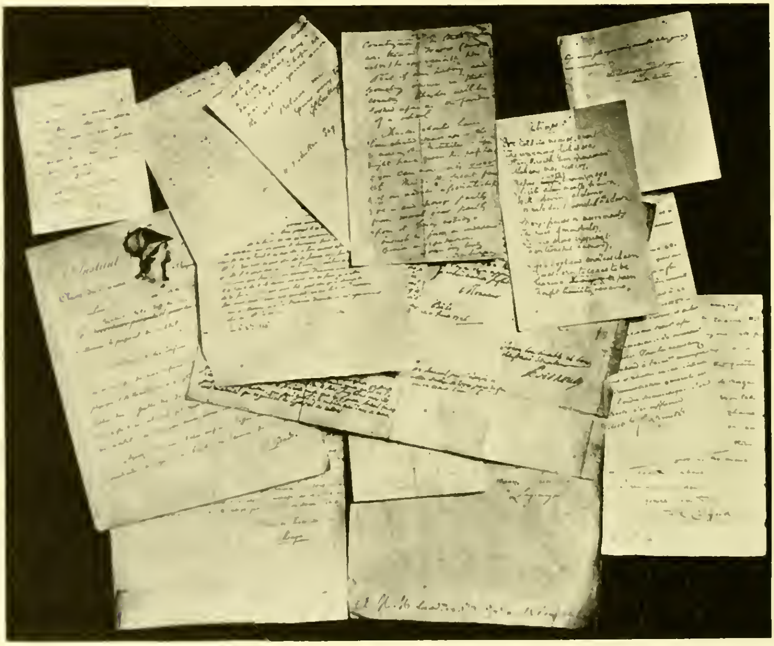
IN THE MAZES OF MATHEMATICS.

A SERIES OF PERPLEXING QUESTIONS.

BY WM. F. WHITE, PH. D.

X. AUTOGRAPHS OF MATHEMATICIANS.

FOR the photograph from which this cut was made the writer is indebted to Prof. David Eugene Smith. As an explorer in the



bypaths of mathematical history and a collector of interesting specimens therefrom, Dr. Smith is, perhaps, without a peer.

The reader will be interested to see a facsimile of the handwriting of Euler and Johann Bernoulli, Lagrange and Laplace and

Legendre, Clifford and Dodgson, and William Rowan Hamilton, and others of the immortals, grouped together on one page. In the upper right corner is the autograph of Moritz Cantor, the historian of mathematics. On the sheet overlapping that, the name over the verses is faint; it is that of J. J. Sylvester, late professor in Johns Hopkins University.

One who tries to decipher some of these documents may feel that he is indeed "In the Mazes of Mathematics." Mathematicians are not as a class noted for the elegance or the legibility of their chirography, and these examples are not submitted as models of penmanship. But each bears the sign manual of one of the builders of the proud structure of modern mathematics.

XI. BRIDGES AND ISLES, FIGURE TRACING, UNICURSAL SIGNATURES, LABYRINTHS.

This section presents a few of the more elementary results of the application of mathematical methods to these interesting puzzle questions.*

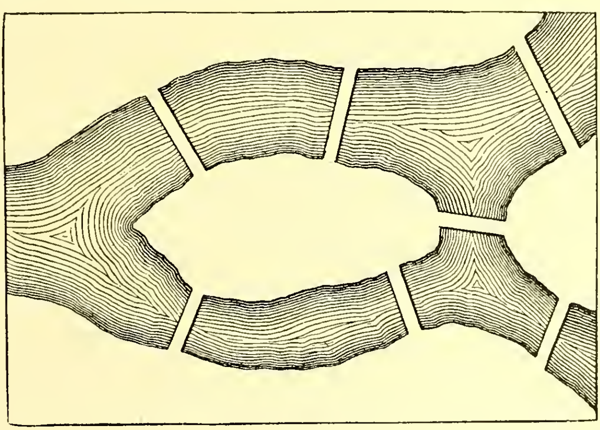


Fig. 1.

The city of Königsberg is near the mouth of the Pregel river, which has at that point an island called Kneiphof. The situation of the seven bridges is shown in the figure. A discussion arose as to whether it is possible to cross all the bridges in a single prom-

* For a more extended discussion, and for proofs of the theorems here stated, see Euler's *Solutio Problematis ad Geometriam Situs Pertinentis,* Listing's *Vorstudien zur Topologie,* Ball's *Mathematical Recreations and Essays,* Lucas's *Récréations Mathématiques,* and the references given in notes by the last two writers named. To these two the present writer is especially indebted.

enade without crossing any bridge a second time. Euler's famous memoir was presented to the Academy of Sciences of St. Petersburg in 1736 in answer to this question. Rather, the Königsberg problem furnished him the occasion to solve the general problem of any number and combination of isles and bridges.

Conceive the isles to shrink to points, and the problem may be stated more conveniently with reference to a diagram as the problem of tracing a given figure without removing the pencil from the paper and without retracing any part; or, if not possible to do so with one stroke, to determine *how many* such strokes are necessary. Fig. 2 is a diagrammatic representation of Fig. 1, the isle Kneiphof being at point K.

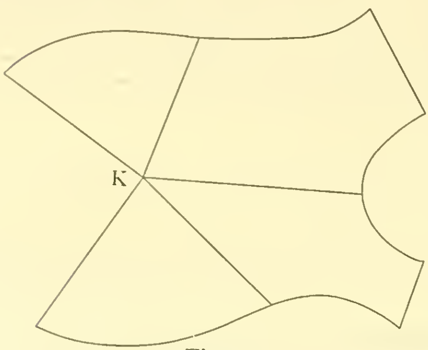


Fig. 2.

The number of lines proceeding from any point of a figure may be called the *order* of that point. Every point will therefore be of either an even order or an odd order. E. g., as there are 3 lines from point A of Fig. 3, the order of the point is odd; the order of point E is even. The well-known conclusions reached by Euler may now be stated as follows:

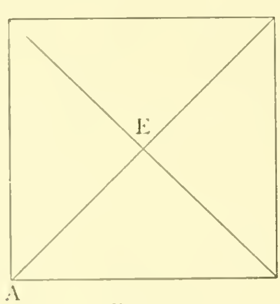


Fig. 3.

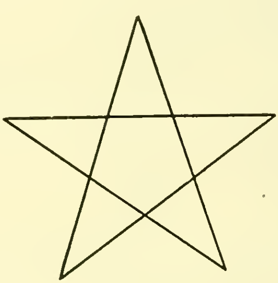


Fig. 4.

In a closed figure (one with no free point or "loose end") *the number of points of odd order is even,* whether the figure is unicursal or not. E. g., Fig. 3, a multicursal closed figure, has *four* points of odd order.

A figure of which every point is of even order can be traced *by one stroke starting from any point of the figure.* E. g., Fig. 4, the magic pentagon, symbol of the Pythagorean school, and Fig. 5, a "magic hexagram commonly called the shield of David and frequently used on synagogues" (Carus), have no points of odd order; each is therefore unicursal.

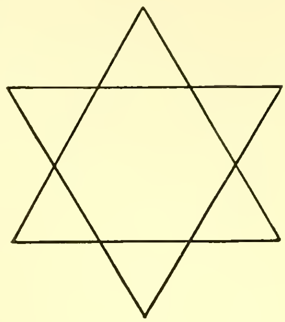


Fig. 5.

A figure with only two points of odd order can be traced by one stroke by starting at one of those points. E. g., Fig. 6 (taken originally from Listing's *Topologie*) has but two points of odd order, A and Z; it may therefore be traced by one stroke beginning at either of these two points and ending at the other. One may make a game of it by drawing a figure, as Lucas suggests, like Fig. 6 but in a larger scale on cardboard, placing a small counter on the middle of each line that joins two neighboring

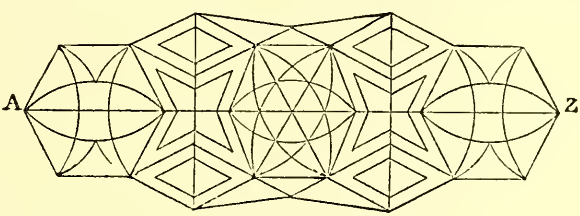


Fig. 6.

points, and setting the problem to determine the course to follow in removing all the counters successively (simply tracing continuously and removing each counter as it is passed, an objective method of recording which lines have been traced).

A figure with more than two points of odd order is multicursal. E. g., Fig. 7 has more than two points of odd order and requires more than one course or stroke, to traverse it.

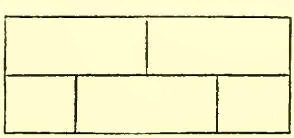


Fig. 7.

The last two theorems just stated are special cases of Listing's:

Let 2n represent the number of points of odd order; then n strokes are necessary and sufficient to trace the figure. E. g., Fig. 6, with 2 points of odd order, requires 1 stroke; Fig. 7, representing a fragment of masonry, has 8 points of odd order and requires 4 strokes.

Return now to the Königsberg problem of Fig. 1. By reference to the diagram in Fig. 2, it is seen that there are 4 points of odd order. Hence it is not possible to cross every bridge once and but once without taking two strolls.

An interesting application of these theorems is the consideration of the number of strokes necessary to describe an n-gon and its diagonals. As the points of intersection of the diagonals are all of even order, we need to consider only the vertexes. Since from each vertex there is a line to every other vertex, the number of lines from each vertex is $n - 1$. Hence, if n is odd, every point is of even order, and the entire figure can be traced unicursally beginning at any point; e. g., Fig. 8, a pentagon with its diagonals. If n is even, $n - 1$ is odd, every vertex is of odd order, the number of points of odd order is n, and the figure can not be described in less than $n/2$ courses; e. g., Fig. 3, quadrilateral, requires 2 strokes.

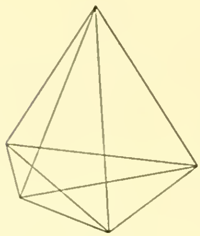


Fig. 8.

Unicursal Signatures. A signature (or other writing) is of course subject to the same laws as are other figures with respect to the number of times the pen must be put to the paper. Since the terminal point could have been connected with the point of starting without lifting the pen, the signature may be counted as a closed figure if it has no free end but these two. The number of points of odd order will be found to be even. The dot over an i, the cross of a t, or any other mark leaving a free point, makes the signature multicursal. There are so many names not requiring separate strokes that one would expect more unicursal signatures than are actually

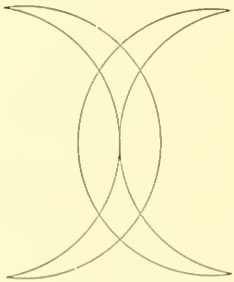


Fig. 9.

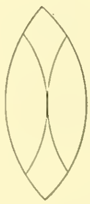


Fig. 10.

found. De Morgan's (as shown in the cut in the preceding section) is one; but most of the signatures there shown were made with several strokes each. Of the signatures to the Declaration of Independence there is not one that is strictly unicursal; though that of

Th Jefferson looks as if the end of the *h* and the beginning of the *J* might often have been completely joined, and in that case his signature would have been written in a single course of the pen.

Fig. 9, formed of two crescents, is "the so-called sign-manual of Mohammed, said to have been originally traced in the sand by the point of his scimetar without taking the scimetar off the ground or retracing any part of the figure," which can easily be done beginning at any point of the figure, as it contains no point of odd order. The mother of the writer suggests that, if the horns of Mohammed's crescents be omitted, a figure (Fig. 10) is left which can not be traced unicursally. There are then four points of odd order; hence two strokes are requisite to describe the figure.

Labyrinths such as the very simple one shown in Fig. 11 (published in 1706 by London and Wise) are familiar, as drawings, to

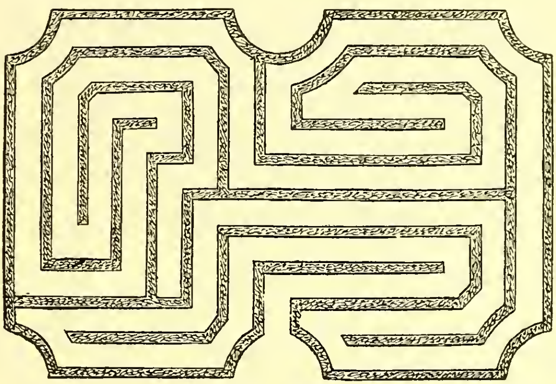


Fig. 11.

every one. In some of the more complicated mazes it is not so easy to thread one's way, even in the drawing, where the entire maze is in sight, while in the actual labyrinth, where walls or hedges conceal everything but the path one is taking at the moment, the difficulty is greatly increased and one needs a rule of procedure.

The mathematical principles involved are the same as for tracing other figures; but in their application several differences are to be noticed in the conditions of the two problems. A labyrinth as it stands, is not a closed figure; for the entrance and the center are free ends, as are also the ends of any blind alleys that the maze may contain. These are therefore points of odd order. There are usually other points of odd order. Hence in a single trip the maze can not be completely traversed. But it is not required to do so. The problem here is to go from the entrance to the center, the shorter the route found the better. Moreover, the rules of the game do not forbid retracing one's course.

It is readily seen (as first suggested by Euler) that by going over each line twice the maze becomes a closed figure, terminating where it begins, at the entrance, including the center as one point in the course, and containing only points of even order. Hence every labyrinth can be completely traversed by going over every path twice —once in each direction. It is only necessary to have some means of marking the routes already taken (and their direction) to avoid the possibility of losing one's way. This duplication of the entire course permits no failure and is so general a method that one does not need to know anything about the particular labyrinth in order to traverse it successfully and confidently. But if a plan of the labyrinth can be had, a course may be found that is shorter.

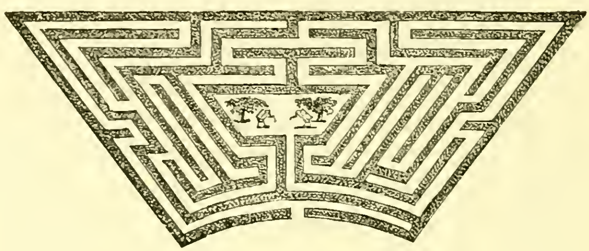


Fig. 12.

Fig. 12 presents one of the most famous labyrinths, though by no means among the most puzzling. It is described in the *Encyclopædia Britannica* (article "Labyrinth") as follows:

"The maze in the gardens at Hampton Court Palace is considered to be one of the finest examples in England. It was planted in the early part of the reign of William III, though it has been supposed that a maze had existed there since the time of Henry VIII. It is constructed on the hedge and alley system, and was, we believe, originally planted with hornbeam, but many of the plants have died out, and been replaced by hollies, yews, etc., so that the vegetation is mixed. The walks are about half a mile in length, and the extent of ground occupied is a little over a quarter of an acre. The center contains two large trees, with a seat beneath each. The key to reach this resting place is to keep the right hand continuously in contact with the hedge from first to last, going around all the stops."