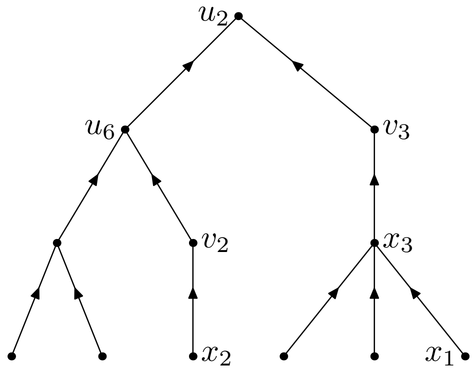


Figure 3.4: The vertices in $T_3^-(u_2)$.

Now, for $j = 2, 3$ we have that v_j is a selfrepeat, as v_2 can only reach v_3 through x_2, and hence will also reach v_2 again and the same for v_3 to reach v_2. From this we also deduce that $R(x_2) = \{x_1, x_2, x_2\}$ and $dist(x_2, x_1) = k - 1$ (as x_2 needs to reach v_1). So $|R^-(v_j)| \geq 1$ for $j = 2, 3$ and $|R^-(x_2)| \geq M(2, k) - 3 + 1$.

Assume there exists a $w \in V(G)\backslash\{S, S'\}$ which is a repeat of x_1, then x_1 appears two times in $T_k^-(w)$. As x_1 must reach w through either v_1 or x_2 (and if through x_2 it must be through v_2 or v_3) we see that

$$\begin{aligned} ||T_k^-(w)|| &\leq 1 + 2 + 4 + \ldots + 2^k - 4 \\ &= M(2, k) - 4. \end{aligned}$$

So the last repeat of x_1 must be in S or S'. But then for $u \in V(G)\backslash\{S, S'\}$ we have $|T_k^-(u)| \geq 1 + 2 + 4 + \ldots + 2^k - 2 = M(2, k) - 3 + 1$ and hence $|R^-(u)| \geq 1$

Now, for v_1 we see that

$$\begin{aligned}|T_k^-(v_1)| &\geq 1+1+4+8+\ldots+2^k-1\\ &= M(2,k)-3+1,\end{aligned}$$

as x_2 can only reach v_1 through one of v_2 or v_3 and $dist(x_2,x_1)=k-1$, thus $|R^-(v_1)|\geq 1$.

Also for x_1 we see that

$$\begin{aligned}|T_k^-(x_1)| &\geq |T_k^-(v_1)|-1+2(2^k-1)-1\\ &= 2(M(2,k)-3)+1,\end{aligned}$$

as $\{x_1,x_1,x_2,x_2\}\subseteq T_k^-(x_1)$, so $|R^-(x_1)|\geq M(2,k)-3+1$.

From (1.2) on page 6, we see that

$$\begin{aligned}3\cdot(M(2,k)-3) &= \sum_{u\in V(G)}|R^-(u)|\\ &= \sum_{u\in V(G)\backslash\{S,S'\}}|R^-(u)|+\sum_{u\in S}|R^-(u)|+|R^-(x_1)|+|R^-(x_2)|\\ &\geq (M(2,k)-3-5)+3+(M(2,k)-3+1)+(M(2,k)-3+1)\\ &= 3(M(2,k)-3),\end{aligned}$$

and hence we must have $|R(x_i)|=M(2,k)-3+1$ for $i=1,2$, $|R^-(u)|=1$ for $u\in V(G)\backslash S'$ and $R(v_i)=\{x_1,x_2,v_i\}$ for $i=1,2,3$, $R(x_1)=\{x_1,x_1,x_2\}$, $R(x_2)=\{x_1,x_2,x_2\}$ and for $u\in V(G)\backslash\{S,S'\}$ we must have $R(u)=\{x_1,x_2,w_u\}$ where $w_u\in V(G)\backslash\{S,S'\}$, each u being a repeat exactly once. □

Lemma 3.8. *Let $|S|=4$, then $S'=\{x_1,x_2\}$ where $d^-(x_1)=d^-(x_2)=4$. Furthermore $R(v)=\{x_1,x_2,v\}$ for $v\in S$, $R(x_1)=\{x_1,x_1,x_2\}$, $R(x_2)=\{x_1,x_2,x_2\}$ and for $u\in V(G)\backslash\{S,S'\}$ we have $R(u)=\{x_1,x_2,w_u\}$ where $w_u\in V(G)\backslash\{S,S'\}$, each u being a repeat exactly once.*

Proof. Let $S=\{v_1,v_2,v_3,v_4\}$ and assume $\{x_1,x_2,x_3\}\in S'$.

According to Lemma 3.1 and 3.5 we can assume $N^+(x_1)=\{v_1,v_2\}$ and $v_3\in N^+(x_2)$, so $\{x_1,x_2\}\in R(u)$ for all $u\in V(G)$ and v_1 and v_2 are selfrepeats, as they need to reach each other. Looking at $T_k^+(x_1)$ we see that it must contain at least three in-neighbors of x_3 (depending on the in-degree of x_3), thus due to the pigeon hole principle, there must be two occurrences of x_3 in $T_k^+(v_i)$ for some $i\in\{1,2\}$, thus making x_3 a repeat of this v_i, a contradiction as v_i already has three repeats.

Now assume $S'=\{x_1,x_2\}$ and thus $N^+(x_1)=\{v_1,v_2\}$ and $N^+(x_2)=\{v_3,v_4\}$. Then $d^-(x_1)=d^-(x_2)=4$, as otherwise we would get $||T_k^-(v_i)||<M(2,k)-3$ for at

least two $i \in \{1,2,3,4\}$. Now, as before we have $R(v_i) = \{x_1, x_2, v_i\}$ as v_1 and v_2 should be able to reach each other, and v_3 and v_4 likewise, so $|R^-(v_i)| \geq 1$ for $i = 1,2,3,4$. With the same argument, we see that $R(x_1) = \{x_1, x_1, x_2\}$ and $R(x_2) = \{x_1, x_2, x_2\}$ and hence $|R^-(x_i)| \geq M(2,k) - 3 + 1$ for $i = 1,2$.

Now, as we know the repeats of the vertices in S and S' we have for $u \in V(G)\backslash\{S, S'\}$ that $|T_k^-(u)| \geq 1 + 2 + 4 + \ldots + 2^k - 2$ as $x \in S'$ can only occur once in $T_k^-(u)$, so $|R^-(u)| \geq 1$. Then we must have that each u is a repeat exactly once, as we know $\{x_1, x_2\} \in R(u)$, hence $R(u) = \{x_1, x_2, w_u\}$ for some $w_u \in V(G)\backslash\{S, S'\}$.

□

To sum up the results of this section, we have proved that if G is not diregular, then there must two, three or four vertices of in-degree 1 and one or two vertices of in-degree 4 or one of in-degree 3 and one of in-degree 4. We can also characterize the in-degrees of the repeat-set of each vertex $u \in V(G)$ according to the in-degree of u.

In the remaining part of this paper, we will be looking at diregular digraphs with defect 3 and the next section states some facts on diregular digraphs of defect δ and 3 in general, whereas the last section will state some result on diregular digraphs of degree 2 and defect 3.

2 Diregular digraphs of defect 3

The following lemma, was first stated for defect 1 in [12], and also believed to be valid for larger values of δ. As we have not been able to identify a proof of this more general statement, we choose to include a proof here, which is a generalization of the proof in [12].

Lemma 3.9 (Neighborhood Lemma)**.** *Let G be a diregular digraph of degree d, diameter k and defect δ, then*

$$N^+(R(v)) = R(N^+(v)) \tag{3.1}$$

for all $v \in V(G)$.

Proof. We will count the number of $\leq (k+1)$-walks starting in the vertex $v \in V(G)$ in two different ways.

First, we use the fact that G is diregular to realize that the number of $\leq (k+1)$-walks from v is

$$1 + d + d^2 + \ldots + d^{k+1}.$$

The second way to count the walks is a bit more complicated. Notice first, that there is the trivial walk of length 0 from v to itself. Furthermore, as all d out-neighbors must reach any y, there is at least $d \leq (k+1)$-walks from v to a given y. Now, if $y \in N^+(R(v))$ there is an additional number of $\leq (k+1)$-walks from v to y according to the number of times y appears in the multiset $N^+(R(v))$. This number can never exceed

δ as each vertex in $R(v)$ must have d different out-neighbors. Similarly, if $y \in R(v_i)$ for some $v_i \in N^+(v)$, there will be an additional number of $\leq (k+1)$-walks from v to y not exceeding δ.

From these two methods of counting, we obtain that

$$1 + d + d^2 + \ldots + d^{k+1} \geq 1 + (M(d,k) - \delta)d + |N^+(R(v)) \cup R(N^+(v))|, \qquad (3.2)$$

where we by the union $N^+(R(v)) \cup R(N^+(v))$ mean the multiset where each element occur the same number of times as in the multiset where it occurs most often. Thus from (3.2) we get

$$\delta d \geq |N^+(R(v)) \cup R(N^+(v))|,$$

and as $|N^+(R(v))| = \delta d$ and $|R(N^+(v))| = \delta d$, we must have $N^+(R(v)) = R(N^+(v))$. $\square$

Now, let G be a diregular digraph of diameter k, degree d and defect 3, then we have the following lemma.

Lemma 3.10. *For all vertices v, all the vertices in $T^+_{k-2}(v)$ are distinct. Furthermore, if $d > 2$ all vertices in $T^+_{k-1}(v)$ are distinct.*

Proof. Assume there exists a vertex u such that $\{u, u\} \subset T^+_{k-1}(v)$. This implies that $\{u, u\} \cup N^+(u) \cup N^+(u) \subset T^+_k(v)$ and hence $\{u\} \cup N^+(u) \subseteq R(v)$. Then we get that

$$\begin{aligned} |\{u\} \cup N^+(u)| &= 1 + d \\ &\leq |R(v)| = 3 \end{aligned}$$

which is only possible if $d = 2$ and $R(v) = \{u\} \cup N^+(u)$, hence one of the occurrences of u must be in $N^+_{k-1}(v)$. $\square$

With the above lemma, we easily observe, that for $d > 2$ the girth must be either k or $k+1$, as the diameter is k. If $d = 2$ the girth could also be $k-1$, but in the next section we will prove that this is not the case.

3 Digraphs of maximum out-degree 2 and defect 3

In the remaining part of this chapter, we will assume G to be a diregular digraph of degree 2, diameter $k \geq 3$ and defect 3. We will prove that there are three possible repeat-sets for each vertex in G and that G must have girth at least k.

Lemma 3.11. *The repeat-set for a vertex 0 is either $R(0) = \{0, 0, r(0)\}$, $R(0) = \{0, r_1(0), r_2(0)\}$ or $R(0) = \{r_1(0), r_2(0), r_3(0)\}$ where $r_i(0) \neq 0$ for $i = 1, 2, 3$ and $r_i(0) \neq r_j(0)$ for $i \neq j$.*

Proof. We need to show that 0 cannot be a TSR and cannot have a TR or a DR.

Assume there exists a vertex 0 which is a TSR and which has out-neighbors 1 and 2. This implies that there should be at least two occurrences of 0 in either $T^+_{k-1}(1)$ or $T^+_{k-1}(2)$. Without loss of generality assume that $\{0, 0\} \in T^+_{k-1}(1)$. This then implies that $R(1) = \{0, 1, 2, 1\}$ a contradiction.

Now, assume 0 has out-neighbors 1 and 2 and a TR $r(0)$ with out-neighbors x and y. As before, we can assume without loss of generality that there is at least two occurrences of $r(0)$ in $T^+_{k-1}(1)$, so $R(1) = \{r(0), x, y\}$. But as $N^+(R(0)) = \{x, y, x, y, x, y\}$ and $r(0) \notin \{x, y\}$ we have a contradiction according to Lemma 3.9.

Finally, assume 0 has out-neighbors 1 and 2 and a DR $r_1(0)$ with out-neighbors x_1 and y_1. Furthermore, assume that the last repeat of 0 is $r_2(0)$ (which could be 0 itself) with out-neighbors x_2 and y_2. As $r_1(0)$ occurs three times in $T^+_k(0)$, one of $r_1(0)$'s in-neighbors must occur at least two times in $T^+_{k-1}(0)$, and hence it must be the last repeat $r_2(0)$. But then the other out-neighbor of $r_2(0)$, y_2 must also occur at least two times in $T^+_k(0)$, and hence $R(0) = \{r_1(0), r_1(0), r_2(0), y_2\}$, a contradiction. □

The following theorem states that the only digraph of degree 2 and defect 3 with girth $k - 1$ is the Kautz digraph with diameter 3 and hence 12 vertices.

Theorem 3.12. *The girth of a digraph of degree* 2*, diameter* $k > 3$ *and defect* 3 *is at least* k*.*

Proof. Assume the girth is $k - 1$, then there exists a vertex 0 which is in a $(k - 1)$-cycle. Let $T^+_k(0)$ be as in Figure 3.5, and let $U = T^+_{k-1}(u_1)$ and $V = T^+_{k-1}(v_1)\backslash\{0, u_1, v_1\}$. As $R(0) = \{0, u_1, v_1\}$ and $R(v_1) = \{v_1, v_2, v_3\}$ we get from Lemma 3.9 and $N^+(R(0)) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ that $R(u_1) = \{u_1, u_2, u_3\}$. Furthermore, as we know every vertex has to be reached by u_1, we get that $\{0, u_1, u_2, u_3\} \cup V \subseteq N^+_k(u_1)$. Now the number of vertices in $N^+_k(u_1)$ is given as

$$|N^+_k(u_1)| = 2 \cdot |N^+_{k-1}(u_1)| = 2 \cdot 2^{k-1} = 2^k,$$

due to diregularity. Also we have that

$$|V| = 2^k - 4,$$

so we must have $N^+_k(u_1) = \{0, u_1, u_2, u_3\} \cup V$. This means that all vertices in V have one in-neighbor from $N^+_{k-1}(u_1)$ and one from $\{0\} \cup T^+_{k-2}(v_1)$, and due to diregularity have no other in-neighbors.

Similarly we get that v_1 has to reach all vertices, so $\{v_2, v_3\} \cup U\backslash\{u_1\} \subseteq N^+_k(v_1)$. Obviously $|N^+_k(v_1)| = 2^k$ and $|U| = 2^k - 1$, so $N^+_k(v_1) = \{v_2, v_3\} \cup U\backslash\{u_1\}$. We already know that $\{u_2, u_3\} = N^+(u_1)$ and $\{v_2, v_3\} = N^+(v_1)$, so we need to distribute $\{u_4, u_5, \ldots, u_{2^k-1}\}$ as out-neighbors to $\{v_{2^{k-1}-1}, v_{2^{k-1}}, \ldots, v_{2^k-4}\}$ and we split this set into two subsets, $A = N^+_{k-1}(v_2) = N^+(\{v_{2^{k-1}-1}, v_{2^{k-1}}, \ldots, v_{3\cdot 2^{k-2}-1}\})$ and $B = N^+_{k-1}(v_3)\backslash N^+(\{u_1, v_1\}) = N^+(\{v_{3\cdot 2^{k-2}}, v_{3\cdot 2^{k-2}+1}, \ldots, v_{2^k-4}\})$.

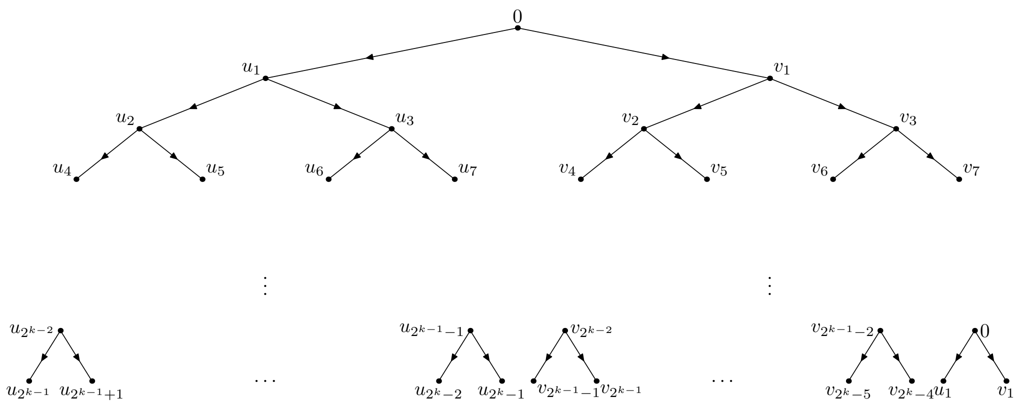


Figure 3.5: $T_k^+(0)$.

Notice, that as v_1 is in a $(k-1)$-cycle, we know that $R(v_2) = \{v_2, v_4, v_5\}$ and $R(v_3) = \{v_3, v_6, v_7\}$. Furthermore, we know that $\{u_4, u_5, u_6, u_7\} \subseteq N_3^+(0)$, so $\{u_4, u_5, u_6, u_7\} \nsubseteq B$, as otherwise v_3 would have too many repeats, so $\{u_4, u_5, u_6, u_7\} \subseteq A$. Now similarly, as $N^+(\{u_4, u_5, u_6, u_7\}) = \{u_8, u_9, \ldots, u_{15}\}$, we must have $\{u_8, u_9, \ldots, u_{15}\} \subseteq B$, as otherwise v_2 would have too many repeats. Repeating this argument for the remaining $N_j^+(u_1)$ with $j = 4, 5, \ldots, k$ we get that:

$$N_{2i+1}^+(u_1) \in B$$

for $1 \le i \le \lceil \frac{k-3}{2} \rceil$ and

$$N_{2i}^+(u_1) \in A$$

for $1 \le i \le \lfloor \frac{k-1}{2} \rfloor$.

Clearly $|A| = 2^{k-1}$ and $|B| = 2^{k-1} - 4$, so $|A| > |B|$. Also, as $\sum_{j=2}^{k-1} |N_j^+(u_1)| = 2^k - 4$ and the way the vertices $\{u_4, u_5, \ldots, u_{2^k-1}\}$ are distributed on A and B, we have $\sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} |N_{2i}^+(u_1)| = |A|$ and $\sum_{i=1}^{\lceil \frac{k-3}{2} \rceil} |N_{2i+1}^+(u_1)| = B$. Calculating, we get

$$\begin{aligned} \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} |N_{2i}^+(u_1)| &= \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} 2^{2i} \\ &= \frac{1}{3}\left(4^{\lfloor \frac{k+1}{2} \rfloor} - 4\right). \end{aligned}$$

Now, if k is odd

$$\begin{aligned}\sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} |N^+_{2i}(u_1)| &= \frac{1}{3}\left(4^{\frac{k+1}{2}} - 4\right)\\ &= \frac{1}{3}\left(2^{k+1} - 4\right)\\ &> 2^{k-1} = |A|\end{aligned}$$

a contradiction, as $k > 3$. If k is even, we get

$$\begin{aligned}\sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} |N^+_{2i}(u_1)| &= \frac{1}{3}\left(4^{\frac{k}{2}} - 4\right)\\ &= \frac{1}{3}\left(2^{k} - 4\right)\\ &< 2^{k-1} = |A|\end{aligned}$$

also a contradiction. This concludes the theorem. $\square$

Chapter 4

The Oriented Maximum Degree/Diameter-Bounded Subgraph

Recently a new problem, called *the maximum degree/diameter-bounded subgraph* problem (MaxDDBS), which is related to the degree/diameter problem for undirected graphs, has been stated in [15] as follows.

Problem 4.1 (MaxDDBS)**.** *Given a connected host graph G, an upper bound d for the maximum degree, and an upper bound k for the diameter, find the largest connected subgraph S of G with maximum degree at most d and diameter at most k.*

The degree/diameter problem for undirected graphs can be given as a special case of the MaxDDBS, with the host graph a sufficiently large complete graph. Other important host graphs are common parallel architectures such as the the square grid, the triangular grid, the honeycomb, the hypercube, the butterfly etc. Upper bounds of the order for the MaxDDBS in the square grid have been given in [15] and [16] for arbitrary dimensions and for the honeycomb network and its cartesian product with infinite paths in [17]. Also these papers give lower bounds of the order in low dimensions for different values of the degree by construction.

Notice that in general the MaxDDBS is NP-hard, whereas we do not know the complexity of the degree/diameter problem for undirected graphs, see [15].

In this chapter we suggest an oriented version of the MaxDDBS, which we denote as *the oriented maximum degree/diameter-bounded subgraph* problem (OMaxDDBS) and state it as follows.

Problem 4.2 (OMaxDDBS)**.** *Given a connected undirected host graph G, an upper bound d for the maximum degree, and an upper bound k for the diameter, find the largest orientation $\bar{S}$ of a subgraph S of G among all connected subgraphs of G, such that $\bar{S}$ has maximum degree at most d and diameter at most k.*

Here one should note that the directed degree/diameter problem is *not* a special case of OMaxDDBS with the complete graph as host graph, as that would be a tournament in OMaxDDBS and we allow digons in the directed degree/diameter problem. One could state a similar version of a directed MaxDDBS, where each edge is replaced by a digon, as follows.

Problem 4.3 (DMaxDDBS)**.** *Given a connected undirected host graph G, an upper bound d for the maximum degree, an upper bound k for the diameter, and the complete biorientation $\overleftrightarrow{G}$ of G, find the largest connected subdigraph S of $\overleftrightarrow{G}$ with maximum degree at most d and diameter at most k.*

So the OMaxDDBS surely seems more complex than the DMaxDDBS and the MaxDDBS as it is not only a question of finding a largest order, but also an orientation which allows this order.

If we for the host graph G momentarily disregard any constraint on the maximum (out-)degree in the above problems, and then assume we have a largest subgraph of order n as the solution to the MaxDDBS for this G, a largest subdigraph of order n' as the solution to the DMaxDDBS and similar the largest orientation of a subgraph of order n'' as the solution to OMaxDDBS, then we obviously have $n'' \leq n' \leq n$. This can help us finding upper theoretical bounds for the OMaxDDBS and DMaxDDBS from the ones for MaxDDBS. The question is then, whether or not it is possible to have $n'' = n$, so the oriented subgraph could just be an orientation of the subgraph with the same diameter. Thus we wish to know whether or not the undirected graph on n vertices is *tightly orientable*. In general checking whether an undirected graph is tightly orientable is NP-complete, see [18], whereas we do not know the complexity of OMaxDDBS.

There are several papers which shows by construction that the hypercubes Q_r are tightly orientable for $r \geq 4$ and Q_1, Q_2 and Q_3 are not tightly orientable, for some of the papers see [19] and [20]. The constructions in [19] are solutions to OMaxDDBS for $d = r - 2$ and $k = r$, whereas the constructions in [20] are solutions for $d = \lceil \frac{r}{2} \rceil$ and $k = r$.

Notice that if d and k are equal to the maximum degree and diameter of the host graph G, then the solutions would just be G for MaxDDBS and $\overleftrightarrow{G}$ for DMaxDDBS, whereas we for OMaxDDBS just as well could restate the problem with the maximum out-degree $d - 1$, as the diameter is finite.

In this chapter we consider the triangular and the square grid in two dimensions as host graphs and find upper theoretical bounds for the order of orientations of subgraphs which solves the OMaxDDBS for these host graphs. We also construct two infinite families of digraphs which are orientations of subgraphs of the triangular grid, one family with odd diameter $k \geq 5$ and the other with even diameter $k \geq 6$ and show that these families are solutions to the OMaxDDBS for sufficiently large d. Furthermore we find the largest possible digraphs with diameter $k \leq 4$ which are orientations of subgraphs of the triangular grid for sufficiently large d using a computer search.

Similarly for the square grid we construct two infinite families of digraphs which are orientations of subgraphs of the square grid, one family with odd diameter $k \geq 9$ and the other with even diameter $k \geq 8$ and show that these families are solutions to the OMaxDDBS for sufficiently large d. We also find the largest possible digraphs with diameter $k \leq 7$ which are orientations of subgraphs of the square grid for sufficiently large d using a computer search.

1 Triangular grid

One way of describing the infinite triangular grid is as three types of lines of infinite length, and all the lines parallel to these, where two parallel lines next to each other are in unit distance, hence an infinite number of lines. The three types of lines could be represented by the horizontal lines, the lines which slant to the left (with an angle of 120 degrees with the horizontal lines) and the lines which slant to the right (with an angle of 60 degrees with the horizontal lines) and whenever at least two lines intersect, exactly three lines intersect. In Figure 4.1 we see such a representation of a part of the infinite triangular grid. When looked upon as a graph, the intersections will be represented by vertices and the line fragments between intersections represented by edges.

We say that every vertex in the triangular grid is contained in one horizontal, one left-slanted and one right-slanted layer, according to the three lines for which the vertex is the intersection. Notice that all edges have the same length due to our construction. The representation seen in Figure 4.1 is just one of the representations we can use for the triangular grid. Others could be rotations of Figure 4.1 or less regular looking representations.

In this section we will only look at digraphs which are orientations of a subgraph of the triangular grid, and thus we will use the notation *t-digraph* for a digraph which is an orientation of a subgraph of the triangular grid. Furthermore we will let $n_t(d,k)$ denote the order of the largest t-digraph of maximum out-degree d and diameter k. Clearly there exist no t-digraph of diameter 1.

Lemma 4.4. *An upper bound for the maximum number of vertices in a t-digraph of diameter $k > 1$ is given by*

$$n_t(d,k) \leq \begin{cases} \frac{3}{4}k^2 + \frac{3}{2}k - \frac{9}{4} & \text{for odd } k \\ \frac{3}{4}k^2 + \frac{3}{2}k - 2 & \text{for even } k \end{cases} \tag{4.1}$$

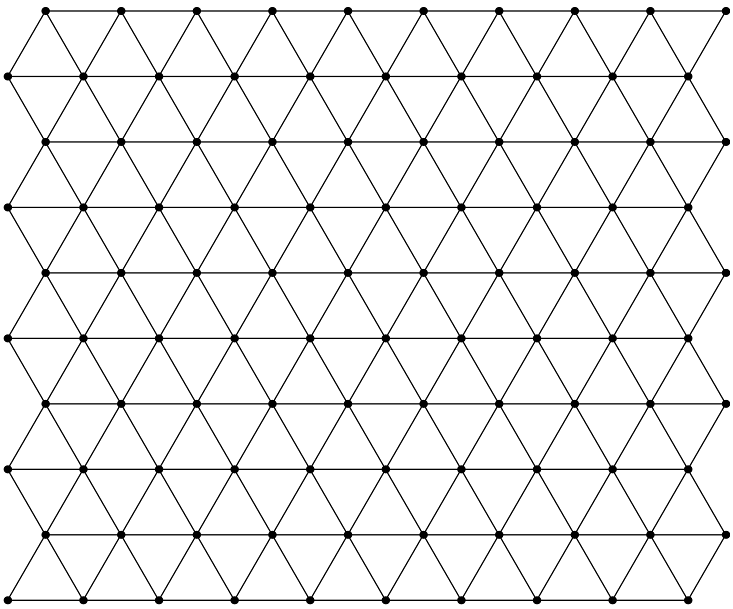


Figure 4.1: A part of the infinite triangular grid.

Proof. Clearly $n_t(d,k)$ cannot exceed the order of a largest undirected subgraph of diameter $k > 1$ in the triangular grid, so determining an upper bound for the order of an undirected subgraph implies an upper bound for the order of a t-digraph.

So assume G is a largest undirected subgraph of the triangular grid and of diameter $k > 1$. As the diameter is finite, we can assume there exists a vertex $u \in V(G)$, such that there do not exist vertices in any of the horizontal layers below the horizontal layer containing u. As we know there exist at least one vertex in the horizontal layer of u and the diameter is k, we have an upper boundary for the horizontal layers of G, which is the horizontal layer in distance k from the bottom layer containing u. In Figure 4.2 the bottom red line represents the horizontal layer containing u and the top red line represents the upper boundary for the area. Repeating this argument for the right-slanted layers, we get a parallelogram as the outer boundary of G seen in Figure 4.2 which is the area bounded by the red and blue lines. Now, when we repeat the argument for the left-slanted layers we get one of the following cases: Either an area bounded by an even-sided triangle (of sidelength at most k edges) or an area bounded by a convex hexagon (not necessarily regular). In Figure 4.2 we see a convex hexagon with the red, blue and green lines as boundary. Translating the green lines to the left gives a triangle and to the right another convex hexagon (but if we translate it far enough to the right we obtain triangles again). Notice that the area is symmetric in the sense that it can be

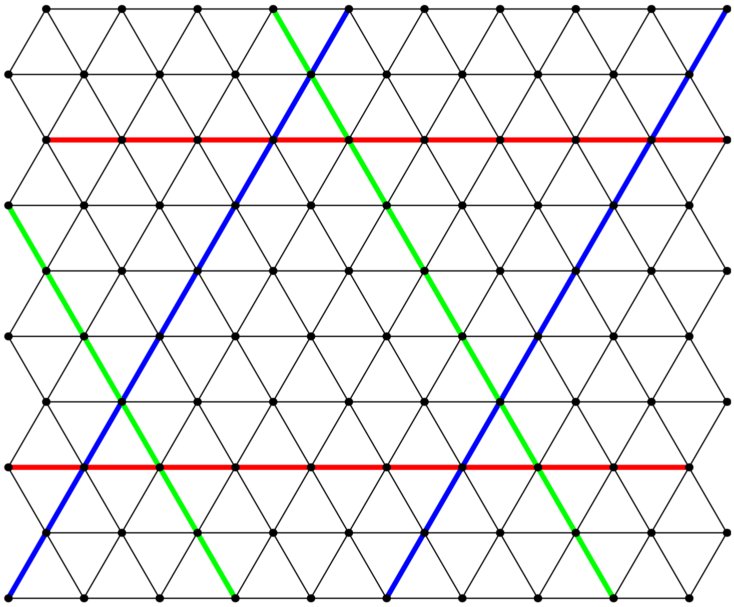


Figure 4.2: A largest undirected graph of diameter $k = 5$ must lie within the blue and red lines.

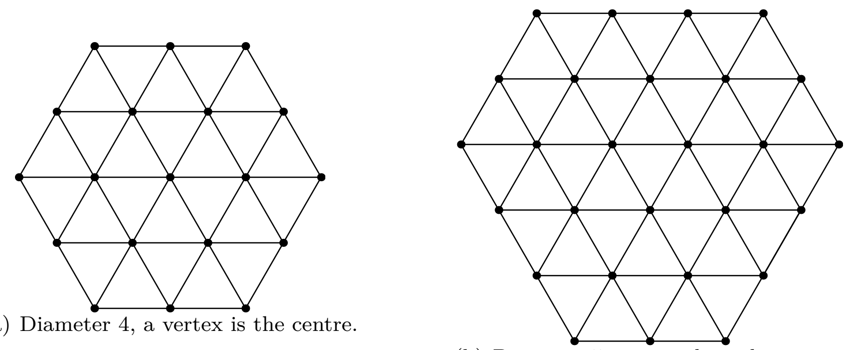
rotated 120 degrees to obtain the same area, and it can be mirrored in the right-slanted diagonal of the before mentioned parallelogram. Vertices which are in the intersection of the lines defining the boundaries, we denote by *corner vertices*.

As we are just interested in finding an upper bound for the number of vertices, we can disregard areas which are subareas of some other area. Hence in this case we look at the triangle of sidelength k edges and the hexagons. The triangle contains $\sum_{i=1}^{k+1} i = \frac{1}{2}k^2 + \frac{3}{2}k + 1$ vertices in total, but notice that if we wish to reach a corner vertex w from another corner vertex v by a directed path of length at most k, then there is no possibility of a directed path from w to v of length at most k. Hence the largest orientation of a subgraph of the area bounded by the triangle can have order at most $\frac{1}{2}k^2 + \frac{3}{2}k - 1$.

Looking at the hexagons, we have the same problem regarding the corners. Between any two corners which are in distance k of each other and in the same layer (opposite corners), it is not possible to have directed paths of length at most k in both directions. Hence the largest orientation of a subgraph of the area bounded by one of the hexagons has order at most three less than the hexagon. As this is the same for all the hexagons, we can look for the hexagon containing the largest number of vertices. The parallelogram from before can be fixed, and then we just have to figure out where the left-slanted area should be, for the area to contain as many vertices as possible. Assume the left-slanted area is as left in the area as possible - notice due to symmetry, that this is the same as it being as much to the right as possible, so in the left-most left-slanted layer we have two vertices, and in the right-most left-slanted layer we have k vertices.

Now, what happens to the number of vertices when we translate these left-slanted layers one edge to the right? We lose the two vertices in the left-most left-slanted layer, but we gain $k-1$ vertices in the new right-most left-slanted layer. This process can be repeated until the hexagon reachs the middle of the parallelogram, every time gaining more vertices than we lose (for odd diameter, we lose as much as we gain, when the hexagon crosses the middle of the parallelogram). What we obtain is, that the largest area is bounded for even k by a regular or for odd k by an almost-regular hexagon, where almost-regular means as close to regular as we can get.

Another way of seeing these hexagons are as balls of radius $\frac{k}{2}$ or $\lfloor \frac{k}{2} \rfloor$ respectively, with the centre a vertex if k is even or a triangle if k is odd. Examples of balls for diameter 4 and 5 (hence both of radius 2 but different centre) are seen in Figure 4.3(a) and 4.3(b).



(a) Diameter 4, a vertex is the centre.

(b) Diameter 5, a triangle is the centre.

Figure 4.3: Two balls of radius 2 in the triangular grid.

Counting the vertices starting from the largest horizontal layer and then counting the layers towards the border, we get

$$n_t(d,k) \leq \begin{cases} (k+1) + 2\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor - 1}(k-i) + (k - \lfloor \frac{k}{2} \rfloor) = \frac{3}{4}k^2 + \frac{3}{2}k + \frac{3}{4} & \text{for odd } k \\ (k+1) + 2\sum_{i=0}^{\frac{k}{2}-1}(k-i) = \frac{3}{4}k^2 + \frac{3}{2}k + 1 & \text{for even } k \end{cases} \tag{4.2}$$

This bound is then decreased by 3, due to the argument from before. Comparing this to the triangular area, we obtain the statement. □

Notice, that if we have a t-digraph D with order

$$n_t(q) = \begin{cases} \frac{3}{4}q^2 + \frac{3}{2}q - \frac{9}{4} & \text{for some odd integer } q \\ \frac{3}{4}q^2 + \frac{3}{2}q - 2 & \text{for some even integer } q \end{cases}$$

then D will have diameter at least q. If D has order as stated, we say that D has *m-diameter* q.

Theorem 4.5. *For all d we have*

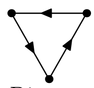
$$n_t(d,2) \leq n_t(1,2) = 3,$$
$$n_t(d,3) \leq n_t(2,3) = 6$$

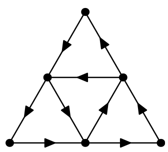
and

$$n_t(d,4) \leq n_t(4,4) = 14.$$

Proof. By Lemma 4.4 we have the upper bounds to be 4, 9 and 16 respectively. However, a computer search for orientations among the complete biorientations of the largest possible balls and all their subgraphs, show that the t-digraphs in Figure 4.4 are the largest t-digraphs of diameter 2, 3 and 4 in an orientation of the triangular grid with order 3, 6 and 13 respectively, see the Appendix.

It can easily be seen that the two t-digraphs with diameter 2 and 3 have both maximum out- and in-degree 1 and 2 respectively, and that the one with diameter 4 has maximum in-degree 5 and maximum out-degree 4.


(a) Diameter 2.


(b) Diameter 3.

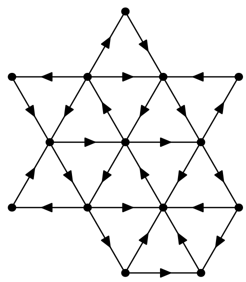
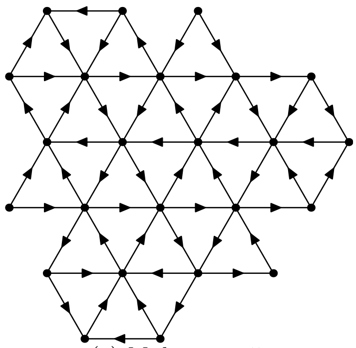

(c) Diameter 4.

Figure 4.4: Some largest t-digraphs of diameter $2, 3$ and 4.

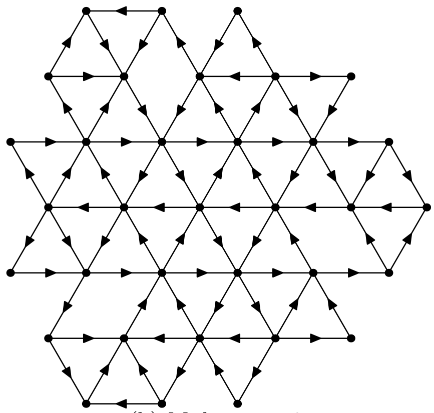
□

In the following we will construct t-digraphs with m-diameter q, and then we will prove they do in fact have diameter $k = q$.

In Figure 4.5(a) and 4.5(b) we see two t-digraphs of m-diameter 5 and 6 respectively. Notice that the constructions are symmetrical in the sense that they can be rotated 120 degrees to obtain isomorphic t-digraphs. This means, that when we wish to investigate distances in the t-digraphs, it is only necessary to consider distances from (or to) approximately one third of the vertices.



(a) M-diameter 5.



(b) M-diameter 6.

Figure 4.5: Some t-digraphs.

Now all t-digraphs of odd m-diameter will be constructed recursively from the t-digraph of m-diameter 5 (Figure 4.5(a)) and all t-digraphs of even m-diameter will be constructed recursively from the t-digraph of m-diameter 6 (Figure 4.5(b)).

Starting from one of the known t-digraphs D with say m-diameter $q \geq 5$ (we call this *the old t-digraph*) we add vertices and arcs to obtain a *new t-digraph* D', and we will also delete a few arcs. We add the vertices which are able to reach a vertex in D through a single arc. An example is seen in Figure 4.6 and 4.7 where all the blue vertices are new vertices and the black vertices and arcs are just the old t-digraphs of m-diameter 5 and 6 respectively. We label some of the vertices, those which are of special interest in the proofs, in the old t-digraph we label with v_i and u_i for $i = 1, 2, 3$. The vertices labeled with u_3 are those situated in every second corner vertices of the hexagon from Lemma 4.4 and the two vertices closest to it are labeled u_1 and u_2. The vertices labeled with v_i for $i = 1, 2, 3$ are the three vertices closest to the opposite corner of the hexagon (where the vertex is missing). We label them according to Figure 4.6 (and 4.7). Similarly this labeling becomes v'_i and u'_i for $i = 1, 2, 3$ in the new t-digraph.

Notice, that due to symmetry we could have labeled any of the three corners of the hexagon and in the remaining part of this section we use this symmetry, in the sense that things proven for a u_i and v_i for $i = 1, 2, 3$ are true for every choice of these.

The arcs are added according to which type of vertex they are connecting to the old t-digraph. The new v'_i and u'_i are connected with arcs to the old v_i and u_i respectively, according to the blue arcs in Figure 4.6 (and 4.7). Recall we have symmetry, so we need to do this a total of three times. All other new vertices are connected to the two closest vertices in the old t-digraph as the interior vertex in a directed 2-path, see Figure 4.6 (and 4.7) with the remaining blue edges. Note that some of the new 2-paths also could be reversed, but doing this we might lose symmetry at these vertices.

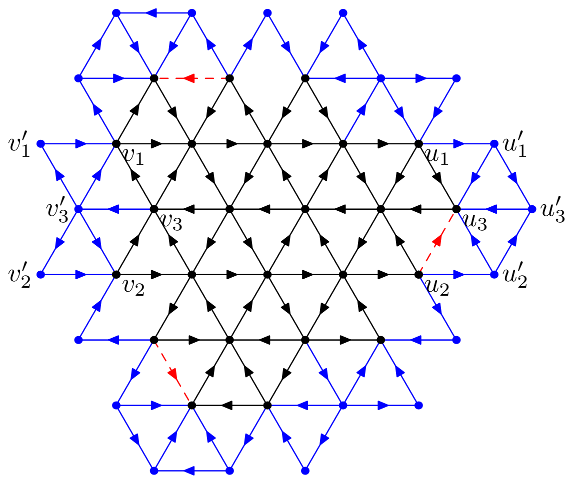


Figure 4.6: A t-digraph of m-diameter 7.

As stated before, we also delete 3 arcs from the old t-digraph, these will be the ones from u_2 to u_3, see the red dotted arcs in Figure 4.6 and 4.7.

Before proving that these t-digraphs do in fact have diameter q, we observe the following.

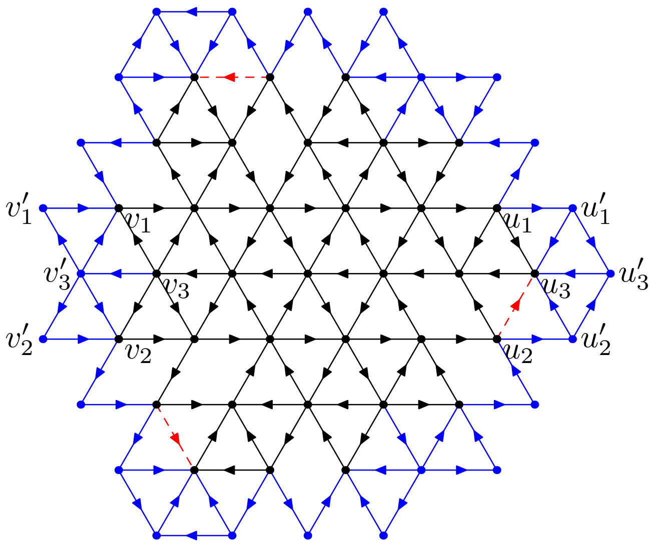


Figure 4.7: A t-digraph of m-diameter 8.

Lemma 4.6. *Let D be one of the constructed t-digraphs with m-diameter $q \geq 5$. The only shortest paths we possibly increase in D when we construct D' are the ones ending in u_3 and these are increased by exactly one.*

Proof. As u_3 only has one out-neighbor and this is also an out-neighbor of u_2, all shortest paths using the arc (u_2, u_3) will end in u_3, as otherwise it would be a contradiction to it being a shortest path. Thus, if we have a shortest path in D using the arc (u_2, u_3) we will in D' use the path (u_2, u_2', u_3) instead of (u_2, u_3), hence increasing the paths to u_3 with exactly one arc. □

Lemma 4.7. *Let D be one of the constructed t-digraphs with m-diameter q, then*

- *all vertices have distance $\leq q-1$ to v_3,*
- *all vertices have distance $\leq q-1$ to either u_1 or u_2.*

Proof. It can easily be checked that the statement is true for the t-digraphs of m-diameter 5 and 6 given in Figure 4.5(a) and 4.5(b). So assume the statement is true for all constructed t-digraphs D with m-diameter $5 \leq q < j-1$ where $j \geq 7$, then we wish to show that the statement follows for D' with even m-diameter j which is constructed from D which has m-diameter $j-2$.

According to the assumption and Lemma 4.6, all vertices in D have distance $\leq j-3$ to v_3 and as the arc (v_3, v_3') is in D', they all have distance $\leq j-2$ to v_3'. As all new vertices have an arc going to an old vertex, this means that all vertices in D' have distance $\leq j-1$ to v'_3.

Similarly, all vertices in D have distance $\leq j-3$ to either u_1 or u_2 and hence, as the arc (u_i, u'_i) exists for $i = 1, 2$, they all have distance $\leq j-2$ to either u_1 or u_2 and as before, all vertices in D' will then have distance $\leq j-1$ to either u'_1 or u'_2. □

Theorem 4.8. *For all $k \geq 5$ there exists a d such that there exists a t-digraph of order*

$$n_t(d,k) = \begin{cases} \frac{3}{4}k^2 + \frac{3}{2}k - \frac{9}{4} & \text{for odd } k \\ \frac{3}{4}k^2 + \frac{3}{2}k - 2 & \text{for even } k \end{cases} \tag{4.3}$$

Proof. We will prove the theorem by proving that the constructed t-digraphs with m-diameter q do in fact have diameter q.

It is a straightforward task to check that the constructed t-digraphs with m-diameter 5 and 6 given in Figure 4.5(a) and 4.5(b) do in fact have diameter 5 and 6 respectively.

Now assume all constructed t-digraphs D with m-diameter $5 \leq q < j-1$, where $j \geq 7$, have diameter q, and let D' be a constructed t-digraph of even m-diameter j, then we easily observe that all new vertices have distance at most:

- $j-1$ to all vertices in D (due to Lemma 4.6 and 4.7)
- j from all old vertices (due to Lemma 4.6)
- j to all new vertices not of type v'_i and u'_i

So we only need to check that all new vertices have distance at most j to new vertices of type v'_i and u'_i.

Due to Lemma 4.7 this is true for all v'_i as $dist(w, v'_3) \leq j-1$ for all $w \in V(D')$ and (v'_3, v'_i) is an arc in D' for $i = 1, 2$.

Similarly we have $\max_{i=1,2}\{dist(w, u'_i)\} \leq \max_{i=1,2}\{dist(w, u_i)\} + 1 \leq j-1+1 = j$ for all $w \in V(D')$ and as (u'_i, u'_3) is an arc in D' for $i = 1, 2$ and $\min_{i=1,2}\{dist(w, u'_i)\} \leq j-1$ due to Lemma 4.7 we see that the above is also true for all u'_3, hence proving the theorem.

□

Corollary 4.9. *For all d: $n_t(d,5) \leq n_t(3,5) = 24$, $n_t(d,6) \leq n_t(4,6) = 31$ and for $k \geq 7$*

$$n_t(4,k) = n_t(5,k) = \begin{cases} \frac{3}{4}k^2 + \frac{3}{2}k - \frac{9}{4} & \text{for odd } k \\ \frac{3}{4}k^2 + \frac{3}{2}k - 2 & \text{for even } k \end{cases} \tag{4.4}$$

Proof. We get $n_t(3,5) = 24$, $n_t(4,6) = 31$ and $n_t(4,7) = 45$ by observing Figure 4.5(a), 4.5(b) and 4.6 respectively. Notice that also the maximum in-degrees are 3 and 4 respectively.

For larger k we see that the only vertices in which we increase the total degree to 6 is v_3. Here the out-degree is 5. The maximum in-degree however is still 4, so by changing the directions on all the arcs we obtain the statement. □

Notice that the t-digraphs in Figs. 4.4(b), 4.4(c), 4.5(a), 4.5(b), 4.6 and 4.7 and all the ones which will be constructed from these are not necessarily unique, in fact for diameter 5 the computer search we used for finding the solution to OMaxDDBS for diameter $k \leq 4$ also finds other t-digraphs of diameter 5 than the one depicted in Figure 4.5(a) which varies in the placement of the corner vertices.

2 Square grid

In this section, we let the host graph in the OMaxDDBS be the infinite planar square grid, which can be represented by an infinite number of horizontal and vertical lines, where two parallel lines next to each other are in unit distance. A horizontal and vertical line intersect exactly once and they are orthogonal/perpendicular. Looked upon as a graph, the intersections can be represented as vertices and the line fragments between intersections can be represented by edges.

Similarly, as in the previous section, we will in the remaining part of this section let a s-digraph denote an orientation of a subgraph of the infinite planar square grid. Now the question is, what is the largest possible order $n_s(d,k)$ of a s-digraph of diameter k and maximum out-degree d? Notice, that we cannot have a s-digraph of diameter $k \leq 2$, as if we did, it would contain an arc (w_1, w_2), but the distance from w_2 to w_1 would then be at least 3 due to the girth of the infinite planar square grid being 4, a contradiction. Also notice that the maximum out-degree in a s-digraph will be at most $d \leq 3$, as the diameter is finite.

Lemma 4.10. *An upper bound for the maximum number of vertices in a s-digraph of even diameter $k > 2$ is given by*

$$n_s(d,k) \leq \frac{1}{2}k^2 + k - 3 \tag{4.5}$$

Proof. As in the triangular grid, we have an obviously upper bound given by the undirected case. Now as the diameter is finite, we can assume there is a right-slanted diagonal (hence a line of infinite length traversing two opposite corners in one of the squares) such that there will be no vertices to the right of this and there will be at least one vertex on it. Then we know, as the diameter is k, that the area the undirected subgraph lies in must be bounded by another right-slanted diagonal in distance k from the first mentioned. Such two right-slanted diagonals can be seen as the green lines in Figure 4.8. Now using the same argument for the left-slanted diagonals, we get two cases, as seen in Figure 4.8 represented by blue lines which intersect the green lines in a vertex and red lines which do not intersect the green lines in a vertex.

Another way of interpreting these areas, are as balls of radius $\frac{k}{2}$ with the center of the ball either at a vertex or in the middle of a square in the grid. Vertices which lies in the intersection of two lines of the boundary, we denote as *corner vertices*.

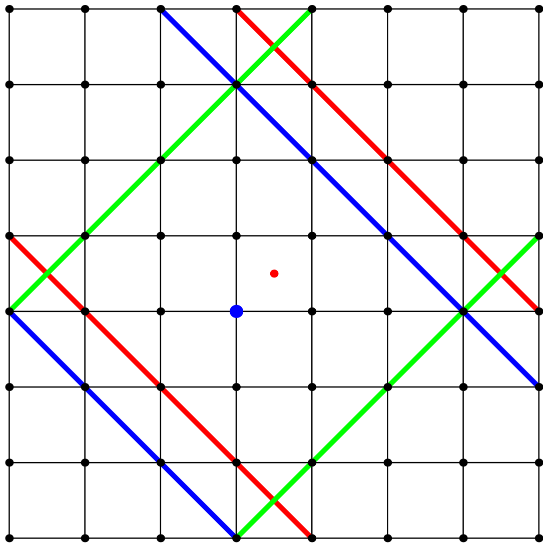


Figure 4.8: Two different balls of diameter 6.

In Figure 4.8 we have marked the possible centres with a blue or red dot, according to the left-slanted areas from before.

By choosing the blue vertex as the center and counting the vertices in layers starting from the middle horizontal layer, we get an upper bound as

$$k + 1 + 2\sum_{i=1}^{\frac{k}{2}}(k + 1 - 2i) = \frac{1}{2}k^2 + k + 1.$$

As there is only one possible arc from the four corner vertices to the remaining vertices of the ball, we disregard these vertices, and hence get an upper bound for the blue ball as $\frac{1}{2}k^2 + k - 3$.

Doing the same for the red ball, we get an upper bound as

$$2\sum_{i=0}^{\frac{k}{2}-1}(k - 2i) = \frac{1}{2}k^2 + k.$$

We notice that this bound is larger than the one for the blue ball, but let us assume we have an orientation of a subgraph with this number of vertices. Then, due to the fact that the diameter is finite, we can assume that three of the arcs are directed as in Figure 4.9. Now, for the vertex a to be able to reach the other vertices in the ball in a $\leq k$-path, we need vertex b to reach all vertices in the ball in $(k-2)$-paths. But b

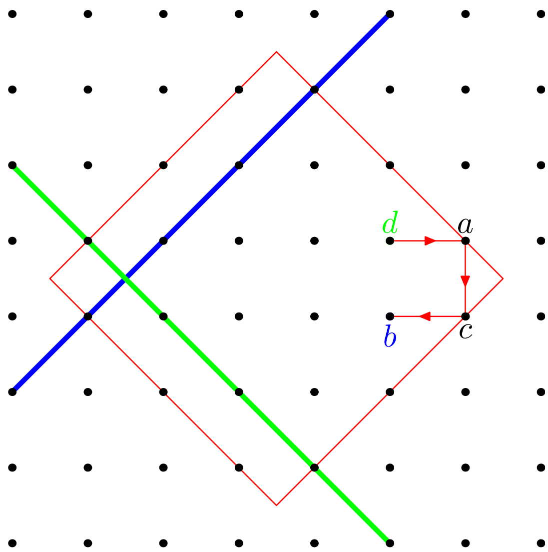


Figure 4.9: Arcs in an orientation of a subgraph of order $\frac{1}{2}k^2 + k$.

cannot reach vertices above the blue line in Figure 4.9 in $\leq (k-2)$-paths, which is a total of $k/2$ vertices. Similar for vertex c to be reached from all vertices in the ball, we need vertex d to be reached by all vertices in $\leq (k-2)$-paths. But d cannot be reached in $\leq (k-2)$-paths by the vertices below the green line, hence another $k/2$ vertices not possible in the orientation of the subgraph. Now we have the area containing a, b, c, d which is bounded by the blue, green and red lines, and the two new corner vertices have one edge connecting it to the remaining area, hence we must delete a total of $k+2$ vertices in order to keep a and c. Deleting these $k+2$ vertices, the resulting ball will just be a subgraph of the blue ball from Figure 4.8 without the four corner vertices. If we instead delete the eight vertices of the types a and c, then we get a ball with $\frac{1}{2}k^2 + k - 8$ vertices. So clearly the upper bound is given by the blue ball, and hence

$$n_s(d, k) \leq \frac{1}{2}k^2 + k - 3.$$

□

Lemma 4.11. *An upper bound for the maximum number of vertices in a s-digraph of odd diameter $k > 3$ is given by*

$$n_s(d, k) \leq \frac{1}{2}k^2 + k - \frac{11}{2} \tag{4.6}$$

Proof. We proceed as in the proof of Lemma 4.11 to obtain a area of odd diameter, and get just one area as seen in Figure 4.10, as any other area would be isomorphic to a subgraph of this area. The vertices which lie in the intersection of a blue and green line, we denote as *corner vertices*.

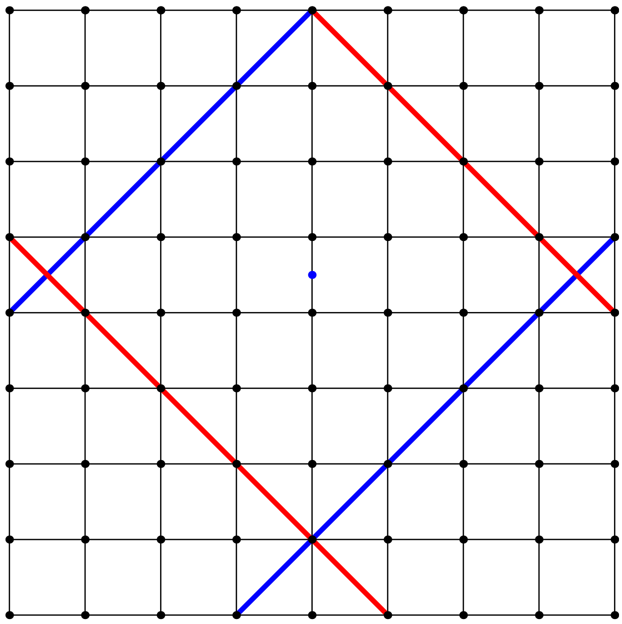


Figure 4.10: A ball of diameter 7 with the blue dot as center.

If we think of the area as a ball, the center will be in the middle of an edge in the grid.

With the same reasoning as with even diameter, we see that the corner vertices are not possible in an orientation of the ball, and for $k > 5$ we also cut our losses by deleting the two vertices in each side, as otherwise we would have to delete a total of $k+1$ vertices instead of just 6 vertices. When $k = 5$ we will have to delete a total of 6 vertices no matter if we choose to keep two vertices in one side or delete all four.

Counting the remaining vertices, we get that

$$2\sum_{i=1}^{\frac{k-1}{2}}(k-2i) - 6 = \frac{1}{2}k^2 + k - \frac{11}{2}.$$

$\square$

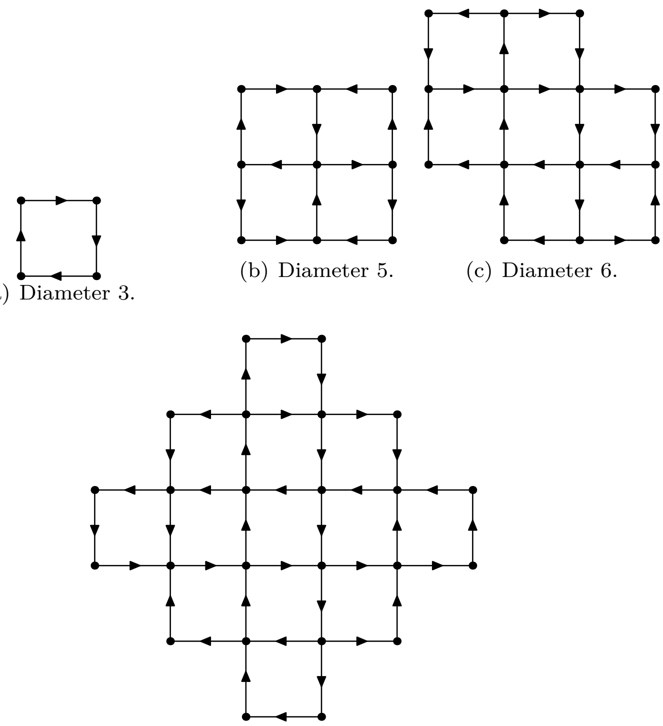
Using the same method as in the above proof, we get the following special case for $k = 3$, as we only need to delete 4 vertices from the ball of diameter 3.

Corollary 4.12. *An upper bound for the maximum number of vertices in a s-digraph of diameter $k = 3$ is given by*

$$n_s(d, 3) \leq n_s(1, 3) = 4, \tag{4.7}$$

see the s-digraph in Figure 4.11(a).

Notice that the upper bound in Lemma 4.10 does not imply that the largest possible order of an orientation of a subgraph should be an orientation of a subgraph of the largest possible area, but that this area just has the largest number of vertices.



(a) Diameter 3. (b) Diameter 5. (c) Diameter 6. (d) Diameter 7.

Figure 4.11: Some largest s-digraphs of diameter 3, 5, 6 and 7.

Theorem 4.13. *There is no s-digraph of diameter exactly* 4 *and for all* d *we have*

$$\begin{aligned} n_s(d,3) &\leq n_s(1,3) = 4, \\ n_s(d,5) &\leq n_s(2,5) = 9, \\ n_s(d,6) &\leq n_s(2,6) = 14 \end{aligned}$$

and

$$n_s(d,7) \leq n_s(3,7) = 24.$$

Proof. Using the same computer search as in the proof of Theorem 4.5 we see there is no s-digraph of diameter excactly 4 and that the s-digraphs in Figure 4.11 are largest s-digraphs of diameter $k = 3, 5, 6$ and 7 respectively. It can easily be checked that they all have maximum out- and in-degree 2 except for the one of diameter 7 which has maximum out- and in-degree 3. □

As in the triangular case, we see that a s-digraph D of an orientation of the grid having order

$$n_s(q) = \begin{cases} \frac{1}{2}q^2 + q - \frac{11}{2} & \text{for an odd integer } q \\ \frac{1}{2}q^2 + q - 3 & \text{for an even integer } q \end{cases}$$

must have diameter at least q, and hence we say that D has *m-diameter* q.

The s-digraph in Figure 4.12 has m-diameter 8. From this we will construct an infinite family of s-digraphs of even m-diameter recursively in the following way.

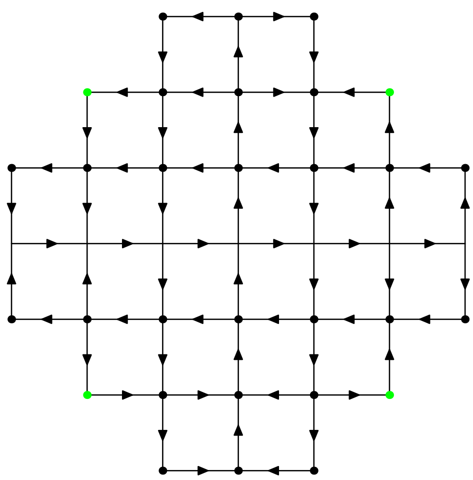


Figure 4.12: A largest s-digraph of m-diameter 8.

Starting from one of the known s-digraphs, D of even m-diameter $q \geq 8$, which we shall also refer to as *the old s-digraph*, we add vertices and arcs to obtain a *new s-digraph* D' which has m-diameter $q + 2$.

We add all the vertices which are able to reach D through a single arc. The ones of special interest to the proof, we label by u_i', v_i', w_i' and z_i' for $i = 1, 2, 3$, and they are the ones which are able to have another new vertex as a neighbor, see Figure 4.13. The vertices in D corresponding to the labeled new vertices we label correspondingly as u_i, v_i, w_i and z_i, see Figure 4.13. The labeled vertices are connected with arcs according to Figure 4.13 and the unlabeled are connected to the old s-digraph such that it is an interior vertex in a directed 2-path between two old vertices, see also Figure 4.13.

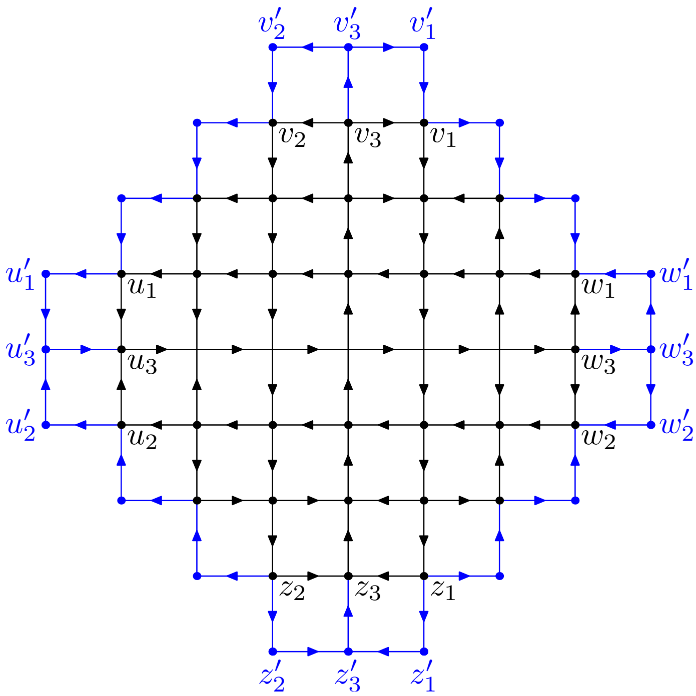


Figure 4.13: A largest s-digraph of diameter 10.

Notice, that due to the construction we have

$$\begin{aligned} dist(x', u_3') &\leq dist(x', u_3) + 1, \\ dist(x', z_3') &\leq dist(x', z_3) + 1, \\ dist(v_3', y') &\leq dist(v_3, y') + 1, \\ dist(w_3', y') &\leq dist(w_3, y') + 1. \end{aligned}$$

for all $x', y' \in D'$.

Lemma 4.14. *Let D be one of the constructed s-digraphs with even m-diameter q, then*

$$\max_{x\in\{u_3,z_3\},y\in\{v_3,w_3\}}\{dist(x,y), dist(u_3,z_3), dist(z_3,u_3), dist(v_3,w_3), dist(w_3,v_3)\} \leq q-2.$$

Proof. It is easy to check that the statement is true for D with m-diameter 8 by looking at Figure 4.12. So assume the statement is true for all constructed D with even m-diameter $8 \leq q < j-1$ and that D' is a s-digraph with even m-diameter j constructed from a s-digraph D with m-diameter $j-2$.

Let $x \in \{u_3, z_3\}$ and $y \in \{v_3, w_3\}$, then

$$\begin{aligned} dist(x', y') &\leq dist(x', x) + dist(x, y) + dist(y, y') \\ &= 1 + (j-4) + 1 = j-2. \end{aligned}$$

We also get

$$\begin{aligned} dist(u_3', z_3') &\leq dist(u_3', u_3) + dist(u_3, z_3') \\ &\leq 1 + dist(u_3, z_3) + 1 \\ &= 1 + (j-4) + 1 = j-2, \end{aligned}$$

and similarly we get $dist(z_3', u_3') \leq j-2$.

Last, we get that

$$\begin{aligned} dist(v_3', w_3') &\leq dist(v_3', w_3) + dist(w_3, w_3') \\ &\leq dist(v_3, w_3) + 1 + 1 \\ &= (j-4) + 1 + 1 = j-2, \end{aligned}$$

and similarly we get $dist(w_3', v_3') \leq j-2$. $\square$

Lemma 4.15. *Let D be one of the constructed s-digraphs with even m-diameter q, then the following is true*

a) $\max_{x\in D}\{dist(x,v_3), dist(x,w_3)\} \leq q-1$,

b) $\max_{x\in D}\{dist(u_3,x), dist(z_3,z)\} \leq q-1$,

c) $\min_{i=1,2}\{dist(x,u_1), dist(x,u_2)\} \leq q-1$ *and* $\min_{i=1,2}\{dist(x,z_1), dist(x,z_2)\} \leq q-1$ *for all* $x \in D$,

d) $\min_{i=1,2}\{dist(v_1,x), dist(v_2,x)\} \leq q-1$ *and* $\min_{i=1,2}\{dist(w_1,x), dist(w_2,x)\} \leq q-1$ *for all* $x \in D$.

Proof. It is easy, but a bit cumbersome to check that the statement is true for the s-digraph in Figure 4.12 with m-diameter 8. Let us assume the statement is true for all s-digraphs D constructed as described and with m-diameter $8 \leq q < j-1$.

Now the statement is obviously satisfied for all x which are vertices in the old s-digraph as all new vertices are connected by $\leq$ 2-paths to and from the old s-digraph. So it remains to show the statement is true for all $x' \in D' \backslash D$. In a) and c) let x be the in-neighbor in D of x', if such a vertex exist and similar in b) and d), let x be the out-neighbor of x'. Then for a) we have according to Lemma 4.14 that

$$\begin{aligned}
dist(x', v_3') &\leq dist(x', v_3) + 1 \\
&\leq \max\{dist(x, v_3) + 2, dist(u_3', v_3) + 2, dist(w_3', v_3) + 1, dist(z_3', v_3) + 2\} \\
&\leq \max\{(j-3) + 2, dist(u_3, v_3) + 3, dist(w_3, v_3) + 2, dist(z_3, v_3) + 3\} \\
&\leq \max\{j-1, (j-4) + 3, (j-4) + 2, (j-4) + 3\} \\
&= j-1,
\end{aligned}$$

and similarly we get that $dist(x', w_3') \leq j-1$.

For b) we have

$$\begin{aligned}
dist(u_3', x') &\leq dist(u_3, x') + 1 \\
&\leq \max\{dist(u_3, x) + 2, dist(u_3, z_3') + 1, dist(u_3, v_3') + 2, dist(u_3, w_3') + 2\} \\
&\leq \max\{(j-3) + 2, dist(u_3, z_3) + 2, dist(u_3, v_3) + 3, dist(u_3, w_3) + 3\} \\
&\leq \max\{j-1, (j-4) + 2, (j-4) + 3, (j-4) + 3\} \\
&\leq j-1,
\end{aligned}$$

and again, we similarly get that $dist(z_3', x') \leq j-1$.

For c) we see

$$\begin{aligned}
\min_{i=1,2}\{dist(x', u_i')\} &\leq \min_{i=1,2}\{dist(x', u_i) + 1\} \\
&\leq \max\left\{\min_{i=1,2}\{dist(x, u_i) + 2\}, \min_{i=1,2}\{dist(z_3', u_i) + 2\},\right. \\
&\qquad \left.\min_{i=1,2}\{dist(v_3', u_i) + 1\}, \min_{i=1,2}\{dist(w_3', u_i) + 1\}\right\} \\
&\leq \max\left\{(j-3) + 2, \min_{i=1,2}\{dist(z_3, u_i) + 3\},\right. \\
&\qquad \left.\min_{i=1,2}\{dist(v_3, u_i) + 2\}, \min_{i=1,2}\{dist(w_3, u_i) + 2\}\right\} \\
&\leq \max\left\{(j-3) + 2, dist(z_3, u_3) + 2\}, (j-3) + 2, (j-3) + 2\right\} \\
&\leq \max\{j-1, (j-4) + 2, j-1, j-1\} \\
&= j-1,
\end{aligned}$$

and similarly we have that $\min_{i=1,2}\{dist(x', z_i')\} \leq j-1$.

Finally, we have that d) is satisfied as

$$\begin{aligned}
\min_{i=1,2}\{dist(v_i', x')\} &\leq \min_{i=1,2}\{dist(v_i, x') + 1\} \\
&\leq \max\Big\{\min_{i=1,2}\{dist(v_i, x) + 2\}, \min_{i=1,2}\{dist(v_i, w_3') + 2\}, \\
&\quad \min_{i=1,2}\{dist(v_i, u_3') + 1\}, \min_{i=1,2}\{dist(v_i, z_3') + 1\}\Big\} \\
&\leq \max\Big\{(j-3) + 2, \min_{i=1,2}\{dist(v_i, w_3) + 3\}, \\
&\quad \min_{i=1,2}\{dist(v_i, u_3) + 2\}, \min_{i=1,2}\{dist(v_i, z_3) + 2\}\Big\} \\
&\leq \max\{j-1, dist(v_3, w_3) + 2, (j-3) + 2, (j-3) + 2\} \\
&\leq \max\{j-1, (j-4) + 2, j-1, j-1\} \\
&= j-1,
\end{aligned}$$

and similarly we get $\min_{i=1,2}\{dist(w_i', x')\} \leq j-1$.

□

Theorem 4.16. *For all even $k \geq 8$ there exists a s-digraph such that*

$$n_s(3,k) = \frac{1}{2}k^2 + k - 3 \tag{4.8}$$

Proof. We wish to prove, that the s-digraphs we have constructed with even m-diameter q do in fact have diameter $k = q$.

It can be checked that the s-digraph in Figure 4.12 is a s-digraph of diameter 8 and that it has $\frac{1}{2} \cdot 8^2 + 8 - 3 = 37$ vertices. A tip for testing the diameter, is to ignore the four green vertices and check the distances among the rest, which you only need to do from half of the vertices due to symmetry. Pay attention to the maximum in- and out-distances of the neighbor vertices of the four green vertices. When you add the four green vertices, you just need to check that their neighbors have in- and out-distance at most 7 and that the distance among the four green vertices are at most 8.

Now assume the statement is true for all constructed D with m-diameter $8 \leq q \leq j-1$ and let D' be the s-digraph with even m-diameter j constructed from a s-digraph D of m-diameter $j-2$. As all new vertices has a ≤ 2-path to and from an old vertex, we have that all new vertices can reach all old vertices and be reached by all old vertices in a $\leq j$-path. So it remains to show that all pair of new vertices can reach each other in a $\leq j$-path. If the pair x, y have x connected to D by a single arc and y connected from D by a single arc, then $dist(x, y) \leq j$. The only vertices for which this is not true, is if $x \in \{u_1', u_2', v_3', z_1', z_2', w_3'\}$ and/or if $y \in \{v_1', v_2', u_3', w_1', w_2', z_3'\}$. Notice, that due to construction $dist(u'_j, u'_3) = dist(v'_3, v'_j) = dist(z'_j, z'_3) = dist(w'_3, w'_j) = 1$ for $j = 1, 2$. The result then follows immediately from Lemma 4.14 and 4.15. □

Now we want to construct s-digraphs of odd diameter recursively, this time from the s-digraph in Figure 4.14, which we see have m-diameter 9. Unfortunately there is no obvious symmetry in this s-digraph.

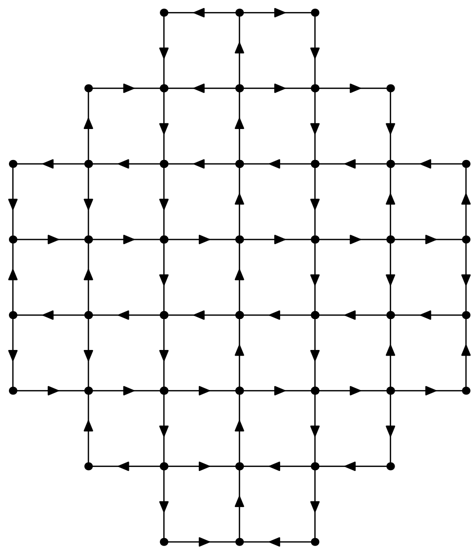


Figure 4.14: A maximum s-digraph of m-diameter 9.

Starting from one of the known s-digraphs, D of odd m-diameter $q \geq 9$, which we shall also refer to as the old s-digraph, we add vertices and arcs to obtain a new s-digraph D' which has m-diameter $q + 2$.

We add all the vertices which are able to reach D through a single arc. The ones of special interest to the proof, we label by u'_j, v'_i, w'_j and z'_i for $j = 1, 2, 3, 4$ and $i = 1, 2, 3$, and they are the ones which are able to have another new vertex as a neighbor, see Figure 4.15. The vertices in D corresponding to the labeled new vertices we label correspondingly as u_j, v_i, w_j and z_i, see Figure 4.15. The labeled vertices are connected with arcs according to Figure 4.15 and the unlabeled are connected to the old s-digraph as an interior vertex of a directed 2-path, see also Figure 4.15.

Lemma 4.17. *Let D be one of the constructed s-digraphs of odd m-diameter q, then*

- $\max_{y \in \{v_3, w_3, u_2\}} \{dist(u_3, y), dist(w_2, u_2), dist(z_3, u_2), dist(v_3, w_3)\} \leq q - 3,$
- $\max_{x \in \{z_3, w_2\}, y \in \{v_3, w_3\}} \{dist(x, y), dist(w_3, u_2), dist(w_2, z_3), dist(v_3, u_2)\} \leq q - 2.$

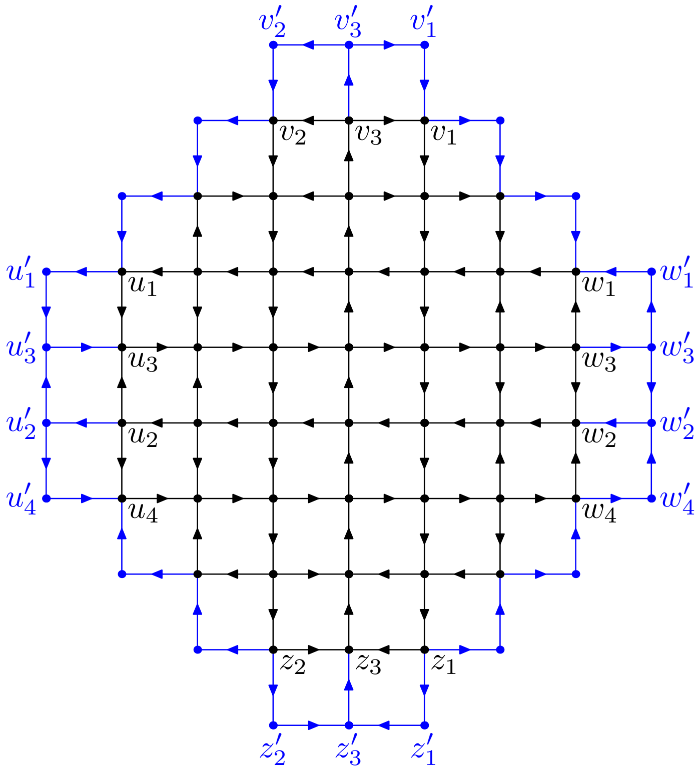


Figure 4.15: A largest s-digraph of m-diameter 11.

Proof. It can easily be checked that the statement is true for m-diameter $q = 9$, so assume it is also true for all constructed s-digraphs of odd m-diameter $9 \leq q \leq j-1$, and let D' be the s-digraph of odd m-diameter j constructed from a s-digraph D of m-diameter $j-2$.

Let $y \in \{v_3, w_3, u_2\}$, then

$$\begin{aligned} dist(u_3', y') &\leq dist(u_3, y) + 2 \\ &\leq j - 5 + 2 = j - 3. \end{aligned}$$

We also see that

$$\begin{aligned} dist(w_2', u_2') &\leq dist(w_2, u_2) + 2 \\ &\leq j - 5 + 2 = j - 3, \end{aligned}$$

$$\begin{aligned} dist(z_3', u_2') &\leq dist(z_3, u_2) + 2 \\ &\leq j - 5 + 2 = j - 3. \end{aligned}$$

and

$$\begin{aligned} dist(v_3', w_3') &\leq dist(v_3, w_3) + 1 \\ &\leq j - 5 + 2 = j - 3 \end{aligned}$$

Now let $x \in \{z_3, w_2\}$ and $y \in \{v_3, w_3\}$, then

$$\begin{aligned} dist(x', y') &\leq dist(x, y) + 2 \\ &\leq j - 4 + 2 = j - 2. \end{aligned}$$

Finally, we see that

$$\begin{aligned} dist(w_3', u_2') &\leq dist(w_3, u_2) + 2 \\ &\leq j - 4 + 2 = j - 2, \end{aligned}$$

$$\begin{aligned} dist(w_2', z_3') &\leq dist(w_2, z_3) + 2 \\ &\leq j - 4 + 2 = j - 2 \end{aligned}$$

and

$$\begin{aligned} dist(v_3', u_2') &\leq dist(v_3, u_2) + 2 \\ &\leq j - 4 + 2 = j - 2, \end{aligned}$$

hence the result follows. □

Lemma 4.18. *Let D be one of the constructed s-digraphs of odd m-diameter q, then*

a) $\max_{x \in V(D)}\{dist(x, u_2), dist(x, v_3), dist(x, w_3)\} \leq q - 1,$

b) $\max_{y \in V(D)}\{dist(u_3, y), dist(z_3, y), dist(w_2, y)\} \leq q - 1,$

c) $\min_{x \in V(D)}\{dist(x, z_1), dist(x, z_2)\} \leq q - 1,$

d) $\min_{y \in V(D)}\{dist(v_1, y), dist(v_2, y)\} \leq q - 1.$

Proof. It can easily be checked that the statement is true for m-diameter $q = 9$, so assume it is also true for all constructed s-digraphs of odd m-diameter $9 \leq q \leq j - 1$, and let D' be the s-digraph of odd m-diameter j constructed from a s-digraph D of m-diameter $j - 2$.

To prove a) we see that for all $x' \in D'$

$$\begin{aligned}
dist(x', u_2') &\le dist(x', u_2) + 1 \\
&\le \max\{dist(x, u_2) + 1, dist(u_3, u_2) + 1, dist(v_3, u_2) + 1, dist(w_2, u_2) + 2, \\
&\quad dist(z_3, u_2) + 2\} + 1 \\
&\le \max\{(j-3) + 1(j-5) + 1, (j-3) + 1, (j-5) + 2, (j-5) + 2\} + 1 \\
&\le \max\{j-4, j-2, j-2, j-3\} + 1 \\
&= j - 1
\end{aligned}$$

and similarly we get $dist(x', v_3') \le j - 1$ and $dist(x', w_3') \le j - 1$.

For b) we have for all $y \in D'$

$$\begin{aligned}
dist(u_3', y') &\le dist(u_3, y') + 1 \\
&\le \max\{dist(u_3, y) + 1, dist(u_3, u_2) + 1, dist(u_3, v_3) + 2, dist(u_3, w_3) + 2, \\
&\quad dist(u_3, z_3) + 1\} + 1 \\
&\le j - 1,
\end{aligned}$$

and similarly we get $dist(z_3, y) \le j - 1$ and $dist(w_2, y) \le j - 1$.

Now for c), we have

$$\begin{aligned}
\min_{i=1,2}\{dist(x', z_i')\} &\le \min_{i=1,2}\{dist(x', z_i)\} + 1 \\
&\le \max\left\{\min_{i=1,2}\{dist(x, z_i)\} + 1, \min_{i=1,2}\{dist(u_3, z_i)\} + 2, \min_{i=1,2}\{dist(v_3, z_i)\} + 1,\right. \\
&\quad \left.\min_{i=1,2}\{dist(w_2, z_i)\} + 2\right\} + 1 \\
&\le \max\{(j-3) + 1, dist(u_3, z_3) + 1, (j-3) + 1, dist(w_2, z_3) + 1\} + 1 \\
&\le j - 1.
\end{aligned}$$

For d) we get

$$\begin{aligned}
\min_{i=1,2}\{dist(v_i', y')\} &\le \min_{i=1,2}\{dist(v_i, y')\} + 1 \\
&\le \max\left\{\min_{i=1,2}\{dist(v_i, y)\} + 1, \min_{i=1,2}\{dist(v_i, u_2)\} + 2, \min_{i=1,2}\{dist(v_i, z_3)\} + 1,\right. \\
&\quad \left.\min_{i=1,2}\{dist(v_i, w_3)\} + 2\right\} + 1 \\
&\le \max\{(j-3) + 1, dist(v_3, u_2) + 1, (j-3) + 1, dist(v_3, w_3) + 1\} \\
&\le j - 1.
\end{aligned}$$

□

Theorem 4.19. *For all odd $k \geq 9$ and there exists a s-digraph such that*

$$n_s(3,k) = \frac{1}{2}k^2 + k - \frac{11}{2} \tag{4.9}$$

Proof. It can be checked that the s-digraph in Figure 4.14 is a s-digraph of diameter 9. Unfortunately there is no symmetry in the s-digraph, so the task is a bit cumbersome if not done by computer.

Now, the s-digraphs we constructed in Figure 4.15 have diameter $k = q$, which follows by Lemma 4.17 and 4.18 and induction on the m-diameter q, and hence the result follows. □

Appendix

The computer searches in the previous sections were executed in Sage, see `http://www.sagemath.org/`.

The following function `find(M,k)`, takes as input a constant k and an adjacency matrix M of a digraph D. If the diameter of D is at most k and there are any digons in D, the function returns two new adjacency matrices $M1$ and $M2$. If (u, v, u) was a digon in D, then $M1$ is be the adjacency matrix for $D - \{(u, v)\}$ and $M2$ the adjacency matrix for $D - \{(v, u)\}$. Then the function is used recursively on $M1$ and $M2$, until we either get that the diameter is larger than k or that there are no more digons, if the latter is the case the function returns a plot of the digraph of diameter $\leq k$ and girth > 2.

```
def find(M,k):
    D=DiGraph(M)
    l=0
    if D.girth()==2 and D.diameter()<=k:
        i=0
while i<M.nrows():
    j=0
            while j<M.ncols():
        if M[i,j]==1 and M[j,i]==1:
                    M1=copy(M)
                    M1[i,j]=0
                    M2=copy(M)
                    M2[j,i]=0
                    return find(M1,k), find(M2,k)
                    i=M.nrows()
                    j=M.ncols()
                j=j+1
```

```
            i=i+1
    elif D.diameter()<=k:
            D=DiGraph(M)
            D.show()
            print D.diameter()
```

The function deleting(M,n,k,p) takes as input the adjacency matrix M of a digraph D, a number of vertices n which we wish to delete from this digraph, the diameter k of the digraph we wish to find and a parameter p used for internal use in the function to avoid deleting the same set of vertices more than once (always set to 0 when executing the function).

The function deletes n vertices from the digraph D in every possible way, thus if N is the order of D, then the function deletes n vertices in $\binom{N}{n}$ different ways. Then for all the new adjacency matrices obtained by deleting n vertices, we use the function find(M,k) to find a subgraph of these digraphs without digons and with diameter at most k.

```
def deleting(M,n,k,p):
    if n>0:
        for i in IntegerRange(p,M.nrows()):
            A=M.delete_rows([i])
            A=A.delete_columns([i])
            deleting(A,n-1,k,i)
    else:
        return find(M,k)
```

When looking for the largest orientation $\bar{S}$ of a subgraph of the triangular grid, such that $\bar{S}$ has diameter 2, the input for the above functions will be the adjacency matrix $T2$ as below.

```
T2=zero_matrix(7,7); T2[0,1]=T2[1,0]=T2[0,2]=T2[2,0]=T2[0,3]=T2[3,0]=
T2[1,3]=T2[3,1]=T2[1,4]=T2[4,1]=T2[2,3]=T2[3,2]=T2[3,4]=T2[4,3]=T2[2,5]=
T2[5,2]=T2[3,5]=T2[5,3]=T2[3,6]=T2[6,3]=T2[4,6]=T2[6,4]=T2[5,6]=T2[6,5]=
1; G2=DiGraph(T2); G2.show()
```

Then the procedure is to execute deleting(T2,0,2,0) and if no output, then increase the value of n until one gets an output, in this case we will get an output when $n = 4$.

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Part II

Papers

Paper A

On k-geodetic digraphs with excess one

Anita Abildgaard Sillasen

The paper has been submitted to
Electronic Journal of Graph Theory and Applications

In peer review

The layout has been revised.

Abstract

A k-geodetic digraph G is a digraph in which, for every pair of vertices u and v (not necessarily distinct), there is at most one walk of length $\leq k$ from u to v. If the diameter of G is k, we say that G is strongly geodetic. Let $N(d,k)$ be the smallest possible order for a k-geodetic digraph of minimum out-degree d, then $N(d,k) \geq 1+d+d^2+\ldots+d^k = M(d,k)$, where $M(d,k)$ is the Moore bound obtained if and only if G is strongly geodetic. Thus strongly geodetic digraphs only exist for $d=1$ or $k=1$, hence for $d,k \geq 2$ we wish to determine if $N(d,k) = M(d,k)+1$ is possible. A k-geodetic digraph with minimum out-degree d and order $M(d,k)+1$ is denoted as a $(d,k,1)$-digraph or said to have excess 1. In this paper we will prove that a $(d,k,1)$-digraph is always out-regular and that if it is not in-regular, then it must have 2 vertices of in-degree less than d, d vertices of in-degree $d+1$ and the remaining vertices will have in-degree d. Furthermore we will prove there exist no $(2,2,1)$-digraphs and no diregular $(2,k,1)$-digraphs for $k \geq 3$.

1 Introduction

A digraph which satisfies that for any two vertices u, v in G, there is at most one walk of length at most k from u to v, is called a *k-geodetic digraph*. If the diameter of a k-geodetic digraph G is k, we say that G is *strongly geodetic*.

Let G be a k-geodetic digraph with minimum out-degree d. What is then the smallest possible order, $N(d,k)$, of such a G? Letting n_i be the number of vertices in distance i from a vertex v for $i = 0, 1, 2, \ldots$, and realizing that $n_i \geq d^i$, we see that a lower bound is given as

$$N(d,k) \geq \sum_{i=0}^{k} n_i \geq \sum_{i=0}^{k} d^i = M(d,k). \tag{A.1}$$

The right hand side of (A.1) is the so called *Moore bound* for digraphs. The Moore bound is an upper theoretical bound for the so called *the degree/diameter problem*, which is the problem of finding the largest possible order of a digraph with maximum out-degree d and diameter k. A digraph with order $M(d,k)$, maximum out-degree d and diameter k is called a *Moore digraph*. If a k-geodetic digraph has $M(d,k)$ vertices, then it must be strongly geodetic, and therefore a Moore digraph. However, the only Moore digraphs are $(k+1)$-cycles ($d=1$) and complete digraphs, K_{d+1} ($k=1$), see [1] or [2], thus for $d \geq 2$ and $k \geq 2$ we are interested in knowing if the order for a k-geodetic digraph with minimum out-degree d could be $M(d,k)+1$. We say that a k-geodetic digraph G of minimum out-degree d and order $M(d,k)+1$ is a *$(d,k,1)$-digraph* or that it has *excess* one.

Notice that $(k+2)$-cycles and $(k+1)$-cycles with a vertex having an arc to a vertex on the $(k+1)$-cycle are $(1,k,1)$-digraphs and that complete digraphs K_{d+2} with at most one arc from each vertex deleted are $(d,1,1)$-digraphs. In the remaining part of this paper we will thus assume $d \geq 2$ and $k \geq 2$.

In this paper we will specify some further properties of the $(d,k,1)$-digraphs, especially we will show that they have diameter $k+1$, and that if a $(d,k,1)$-digraph is not diregular, then it is out-regular and there will be exactly d vertices of in-degree $d+1$, two vertices of in-degree less than d and the remaining vertices will have in-degree d. In the last section we will show that there exist no $(2,2,1)$-digraphs and no diregular $(2,k,1)$-digraphs.

2 Results

Let an i-walk denote a walk of length i and a $\leq i$-walk denote a walk of length at most i. Furthermore, let $N_i^+(u)$ denote the multiset of all vertices which are end vertices in an i-walk starting in the vertex u, notice that $N_0^+(u) = \{u\}$ and $N_1^+(u) = N^+(u)$. Also let $T_i^+(u) = \cup_{j=0}^{i} N_j^+(u)$, thus it is the multiset of all vertices which are end vertices in a $\leq i$-walk. Notice that for k-geodetic digraphhs $N_i^+(u)$ and $T_i^+(u)$ are sets when $i \leq k$. Looking at $(d,k,1)$-digraphs, we will often depict all the $\leq (k+1)$-paths from some arbitrary vertex u, thus the vertices in the multiset $T_{k+1}^+(u)$.

The first important result is that a $(d,k,1)$-digraph G is in fact out-regular, as if we assume the contrary, that there is a vertex $u \in V(G)$ with $d^+(u) \geq d+1$, we get that

$$\begin{aligned}|V(G)| &\geq |T_k^+(u)| \\ &= 1 + (d+1) + (d+1)d + (d+1)d^2 + \ldots + (d+1)d^{k-1} \\ &= M(d,k) + M(d,k-1),\end{aligned}$$

a contradiction as $M(d,k-1) > 1$ for $k \geq 2$.

An immediate consequence of a $(d,k,1)$-digraph being out-regular, is that it has diameter $k+1$ which follows in the following lemma.

Lemma A.1. *Let G be a $(d,k,1)$-digraph, then*

- *for each vertex $u \in V(G)$ there exists exactly one vertex $o(u) \in V(G)$ such that $dist(u, o(u)) = k+1$,*
- *for any two vertices, $u, v \neq o(u)$ there is exactly one $\leq k$-path from u to v.*

Proof. As we know G is out-regular and the order is $M(d,k)+1$, the second statement follows. Let $u \in V(G)$ be any vertex and let $o(u)$ be the unique vertex not reachable with a $\leq k$-path from u, then we just need to prove $d^-(o(u)) > 0$. Assume the contrary, that $d^-(o(u)) = 0$, then $o(u) = o(v)$ for all $v \in V(G)\backslash\{o(u)\}$. But then $G\backslash\{o(u)\}$ will be a Moore digraph of degree $d \geq 2$ and diameter $k \geq 2$, a contradiction. Hence $d^-(o(u)) > 0$ for all $u \in V(G)$ and thus $dist(u, o(u)) = k+1$. □

The unique vertex $o(u)$ with $dist(u, o(u)) = k + 1$ will be called the *outlier* of u. So a $(d, k, 1)$-digraph is out-regular of out-degree d and has diameter $k + 1$. Showing that a $(d, k, 1)$-digraph G is also in-regular is not as straightforward. We will prove that if it is not in-regular, then there are exactly two vertices of in-degree less than d, d vertices of in-degree $d + 1$ and the remaining vertices are of in-degree d. Let $S' = \{v \in V(G)|d^-(v) > d\}$ and $S = \{v \in V(G)|d^-(v) < d\}$, then we get the following lemmas and theorem.

Lemma A.2. *Let G be a $(d, k, 1)$-digraph, then*

- $|S'| \leq d$ *and* $d^-(v) = d + 1$ *for all* $v \in S'$,
- $S' \subseteq N^+(o(u))$ *for all* $u \in V(G)$.

Proof. Assume $u \in V(G)$ and $v \notin N^+(o(u))$, then as u must reach all in-neighbors of v in $\leq k$-paths, we must have $d^+(u) \geq d^-(v)$. If not, then there will exist an out-neighbor u' of u which has two $\leq k$-paths to v, a contradiction. Now, if $v \in N^+(o(u))$, then u must reach all in-neighbors of v, except $o(u)$, in a $\leq k$-path. Thus with the same arguments as before, we must have $d^+(u) \geq d^-(v) - 1$. Thus all vertices in S' must have in-degree $d + 1$ and both statements follows, as $|N^+(o(u))| = d$. $\square$

Lemma A.3. *If $S' \neq \emptyset$, then $|S'| = d$.*

Proof. As a $(d, k, 1)$-digraph is out-regular, its average in-degree must be d and thus $\sum_{v \in S'}(d^-(v) - d) = \sum_{v \in S}(d - d^-(v)) = |S'|$. Now let $v \in S'$, then we know $|N^-(v)| = |N_1^-(v)| = d+1$ and $|N_t^-(v)| \geq d|N_{t-1}^-(v)| - \epsilon_t$ for $1 < t \leq k$, where $\epsilon_2 + \epsilon_3 + \ldots + \epsilon_k \leq |S'|$. As all vertices in $T_k^-(v)$ are distinct, it implies that

$$|V(G)| \geq \sum_{i=0}^{k} |N_i^-(v)|. \tag{A.2}$$

Estimating the above sum, we get a safe lower bound by letting $\epsilon_2 = |S'|$ and $\epsilon_t = 0$ for all $3 \leq t \leq k$, thus

$$\begin{aligned}
|V(G)| &\geq 1 + |N^-(v)| + |N_2^-(v)| + |N_3^-(v)| + \ldots + |N_k^-(v)| \\
&\geq 1 + (d + 1) + ((d + 1)d - |S'|)(1 + d + \ldots + d^{k-2}) \\
&= 2 + d + d^2 + \ldots + d^k + (d - |S'|)(1 + d + \ldots d^{k-2}) \\
&= M(d, k) + 1 + (d - |S'|)M(d, k - 2).
\end{aligned}$$

But as G is a $(d, k, 1)$-digraph, we have $|V(G)| = M(d, k) + 1$, which together with the preceding inequality and Lemma A.2 gives $|S'| = d$. $\square$

A consequence of the above proof, is also that $S \subseteq N^-(v)$ for all $v \in S'$.

Theorem A.4. *Let G be a $(d,k,1)$-digraph. Then, if G is not diregular, we have $S = \{z, z'\}$ where $o(u) \in S$ for all $u \in V(G)$.*

Proof. Assume G is not diregular, thus we can assume $S' = \{u_1, u_2, \ldots, u_d\}$ where $d^-(u_i) = d+1$ and $o(u) \in N^-(u_j)$ for all $u \in V(G)$ and $j = 1, 2, \ldots, d$ according to Lemmas A.2 and A.3. Moreover, from the proof of Lemma A.3 we see that $dist(v, u_i) \leq k$ for all $v \in G$ and $i = 1, 2, \ldots, d$.

Now let $N^-(u_1) = \{z_1, z_2, \ldots, z_{d+1}\}$ where $z_1 = o(u_1)$. Then $S' \cap T^-_{k-1}(z_1) = \emptyset$, as otherwise $(z_1, u_j, \ldots, z_1)$ will be a $\leq k$-cycle for some $j = 1, 2, \ldots, d$. Also, no two vertices u_i and u_j can belong to the same $T^-_{k-1}(z_l)$ for $1 \leq l \leq d+1$, as if they did, $(z_1, u_i, \ldots, z_l)$ and $(z_1, u_j, \ldots, z_l)$ would be two distinct $\leq k$-paths. Thus we can assume $S' \cap T^-_{k-1}(z_l) = \{u_l\}$ for $2 \leq l \leq d$ and $dist(u_l, z_l) = k-1$, as otherwise there will be two $\leq k$-walks $(z_1, u_l, \ldots, z_l, u_1)$ and (z_1, u_1). As $(o(u), u_i)$ is an arc for all $u \in V(G)$ and $i = 1, 2, \ldots, d$ none of the vertices $z_2, z_3, \ldots, z_d$ can be the outlier of any vertex in G, as otherwise $(o(u) = z_l, u_l, \ldots, z_l)$ will be a k-cycle. Thus $o(u) \in \{z_1, z_{d+1}\}$ for all $u \in V(G)$.

Finally we wish to show that $S = \{z_1, z_{d+1}\}$. Assume the contrary, thus for some $2 \leq l \leq d$ we have $d^-(z_l) < d$ and $o(u) \neq z_l$ for all $u \in V(G)$, as $S \subseteq N^-(u_1)$. But then

$$\begin{aligned} |V(G)| &\leq 1 + (d-1)(1 + d + d^2 + \ldots + d^{k-1}) + 1 \\ &= M(d,k) - M(d,k-1) + 1 \\ &< M(d,k) + 1 \end{aligned}$$

as $dist(u_l, z_l) = k-1$ and $dist(u_j, z_l) \geq k$ for all $j \neq l$. Thus $S \subseteq \{z_1, z_{d+1}\}$ and as $\sum_{v \in S'}(d^-(v) - d) = d = \sum_{v \in S}(d - d^-(v))$ and $d^-(u) > 0$ for all $u \in V(G)$ the result follows.

□

If G is diregular, we get the following useful lemma.

Lemma A.5. *Let G be a diregular $(d,k,1)$-digraph, then the mapping $o : V(G) \mapsto V(G)$ is an automorphism.*

Proof. Let A be the adjacency matrix of G, then due to the properties of G we get

$$I + A + A^2 + \ldots + A^k = J - P, \tag{A.3}$$

where J is the matrix with all entries equal to 1 and P is a permutation matrix with entry $P_{ij} = 1$ if $o(i) = j$ and $P_{ij} = 0$ otherwise.

Now, as we know G is diregular, we know that $AJ = JA$, and as the left hand side of (A.3) is a polynomial in A, we must also have $PA = AP$, thus o is an automorphism. □

Notice that if G is diregular there will be exactly d $(k+1)$-paths from a given vertex u to $o(u)$, as all u's out-neighbors must reach $o(u)$ in k-paths and if there were more than d $(k+1)$-paths, one of u's out-neighbors would have more than one $\leq k$-path to $o(u)$, a violation of the definition of $(d,k,1)$-digraphs.

3 $(2, k, 1)$-digraphs

In this section we will assume $d = 2$ and prove the none-existence of $(2, 2, 1)$-digraphs and diregular $(2, k, 1)$-digraphs.

Theorem A.6. *There are no* $(2, 2, 1)$*-digraphs.*

Proof. Assume G is a $(2, 2, 1)$-digraph, then it has 8 vertices and we can depict the relationship between the vertices in $T_3^+(1)$ as in Figure A.1, where we can see $o(1) = 8$.

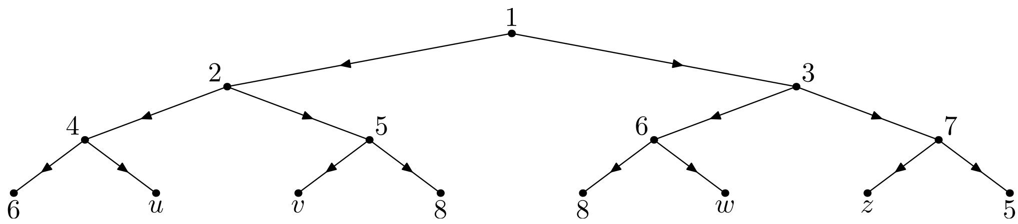

Figure A.1: $T_3^+(1)$.

Assume G isn't diregular, then we know from Theorem A.4 that $d^-(8) = 1$ and there exist another vertex $z \in V(G)$ with $d^-(z) = 1$ and $o(3) = o(6) = z$. Furthermore we know $N^+(8) = N^+(z) = \{u_1, u_2\}$ with $d^-(u_i) = 3$ for $i = 1, 2$. Notice that $6 \notin \{u_1, u_2\}$, as otherwise G would contain a 2-cycle, $(6, 8, 6)$. As the diameter of G is 3, we must have $dist(2, 6) = 2$ for 2 to reach 8 and thus $o(2) = 8$. Assume without loss of generality that $6 \in N^+(4)$. Then for 5 to reach 8 we must have $3 \in N^+(5)$, as $N^-(6) = \{3, 4\}$ and $4 \notin N^+(5)$, as otherwise $(2, 4)$ and $(2, 5, 4)$ will be two distinct ≤ 2-paths. The only vertices which 2 cannot reach are 1 and 7. If $7 \in N^+(5)$ we have $(5, 7)$ and $(5, 3, 7)$ as ≤ 2-paths, which is a contradiction. If instead $1 \in N^+(5)$ then we have the ≤ 2-paths $(5, 1, 3)$ and $(5, 3)$ another contradiction.

Now assume that G is diregular and recall that then o is an automorphism, thus we can assume $8 \in N^+(5)$ as $o(2) \neq 8$. Then we see that $o(2) \neq 6$, as otherwise there would be a 2-cycle $(6, 8, 6)$ as o is an automorphism, a contradiction. So there will be a ≤ 2-path from 2 to 6, but $6 \notin N^+(5)$ as otherwise there are two ≤ 2-paths from 5 to 8, namely $(5, 8)$ and $(5, 6, 8)$. Thus $6 \in N^+(4)$, and in the same manner we see that $5 \in N^+(7)$. Let u and v be the other out-neighbor of 4 and 5 respectively, and w and z the other out-neighbor of 6 and 7 respectively.

As 2 has to reach vertex $1, 3$ and 7 and at most one of them can be the outlier of 2, we must have $u \in \{1, 7\}$ and $v \in \{1, 3\}$, as if $u = 3$ there will exist two ≤ 2-paths from 4 to 6, namely $(4, 6)$ and $(4, 3, 6)$ and if $v = 7$ we will get a 2-cycle, $(7, 5, 7)$. Similar we see $z \in \{1, 4\}$ and $w \in \{1, 2\}$.

Now assume $o(2) = 1$, hence $o(3) \neq 1$ and $(o(1), o(2)) = (8, 1)$ is an arc. Then $u = 7$ and $v = 3$, and as o is an automorphism, we must have $z = 1$, as if $w = 1$ we will have the two ≤ 2-paths, $(6, 1)$ and $(6, 8, 1)$. But then $(7, 1, 3)$ and $(7, 5, 3)$ are both 2-paths from 7 to 3, a contradiction.

Instead assume $o(2) = 3$, thus $u = 7$ and $v = 1$ and $(o(1), o(2)) = (8, 3)$ is an arc. But then $(5, 1, 3)$ and $(5, 8, 3)$ are both 2-paths from 5 to 1. So we can safely assume $o(2) = 7$, thus $u = 1$ and $v = 3$, but then $(5, 3, 7)$ and $(5, 8, 7)$ are both 2-paths from 5 to 7, another contradiction.

□

Theorem A.7. *No diregular $(2, k, 1)$-digraph exists for $k \geq 2$.*

Proof. Due to Theorem A.6 we can assume $k > 2$ and we label the vertices in $T^+_{k+1}(1)$ as in Figure A.2. First of all, notice that for all $u \in V(G)$ we obviously have $o(u) \notin T^+_k(u)$, so we must have $o(2) \in T^+_{k-1}(3) \cup \{1\}$. We also see that $o(2) \notin T^+_{k-2}(6)$, as otherwise there will be two $\leq k$-paths from 6 to $o(2)$, the one in $T^+_{k-2}(6)$ and $(6, 12, \ldots, 3 \cdot 2^{k-1}, 2^{k+1} = o(1), o(2))$, a contradiction.

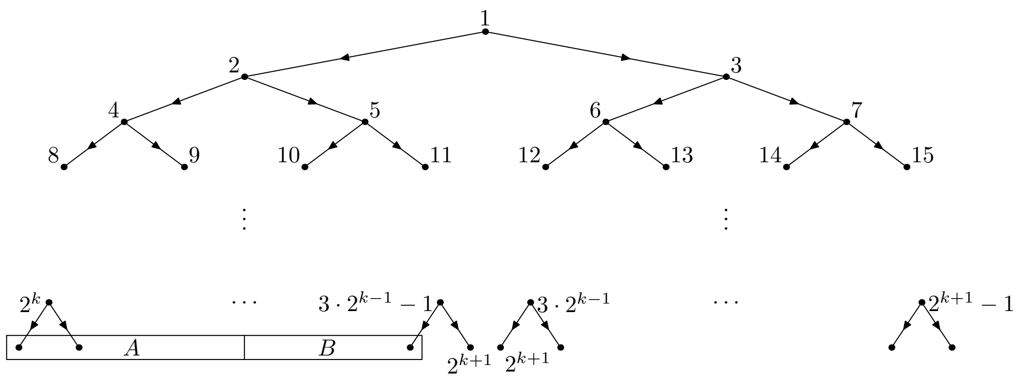

Figure A.2: $T^+_{k+1}(1)$.

Now, let $A = N^+_{k-1}(4)$ and $B = N^+_{k-1}(5) \backslash \{2^{k+1}\}$, so $|A| = 2^{k-1}$ and $|B| = 2^{k-1} - 1$. Then we will look at how $(\{1\} \cup T^+_{k-1}(3)) \backslash o(2)$ is distributed on A and B. For any arc (u, v) in G, we must have that u and v will not both be in A and not both in B, as otherwise there would be two $\leq k$-paths from either 4 or 5 to v. We observe that $3 \cdot 2^{k-1} \notin B$, as otherwise there would be two $\leq k$-paths from 5 to 2^{k+1}, namely $(5, 11, \ldots 3 \cdot 2^{k-1} - 1, 2^{k+1})$ and $(5, \ldots, 3 \cdot 2^{k-1}, 2^{k+1})$. So we must have $3 \cdot 2^{k-1} \in A$, $3 \cdot 2^{k-2} \in B$, $3 \cdot 2^{k-3} \in A$, and so on, until we reach vertex 6. A consequence of this is that $N^+_{k-2}(6) \in A$, $N^+_{k-3}(6) \in B$, $N^+_{k-4}(6) \in A$ and so on, until we get either $6 \in A$ if k is even or $6 \in B$ if k is odd.

Let $a = |A \cap T^+_{k-2}(6)|$ and $b = |B \cap T^+_{k-2}(6)|$, so $a + b = 2^{k-1} - 1$. Now, if k is even we let

$$a_e = a = \sum_{i=0}^{\frac{k}{2}-1} 2^{2i} = -\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}$$

and

$$b_e = b = \sum_{i=0}^{\frac{k}{2}-2} 2^{2i+1} = -\frac{2}{3} + \frac{1}{3} \cdot 2^{k-1}.$$

Similarly, if k is odd we let

$$a_o = a = \sum_{i=0}^{\frac{k-3}{2}} 2^{2i+1} = -\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}$$

and

$$b_o = b = \sum_{i=0}^{\frac{k-3}{2}} 2^{2i} = -\frac{1}{3} + \frac{1}{3} \cdot 2^{k-1} = \frac{1}{2} a_o.$$

We start by assuming that $o(2) = 1$, then if k is even we see that vertex 3 must be in B, so $7 \in A$, $\{14, 15\} \subseteq B, \ldots, N^+_{k-2}(7) \subseteq A$. Thus

$$|A| = 2 \cdot a_e = 2 \left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1} \right) > 2^{k-1}$$

as $k > 2$, a contradiction. If k is odd, we see that vertex 3 must be in A, so $7 \in B$, $\{14, 15\} \subseteq A, \ldots, N^+_{k-2}(7) \subseteq A$, thus

$$|A| = 2a_o + 1 = 2 \left(-\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1} \right) + 1 > 2^{k-1}$$

as $k > 2$, yet a contradiction. So we know due to symmetry that $1 \notin \{o(2), o(3)\}$.

Now, assume that $o(2) \neq 3$. Then we know the distribution of all the vertices in $T^+_{k-1}(3) \cup \{1\}$ except for those in $T^+_i(o(2))$, where i is given by $dist(3, o(2)) = k - 1 - i$. Assume $i = 0$, thus $o(2) \in N^+_{k-2}(7)$, or that $N^+(o(2))$ is in the same set (A or B) as $N^+_{k-1-i}(6)$, then we see that $|A| \geq 2a > 2^{k-1}$, a contradiction. So we can assume there exist vertices u and v, such that $N^+(o(2)) = \{u, v\} \subseteq T^+_{k-2}(7)$ and that not both u and v are in the same set (A or B) as $N^+_{k-1-i}(6)$.

Let for i even c_e denote the number of vertices in every second layer of $T_i^+(o(2))$ such that $N_i^+(o(2))$ is not one of those layers, then

$$c_e = \sum_{j=0}^{\frac{i}{2}-1} |N_{2j+1}^+(o(2))| = 2(1 + 2^2 + \ldots + 2^{i-2}) = \frac{2}{3} \cdot 2^i - \frac{2}{3}.$$

Let d_e denote the number of vertices in the remaining layers, thus

$$d_e = \sum_{j=0}^{\frac{i}{2}-1} |N_{2j+2}^+(o(2))| = 2c_e.$$

Similar for i odd we let c_o denote the number of vertices in every second layer, where $N_i^+(o(2))$ is not one of those layers, thus

$$c_o = \sum_{j=0}^{\frac{i-3}{2}} |N_{2j+2}^+(o(2))| = \frac{1}{3}(2^{i+1} - 1) - 1 = \frac{1}{3} \cdot 2^{i+1} - \frac{4}{3}$$

and the number of vertices in the remaining layers are then

$$d_o = \sum_{j=0}^{\frac{i-1}{2}} |N_{2j+1}^+(o(2))| = 2c_o + 2.$$

We will now count the number of vertices in A depending on whether k and i are even or odd, and which set (A or B) u and v are in, a total of 8 different scenarios. Notice that exactly one of 1 and 3 will be in A. We will obtain contradictions in some of the scenarios and in the remaining we will obtain that $o(2) = 7$. Thus we will have proven that $o(2) \in \{3, 7\}$.

If k is even, we get following scenarios:

- i **even:**
  - $u, v \in A$**:** Then

$$\begin{aligned} |A| &= 2a_e + 1 + c_e - d_e - 1 \\ &= 2\left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}\right) - c_e \\ &= \frac{2}{3} \cdot 2^k - \frac{2}{3} \cdot 2^i. \end{aligned}$$

    Now as we already know $|A| = 2^{k-1}$, we must have $i = k - 2$, and thus $o(2) = 7$.

- **$u \in A$, $v \in B$:** Then half of the vertices in $T_i^+(o(2))\backslash\{o(2)\}$, thus 2^i-1 vertices, will be in A and the other in B, hence

$$\begin{aligned}|A| &= 2a_e + 1 - d_e - 1 + 2^i - 1\\ &= 2\left(-\frac{1}{3}+\frac{2}{3}\cdot 2^{k-1}\right) - \frac{4}{3}(2^i-1) + 2^i - 1\\ &= -\frac{1}{3}+\frac{2}{3}\cdot 2^k - \frac{1}{3}\cdot 2^i\end{aligned}$$

a contradiction with $|A| = 2^{k-1}$.

- **i odd:**
  - **$u,v \in B$:** Similar to before, we see that

$$\begin{aligned}|A| &= 2a_e + 1 + c_o - d_o\\ &= 2\left(-\frac{1}{3}+\frac{2}{3}\cdot 2^{k-1}\right) + 1 + c_o - 2c_o - 2\\ &= -\frac{2}{3}+\frac{4}{3}\cdot 2^{k-1} - \left(\frac{1}{3}\cdot 2^{i+1} - \frac{4}{3}\right) - 1\\ &= -\frac{1}{3}+\frac{4}{3}\cdot 2^{k-1} - \frac{1}{3}\cdot 2^{i+1},\end{aligned}$$

again a contradiction to the fact that $|A| = 2^{k-1}$.

  - **$u \in A$, $v \in B$:** We see

$$\begin{aligned}|A| &= 2a_e + 1 + 2^i - 1 - d_o\\ &= 2\left(-\frac{1}{3}+\frac{2}{3}\cdot 2^{k-1}\right) + 1 + 2^i - 1 - \frac{2}{3}(2^{i+1}-1)\\ &= \frac{2}{3}\cdot 2^k - \frac{1}{3}\cdot 2^i.\end{aligned}$$

As $|A| = 2^{k-1}$, this implies $i = k-1$, but then $o(2) = 3$, a contradiction to our assumption.

If k is odd we have:

- **i even:**
  - **$u,v \in A$:** Then

$$\begin{aligned}|A| &= 2a_o + 1 + c_e - d_e - 1\\ &= 2\left(-\frac{2}{3}+\frac{2}{3}\cdot 2^{k-1}\right) - c_e\\ &= -\frac{2}{3}+\frac{2}{3}\cdot 2^k - \frac{2}{3}\cdot 2^i,\end{aligned}$$

yet a contradiction to $|A| = 2^{k-1}$.

- $u \in A$, $v \in B$: We see

$$\begin{aligned}|A| &= 2a_o + 1 - d_e - 1 + 2^i - 1\\ &= 2\left(-\frac{2}{3} + \frac{2}{3}\cdot 2^{k-1}\right) - \frac{4}{3}(2^i - 1) + 2^i - 1\\ &= -1 + \frac{2}{3}\cdot 2^k - \frac{1}{3}\cdot 2^i,\end{aligned}$$

a contradiction to $|A| = 2^{k-1}$ and $i \neq 0$.

- i **odd:**
  - $u, v \in B$: Similar to before, we see that

$$\begin{aligned}|A| &= 2a_o + 1 + c_o - d_o\\ &= 2\left(-\frac{2}{3} + \frac{2}{3}\cdot 2^{k-1}\right) + 1 + c_o - 2c_o - 2\\ &= -\frac{4}{3} + \frac{4}{3}\cdot 2^{k-1} - \left(\frac{1}{3}\cdot 2^{i+1} - \frac{4}{3}\right) - 1\\ &= -1 + \frac{4}{3}\cdot 2^{k-1} - \frac{1}{3}\cdot 2^{i+1},\end{aligned}$$

yet another contradiction to the fact that $|A| = 2^{k-1}$.

  - $u \in A$, $v \in B$: We see

$$\begin{aligned}|A| &= 2a_o + 1 + 2^i - 1 - d_o\\ &= 2\left(-\frac{2}{3} + \frac{2}{3}\cdot 2^{k-1}\right) + 2^i - \frac{2}{3}(2^{i+1} - 1)\\ &= -\frac{2}{3} + \frac{2}{3}\cdot 2^k - \frac{1}{3}\cdot 2^i.\end{aligned}$$

Then we must have $k = 3$ and $i = 1$, thus $o(2) = 7$.

To summarize the above, we have $o(2) \in \{3, 7\}$ and $o(3) \in \{2, 4\}$. Using similar arguments we observe $o(4) \in \{5, 10\}$, as $(11, \ldots, 2^{k+1} = o(1), o(2), o(4))$ is a k-path. Now, if $o(2) = 3$ we get $o(4) \in N^+(o(2)) = \{6, 7\}$, but this is a contradiction to our observation. On the other hand, if $o(2) = 7$ we must have $o(4) \in \{14, 15\}$ again a contradiction.

□

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Paper B

Subdigraphs of almost Moore digraphs induced by fixpoints of an automorphism

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Abstract

The degree/diameter problem for directed graphs is the problem of determining the largest possible order for a digraph with given maximum out-degree d and diameter k. An upper bound is given by the Moore bound $M(d,k) = \sum_{i=0}^{k} d^i$ and almost Moore digraphs are digraphs with maximum out-degree d, diameter k and order $M(d,k) - 1$.

In this paper we will look at the structure of subdigraphs of almost Moore digraphs, which are induced by the vertices fixed by some automorphism φ. If the automorphism fixes at least three vertices, we prove that the induced subdigraph is either an almost Moore digraph or a diregular k-geodetic digraph of degree $d' \leq d-2$, order $M(d',k)+1$ and diameter $k+1$.

As it is known that almost Moore digraphs have an automorphism r, these results can help us determine structural properties of almost Moore digraphs, such as how many vertices of each order there are with respect to r. We determine this for $d = 4$ and $d = 5$, where we prove that except in some special cases, all vertices will have the same order.

1 Introduction

Let G be a digraph and u be a vertex of maximum out-degree d in G, and let n_i denote the number of vertices in distance i from u. Then we have $n_i \leq d^i$ for $i = 0, 1, \ldots, k$, and thus the order n of G is bounded by

$$n = \sum_{i=0}^{k} n_i \leq \sum_{i=0}^{k} d^i. \tag{B.1}$$

If equality is obtained in (B.1) we say that G is a *Moore digraph* of degree d and diameter k, and the right-hand side of (B.1) is called the *Moore bound* denoted by $M(d,k) = \sum_{i=0}^{k} d^i$. Moore digraphs are known to be diregular and exist only when $d = 1$ (cycles of length $(k+1)$) or $k = 1$ (complete digraphs with order $d+1$), see [1] or [2]. So we are interested in knowing how close the order can get to the Moore bound for $d > 1$ and $k > 1$. Let G be a digraph of maximum out-degree d, diameter k and order $M(d,k) - \delta$, then we say G is a $(d,k,-\delta)$-*digraph* or alternatively a (d,k)-*digraph of defect* δ. When $\delta < M(d,k-1)$ we have out-regularity, see [3], whereas it in general is not known if we also have in-regularity. Of special interest is the case $\delta = 1$, and a $(d,k,-1)$-digraph is also denoted as an *almost Moore digraph*. Almost Moore digraphs do exist for $k = 2$ as the line digraphs of K_{d+1} for any $d \geq 2$, see [4], whereas $(2,k,-1)$-digraphs for $k > 2$, $(3,k,-1)$-digraphs for $k > 2$, $(d,3,-1)$-digraphs for $d > 1$ and $(d,4,-1)$-digraphs for $d > 1$ do not exist, see [5], [6], [7] and [8]. We do know that almost Moore digraphs are diregular for $d > 1$ and $k > 1$, see [3].

In the last section of the paper, we will be needing the following theorem which summarises some of the above results.

Theorem B.1 ([5],[6])**.** *Almost Moore digraphs of degree* 2 *and* 3 *and diameter* $k > 2$ *do not exist.*

Furthermore, almost Moore digraphs satisfies the following properties, where a $\leq k$-walk is a walk of length at most k.

Lemma B.2 ([9])**.** *Let G be an almost Moore digraph, then*

- *for each pair of vertices $u, v \in V(G)$ there is at most one $< k$-walk from u to v,*
- *for every vertex $u \in V(G)$ there exist a unique vertex $r(u)$ such that there are two $\leq k$-walks from u to $r(u)$.*

The mapping $r : V(G) \mapsto V(G)$ is in fact an automorphism, see [9] and thus the two $\leq k$-walks from u to $r(u)$ are internally disjoint. The vertex $r(u)$ is said to be the *repeat* of u. If we have $u = r(u)$, thus u has order 1 with respect to r, u is said to be a *selfrepeat*. If there is a selfrepeat in G, then there are exactly k selfrepeats, which lie on a k-cycle, see [10].

In this paper we will give some conditions for the existence of an almost Moore digraph G with respect to some automorphism $\varphi : V(G) \mapsto V(G)$. These results can then be used to investigate the orders of the vertices with respect to the automorphism r. Before stating the core result of this paper, we will introduce another type of digraph which shows to be important when characterizing induced subdigraphs of almost Moore digraphs.

Let D be a digraph such that for each pair of vertices $u, v \in V(D)$ we have at most one $\leq k$-walk from u to v, then we say D is *k-geodetic*. Let u be a vertex of minimum out-degree d, and let n_i be the number of vertices in distance i from u for $i = 0, 1, \ldots, k$. Then $n_i \geq d^i$ and the order n of D is bounded by

$$n \geq \sum_{i=0}^{k} n_i \geq \sum_{i=0}^{k} d^i. \tag{B.2}$$

Notice that the right-hand side is the Moore bound, $M(d, k)$ and that the diameter for a k-geodetic digraph is at least k. As we already know, Moore digraphs do only exist for $d = 1$ or $k = 1$, we wish to know how close the order of a k-geodetic digraph can get to the Moore bound. By a *(d, k, ϵ)-digraph* we understand a k-geodetic digraph of minimum out-degree d and order $M(d, k) + \epsilon$. Alternatively we say that we have a *(d, k)-digraph of excess ϵ*. The first case which is interesting is when $\epsilon = 1$. A $(d, k, 1)$-digraph has diameter $k + 1$, and for each vertex u there is exactly one vertex, the *outlier* $o(u)$ such that $dist(u, o(u)) = k + 1$, see [11].

A $(d, k, 1)$-digraph is diregular if and only the mapping $o : V(D) \mapsto V(D)$ is an automorphism, see [11]. From [11] we also have the following therem.

Theorem B.3 ([11])**.** *No diregular $(2, k, 1)$-digraphs exist for $k > 1$.*

2 Results

For simplicity, we will, in the remaining part of this paper, let a $(d,k,-1)$-digraph (almost Moore digraphs) denote any digraph which has degree $d > 0$, diameter $k > 0$ and order $M(d,k) - 1$, thus we will let k-cycles be included in this class. Similar, a $(d,k,1)$-digraph will denote any k-geodetic digraph of minimum out-degree $d > 0$ and order $M(d,k) + 1$.

The scope of this paper is to prove the following theorem.

Theorem B.4. *Let G be an almost Moore digraph of degree $d \geq 4$ and diameter $k \geq 3$ and let H be a subdigraph induced by the vertices which are fixed by some automorphism $\varphi : V(G) \mapsto V(G)$. Then H is either*

- *the empty digraph,*
- *two isolated vertices,*
- *an almost Moore digraph of degree $d' \leq d$ and diameter k or*
- *a diregular $(d',k,1)$-digraph where $d' \leq d-2$.*

In the remaining part of this paper we will assume G to be an almost Moore digraph of degree $d \geq 4$ and diameter $k \geq 3$, and H to be a subdigraph of G induced by the fixpoints of some automorphism $\varphi : V(G) \mapsto V(G)$.

We start by stating some properties of the fixpoints of G.

Lemma B.5. *Let u and v be fixpoints of G with respect to the automorphism φ, then*

- *$r(u)$ is a fixpoint,*
- *if there is a $\leq k$-walk P from u to v and $v \neq r(u)$, all vertices $w \in P$ are fixpoints*
- *if $v = r(u)$ and P and Q are the two $\leq k$-walks from u to v, either all internal vertices on P and Q are fixpoints, or none of them are. Furthermore, if $dist(u, r(u)) < k$, then all vertices on P and Q are fixpoints.*

Proof.

- We know there are two $\leq k$-walks, P and Q, from u to $r(u)$. Now, $\varphi(P)$ and $\varphi(Q)$ are two $\leq k$-walks from u to $\varphi(r(u))$, and hence $\varphi(r(u))$ is a repeat of u. As u only has one repeat, the statement follows.
- Let P be the unique $\leq k$-walk from u to v. Then $\varphi(P)$ will also be a $\leq k$-walk from u to v, and hence $P = \varphi(P)$.
- Assume not all vertices on the $\leq k$-walk P are fixpoints, hence there exists a vertex $w \in P$ such that $w \neq \varphi(w)$ and thus $\varphi(P) \neq P$ is also a $\leq k$-walk from u to $v = r(u)$. As there are only two $\leq k$-walks from u to $v = r(u)$, we must have

$\varphi(P) = Q$ and thus none of the internal vertices of P are fixpoints, as P and Q are internally disjoint. Now if $dist(u, r(u)) < k$, then P and Q are obviously of different length, so we must have all vertices on P and Q as fixpoints.

□

Corollary B.6. *Let φ be an automorphism of G, then all $\leq k$-walks among the fixpoints of φ in G are preserved to H, except for possibly the k-walks from a vertex to its repeat.*

Notice, that if u and v are selfrepeats fixed by φ, then there are exactly d internally disjoint $\leq (k+1)$-walks from u to v, $(u, u_i, \ldots, v_i, v)$ for $i = 1, 2, \ldots, d$. Hence if the order of u_i with respect to φ is p, and the order of v_i with respect to φ is q, then $(u, u_i = \varphi^p(u_i), \ldots, \varphi^p(v_i), v)$ and $(u, u = \varphi^q(u_i), \ldots, v_i = \varphi^q(v_i), v)$ are both $\leq (k+1)$-walks, and thus we must have $p = q$. Said in another way, the permutation cycles with respect to some automorphism φ of the vertices in $N^+(u)$ and $N^-(v)$ are the same when u and v are selfrepeats.

The following lemma is a more general result than that of [12].

Lemma B.7. *If G has a selfrepeat which is fixed by φ, then H is an almost Moore digraph with selfrepeats of degree $d' \leq d$ and diameter k.*

Proof. Let $z = r(z) = \varphi(z)$, then according to Lemma B.5 we must have all vertices on the two $\leq k$-walks from z to $r(z)$ as fixpoints, and all the selfrepeats lie on the non-trivial walk from z to z, so H contains a k-cycle.

Notice that $d^+_H(z) = d^-_H(z) = d' \leq d$ for all $z = r(z) \in V(H)$, as the permutation cycles in $N^+(z)$ and $N^-(z)$ are the same. Now, if we have a vertex $u = \varphi(u) \neq r(u)$, then we can pick a selfrepeat z such that $r(u) \notin N^-(z)$, as otherwise we would have $r(u) \in N^-(z')$ for all selfrepeats z' of G, and therefore $r(r(u))$ would be a selfrepeat, a contradiction as u is not a selfrepeat. Thus for this u and z we have d internally disjoint $\leq (k+1)$-walks $(u, u_i, \ldots, z_i, z)$ in G. Then d' of the internally disjoint $\leq (k+1)$-walks from u to z will also be in H, due to Lemma B.5, and thus $d^+(u) \geq d'$. Assume that $d^+(u) > d'$, then there exists a $j \in \{1, 2, \ldots d\}$ such that $u_j = \varphi(u_j)$ and $z_j \neq \varphi(z_j)$. But then $(u_j, \ldots, z_j, z)$ and $(u_j, \ldots, \varphi(z_j), z)$ are two distinct $\leq k$-walks from u_j to z, a contradiction as z is a selfrepeat.

So H is a diregular digraph of degree d'. Now, assume H has diameter $k+1$, this implies that there exists a vertex v such that $dist_H(v, r(v)) = k+1$ thus the order of H is $n = 1 + d' + d'^2 + \ldots + d'^k + 1 = M(d', k) + 1$, according to Corollary B.6. However, looking at a selfrepeat $z \in H$, we get the order as $n = 1 + d' + d'^2 + \ldots + d'^k - 1 = M(d', k) - 1$, a contradiction.

So H must be diregular with degree $d' \leq d$, diameter k and its order must be $M(d, k) - 1$, hence it is an almost Moore digraph with selfrepeats, as the girth of H is k. □

Lemma B.8. *Let φ fix at least three vertices, then H is diregular of degree d' and either*

- *H is an almost Moore digraph of degree $d' \leq d$ and diameter k, or*
- *H is a $(d', k, 1)$-digraph of degree $d' \leq d - 2$.*

Proof. If φ fixes a selfrepeat, then we have the first case of the statement according to Lemma B.7. Thus we can assume φ does not fix any selfrepeats.

Let u and v be any two fixed vertices in G, thus they are not selfrepeats, and let $N^+(u) = \{u_1, u_2, \ldots, u_d\}$ and $N^-(v) = \{v_1, v_2, \ldots, v_d\}$. Assume $r(u) \neq v_j$ for $j = 1, 2, \ldots, d$. Then in G we have internally disjoint $\leq (k+1)$-walks $(u, u_i, \ldots, v_i, v)$ for $i = 1, 2, \ldots, d$. As r is an automorphism, we get $r(u_i) \neq v$ for $i = 1, 2, \ldots, d$. Now, we have $u_i = \varphi(u_i)$ if and only if $v_i = \varphi(v_i)$ due to Lemma B.5, hence $d^+_H(u) = d^-_H(v)$. As we could have $v = r(u)$, we see that each vertex in H is balanced, as $d^+(u) = d^+(r(u))$ and $d^-(u) = d^-(r(u))$.

Now, assume H is not diregular, thus for each vertex $u \in V(H)$ we must have a vertex $v \in N^+(r(u)) \cap V(H)$ such that $d^+_H(u) \neq d^-_H(v)$. Let $u \in V(G)$ be a vertex of minimum degree $d_1 \leq d$ in H, and let $v \in V(H)$ be a vertex with $d^-_H(v) > d_1$. Then $d^-_H(v) = d_1 + 2$ as we must have $v \in N^+(r(u))$ with $dist_H(u, r(u)) = k + 1$ and $dist_H(r^-(v), v) \leq k$. But then there must be at most d_1 vertices of degree different from d_1 in H and at most $d_1 + 2$ vertices of degree different from $d_1 + 2$, hence $|V(H)| \leq d_1 + (d_1 + 2)$. This is a contradiction to the fact that $|V(H)| \geq d_1 + d_1^2 + \ldots + d_1^k$ as the diameter of H is at least $k \geq 3$. So, obviously H is diregular. If $dist(u, r(u)) = k + 1$, then each vertex in H must have at least two out-neighbors of order two with respect to φ and thus the statement follows. $\square$

Theorem B.4 now follows directly from Lemmas B.7 and B.8.

3 Almost Moore digraphs of degree 4 and 5

In this section we will look at almost Moore digraphs of degree 4 and 5 and specify the order of the vertices with respect to the automorphism r.

Lemma B.9. *Let $u \in V(G)$ be a vertex with $\varphi(u) = u \neq r(u)$, then if H is two isolated vertices or has diameter $(k+1)$ we must have two vertices in $N^+_G(u)$ which have order 2 with respect to φ.*

Proof. In G we have two $\leq k$-paths, P and Q from u to $r(u)$. If H is either two isolated vertices or has diameter $k+1$, we must have that the internal vertices on P and Q are not in H. Thus $\varphi(P) = Q$ and $\varphi(Q) = P$, and hence $\varphi^2(v) = v$ and $\varphi(v) \neq v$ for all internal vertices v on P and Q. $\square$

The following theorem is a more general result than that of [13] and [12].

Theorem B.10. *Let G be an almost Moore digraph of degree 4, then the vertices of G have orders with respect to the automorphism r according to one of the following:*

- *there are k vertices of order 1 and $M(4,k)-1-k$ of order 3 or*
- *all vertices are of the same order $p \geq 2$.*

Proof. Assume throughout that not all vertices are of the same order. Let u be a vertex of G of the smallest order p with respect to r in G. Let $N^+(u) = \{u_1, u_2, u_3, u_4\}$, then we can split $N^+(u)$ into permutation cycles with respect to r^p in one of the following ways: $(u_1)(u_2)(u_3, u_4)$, $(u_1)(u_2, u_3, u_4)$, (u_1, u_2, u_3, u_4) or $(u_1, u_2)(u_3, u_4)$. Notice however that the splitting
$(u_1)(u_2)(u_3, u_4)$ is not possible, as there according to Theorem B.4 where $\varphi = r^p$ would exist a $(2, k, -1)$- or $(2, k, 1)$-digraph as an induced subdigraph of G, a contradiction to Theorems B.1 and B.3.

First assume there is a vertex u of order 1, thus u is a selfrepeat and hence there are exactly k vertices of order 1 inducing a k-cycle in G. Thus among the above ways of having permutation cycles, the only possibility is $(u_1)(u_2, u_3, u_4)$. Then all vertices which are not selfrepeats must have order 3 according to Lemma B.7 by letting $\varphi = r^3$.

Now assume $u \in V(G)$ has the smallest possible order $p \geq 2$, then according to Lemma B.9 the only possible permutation cycles are $(u_1, u_2)(u_3, u_4)$. In turn, this is only possible if $p = 2$, as there will always be at least p vertices of order p in G.

Thus G will contain $M(4,k)-3$ vertices of order 4, thus 4 should divide $M(4,k)-3$. But in fact

$$M(4,k) - 3 \equiv -2 + 4 + 4^2 + \dots 4^k \equiv 2 \mod 4,$$

a contradiction.

□

Theorem B.11. *Let G be an almost Moore digraph of degree 5, then the vertices of G have orders with respect to the automorphism r according to one of the following:*

- *there are $M(3,k)+1$ vertices of order $p \geq 2$ and $M(5,k)-M(3,k)-2$ of order $2p$*
- *there are $k+2$ vertices of order $p \geq 2$ and $M(5,k)-3-k$ of order $2p$*
- *there are k vertices of order 1 and either $M(5,k)-1-k$ of order 2 or $M(5,k)-1-k$ of order 4*
- *all vertices are of the same order $p \geq 2$.*

Proof. Assume throughout that not all vertices are of the same order. Let u be a vertex of G of the smallest order p. Let $N^+(u) = \{u_1, u_2, u_3, u_4, u_5\}$, then we can split $N^+(u)$ into permutation cycles with respect to r^p in one of the following ways: $(u_1)(u_2, u_3, u_4, u_5)$, $(u_1)(u_2)(u_3)(u_4, u_5)$ or
$(u_1)(u_2, u_3)(u_4, u_5)$ due to Lemma B.9 and Theorems B.1 and B.3.

If the permutation cycles are $(u_1)(u_2, u_3, u_4, u_5)$, then due to Lemma B.9 we must have u is a selfrepeat, hence there is k vertices of order 1 and $M(5, k) - k - 1$ of order 4. If instead the permutation cyles are $(u_1)(u_2, u_3)(u_4, u_5)$, then we could have k vertices of order 1 and $M(5, k) - k - 1$ of order 2 or $k+2$ vertices of order $p \geq 2$ and $M(5, k) - k - 3$ of order $2p$.

Finally, if the permutation cycles are $(u_1)(u_2)(u_3)(u_4, u_5)$, then if $\varphi = r^p$, we would have H to be either a $(3, k, -1)$-digraph or a $(3, k, 1)$-digraph. But $(3, k, -1)$-digraphs do not exist according to Theorem B.1, thus we must have $M(3, k) + 1$ vertices of order $p \geq 2$ and $M(5, k) - M(3, k) - 2$ of order $2p$. □

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Paper C

On the Existence of Friendship Hypergraphs

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In the submitted version there is no proof of Lemma C.18.

The layout has been revised.

Abstract

A 3-uniform friendship hypergraph is a 3-uniform hypergraph in which, for all triples of vertices x, y, z there exists a unique vertex w, such that xyw, xzw and yzw are edges in the hypergraph. Sós showed that such 3-uniform friendship hypergraphs on n vertices exist with a so called universal friend if and only if a Steiner triple system, $S(2,3,n-1)$ exists. Hartke and Vandenbussche used integer programming to search for 3-uniform friendship hypergraphs without a universal friend and found one on 8, three non-isomorphic on 16 and one on 32 vertices. So far, these five hypergraphs are the only known 3-uniform friendship hypergraphs. In this paper we construct an infinite family of 3-uniform friendship hypergraphs on 2^k vertices and $2^{k-1}(3^{k-1}-1)$ edges. We also construct 3-uniform friendship hypergraphs on 20 and 28 vertices using a computer. Furthermore, we define r-uniform friendship hypergraphs and state that the existence of those with a universal friend is dependent on the existence of a Steiner system, $S(r-1,r,n-1)$. As a result hereof, we know infinitely many 4-uniform friendship hypergraphs with a universal friend. Finally we show how to construct a 4-uniform friendship hypergraph on 9 vertices and with no universal friend.

1 Introduction

A *hypergraph* is a pair $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ where $\mathcal{V}(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ is the set of *vertices* of $\mathcal{H}$ and $\mathcal{E}(\mathcal{H})$ is a subset of the powerset of V without the empty set. Each element in $\mathcal{E}(\mathcal{H})$ is denoted as an *edge*. The number of vertices n is also denoted as the *order* of the hypergraph. If each edge in $\mathcal{E}(\mathcal{H})$ contains exactly r vertices, then we say the hypergraph is *r-uniform*. If the hypergraph is 2-uniform, we just denote it as a *graph* and write $H = (V(H), E(H))$. Two vertices x and y are said to be neighbors in a graph H if and only if $\{x, y\} \in E(H)$. When possible, we will denote an edge $\{x_1, x_2, \ldots, x_r\}$ by $x_1x_2 \ldots x_r$ for short. When we do not specify a hypergraph to be r-uniform in this paper, we will assume it to be 3-uniform.

A *friendship graph* is a graph in which every pair of vertices has a unique common neighbor. The Friendship Theorem states that if G is a friendship graph, then there exists a single vertex joined to all others. Also, friendship graphs exist only for odd number of vertices, and they are unique in the sense that the graphs consisting of $(n-1)/2$ triangles joined at a single vertex, so called windmill graphs, are the only type of friendship graphs, which was proved by Erdős, Rényi and Sós in 1966, see [1].

In this paper we will consider a known generalization of the friendship graphs, which concerns 3-uniform hypergraphs. We say that a 3-uniform hypergraph is a *3-uniform friendship hypergraph*, if it satisfies the *friendship property* that

Definition C.1 (Friendship Property)**.** *For every three vertices x, y and z, there exists a unique vertex w such that xyw, xzw and yzw are edges in the hypergraph.*

In the remaining part of this paper, we will denote such a w as the *completion* of x, y, z. Sós was the first one to consider this generalization in 1976, see [2]. She actually just considered 3-uniform hypergraphs with edge set $\{v_i, v_j, v_n\}$ for all $1 \leq i < j < n$ and a Steiner triple system on the vertices $\{v_1, v_2, \ldots, v_{n-1}\}$ and observed that they satisfy the friendship property. The vertex v_n will be denoted as a *universal friend.* As Steiner triple systems $S(2, 3, n-1)$ are known to exist if and only if $n \equiv 2 \mod 6$ or $n \equiv 4 \mod 6$, we see that these are the only orders for which there exist friendship hypergraphs with a universal friend.

Sós then asked whether there exist other 3-uniform hypergraphs satisfying the friendship property other than the ones with a universal friend. This was answered by Hartke and Vandenbussche in 2008, see [3], where they used integer programming to prove that for $n = 8, 16$ and 32 there exist hypergraphs satisfying the friendship property without containing a universal friend. The integer programming also showed that for $n \leq 10$ and $n \neq 8$ the only 3-uniform friendship hypergraphs are the hypergraphs with a universal friend. They also succeeded in showing that the 3-uniform friendship hypergraph with $n = 8$ vertices they constructed is the only one of its kind without a universal friend. For $n = 16$ they showed there are at least three nonisomorphic constructions.

All friendship hypergraphs can be characterized by using complete 3-uniform hypergraphs on four vertices. Such a complete 3-uniform hypergraph on four vertices we will denote as a *quad.* We see why we can use quads to describe the friendship hypergraphs instead of edges in the following lemma, for which we include the proof from [3].

Lemma C.2. *[3] The following is true for every 3-uniform friendship hypergraph $\mathcal{H}$.*

(a) Every pair of vertices appears in at least one edge together.

(b) Every edge must be contained in a unique quad.

Proof. (a) Let $x, y \in \mathcal{V}(\mathcal{H})$ and $z \neq x, y$. Then the triple x, y, z has a completion w such that xyw, xzw and yzw are edges in $\mathcal{H}$. Hence x, y are in an edge together.

(b) Let $xyz \in \mathcal{E}(\mathcal{H})$, then the triple x, y, z has a completion w such that xyw, xzw and yzw are also edges in $\mathcal{H}$. These four edges form a quad. The uniqueness of the quad follows from the uniqueness of the completion w.

□

Observation (b) implies that the edges of the friendship hypergraph can be partitioned into quads. This allows one to solely use quads to describe the friendship hypergraph, as the quad structure tells us everything about the edge structure. As the number of edges are at most $\binom{n}{3}$, this also gives an upper bound on the number of quads in a friendship hypergraph as $\binom{n}{3}/4$. A lower bound on the number of edges was proved to be $n(n-2)/2$ in [3].

With the exception of one of the friendship hypergraphs on 16 vertices, all of the friendship hypergraphs found in [3] also satisfy three properties, which were actually used in describing the IP and later relaxed to try to obtain more friendship hypergraphs. The computation time was greatly improved by having specified these properties, which are the *inductive, pair and automorphism property.* The friendship hypergraphs on $n = 16$ and 32 vertices satisfy that they contain two disjoint copies of a friendship hypergraph on $n/2$ vertices, also called the inductive property. Except for one of the friendship hypergraphs on 16 vertices, they also satisfy the pair property, which is that the vertices can be divided into pairs, such that each pair appears in a quad with each other pair. Finally if we view the vertices as binary $\log(n)$-tuples, then any map that flips a fixed subset of the $\log(n)$ bits is an automorphism, hence the friendship hypergraph satisfies the automorphism property. Note that this means that the friendship hypergraph is regular and vertex-transitive and also that the number of quads is a multiple of $n/4$. Hartke and Vandenbussche [3] conjectured that for all positive integers $k \geq 4$ friendship hypergraphs with $n = 2^k$ vertices satisfying the three properties exist.

Hartke and Vandenbussche's results were improved upon by Li, van Rees, Seo and Singhi in [4], where it was stated that no 3-uniform friendship hypergraph on 11 or 12 vertices exists. They also showed that the three friendship hypergraphs on 16 vertices which where found by Hartke and Vandenbussche are the only friendship hypergraphs on 16 vertices which satisfy that the vertex-set can be partitioned into groups of disjoint quads and for which the corresponding friendship design can be embedded into an affine geometry. The lower bound on the number of edges were also improved, as they showed that if n is odd, then there are at least (roughly) $2n^2/3$ edges and if n is even there are at least $n^2/2$ edges. Also the upper bound was improved, to obtain that there are at most $\binom{n}{3}(2n-6)/(3n-10)$ edges.

In this paper we first construct an infinite family of 3-uniform hypergraphs given in the following definition.

Definition C.3. *Let $k \geq 2$ and $H = (V, E)$ be a hypercube on $n = 2^k$ vertices, where the vertices are labelled with the k-bit binary strings from 0 to $2^k - 1$, such that two neighboring vertices differs in exactly one bit. Then we define* ***the cubeconstructed hypergraph*** *$\mathcal{H}$ as the 3-uniform hypergraph with vertex-set V and with the triple xyz in the edge-set if and only if $dist_H(x, y) + dist_H(x, z) + dist_H(y, z) = 2k$.*

In Figure C.1 we see the hypercube on $8 = 2^3$ vertices and in Table C.1 the quads of the corresponding cubeconstructed hypergraph on 8 vertices.

In Section 2 we show that the cubeconstructed hypergraphs are in fact 3-uniform friendship hypergraphs without a universal friend on $n = 2^k$ vertices for all $k \geq 2$, and that they satisfy the conjecture of [3] for all $k \geq 4$. We also show that they have $2^{k-1}(3^{k-1} - 1)$ edges.

Furthermore, in Section 3 we construct friendship hypergraphs on 20 and on 28 vertices, hence disprove a conjecture of [4] saying that all friendship hypergraphs without a universal friend must be on 2^k vertices.

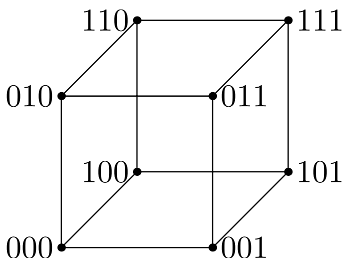

Figure C.1: The hypercube of dimension $k = 3$ with the described labels.

{000,001,110,111}
{000,010,101,111}
{000,100,011,111}
{000,110,101,011}
{010,100,101,011}
{010,001,110,101}
{001,100,011,110}
{001,100,010,111}

Table C.1: Quads in the cubeconstructed hypergraph on 8 vertices.

In Section 4 we generalize the concept of friendship graphs and 3-uniform friendship hypergraphs to r-uniform friendship hypergraphs and observe that the existence of those with a universal friend are dependent on the existence of a Steiner system $S(r-1, r, n-1)$ similar to the case of the 3-universal friendship hypergraphs. Finally we construct a 4-uniform friendship hypergraph on 9 vertices using the Steiner system $S(5, 6, 12)$.

2 3-Uniform Friendship Hypergraphs on 2^k vertices

Theorem C.4. *The cubeconstructed hypergraphs are* 3*-uniform friendship hypergraphs.*

Proof. To prove the theorem, we only need to prove that the friendship property is satisfied in the cubeconstructed hypergraphs, hence that for all three vertices x, y and z exists a unique vertex w such that

$$\begin{aligned} dist_H(x,y) + dist_H(x,w) + dist_H(y,w) = 2k,\\ dist_H(x,z) + dist_H(x,w) + dist_H(z,w) = 2k,\\ dist_H(y,z) + dist_H(y,w) + dist_H(z,w) = 2k. \end{aligned} \tag{C.1}$$

Due to vertex-transitivity of H, we only need to consider sets containing the vertex $0\dots0$. So let $x = 0\dots0$ and let y and z be two arbitrary vertices in H. Let a, b, c and

d be the non-negative integers such that there are a bits where y has a 1 and z has a 0, b bits where they both have a 1, c bits where y has a 0 and z has a 1 and finally d bits where they both have 0. Without loss of generality, we can assume x, y and z to be as in (C.2), as the distribution of the corresponding bits in y and z are irrelevant.

$$\begin{aligned} x &= 0\ldots0\,0\ldots0\,0\ldots0\,0\ldots0, \\ y &= 1\ldots1\,1\ldots1\,0\ldots0\,0\ldots0, \\ z &= \underbrace{0\ldots0}_{a}\underbrace{1\ldots1}_{b}\underbrace{1\ldots1}_{c}\underbrace{0\ldots0}_{d}. \end{aligned} \tag{C.2}$$

Now let r, s, t and u be non-negative integers such that w has r bits of value 1 among the first a bits, s bits of value 1 among the next b bits, t bits of value 1 among the following c bits and finally u bits of value 1 among the last d bits, hence w consists of $r+s+t+u$ bits of value 1 and $k-(r+s+t+u)$ bits of value 0.

As we wish to determine w, we need to solve (C.1) with respect to r, s, t and u.

Given the method we used to label the vertices in the hypercube, we get

$$\begin{aligned} dist_H(x,y) &= a+b, \\ dist_H(x,z) &= b+c, \\ dist_H(y,z) &= a+c, \\ dist_H(x,w) &= r+s+t+u, \\ dist_H(y,w) &= a-r+b-s+t+u \end{aligned}$$

and

$$dist_H(z,w) = r+b-s+c-t+u$$

and hence, we get from (C.1) that

$$\begin{aligned} a+b+r+s+t+u+a-r+b-s+t&+u \\ &= 2a+2b+2t+2u = 2k, \\ b+c+r+s+t+u+r+b-s+c-t&+u \\ &= 2b+2c+2r+2u = 2k, \\ a+c+a-r+b-s+t+u+r+b-s+c-t&+u \\ &= 2a+2b+2c-2s+2u = 2k. \end{aligned}$$

Clearly $k = a+b+c+d$, so from the above we get

$$\begin{aligned} t+u &= c+d, \\ r+u &= a+d, \\ -s+u &= d \end{aligned}$$

and as $r \leq a$, $s \geq 0$, $t \leq c$ and $u \leq d$, the unique solution is $r = a$, $s = 0$, $t = c$ and $u = d$.

So

$$w = \underbrace{1 \ldots 1}_{a}\underbrace{0 \ldots 0}_{b}\underbrace{1 \ldots 1}_{c}\underbrace{1 \ldots 1}_{d}$$

is the unique solution to (C.1) we were looking for, and hence the cubeconstructed hypergraphs satisfy the friendship property. □

We wish to determine the number of edges in a cubeconstructed hypergraph, but before doing so, we need to make the following observations.

Lemma C.5. *Let x, y and z be vertices in a cubeconstructed hypergraph $\mathcal{H}$. Then xyz is an edge of $\mathcal{H}$ if and only if in each bit exactly two of the three vertices share the same value.*

Proof. First, assume xyz is an edge, then we know

$$dist_H(x, y) + dist_H(x, z) + dist_H(y, z) = 2k. \tag{C.3}$$

As stated in Lemma C.2 each pair of vertices appears in at least one edge together, so without loss of generality, we can assume that x and y are as below, with $0 \leq a, b, c, d \leq k$,

$$\begin{aligned} x &= 1 \ldots 1\, 1 \ldots 1 0 \ldots 0 0 \ldots 0 \\ y &= \underbrace{0 \ldots 0}_{a}\underbrace{1 \ldots 1}_{b}\underbrace{1 \ldots 1}_{c}\underbrace{0 \ldots 0}_{d} \end{aligned}$$

which means $dist_H(x, y) = a + c$. Now, no matter which value we have for z in each of the bits corresponding to the $a + c$ bits in x and y, we will get a contribution of exactly $a + c$ to the left-hand-side of (C.3), as z in each bit differs either from x or y. We are now $2k - 2(a + c) = 2b + 2d$ short of satisfying (C.3), and the only possibility to obtain this, is if z differs from both x and y in the corresponding $b + d$ bits, hence in the bits where x and y have the same value, z has to be the other value, proving the implication.

Now assume x, y and z are vertices such that in each bit exactly two of them have the same value. Thus there is a contribution of two for each bit to $dist_H(x, y) + dist_H(x, z) + dist_H(y, z)$, hence a total of $2k$ is obtained, which proves that xyz is an edge. □

From Lemma C.5 we get the following Corollary.

Corollary C.6. *The four vertices x, y, z and w form a quad in the cubeconstructed hypergraph if and only if in each bit exactly two of the four vertices have a* 0 *and the two others have a* 1*.*

Lemma C.7. *Each pair x, y of vertices with $dist_H(x, y) = i$ is in exactly 2^i edges if $i < k$ and in exactly $2^k - 2$ edges if $i = k$.*

Proof. Without loss of generality, let x and y be as in the proof of Lemma C.5, so $i = a + c$. If $i < k$ we can choose a z such that xyz is an edge in 2^i ways according to the proof of Lemma C.5. If $i = k$ then the only vertices we cannot choose as z are x and y, hence there are $2^k - 2$ vertices to choose from. □

Theorem C.8. *The number of edges in the cubeconstructed hypergraph on $n = 2^k$ vertices is $2^{k-1}(3^{k-1} - 1)$.*

Proof. Due to Lemma C.7 we just need to calculate how many pairs of distance i there are in H.

First assume $i = k$, then for each vertex, there is exactly one vertex with distance k, hence the total number of pairs with distance k is $2^k/2 = 2^{k-1}$. Now let $i < k$, then every pair with distance i has exactly i bits where they differ in value, and $k - i$ bits where they are equal in value. In each of these bits we have two choices, as in the i bits where they differ, we can choose which one of the vertices should have a 1 and in the bits where they are equal, can choose if they should be both 1 or both 0. Also the placement of the i bits where they differ can be done in $\binom{k}{i}$ ways, so in total we get $2^i 2^{k-i}\binom{k}{i}/2 = 2^{k-1}\binom{k}{i}$ pairs with distance i, as we have divided by 2 to avoid counting the pairs twice.

Combining this with Lemma C.7 and the fact that we count an edge three times, once for each of the three pairs in the edge, we get the total number of edges in the hypergraph as

$$\frac{1}{3}\left(\sum_{i=1}^{k-1} 2^{k-1}2^i\binom{k}{i} + 2^{k-1}(2^k - 2)\right) = 2^{k-1}(3^{k-1} - 1). \tag{C.4}$$

□

As each edge is contained in a unique quad (Lemma C.2) and there are four edges in each quad, we obtain the following corollary.

Corollary C.9. *The number of quads in the cubeconstructed hypergraph on $n = 2^k$ vertices is $2^{k-3}(3^{k-1} - 1)$.*

The next theorem states, that if we know the cubeconstructed hypergraph on 8 vertices, then the remaining cubeconstructed hypergraphs can be constructed inductively.

Theorem C.10. *Let $k > 3$, then the cubeconstructed hypergraph on 2^k vertices is isomorphic to the union of $k^2 - k$ copies of a cubeconstructed hypergraph on 2^{k-1} vertices.*

Proof. First, let H be a hypercube of dimension $k - 1$. Then we define a new labeling on H with k-bit binary labels from the old labeling of H to obtain a copy of H which we denote by H', and we show there are $k^2 - k$ different labelings of this kind. We then show that from this H' we get a hypergraph from the cubeconstructed hypergraph on

2^{k-1} vertices, which is in fact a subhypergraph of the cubeconstructed hypergraph on 2^k vertices.

In the vertices of H we know that in each bit, half of the vertices have a 0 and the other half have a 1. We split the vertex set into two equal size parts by fixing a bit and letting H_1 be the vertices with a 0 in this bit and H_2 be the vertices with a 1 in this bit. For example, if $k = 4$, then if we fix the second bit, we get a division into $H_1 = \{000, 001, 100, 101\}$ and $H_2 = \{111, 110, 011, 010\}$. We can choose the bit in $k - 1$ ways. Notice, that according to Lemma C.5 all edges in the corresponding cubeconstructed hypergraph will have at most two vertices in H_1 and H_2 respectively. Now we add an extra bit to each vertex in H, to obtain H' with H'_1 and H'_2 such that the value of the bit is distinct in H'_1 and H'_2 respectively. Using the example from before, and placing the new bit between the second and third bit, we get, by letting the value of the new bit be 1 in H_1 and 0 in H_2, that $H_1 = \{0010, 0011, 1010, 1011\}$ and $H_2 = \{1101, 1100, 0101, 0100\}$. Choosing which position to place the new bit in, can be done in k ways, and choosing the value can be done in two different ways. Then the vertices in H' correspond to half of the vertices in the hypercube of dimension k, namely two disjoint subcubes of dimension $k - 2$ where one is just the vertices in the other with all bits flipped. Going through every possible choice of the bit that splits H and added bit, we obtain the same H' two times, as in the new labeling we cannot distinguish between whether a bit has been chosen in the splitting of H or whether it has been added to obtain H'. Therefore we get that there are a total of $(k-1)k = k^2 - k$ copies of a H' in the hypercube of dimension k.

Let x, y, z be vertices in H. If xyz is an edge in the corresponding cubeconstructed hypergraph, we know

$$dist_H(x, y) + dist_H(x, z) + dist_H(y, z) = 2(k - 1)$$

and that at most two of x, y and z are in H_1 and H_2 respectively. Hence according to the construction of H' above, we add a bit to x, y and z to obtain x', y' and z', such that one, let's say x', differs from the others, which is then y' and z'. Hence in the hypercube of dimension k we will get

$$\begin{aligned}
dist_{H'}(x', y') &+ dist_{H'}(x', z') + dist_{H'}(y', z') \\
&= dist_H(x, y) + 1 + dist_H(x, z) + 1 + dist_H(y, z) \\
&= 2(k - 1) + 2 \\
&= 2k.
\end{aligned}$$

Thus $x'y'z'$ is an edge in the cubeconstructed hypergraph with 2^k vertices. Similarly, if xyz is not an edge, we know

$$dist_H(x, y) + dist_H(x, z) + dist_H(y, z) < 2(k - 1),$$

and we get the largest addition to the sum of the distances, when one is in say H_1' and the two others in H_2'. So in total

$$\begin{aligned} dist_{H'}(x', y') + dist_{H'}(x', z') + dist_{H'}(y', z') \\ &\leq dist_H(x, y) + dist_H(x, z) + dist_H(y, z) + 2 \\ &< 2(k-1) + 2 \\ &= 2k \end{aligned}$$

and hence $x'y'z'$ is not an edge in the cubeconstructed hypergraph with 2^k vertices.

So clearly, the $k^2 - k$ copies of the cubeconstructed hypergraphs on 2^{k-1} vertices are isomorphic to a subhypergraph of the cubeconstructed hypergraph on 2^k vertices.

Now assume we have a cubeconstructed hypergraph on 2^k vertices, and let xyz be an edge therein. Then we wish to prove that xyz corresponds to an edge in a cubeconstructed hypergraph with 2^{k-1} vertices as the ones above. First we see that there are at least two of the three vertices which are with distance no more than $k - 2$ from each other, otherwise we would have

$$\begin{aligned} 2k &= dist_H(x, y) + dist_H(x, z) + dist_H(y, z) \\ &\geq 3(k-1) \\ &= 3k - 3 \end{aligned}$$

a contradiction as $k > 3$. Let's assume $dist_H(x, y) \leq k - 2$, this means x and y share at least two bits, which determines whether they are in a copy of some H_1 or H_2 as given above, let us say without loss of generality they are in a copy of H_1. As xyz is an edge, we have according to Lemma C.5 that z must differ in these two bits, which means z must be in a copy of the corresponding H_2.

So the cubeconstructed hypergraph on 2^k vertices is isomorphic to a subhypergraph of the $k^2 - k$ copies of a cubeconstructed hypergraphs on 2^{k-1} vertices, proving the theorem. □

Notice that the friendship hypergraph on 8 vertices and one of the ones on 16 vertices ($\mathcal{F}_1^{16}$ found in [3]) are actually just cubeconstructed hypergraphs. The cubeconstructed hypergraph on 32 vertices and 320 quads is however not isomorphic to the friendship hypergraph on 32 vertices found in [3], as this contains 344 quads.

The following corollary gives an affirmation of the conjecture from [3].

Corollary C.11. *The cubeconstructed hypergraphs satisfy the inductive, pair and automorphism property.*

Proof. According to Theorem C.10 and [3] the inductive property is satisfied. Also the automorphism property is satisfied, due to the hypercube being vertex-transitive.

Regarding the pair property, we see that the division of the vertex set of the cubeconstructed hypercube on 2^k vertices into pairs where each pair contains vertices with distance k, results in this property being satisfied as well. □

3 Other friendship hypergraphs

In [4] it was conjectured that no other friendship hypergraphs than the ones with a universal friend and the ones on 2^k vertices exist. Our next theorem will show, that this is not true. But first we state a necessity for a friendship hypergraph to be vertex transitive.

Lemma C.12. *Let $\mathcal{H}$ be a friendship hypergraph on n vertices which is vertex-transitive. Then $(n-1)$ or $(n-2)$ must be divisible by 3.*

Proof. Due to the friendship property, we know that for each set of three vertices, there exists a unique completion of these. As $\mathcal{H}$ is vertex-transitive, we know that each vertex must be a completion to such a set, the same number of times.

Hence this number is given by

$$\frac{\binom{n}{3}}{n} = \frac{(n-1)(n-2)}{3 \cdot 2},$$

and as 2 divides either $n-1$ or $n-2$, 3 must also be a prime divisor of one of them. □

Theorem C.13. *There exist at least three non-isomorphic friendship hypergraphs on 20 vertices and 420 edges and at least one friendship hypergraph on 28 vertices and 1036 edges.*

Proof. Let the n vertices be represented by the elements in $\mathbb{Z}_n$. The constructions arise from some fixed quads $\{a, b, c, d\}$ where $a, b, c, d \in \mathbb{Z}_n$ and given these, all the quads of the type $\{a, b, c, d\} + i = \{a + i \mod n, b + i \mod n, c + i \mod n, d + i \mod n\}$ where $i \in \mathbb{Z}_n$.

By an exhaustive computer search among all cyclic hypergraphs (i.e., hypergraphs consisting of all cyclic shifts of a number of fixed quads), we have found three non-isomorphic friendship hypergraphs on $n = 20$ vertices, all of which contain the following five fixed quads:

$$\{0, 1, 10, 11\}, \{0, 2, 10, 12\}, \{0, 3, 10, 13\}, \{0, 4, 10, 14\}, \{0, 5, 10, 15\}.$$

Except for $\{0, 5, 10, 15\}$ which represents a total of 5 quads, all these fixed quads represent 10 quads each.

The remaining fixed quads are specific to the different friendship hypergraphs as given below:

a) $\{0, 1, 3, 14\}, \{0, 1, 9, 15\}, \{0, 2, 4, 7\}$,

b) $\{0, 1, 4, 13\}, \{0, 1, 6, 12\}, \{0, 2, 4, 7\}$,

c) $\{0, 1, 4, 13\}, \{0, 1, 9, 15\}, \{0, 2, 4, 17\}$.

These fixed quads in a)-c) all represent 20 quads each, so all three hypergraphs contain a total of 105 quads. Hence, they all contain 420 edges.

For $n = 28$ we also found a friendship hypergraph using computer search. A search criteria among all cyclic hypergraphs was that it had to contain seven fixed quads which share similarities with the five fixed quads in all the found friendship hypergraphs on 20 vertices. The seven fixed quads are as follows:

$$\{0,1,14,15\}, \{0,2,14,16\}, \{0,3,14,17\}, \{0,4,14,18\},$$
$$\{0,5,14,19\}, \{0,6,14,20\}, \{0,7,14,21\}.$$

Expect for {0,7,14,21} which represents 7 quads, they all represent 14 quads each.

Furthermore the found hypergraph contains the six fixed quads

$$\{0,1,4,17\}, \{0,1,5,20\}, \{0,1,7,13\}, \{0,2,4,23\}, \{0,3,8,19\}, \{0,3,10,20\},$$

which all represent 28 quads each. Hence a total of 259 quads and 1036 edges. □

Inspired by Theorem C.13 and the other known vertex-transitive friendship hypergraphs presented in this paper, we conjecture the following.

Conjecture C.14. *For all n which is divisible by 4 and not divisible by 3, there exists a vertex-transitive friendship hypergraph.*

4 r-uniform friendship hypergraphs

In this section we will give another generalization of the friendship graphs, the so called *r-uniform friendship hypergraphs* for $r \geq 2$, which satisfy the following property.

Definition C.15 (Friendship property for r-uniform hypergraphs)**.** *For every r vertices $x_1, x_2, \ldots, x_r$, there exists a unique vertex w such that*

$$\{x_1, x_2, \ldots, x_r, w\} - \{x_i\}$$

is an edge in the hypergraph for all $i = 1, 2, \ldots, r$.

Notice that for $r = 2$ the above definition corresponds to that of friendship graphs and for $r = 3$ it corresponds to Definition C.1. Similar to before, we will denote a r-uniform hypergraph which satisfies the friendship property for r-uniform hypergraphs as a *r-uniform friendship hypergraph.* Also w will be denoted as the *completion* of $x_1, x_2, \ldots, x_r$.

Similarly to Lemma C.2, we have the following observations for r-uniform friendship hypergraphs.

Lemma C.16. *The following is true for every r-uniform friendship hypergraph $\mathcal{H}$.*

(a) *Every set of at most $r-1$ vertices appears in at least one edge together.*

(b) *Every edge must be contained in a unique complete r-uniform hypergraph on $r+1$ vertices.*

The proof is similar to that of Lemma C.2.

We also observe that a friendship hypergraph on n vertices with a universal friend exists, if and only if a Steiner system $S(r-1, r, n-1)$ exists, as it has edges

$$\{v_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}, v_n\}$$

for all $1 \leq i_1 < i_2 < \ldots < i_{r-1} < n$ and the remaining edges described by a Steiner system on $\{v_1, v_2, \ldots, v_{n-1}\}$. From this we know it has

$$\binom{n-1}{r-1} + \frac{\binom{n-1}{r-1}}{\binom{r}{r-1}} = (1 + \frac{1}{r})\binom{n-1}{r-1}$$

edges.

The only $r \geq 4$, for which we know in general that Steiner systems exist, is $r = 4$ and the Steiner systems $S(3, 4, n-1)$ are referred to as Steiner quadruple systems. They exist if and only if $n \equiv 3 \mod 6$ or $n \equiv 5 \mod 6$. For all other values of r we only know a finite number of Steiner systems, see [5] for an overview of which Steiner systems are known, and some of the properties of Steiner systems.

The last r-uniform friendship hypergraph we will present in this paper, is the one given in the following definition.

Definition C.17 (4-uniform hypergraph on 9 vertices)**.** *Let a, b, c be three of the elements in the Steiner system $S(5, 6, 12)$ and let the remaining 9 elements represent the vertex set V in a 4-uniform hypergraph. The quadruple $v_1v_2v_3v_4$ is an edge in the 4-uniform hypergraph if and only if $\{a, v_1, v_2, v_3, v_4, x\} \in S(5, 6, 12)$ for some $x \in V$.*

In the next lemma, we will use the block intersection numbers $\lambda_{i,j}$ of a Steiner system $S(t, k, v)$ which, given two disjoint sets I of size i and J of size j, determines the number of blocks which contain I but do not intersect J. Due to the properties of Steiner systems, this number only depends on i and j, and hence we can calculate any $\lambda_{i,j}$ from the following:

$$\lambda_{i,0} = \begin{cases} \binom{v-i}{t-i}/\binom{k-i}{t-i} & \text{for } 0 \leq i \leq t \\ 1 & \text{for } t < i \leq k \end{cases}$$
$$\lambda_{i,j} = \lambda_{i,j-1} - \lambda_{i+1,j-1}.$$

Lemma C.18. *The 4-uniform hypergraph given in Definition C.17 has 90 edges*

Proof. For each 6-block in $S(5,6,12)$ which contains a and neither b or c, we get a total of 5 edges due to the definition of the 4-uniform hypergraph. So we wish to determine this number of blocks, which can be done by using the block intersection number, $\lambda_{1,2}$. According to the above we get

$$\begin{aligned}\lambda_{1,2} &= \lambda_{1,0} - 2\lambda_{2,0} + \lambda_{3,0} \\ &= \frac{\binom{11}{4}}{5} - 2 \cdot \frac{\binom{10}{3}}{4} + \frac{\binom{9}{2}}{3} \\ &= 18.\end{aligned}$$

So the number of edges in the hypergraph is $5 \cdot 18 = 90$. □

Theorem C.19. *The hypergraph given in Definition C.17 is a* 4*-uniform friendship hypergraph without a universal friend.*

Proof. Due to the fact that the 4-uniform friendship hypergraph with a universal friend on 9 vertices has $(1 + 1/4)\binom{8}{3} = 70$ edges and the 4-uniform hypergraph given in Definition C.17 has 90 edges, the one from Definition C.17 cannot have a universal friend.

It remains to show that the friendship property is in fact satisfied in the 4-uniform hypergraph given in Definition C.17, hence that for all quadruples v_1, v_2, v_3, v_4 there exists a unique vertex w (the completion) such that $\{a, v_1, v_2, v_3, w, x_4\}$, $\{a, v_1, v_2, v_4, w, x_3\}$, $\{a, v_1, v_3, v_4, w, x_2\}$ and
$\{a, v_2, v_3, v_4, w, x_4\} \in S(5,6,12)$ for some $x_i \in V$ for all $i = 1,2,3,4$.

As we know $S(5,6,12)$ is a Steiner system, we know that there exists some unique element x such that $\{a, v_1, v_2, v_3, v_4, x\} \in S(5,6,12)$. If $x \in V$, then this x is the unique completion of v_1, v_2, v_3, v_4. Assume the contrary, thus there exist another vertex $x' \neq x$ with $x' \in V$ and vertices $z_i \neq x$, $z_i \in V$ for $i = 1,2,3,4$ such that

$$\begin{aligned}\{a, v_1, v_2, v_3, x', z_4\}, &\{a, v_1, v_2, v_4, x', z_3\}, \{a, v_1, v_3, v_4, x', z_2\}, \\ &\{a, v_2, v_3, v_4, x', z_1\} \in S(5,6,12)\end{aligned}$$

where $z_i \neq z_j$ for $i \neq j$. Thus we have at least 10 vertices in V, a contradiction.

Now assume $x \notin V$, without loss of generality we can assume that $x = b$, so $\{a, b, v_1, v_2, v_3, v_4\} \in S(5,6,12)$. Now we know that $\{a, c, v_1, v_2, v_3, y_4\} \in S(5,6,12)$ for some element y_4, and we see that $y_4 \neq b$ as otherwise it would be a contradiction to $S(5,6,12)$ being a Steiner system. So we must have $y_4 \in V$. Similarly we see that

$$\{a, c, v_1, v_2, v_4, y_3\}, \{a, c, v_1, v_3, v_4, y_2\}, \{a, c, v_2, v_3, v_4, y_1\} \in S(5,6,12)$$

where $y_i \neq y_j$ for $i \neq j$ and $y_i \neq b$ for $i = 1,2,3$. Notice that y_1 cannot be the completion of v_1, v_2, v_3, v_4, as this would imply $\{a, x, v_2, v_3, v_4, y_1\} \in S(5,6,12)$ for some $x \in V$, a contradiction as $\{a, c, v_2, v_3, v_4, y_1\} \in S(5,6,12)$. Similarly we see that y_2, y_3 and y_4 cannot be the completion.

Now we have a unique vertex $w \in V$ given by $w \neq y_i, v_i$ for all $i = 1, 2, 3, 4$ which satisfies $\{a, v_1, v_2, v_3, w, x_4\}$, $\{a, v_1, v_2, v_4, w, x_3\}$, $\{a, v_1, v_3, v_4, w, x_2\}$ and $\{a, v_2, v_3, v_4, w, x_4\} \in S(5, 6, 12)$ for some $x_i \in V$ for all $i = 1, 2, 3, 4$. So this w is the completion, hence proving the friendship property for a 4-uniform hypergraph is satisfied. $\square$

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