EUCLID'S PARALLEL POSTULATE.*

BY OSWALD VEBLEN.

MATHEMATICIANS are in possession of several bodies of theory which they call geometries. A geometry (and, indeed, a mathematical science in general) is a set of propositions stated in terms of symbols some of which are defined in terms of others, but some of which are necessarily undefined. The majority of the propositions (those called theorems) are logical consequences of other propositions, but some of the propositions are necessarily unproved. The latter are called axioms or postulates or, more plainly, *unproved propositions*. In its mathematical aspect, a geometry is rather completely characterized by its undefined symbols and its unproved propositions since all other features of the science are derived from these by the two processes of definition and deduction.

Geometries might have, but actually have not, been created in an accidental or artificial manner. The symbols (in particular the undefined symbols) of geometry stand for the words that we use in describing that complex of sensations, perceptions, etc., called space, and its propositions are statements which one makes (or may make if learned enough) about space. Thus there are two questions which may be asked about a geometrical proposition: (1) Is it an axiom or a consequence of the axioms of a certain geometry? (2) Is it true of space? The first of these questions is strictly mathematical. The second belongs perhaps to mathematics, perhaps to natural science, but probably to philosophy. The two questions were formerly jumbled into one and it is only in recent years that the mathematicians have fully separated them.

For a long time, there existed only one geometry, that of Euclid, and this geometry because of its uniqueness occupied a post of peculiar sanctity. Its propositions were not only held to be true of space,

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but they were supposed by many (e. g. Kant) to be necessary laws of thought. In the last century, however, there appeared on the scene first one, and then many, geometries which contained propositions different from those of Euclid. These geometries are in the first place so logically consistent that if one of them contains a self-contradiction, so does Euclid, and in the second place certain of them, notably those of Lobatchewsky and Riemann, have claims to truth that rival those of Euclid.

The philosophical importance of a theory which, on the face of the returns, seems to destroy Kant's main example of an *a priori* synthetic judgment will hardly be questioned. But on account of the difficulty of the technical language of the philosophers for the mathematicians and *vice versa,* the subject has not yet had an adequate discussion.

Mr. Withers is one of the first who comes to the subject as a philosopher and yet is in possession of the necessary mathematics. His book, which is a Yale Doctor's Thesis, begins with a history of the mathematical researches that is probably clearer than any available to non-mathematicians in English. It does not contain a complete account of the corresponding philosophical discussions—an omission which probably makes for clearness since many of the discussions were beclouded by misunderstandings between the mathematicians and philosophers.

The historical introduction is followed by a couple of chapters which, waiving for a moment the notion that no thought is possible which does not presuppose a Euclidean space, discuss the claims of the geometries of Euclid, Lobatchewsky, and Riemann to validity as exponents of our geometrical experience. Mr. Withers reaches the conclusion, familiar to mathematicians, that we cannot at present decide; that a decision against Euclid is possible; that one absolutely in his favor probably is not. In the discussion leading to this result, by some remarks on the empirical origin and the psychology of certain conceptions like that of direction he successfully disposes of several of the usual errors.

On the other hand, a mathematician is pretty sure to feel the need of a few more "ifs" and "buts." For example, on pages 106-107 where the author very clearly exposes the "shortest distance" fallacy, he ought also to note that distance can be defined analytically so as to avoid the difficulty. Without citing further instances we will assert that throughout the book there are statements uttered directly that a mathematician would prefer to see qualified. We will not deny, however, that for the purpose of conveying the right emphasis the methods of Mr. Withers may be better than the attempt at literal accuracy of a mathematician.

There are places where Mr. Withers seems to overlook temporarily the nature of an abstract science. For example, he regards it as a difficulty (page 112) that Pieri should use undefined symbols and unproved propositions which involve metrical ideas in making a definition of metrical terms; and of Riemann he says (pp. 112, 113): "In other words by assuming metrical properties in his *ds* and then proceeding to determine these properties upon the basis of this assumption, he easily draws out at the faucet what he has already poured in at the bung." But this is what we always do in mathematics. In geometry no more than elsewhere do we expect to get something for nothing. The axioms of a science must necessarily involve the whole structure. We never expect to *generate* anything by a logical process. By mathematical language we can never tell the meaning, say of a straight line, (cf. Chap. IV), in any other sense than that we utter a set of propositions, logically related and including the statements that can be made about straight lines.

It seems that by being more explicit in his statements about abstract science in general, Mr. Withers might have considerably abbreviated and improved his statements about curvature of space and the necessity or lack of necessity of assuming a Euclidean space of higher dimensions in order to realize a space of constant positive or negative curvature. Presumably for a like reason, the discussion of Peano's work on pages 107-108 seems to confuse two separate studies in one of which "distance" was the undefined symbol and in the other of which the notion of "betweenness" was fundamental.*

After having shown that Euclid's geometry cannot be proved true by any appeal to experience, Mr. Withers decides in the last two chapters that there is no way of accomplishing this result by an *a priori* method. We have remarked above on the details of this argument and here raise only one further question—perhaps without putting it in a clear-cut form. How shall we use the word exist? There is a technical usage which says that a mathematical science (cf. our first paragraphs) exists if no two propositions deducible from its hypotheses are in contradiction. In this sense (due to

* We note in passing that the second footnote reference on page 108 is incorrect; that in the bibliography under the single head, Moore, appear works of two men, one an American and the other an Englishman; that on page 96, line 7, the word "of" should be deleted; that on page 142, "motion" is printed for "notion."

Hilbert) we are able to say that all mathematical sciences exist if arithmetic exists—i. e., the science of the positive whole numbers. One is tempted to say that surely the whole numbers, 1, 2, 3....etc. exist. But what would be the content of such a statement? and do we know these numbers except by the propositions which we wish to prove consistent?

A more difficult form of the same question would be to ask what Mr. Withers means by such language as this: "....nor is it maintained that a merely formal world could really exist or be truly known if it did exist" (page 147). Or the following from pages 160-161: "We cannot in any *a priori* fashion dogmatically deny the existence of a four-dimensional space-world any more than our two-dimensional beings could deny that our world exists." Altogether the discussion in Mr. Withers' last chapter is obscured by the lack of a satisfactory meaning for the word "exist."

We have taken pains to warn the reader not to accept all the statements of Mr. Withers as representing a mathematical point of view with strict accuracy because we believe that the book, on account of its general clearness, ought to have a wide circle of readers. It might well be read as an introduction to the large work of Russell on the *Principles of Mathematics.*