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PhD Thesis

On the Construction of Flexible Frames and Bases for Decomposition Spaces

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Preface

This PhD thesis is the result of work carried out at the Department of Mathematical Sciences, Aalborg University, and during a visit to NuHAG, University of Vienna, in the spring of 2011. It is presented mainly in the form of three published journal papers:

K. N. Rasmussen. Orthonormal bases for anisotropic α-modulation spaces. Collectanea Mathematica, vol. 63(1), pp. 109-121, 2012.

M. Nielsen and K. N. Rasmussen. Compactly supported frames for decomposition spaces. Journal of Fourier Analysis and Applications, vol. 18(1), pp. 87-117, 2012.

K. N. Rasmussen and M. Nielsen. Compactly supported curvelet-type systems. Journal of Function Spaces and Applications, vol. 2012, 2012.

These can be found in Chapters 2-4 with an introduction in Chapter 1 and a further discussion in Chapter 5. The papers have been preserved in their original journal form apart from some minor corrections and a condensation of the bibliographies to a single bibliography at the end of the thesis.

I am indebted to my supervisor Morten Nielsen for being the steady rock I could always fall back on no matter how lost. Thanks for setting me on the path and then keeping up with me for three years. I would also like to thank Hans. G. Feichtinger for hosting my visit to NuHAG and showing me another aspect of harmonic analysis.

Kenneth N. Rasmussen
Aalborg, February 2012

Summary

The topic of this thesis is harmonic analysis more specifically generalized wavelet systems. While wavelets have proven a very useful tool for representing images and sound signals, new generalized wavelet systems perform even better in certain cases. This thesis focuses on the construction of flexible generalized wavelet systems with a prescribed nature such as compact support.

In Chapter 1 we introduce the framework on which the following chapters were build. We motivate the search for new representations by looking at n-term nonlinear approximation and compress an image with wavelets to show the advantage of flexibility.

In Chapter 2 we construct orthonormal bases for bivariate anisotropic α-modulation spaces. The construction is based on generating a nice anisotropic α-covering and using carefully selected tensor products of univariate brushlet functions with regards to this covering. As an application, we show that n-term nonlinear approximation with these orthonormal bases in certain anisotropic α-modulation spaces can be completely characterized.

In Chapter 3 we study a construction of flexible representations for decomposition spaces of Triebel-Lizorkin type and for the associated modulation spaces. The new representations are constructed by extending the machinery of almost diagonal matrices to Triebel-Lizorkin type spaces and the associated modulation spaces. Furthermore, an already known representation for these spaces is approximated by finite linear combinations of shifts and dilates of a single function with sufficient decay in both the direct and the frequency space to obtain the new representations.

In Chapter 4 we study a construction of flexible curvelet type representations. These curvelet type systems have the same sparse representation properties as curvelets for appropriate classes of smooth functions. We use the machinery of almost diagonal matrices to show that a system of curvelet molecules which is sufficiently close to curvelets constitutes a frame for curvelet type spaces. Such a system of curvelet molecules can then be constructed using finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay.

In Chapter 5 we look at some of the open problems which present themselves in extension of the previous chapters.

Danish summary

Emnet for denne afhandling er harmonisk analyse mere specifikt generaliserede wavelet systemer. Wavelets har vist sig at være et meget nyttigt værktøj til at repræsentere billeder og lydsignaler, men bedre resultater er i visse tilfælde opnået med nye generaliserede wavelet systemer. Denne afhandling fokuserer på konstruktionen af fleksible generaliserede wavelet systemer med foreskrevne egenskaber såsom kompakt støtte.

I kapitel 1 introducerer vi den ramme hvorpå de efterfølgende kapitler er baseret. Vi motiverer jagten på nye repræsentationer ved at se på n-led ikke-lineær approksimation og komprimere et billede med wavelets for at vise fordelen ved fleksibilitet.

I kapitel 2 konstruerer vi orthonormale baser for bivariate anisotrope α-modulationsrum. Denne konstruktion er baseret på at danne en pæn anisotrope α-overdækning og bruge omhyggeligt udvalgte tensor produkter of univariate brushlet funktioner i forhold til denne overdækning. Som anvendelsesmulighed viser vi at n-led ikke-linær approksimation med disse orthonormale baser i visse anisotrope α-modulationsrum kan karakteriseres fuldstændigt.

I kapitel 3 studerer vi konstruktionen af fleksible repræsentationer for dekompositionsrum af Triebel-Lizorkin typen og for de tilhørende modulationsrum. De nye repræsentationer er konstrueret ved at udvide næsten diagonale matricer maskineriet til Triebel-Lizorkin lignende rum og de tilhørende modulationsrum. Derudover bliver en allerede kendt repræsentation for disse rum approksimeret med endelige linearkombinationer af translationer og dilationer af en enkelt funktion med tilstrækkelig henfald i både det direkte rum og frekvensrummet for at opnå de nye repræsentationer.

I kapitel 4 studerer vi konstruktionen af fleksible curvelet lignende repræsentationer. Disse curvelet systemer har samme effektive repræsentationsegenskaber som curvelets for en passende klasse af glatte funktioner. Vi bruger næsten diagonale matricer maskineriet til at vise at et system af curvelet molekyler som er tilstrækkeligt tæt på curvelets danner en frame for curvelet lignende rum. Sådan et system af curvelet molekyler kan så blive konstrueret ved at bruge endelige linearkombinationer af translationer og dilationer af en enkelt funktion med tilstrækkelig glathed og henfald.

I kapitel 5 ser vi på nogle af de åbne problemer som viser sig i forlængelse af de foregående kapitler.

Contents

CHAPTER 1

Prologue

If I have seen further it is by standing on ye sholders of Giants.

Sir Isaac Newton

In this chapter we set the scene for the story to come by presenting the framework on which the papers in Chapters 2-4 were built as well as placing them in the context of harmonic analysis. The fundamental idea is to decompose a function f in terms of basic functions (atoms) ψ_j so that we can simplify the analysis of f and operators which act on it. This can be done by either a discrete or continuous representation, and we will here restrict ourselves to the discrete representation,

$$f = \sum_j c_j \psi_j, \tag{1.1}$$

with convergence in some suitable sense. We start out with the following application to motivate expansions of the type (1.1).

Nonlinear approximation. In n-term nonlinear approximation we approximate a complicated function f with a linear combination of n simpler functions from the family $\{\psi_j\}$ (for an overview see e.g. [**12**, **15**]). An approximant which better resolves f can generally be constructed by increasing n. The understanding of this tradeoff between complexity and resolution is the main goal of constructive approximation. More specifically, we have on one side f with complexity typically measured by its membership in a certain smoothness space, and on the other side we have an approximation space which includes all functions which can be approximated at a certain rate asymptotically by n-term approximation with $\{\psi_j\}$. A well-known example where this match-up is possible is n-term wavelet approximation where, in certain cases, the approximation space is a Besov space [**14**]. The method used to prove this is particularly interesting because it relies heavily on the machinery of n-term nonlinear approximation and very little on wavelets specifically. Wavelets constitute an unconditional basis for a Besov space, and furthermore the norm in the Besov space can be characterized by an associated sequence norm applied to the wavelet coefficients. The norm characterization gives a so-called Jackson inequality which can be used to show that the Besov space is included in the approximation space, and the linear independence of the wavelets gives a so-called Bernstein inequality which can be used to show that the Besov space is not only included, but equal to the approximation space. So one possible idea for finding an approximation space associated with a general smoothness space is to construct an unconditional basis for the smoothness space or, lacking linear independence, an atomic decomposition. Next, we will look at smoothness spaces where this is possible.

Decomposition spaces. A very broad class of smoothness spaces can be constructed by considering structured decompositions of the frequency space $\mathbb{R}^d$. This was done by Feichtinger and Gröbner with the introduction of decomposition spaces [**17, 19**] (strictly speaking the smoothness spaces are decomposition spaces on the Fourier side). With this perspective, Besov and Triebel-Lizorkin spaces correspond to smoothness spaces with a dyadic decomposition of the frequency space. Furthermore, Feichtinger introduced the classical modulation spaces [**18**] which correspond to a uniform decomposition of the frequency space. Gröbner then used the decomposition methods to define α-modulation spaces [**28**] as a family of intermediate spaces between modulation and Besov spaces with a polynomial type decomposition of the frequency space. While classical modulation spaces are well understood and have become a standard tool in time-frequency analysis (see e.g. [**29**]), α-modulation spaces are still on the verge of a breakthrough. Atomic decompositions for α-modulation spaces were considered by Fornasier [**23**] and later Nielsen constructed an orthonormal basis for bivariate α-modulation spaces and used it to characterize n-term nonlinear approximation in [**44**]. The first main contribution of this thesis is the extension of the orthonormal basis and characterization to anisotropic α-modulation spaces which can be found in Chapter 2. This is done by simplifying the underlying decomposition which then allows for a generalization to the anisotropic case. Feichtinger and Gröchenig also introduced a more restrictive class of smoothness spaces called coorbit spaces [**20–22**] which we will discuss further in Chapter 5.

In the perspective of decomposition spaces, anisotropic α-modulation spaces are a prime example in a broad subclass of decomposition spaces which we get by taking Besov spaces and extending the decomposition of the frequency space to more general decompositions. This was done by Borup and Nielsen in [**4**] where they constructed an atomic decomposition for these modulation spaces and used it to derive Jackson inequalities. Later in [**5**] they extended the atomic decomposition to similarly constructed Triebel-Lizorkin (T-L) type spaces. Candès and Donoho introduced a fundamentally different frame called curvelets with a decomposition of the frequency space which is described by a parabolic scaling relation. Curvelets have proven quite useful in providing sparse representations for certain natural images [**9**];

moreover, curvelets provide an optimally sparse representation of Fourier integral operators [7] and an optimally sparse and well organized solution operator for a wide class of linear hyperbolic differential equations [8]. In [4] Borup and Nielsen also showed that their general frame construction could be adapted to a curvelet type frame. However, this general frame construction lacks flexibility: the frame is compactly supported in the frequency space which prohibits compact support in the direct space. Compact support will be discussed further in the last section, but first we look at how to construct flexible systems.

Perturbation principle. A well-known perturbation principle is that given a basis $\{\eta_j\}$ for some Banach space X, suppose the functions $\{\psi_j\}$ approximate $\{\eta_j\}$ well enough, then $\{\psi_j\}$ will also be a basis for X. A classical way of doing this is by taking $\|\eta_j - \psi_j\|_X$ small enough (see e.g. [39]). However, this approach leaves little room when selecting $\{\psi_j\}$. Kyriazis and Petrushev instead took a wavelet basis $\{\eta_j\}$ for T-L and Besov spaces and approximated the derivatives of η_j with functions ψ_j with sufficient vanishing moments [34, 45]. By using the machinery of almost diagonal matrices developed by Frazier and Jawerth [25], they then showed that $\{\eta_j\}$ is also a basis for T-L and Besov spaces. The relatively simple requirements on ψ_j allowed a construction with linear combinations of a fixed number of shifts and dilates of a single function g with sufficient smoothness and decay. Prime examples of g being the Gaussian or a spline with compact support. Later Kyriazis and Petrushev extended the perturbation principle to atomic decompositions for T-L and Besov spaces [35] and Bownik and Ho extended almost diagonal matrices to anisotropic T-L and Besov spaces. The second main contribution of this thesis is the construction of a compactly supported atomic decomposition for T-L type spaces and the associated modulation spaces which can be found in Chapter 3. This is done using the above mentioned perturbation principle on the atomic decomposition in [4] with the twist that ψ_j instead approximates η_j sufficiently well in both the direct and the frequency space. The last main contribution can be found in Chapter 4 where we similarly construct a compactly supported curvelet type frame. This case relies on work by Candès and Demanet on almost diagonal matrices [8]. Shearlets are a directional representation system which resembles curvelets and for them compactly supported frames were recently constructed by Kittipoom, Kutyniok and Lim [33].

Compact support. To see why a flexible frame construction is important, let us look at Meyer and Daubechies wavelets. Meyer wavelets [42] marked the beginning of modern wavelet theory and their Fourier transforms are infinitely smooth and compactly supported away from origo. Hence they decay faster than the inverse of any polynomial, have vanishing moments of all orders, but also cannot have exponential decay. Daubechies wavelets [11], on the other hand, are a family of wavelets with compact support and fixed smoothness and vanishing moments depending on the size of the support. In Figure 1 we compressed the image ″Cameraman″ with Meyer and Daubechies wavelets by decomposing the image and only keeping the 15% biggest coefficients. With Meyer wavelets we see in Figure 1(a) that a lack of spatial localization causes so-called ringing artifacts around the edge of the man which are far less prominent with the Daubechies wavelet in Figure 1(b). Here we chose a Daubechies wavelet with as much spatial localization as possible while still having sufficient smoothness to represent the image well. We see that the asymptotical behavior imposed by approximation spaces does not reveal everything and it pays off to have a bit of flexibility. In general, compact support is important for wavelets as it allows for efficient computation with the fast wavelet transform by Mallat **[40]** (as Matlab uses this transform we actually used a discrete approximation of the Meyer wavelet with compact support in Figure 1(a)).

(a) Using the wavelet ″dmey″. (b) Using the wavelet ″db10″.

Figure 1. Level 8 wavelet decomposition of the image ″Cameraman″ with 15% non-zeroes with Matlab Wavelet Toolbox.

CHAPTER 2

Orthonormal bases for anisotropic α-modulation spaces

Publication details:
Published in Collectanea Mathematica, vol. 63(1), pp. 109-121, 2012.

Abstract: In this paper we construct orthonormal bases for bivariate anisotropic α-modulation spaces. The construction is based on generating a nice anisotropic α-covering and using carefully selected tensor products of univariate brushlet functions with regards to this covering. As an application, we show that n-term nonlinear approximation with these orthonormal bases in certain anisotropic α-modulation spaces can be completely characterized.

2.1. Introduction

The construction of unconditional bases for a given smoothness space is important as it often leads to simple characterizations of the space. For example, smoothness measured in a Besov space is equivalent to a certain sparseness of a wavelet expansion [**43**]. More generally, norm characterizations allow us to identify certain smoothness spaces as nonlinear approximation spaces (see e.g. [**27, 34**]). As a consequence we gain better understanding of how sufficiently smooth functions can be compressed by thresholding the expansion coefficients for a sparse representation of the function [**13, 14**].

The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$, $\alpha \in [0,1]$, were introduced by Gröbner [**28**] and include the Besov and modulation spaces as special cases corresponding to $\alpha = 1$ and $\alpha = 0$, respectively. They are part of a much more general construction introduced by Feichtinger and Gröbner called decomposition spaces [**17, 19**]. Decomposition spaces are based on structured coverings of the frequency space $\mathbb{R}^d$ and in the case of the α-modulation spaces the α-parameter determines the nature of the covering. The Besov spaces ($\alpha = 1$) correspond to a dyadic covering, the modulation spaces ($\alpha = 0$) correspond to a uniform covering and the intermediate cases correspond to "polynomial type" coverings of the frequency space. So far frames have been constructed for a broad subclass of the decomposition spaces [**4**], but the author is not aware of any general method for constructing bases for decomposition spaces. On the other hand, a orthonormal basis for bivariate α-modulation spaces was constructed in [**44**].

The goal of this paper is to construct an orthonormal basis for bivariate anisotropic α-modulation spaces. Building on the work in [**44**] the orthonormal basis is constructed by using carefully selected tensor products of univariate brushlet functions. Brushlets are the image of a local trigonometric basis under the Fourier transform, and such systems were introduced by Laeng [**37**]. Later Coifman and Meyer used brushlets as a tool for image compression [**41**]. By using the constructed orthonormal basis, we also identity certain anisotropic α-modulation spaces as approximation spaces associated with nonlinear n-term approximation.

The outline of the paper is as follows. In Section 2.2 univariate brushlets are defined, and bivariate brushlet bases are constructed for a flexible covering of $\mathbb{R}^2$. In Section 2.3 anisotropic α-modulation spaces are defined, and an anisotropic α-covering is constructed. Furthermore, by applying the constructed α-covering to the bivariate brushlet bases from Section 2.2, we show that unconditional bases for the anisotropic α-modulation spaces are generated. In Section 2.4 we apply the constructed basis to nonlinear n-term approximation. Finally, there is an appendix where we prove that anisotropic α-modulation spaces are independent of the α-covering used.

2.2. Brushlet bases

In this section we introduce orthonormal brushlet bases for $L_2(\mathbb{R})$, and use them to construct bivariate brushlet bases associated with a flexible covering of the frequency space $\mathbb{R}^2$ (see e.g. [**2**]). In the following section, by choosing a covering that fits to the anisotropic α-modulation spaces, we will then be able to show that the constructed bivariate brushlet bases form unconditional bases for the α-modulation spaces.

Each univariate brushlet basis is associated with a partition of the frequency axis. The partition can be chosen with almost no restrictions, but in order to have good properties of the associated basis we need to impose some growth conditions on the partition.

DEFINITION 2.1.
A family of intervals $\mathbb{I}$ is called a *disjoint covering* of $\Omega = [\omega, \omega') \subseteq \mathbb{R}$, $\omega < \omega'$, if it consists of a countable set of pairwise disjoint half-open intervals $I = [\alpha_I, \alpha_I')$, $\alpha_I < \alpha_I'$, such that $\cup_{I \in \mathbb{I}} I = \Omega$. If, furthermore, each interval in $\mathbb{I}$ has a unique adjacent interval in $\mathbb{I}$ to the left and to the right, and there exists a constant $A > 1$ such that

$$A^{-1} \leq \frac{|I|}{|I'|} \leq A, \text{ for all adjacent } I, I' \in \mathbb{I}, \tag{2.1}$$

we call $\mathbb{I}$ a *moderate disjoint covering* of Ω. $\diamond$

Given a moderate disjoint covering $\mathbb{I}$ of Ω, we can easily assign to each interval $I \in \mathbb{I}$ a cutoff radius $\varepsilon_I > 0$ at the left endpoint and a cutoff radius ε_I' at the right endpoint, satisfying

$$\begin{cases} \varepsilon'_I = \varepsilon_{I'}, \text{whenever } \alpha'_I = \alpha_{I'} \\ \varepsilon_I + \varepsilon'_I \le |I| \\ \varepsilon_I \ge C|I|, \end{cases} \tag{2.2}$$

with $C > 0$ independent of I.

We are now ready to define the brushlet system. For each $I \in \mathbb{I}$, we first construct a smooth bell function localized in a neighborhood of I. Take a non-negative ramp function $\rho \in C^\infty(\mathbb{R})$ satisfying

$$\rho(\xi) = \begin{cases} 0 \text{ for } \xi \le -1 \\ 1 \text{ for } \xi \ge \ \ 1, \end{cases}$$

with the property that

$$\rho(\xi)^2 + \rho(-\xi)^2 \equiv 1.$$

Define for each $I = [\alpha_I, \alpha'_I) \in \mathbb{I}$ the *bell function*

$$b_I(\xi) := \rho\left(\frac{\xi - \alpha_I}{\varepsilon_I}\right)\rho\left(\frac{\alpha'_I - \xi}{\varepsilon'_I}\right).$$

Notice that $\mathrm{supp}(b_I) \subset (\alpha_I - \varepsilon_I, \alpha'_I + \varepsilon'_I)$ and $b_I(\xi) = 1$ for $\xi \in [\alpha_I + \varepsilon_I, \alpha'_I - \varepsilon'_I]$. Let $\hat{f}(\xi) := \mathcal{F}(f)(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $f \in L_2(\mathbb{R}^d)$. Now if $\mathbb{I}$ is a moderate disjoint covering of $\mathbb{R}$ then the set of local cosine functions

$$\hat{w}_{m,I}(\xi) := \sqrt{\frac{2}{|I|}}b_I(\xi)\cos\left(\pi(m+\tfrac{1}{2})\frac{\xi - \alpha_I}{|I|}\right),\ m \in \mathbb{N}_0,\ I \in \mathbb{I}, \tag{2.3}$$

constitute an orthonormal basis for $L_2(\mathbb{R})$, see e.g. [**1**]. We call the collection $\{w_{m,I}\}_{m\in\mathbb{N}_0, I\in\mathbb{I}}$ a *brushlet system*. There is also a more explicit representation of brushlets in the direct space. Define $\hat{g}_I(\xi) := b_I(|I|\xi + \alpha_I)$ and $e_{m,I} := \pi(m+\frac{1}{2})|I|^{-1}$, we then have

$$w_{m,I}(x) = \sqrt{\frac{|I|}{2}}e^{i\alpha_I x}\left[g_I\big(|I|(x+e_{m,I})\big) + g_I\big(|I|(x-e_{m,I})\big)\right]. \tag{2.4}$$

It can easily be verified that for $r \ge 1$ there exists $C > 0$ such that

$$|g_I(x)| \le C(1+|x|)^{-r} \tag{2.5}$$

independent of $I \in \mathbb{I}$.

To later generate bivariate brushlet bases, we define the operator $\mathcal{P}_I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as

$$\widehat{\mathcal{P}_I f}(\xi) := b_I(\xi)[b_I(\xi)\hat{f}(\xi) + b_I(2\alpha_I - \xi)\hat{f}(2\alpha_I - \xi) - b_I(2\alpha'_I - \xi)\hat{f}(2\alpha'_I - \xi)].$$

By straight forward calculations it can be verified that $\mathcal{P}_I$ is an orthogonal projection, mapping $L_2(\mathbb{R})$ onto $\overline{\mathrm{span}}(\{w_{m,I}\}_{m\in\mathbb{N}_0})$. We shall list some properties of $\mathcal{P}_I$ here and refer to [**30**, Chap. 1] for a more detailed discussion of local trigonometric bases.

If I and J are two adjacent intervals in $\mathbb{I}$ then for $f \in L_2(\mathbb{R})$,

$$\widehat{\mathcal{P}_I f}(\xi) + \widehat{\mathcal{P}_J f}(\xi) = \hat{f}(\xi), \ \xi \in [\alpha_I + \varepsilon_I, \alpha'_J - \varepsilon'_J]. \tag{2.6}$$

Furthermore,

$$\mathcal{P}_I + \mathcal{P}_J = \mathcal{P}_{I \cup J} \tag{2.7}$$

with the ε-values ε_I and ε'_J. It follows that $\{w_{m,I'}\}_{m \in \mathbb{N}_0, I' \in \{I,J\}}$ is an orthonormal basis for functions bandlimited to $[\alpha_I + \varepsilon_I, \alpha'_J - \varepsilon'_J]$ on $L_2(\mathbb{R}^2)$, and by repeating the argument, a basis for all functions in $L_2(\mathbb{R})$ can be constructed by using a moderate disjoint covering of $\mathbb{R}$. This will be the key idea for constructing nice bivariate brushlet bases.

For later use, we introduce $P_Q := \mathcal{P}_I \otimes \mathcal{P}_J$, $Q := I \times J \subset \mathbb{R}^2$. By using the univariate case, we have that P_Q is an orthogonal projection, mapping $L_2(\mathbb{R}^2)$ onto $\overline{\text{span}}(\{w_{m_1,I} \otimes w_{m_2,J}\}_{m_1,m_2 \in \mathbb{N}_0})$.

Construction of bivariate brushlet bases. A simple way of constructing bivariate brushlet bases is to use the tensor product on a univariate brushlet basis. Although this gives us a basis for $L_2(\mathbb{R}^2)$, we lose the ability to generate a structured anisotropic covering of the frequency plane. An example of this in the isometric case can be seen with tensor products of orthonormal wavelets. Here we end up with hyperbolic bivariate wavelet systems which offer no characterizations of isotropic smoothness spaces. Instead we take the tensor product of two brushlet bases, extract the brushlets on the diagonal with regards to the frequency index, and then repopulate this subsystem in a structured way.

We saw earlier that the univariate brushlet bases were constructed from an moderate disjoint covering of $\mathbb{R}$, and the operator $\mathcal{P}_I$ could be seen as a building block associated with the bandlimited functions on I. We shall use the same idea here, and first construct a covering of $\mathbb{R}^2$.

Let $\{I_n\}_{n \in \mathbb{Z}}$ and $\{J_n^l\}_{n \in \mathbb{N}_-, k_n \le l \le 0} \cup \{J_0^0\} \cup \{J_n^l\}_{n \in \mathbb{N}_+, 0 \le l \le k_n}$ be moderate disjoint coverings of $\mathbb{R}$ such that $n < n'$ implies $\alpha_{I_n} < \alpha_{I_{n'}}$ and $\alpha_{J_n^l} < \alpha_{J_{n'}^{l'}}$, and $l < l'$ implies $\alpha_{J_n^l} < \alpha_{J_n^{l'}}$. This gives us the diagonal part of our covering and works as a "scaffold" for the rest of the covering, see Figure 1. Next, let $\{I_{n,i}\}_{1 \le i \le m_n^I}$, $n \ge 1$, be moderate disjoint coverings of $\cup_{n'=-n}^{n} I_{n'}$ with the same constant A from (2.1) as the covering $\{I_n\}_{n \in \mathbb{Z}}$; furthermore, we require that $I_{n,1} = I_{-n}$ and $I_{n,m_n^I} = I_n$. Define $\{J_{n,j}^0\}_{1 \le j \le m_n^J}$ similarly. We introduce a covering of $\mathbb{R}^2$ with the help of the hollow rectangles $\cup_{Q \in \mathbb{P}_n} Q$, $\mathbb{P}_n := \mathbb{P}_n^b \cup \mathbb{P}_n^t \cup \mathbb{P}_n^l \cup \mathbb{P}_n^r$, $n \ge 1$,

$$\mathbb{P}_n^b = \left\{ I_{n,i} \times J_{-n}^l | 1 \le i \le m_n^I, k_{-n} \le l \le 0 \right\}$$

$$\mathbb{P}_n^t = \left\{ I_{n,i} \times J_n^l | 1 \le i \le m_n^I, 0 \le l \le k_n \right\}$$
$$\mathbb{P}_n^l = \left\{ I_{-n} \times J_{n,j}^0 | 2 \le j \le m_n^J - 1 \right\}$$
$$\mathbb{P}_n^r = \left\{ I_n \times J_{n,j}^0 | 2 \le j \le m_n^J - 1 \right\},$$

and the center rectangle $\mathbb{P}_0$,

$$\mathbb{P}_0 = \{I_0 \times J_0^0\}.$$

It follows that $\cup_{Q\in\mathbb{P}} Q = \mathbb{R}^2$, $\mathbb{P} := \cup_{n=0}^{\infty}\mathbb{P}_n$ and the sets in $\mathbb{P}$ are disjoint.

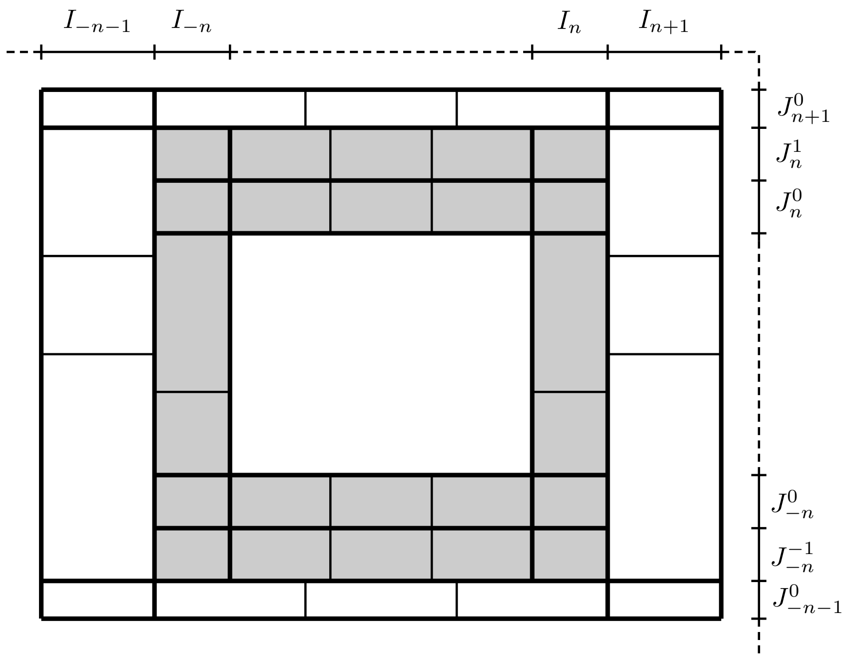

Figure 1. *Covering of $\mathbb{R}^2$ by $\mathbb{P}$. The shaded area is the sets in $\mathbb{P}_n$.*

With the covering $\mathbb{P}$, we can now define our bivariate brushlet system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$,

$$w_{m,Q}(x,y) := w_{m_1,I}(x)w_{m_2,J}(y),\ m = (m_1,m_2),\ Q = I \times J,$$

where $w_{m_1,I}$ was defined in (2.3). With this notation, we have that P_Q denotes the orthogonal projection onto $\overline{\text{span}}(\{w_{m,Q}\}_{m\in\mathbb{N}_0^2})$,

$$P_Q f = \sum_{m\in\mathbb{N}_0^2} \langle f, w_{m,Q}\rangle w_{m,Q},\ f \in L_2(\mathbb{R}^2).$$

Next, we use the orthogonal projections P_Q to prove that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$.

PROPOSITION 2.2.
The system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$.

Proof:
To prove that the system is complete in $L_2(\mathbb{R}^2)$, we first observe that only adjacent rectangles in $\mathbb{P}$ overlap. It follows that there exists a family of open sets $\{U_n\}_{n\in\mathbb{Z}^2}$ such that for $f \in L_2(\mathbb{R}^2)$, $\sum_{Q\in\mathbb{P}} \widehat{P_Q f}(\xi)$, $\xi \in \bar{U}_n$, contains at most four non-zero elements and $\cup_{n\in\mathbb{Z}^2}\bar{U}_n = \mathbb{R}^2$. This can be used to show that $\sum_{Q\in\mathbb{P}} P_Q$ converges strongly to a bounded operator on $L_2(\mathbb{R}^2)$, and it suffices to prove pointwise that

$$\sum_{Q\in\mathbb{P}} \widehat{P_Q s} = \hat{s} \tag{2.8}$$

for functions s in a suitable dense subset of $L_2(\mathbb{R}^2)$. Since finite linear combinations of separable functions are dense in $L_2(\mathbb{R}^2)$, we only need to verify (2.8) for a separable function $s(x,y) = g(x)h(y)$ with $g, h \in L_2(\mathbb{R})$.

We begin with the projections associated with $\mathbb{P}_0$ and $\mathbb{P}_1$. By using (2.7) on the second coordinate, we sum up the projections associated with $\mathbb{P}_1^l$ and $\mathbb{P}_1^r$,

$$\sum_{Q\in\mathbb{P}_1^l} P_Q = P_{I_{-1}\times J_0^0} \tag{2.9}$$

$$\sum_{Q\in\mathbb{P}_1^r} P_Q = P_{I_1\times J_0^0}. \tag{2.10}$$

Next, we use (2.7) on the first coordinate to sum (2.9) and (2.10) together with the projection associated with the center rectangle $I_0 \times J_0^0$,

$$\sum_{Q\in\mathbb{P}_1^l\cup\mathbb{P}_0\cup\mathbb{P}_1^r} P_Q = P_{\cup_{i=-1}^1 I_i\times J_0^0}.$$

Finally, we add the projections associated with $\mathbb{P}_1^b$ and $\mathbb{P}_1^t$ to get

$$\sum_{Q\in\cup_{n=0}^1\mathbb{P}_n} P_Q = P_{\cup_{i=-1}^1 I_i\times\cup_{j=-1}^1\cup_l J_j^l}.$$

By repeating the procedure N times we end up with

$$\sum_{Q\in\cup_{n=0}^N\mathbb{P}_n} P_Q = P_{\cup_{i=-N}^N I_i\times\cup_{j=-N}^N\cup_l J_j^l}.$$

It then follows from (2.6) that as N goes to infinity,

$$\sum_{Q\in\cup_{n=0}^N\mathbb{P}_n} \widehat{P_Q s}$$

converges pointwise to $\hat{s}$ which proves (2.8). Hence, $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ is complete in $L_2(\mathbb{R}^2)$.

That the system is orthonormal will follow from the fact that it consists of carefully selected tensor products of univariate brushlets. One can check that two distinct brushlets associated with the same hollow rectangle $\mathbb{P}_n$ are orthogonal. If $|n-m| \geq 2$ then two brushlets associated with $\mathbb{P}_n$ and $\mathbb{P}_m$, respectively, do not overlap in the frequency space. This leaves us with brushlets that are associated with $Q \in \mathbb{P}_n$ and $P \in \mathbb{P}_{n+1}$, respectively. If we look at the $\mathbb{R}_+ \times \mathbb{R}_+$ part of the frequency space, then the brushlets only overlap if $Q = I_n \times Q_2$, $P = I_{n+1} \times P_2$ or $Q = Q_1 \times J_n^{k_n}$, $P = P_1 \times J_{n+1}^0$ (see Figure 1). In which case we have from the univariate brushlets that the brushlets are orthogonal. The rest of the frequency space follows similarly. ■

2.3. Anisotropic α-modulation spaces

In this section we define the anisotropic α-modulation spaces and show that our brushlet system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ can constitute bases for them. To define the anisotropic α-modulation spaces, we need a nice partition of unity and this partition is based on a covering of the frequency plane which again is based on an anisotropic quasi-norm.

First we define an anisotropic quasi-norm $|\cdot|_a$,

$$|\xi|_a := |\xi_1|^{1/a_1} + |\xi_2|^{1/a_2},\ \xi = (\xi_1,\xi_2) \in \mathbb{R}^2,$$

where $a = (a_1,a_2)$, $a_1, a_2 > 0$ and $a_1 + a_2 = 2$. We also define $\langle\xi\rangle_a := (1 + |\xi|_a^2)^{1/2}$ and the balls

$$\mathcal{B}_a(\xi,r) := \{\zeta \in \mathbb{R}^2 : |\xi-\zeta|_a < r\}.$$

Notice that $|\mathcal{B}_a(\xi,r)| = r^2\lambda_a$, $\lambda_a := |\mathcal{B}_a(0,1)|$.

With such an quasi-norm $|\cdot|_a$, we can define anisotropic α-coverings.

Definition 2.3.
A countable set $\mathcal{Q}$ of measurable connected subsets $Q \subset \mathbb{R}^2$ is called a *connected admissible covering* if $\mathbb{R}^2 = \cup_{Q\in\mathcal{Q}} Q$ and there exists $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q} : \bar{Q} \cap \bar{Q}' \neq \emptyset\} \leq n_0$ for all $Q \in \mathcal{Q}$. Let

$$r_Q = \sup\{r \in \mathbb{R} : \mathcal{B}_a(c_r,r) \subset Q,\ c_r \in \mathbb{R}^2\},$$
$$R_Q = \inf\{R \in \mathbb{R} : Q \subset \mathcal{B}_a(c_R,R),\ c_R \in \mathbb{R}^2\}$$

denote the radius of the inscribed and circumscribed disc of $Q \in \mathcal{Q}$, respectively. A connected admissible covering $\mathcal{Q}$ is called an *anisotropic α-covering* of $\mathbb{R}^2$, $0 \leq \alpha \leq 1$, if $|Q| \asymp \langle\xi\rangle_a^{2\alpha}$ for some $\xi \in Q$ and all $Q \in \mathcal{Q}$, and there exists $K < \infty$ such that $R_Q/r_Q \leq K$ for all $Q \in \mathcal{Q}$. ◇

REMARK 2.4.
Notice that $|Q| \asymp \langle\xi\rangle_a^{2\alpha}$ for some $\xi \in Q$ implies the same for all $\xi \in Q$ with constants independent of ξ and Q. Also we have restricted ourself to connected sets to later use the general theory of decomposition spaces to show that anisotropic α-modulation spaces are well-defined (see [**4**]). However, by generalizing [**3**, Theorem 3.1] one can drop the requirement that the sets need to be connected. $\circ$

For technical reasons we shall require our partitions of unity to satisfy the following.

DEFINITION 2.5.
Given $0 \leq \alpha \leq 1$, let $\mathcal{Q}$ be an anisotropic α-covering. A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\{\psi_Q\}_{Q\in\mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^2)$ satisfying:

- $\operatorname{supp}(\psi_Q) \subseteq Q$
- $\sum_{Q\in\mathcal{Q}} \psi_Q \equiv 1$
- $\sup_{Q\in\mathcal{Q}} |Q|^{1/p-1} \|\mathcal{F}^{-1}\psi_Q\|_{L_p(\mathbb{R}^2)} < \infty$, $p \in (0,1]$.

$\diamond$

It was proven in [**4**, Section 6] that an anisotropic α-covering with a corresponding BAPU exists for every $\alpha \in [0,1]$. We define the multiplier $\psi_Q(D)f := \mathcal{F}^{-1}(\psi_Q \mathcal{F} f)$, $f \in L_2(\mathbb{R}^2)$. A standard result on band-limited multipliers [**51**, Proposition 1.5.1] ensures that if $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a BAPU, then $\psi_Q(D)$ extends to a bounded operator on band-limited functions in $L_p(\mathbb{R}^2)$, $0 < p \leq \infty$, uniformly in $Q \in \mathcal{Q}$.

We are now ready to define anisotropic α-modulation spaces.

DEFINITION 2.6.
Given $0 \leq \alpha \leq 1$, let $\mathcal{Q}$ be an anisotropic α-covering of $\mathbb{R}^2$ with a corresponding BAPU $\{\psi_Q\}_{Q\in\mathcal{Q}}$. For $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q < \infty$, we define the *anisotropic α-modulation space*, $M_{p,q}^{s,\alpha}(\mathbb{R}^2)$, as the set of distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ satisfying

$$\|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^2)} := \Big(\sum_{Q\in\mathcal{Q}} \langle\xi_Q\rangle_a^{qs} \|\psi_Q(D)f\|_{L_p}^q \Big)^{1/q} < \infty,$$

where $\xi_Q \in Q$. $\diamond$

We show in the Appendix that anisotropic α-modulation spaces are independent of which α-covering is used. Furthermore, it can be shown that $M_{p,q}^{s,\alpha}(\mathbb{R}^2)$ is a quasi-Banach space (Banach space for $p,q \geq 1$), and $\mathcal{S}(\mathbb{R}^2)$ is dense in $M_{p,q}^{s,\alpha}(\mathbb{R}^2)$ [**4**, **19**]. For more information on quasi-Banach spaces, we refer the reader to [**31**, **32**].

Orthonormal bases for anisotropic α-modulation spaces. With the anisotropic α-modulation spaces in place, we need to adapt the covering $\mathbb{P}$ such that the associated brushlet system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ constitutes bases for them. The natural choice would be to make $\mathbb{P}$ an α-covering, and as we shall see, this will suffice.

First we need to make $\mathbb{P}$ an α-covering. We will focus on $\alpha \in [0,1)$ since $\alpha = 1$ corresponds to a dyadic covering, and we use a polynomial type covering. Without loss of generality we will also assume that $a_2 \geq a_1$. Let $I_0 := [-1,1)$, $I_n := [n^{\beta a_1}, (n+1)^{\beta a_1})$, and $I_{-n} := -I_n$, $n \geq 1$, $\beta \geq 1$. Next, we introduce the sequence $\{y_m\}_{m\in\mathbb{N}}$, $y_0 := 1$, $y_m := y_{m-1} + n^{\beta a_2 - a_2/a_1}$, where $n \in \mathbb{N}$ is chosen such that $n^{\beta a_2} \leq y_{m-1} < (n+1)^{\beta a_2}$. We can then define $J_n^l := [y_{m-1}, y_m)$, $m := n + l + \sum_{i=1}^{n-1} k_i$, $0 \leq l \leq k_n$, where $k_n \in \mathbb{N}_0$ are chosen such that $|J_n^l| = n^{\beta a_2 - a_2/a_1}$, see figure 2. Furthermore, let $J_0^0 := [-1,1)$ and $J_{-n}^{-l} := -J_n^l$. To make sure that J_n^l is defined for all $n \in \mathbb{N}$, we notice that

$$y_m - y_{m-1} = n^{\beta a_2 - a_2/a_1} \leq n^{\beta a_2 - 1} < (n+2)^{\beta a_2} - (n+1)^{\beta a_2}.$$

In fact, we have $k_n + 1 \asymp n^{a_2/a_1 - 1}$.

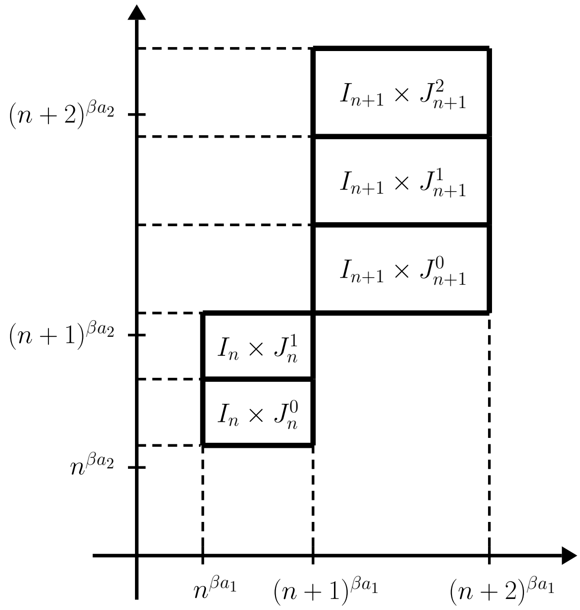

Figure 2. *Choosing $I_n \times J_n^l$ such that $\mathbb{P}$ is an α-covering.*

One can check that $\{I_n\}$ and $\{J_n^l\}$ are moderate disjoint coverings of $\mathbb{R}$. To generate $\mathbb{P}$, we choose $\{I_{n,i}\}$ and $\{J_{n,j}^0\}$ such that $|I_{n,i}| \asymp |I_n|$ and $|J_{n,j}^0| \asymp |J_n^0|$. As the sets in $\mathbb{P}$ are disjoint it follows easily that $\mathbb{P}$ is a connected admissible covering of $\mathbb{R}^2$.

Next, to show that $\mathbb{P}$ is a an anisotropic α-covering, we notice that $\mathbb{P}$ is constructed such that we only need to check the requirements for $I_n \times J_n^l$. We have

$$|I_n|^{1/a_1} \asymp n^{(\beta a_1 - 1)/a_1} = n^{(\beta a_2 - a_2/a_1)/a_2} = |J_n^l|^{1/a_2}.$$

For $Q = I_n \times J_n^l$ it follows that $r_Q \geq C|I_n|^{1/a_1}$ and $R_Q \leq |I_n|^{1/a_1} + |J_n^l|^{1/a_2}$ which gives $R_Q/r_Q \leq K < \infty$, $Q \in \mathbb{P}$. Finally, given $\alpha \in [0,1)$ we need to define β such that $|Q| \asymp \langle \xi \rangle_a^{2\alpha}$ for some $\xi \in Q$, $Q \in \mathbb{P}$. By choosing

$$\beta := \frac{1 + \frac{a_2}{a_1}}{2(1-\alpha)}, \tag{2.11}$$

we get $2\alpha\beta = 2\beta - 1 - a_2/a_1$, and it follows that

$$|I_n \times J_n^l| \asymp n^{\beta a_1 - 1 + \beta a_2 - a_2/a_1} = n^{2\beta - 1 - a_2/a_1} = n^{2\alpha\beta} \asymp \langle \xi \rangle_a^{2\alpha},$$

where ξ is the corner of $I_n \times J_n^l$ closest to origo.

We now have that $\mathbb{P}$ is an anisotropic α-covering, and from Proposition 2.2 we know that $\{w_{m,Q}\}_{m \in \mathbb{N}_0^2, Q \in \mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$. Next, we show that these conditions are sufficient to prove that $\{w_{m,Q}\}_{m \in \mathbb{N}_0^2, Q \in \mathbb{P}}$ is an unconditional basis for the corresponding anisotropic α-modulation space.

First we need the following definition and lemma.

Definition 2.7.
Let $\mathcal{Q}$ be a covering of $\mathbb{R}^2$ and G a subset of $\mathbb{R}^2$. We define

$$A_G^{\mathcal{Q}} := \{Q \in \mathcal{Q} : \bar{Q} \cap \bar{G} \neq \emptyset\}$$

and the sets

$$\widetilde{Q} := \bigcup_{Q' \in A_Q^{\mathcal{Q}}} \bar{Q}', \; Q \in \mathcal{Q}. \tag{2.12}$$

$\diamond$

Notice that a connected admissible covering $\mathcal{Q}$ fulfills $\# A_Q^{\mathcal{Q}} \leq n_0$, $Q \in \mathcal{Q}$. One can also check that if $\mathcal{Q}$ is an anisotropic α-covering then $\{\widetilde{Q}\}_{Q \in \mathbb{P}}$ is also an anisotropic α-covering.

Lemma 2.8.
Given $f \in L_2(\mathbb{R}^2)$, $0 \leq \alpha < 1$ and $0 < p \leq \infty$. If $\{\widetilde{\psi}_Q\}_{Q \in \mathbb{P}}$ is a partition of unity for $\{\widetilde{Q}\}_{Q \in \mathbb{P}}$ which satisfies

$$\widetilde{\psi}_Q(x) = 1, \; x \in \operatorname{supp}(\hat{w}_{0,Q}),$$

and $\{\psi_Q\}_{Q\in\mathbb{P}}$ is a partition of unity for $\{\mathrm{supp}(\hat{w}_{0,Q})\}_{Q\in\mathbb{P}}$, then there exists $C, C' > 0$, independent of $Q \in \mathbb{P}$, such that

$$\Big(\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q}\rangle|^p\Big)^{1/p} \leq C|Q|^{\frac{1}{p}-\frac{1}{2}}\|\widetilde{\psi}_Q(D)f\|_{L_p}, \text{ and}$$

$$\|\psi_Q(D)f\|_{L_p} \leq C'|Q|^{\frac{1}{2}-\frac{1}{p}} \sum_{Q'\in A_Q^{\mathbb{P}}} \Big(\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q'}\rangle|^p\Big)^{1/p}.$$

When $p = \infty$ the sum over $m \in \mathbb{N}_0^2$ is changed to sup.

Proof:
Notice that (2.4) together with (2.5) yield the following estimates,

$$\sup_{x\in\mathbb{R}^2}\sum_{m\in\mathbb{N}_0^2} |w_{m,Q}(x)|^p \leq C_p|Q|^{\frac{p}{2}} \text{ and } \sup_{m\in\mathbb{N}_0^2}\|w_{m,Q}\|_{L_p}^p \leq C_p'|Q|^{\frac{p}{2}-1}. \tag{2.13}$$

Take $f \in L_2(\mathbb{R}^2)$ and let us first assume that $p \leq 1$. We then have (see, e.g. [**51**, p. 18])

$$\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q}\rangle|^p = \sum_{m\in\mathbb{N}_0^2} |\langle \widetilde{\psi}_Q(D)f, w_{m,Q}\rangle|^p \leq \sum_{m\in\mathbb{N}_0^2} \|(\widetilde{\psi}_Q(D)f)w_{m,Q}\|_{L_1}^p$$
$$\leq C|Q|^{1-p}\sum_{m\in\mathbb{N}_0^2} \|(\widetilde{\psi}_Q(D)f)w_{m,Q}\|_{L_p}^p \leq C|Q|^{1-\frac{p}{2}}\|\widetilde{\psi}_Q(D)f\|_{L_p}^p.$$

By using that $\psi_Q(D)$ is bounded on band-limited functions in L_p, we have the second inequality in the lemma,

$$\|\psi_Q(D)f\|_{L_p}^p \leq C' \sum_{Q'\in A_Q^{\mathbb{P}}}\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q'}\rangle|^p \|w_{m,Q'}\|_{L_p}^p$$
$$\leq C'|Q|^{\frac{p}{2}-1}\sum_{Q'\in A_Q^{\mathbb{P}}}\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q'}\rangle|^p.$$

For $1 < p < \infty$ the lemma follows by using the two estimates in (2.13) with $p = 1$ together with Hölder's inequality (see e.g. [**43**, §2.5]). The case $p = \infty$ follows similar to $p \leq 1$. ■

By taking the l_q-norm in Lemma 2.8, we can derive our main result.

THEOREM 2.9.
Given $0 < p \leq \infty$, $0 < q < \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha < 1$. With the system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$, we have the following characterization

$$\|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^2)} \asymp \Big(\sum_{n=0}^{\infty} n^{q\beta(s+\alpha-\frac{2\alpha}{p})}\sum_{Q\in\mathbb{P}_n}\Big(\sum_{m\in\mathbb{N}_0^2} |\langle f, w_{m,Q}\rangle|^p\Big)^{q/p}\Big)^{1/q},$$

where β was defined in (2.11). Furthermore, $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ constitutes an unconditional basis for $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$.

Proof:
The norm characterization follows by taking the l_q-norm in Lemma 2.8 and using that $|Q| = n^{2\alpha\beta} \asymp \langle\xi_Q\rangle_a^{2\alpha}$, $\xi_Q \in Q$, $Q \in \mathbb{P}_n$. That $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ constitutes a unconditional basis for $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$, follows by standard results using the norm characterization, that $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ is a quasi-Banach space in which $\mathcal{S}(\mathbb{R}^2)$ is dense and that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ is an orthonormal basis for $L_2(\mathbb{R}^2)$. ■

Remark 2.10.
By Remark 3.21, one can use $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ to construct a compactly supported basis for $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ with the same norm characterization as $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$. ○

Theorem 2.9 also shows that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ induces a natural isomorphism between $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ and the sequence space $m^{s,\alpha}_{p,q}$ defined by:

Definition 2.11.
Given $0 < p \le \infty$, $0 < q < \infty$, $s \in \mathbb{R}$, $0 \le \alpha < 1$, we define the sequence space $m^{s,\alpha}_{p,q}$ as the set of sequences $c := \{c_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}} \subset \mathbb{C}$ satisfying

$$\|c\|_{m^{s,\alpha}_{p,q}} := \Big(\sum_{n=0}^{\infty} n^{q\beta(s+\alpha-\frac{2\alpha}{p})} \sum_{Q\in\mathbb{P}_n} \Big(\sum_{m\in\mathbb{N}_0^2} |c_{m,Q}|^p \Big)^{q/p} \Big)^{1/q} < \infty,$$

where β was defined in (2.11). ◇

2.4. An application to nonlinear approximation

We finish this paper with applying $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2,Q\in\mathbb{P}}$ to n-term nonlinear approximation in certain anisotropic α-modulation spaces.

First, we need some notation regarding nonlinear approximation. Let $\mathcal{D} := \{g_k\}_{k\in\mathbb{N}}$ be a Schauder basis in a quasi-Banach space X. We consider the collection of all possible n-term expansions with elements from $\mathcal{D}$:

$$\Sigma_n(\mathcal{D}) := \Big\{ \sum_{i\in\Lambda} c_i g_i \Big| c_i \in \mathbb{C}, \#\Lambda \le n \Big\}.$$

The error of the best n-term approximation to an element $f \in X$ is then

$$\sigma_n(f,\mathcal{D})_X := \inf_{f_n\in\Sigma_n(\mathcal{D})} \|f - f_n\|_X.$$

Next, we introduce the approximation spaces $\mathcal{A}^\gamma_q(X,\mathcal{D})$ which essentially consists of the elements f for which $\sigma_n(f,\mathcal{D})_X = \mathcal{O}(n^{-\gamma})$.

Definition 2.12.
Let $0 < \gamma, q < \infty$. We define the approximation space $\mathcal{A}^{\gamma}_{q}(X, \mathcal{D})$ as the set of distributions $f \in X$ satisfying

$$|f|_{\mathcal{A}^{\gamma}_{q}(X,\mathcal{D})} := \Big(\sum_{n=1}^{\infty} (n^{\gamma}\sigma_n(f,\mathcal{D})_X)^q \frac{1}{n} \Big)^{1/q} < \infty,$$

and quasi-norm it with $\|f\|_{\mathcal{A}^{\gamma}_{q}(X,\mathcal{D})} := \|f\|_X + |f|_{\mathcal{A}^{\gamma}_{q}(X,\mathcal{D})}$. $\diamond$

As Theorem 2.9 showed that $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ induces an isomorphism between $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ and $m^{s,\alpha}_{p,q}$, we can apply **[26]** to get a complete characterization of certain nonlinear approximation spaces associated with anisotropic α-modulation spaces:

Theorem 2.13.
Let $0 < \gamma, p < \infty$, $0 \le \alpha < 1$, $s \in \mathbb{R}$, $\tau^{-1} := \gamma + p^{-1}$ and $\rho := 2\alpha\gamma + s$. If $\mathcal{D}$ is the system $\{w_{m,Q}\}_{m\in\mathbb{N}_0^2, Q\in\mathbb{P}}$ normalized in $M^{s,\alpha}_{p,p}(\mathbb{R}^2)$, then we have the characterization

$$\mathcal{A}^{\gamma}_{\tau}(M^{s,\alpha}_{p,p}(\mathbb{R}^2), \mathcal{D}) = M^{\rho,\alpha}_{\tau,\tau}(\mathbb{R}^2)$$

with equivalent norms. $\square$

Remark 2.14.
By using Remark 2.10, we can also get the characterization in Theorem 2.13 for a compactly supported basis for $M^{s,\alpha}_{p,p}(\mathbb{R}^2)$. $\circ$

Appendix

In this appendix we show that $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ only depends on the α-covering up to equivalence of the norms. First we extend Definition 2.7.

Definition 2.15.
Let $\widetilde{Q}^{(0)} := \bar{Q}$, and define inductively $\widetilde{Q}^{(k+1)} := \widetilde{\widetilde{Q}^{(k)}}$, $k \ge 0$. Finally let $\widetilde{\mathcal{Q}}^{(k)} := \{\widetilde{Q}^{(k)}\}_{Q\in\mathcal{Q}}$. $\mathcal{P}$ is called *almost subordinate* to $\mathcal{Q}$ (written $\mathcal{P} \le \mathcal{Q}$) if there exists $k \in \mathbb{N}$ such that for all $P \in \mathcal{P}$, we have $P \subseteq \widetilde{Q}^{(k)}$ for some $Q \in \mathcal{Q}$. $\diamond$

Let $\mathcal{Q}$ and $\mathcal{P}$ be two anisotropic α-coverings. If $\bar{Q} \cap \bar{P} \neq \emptyset$, $Q \in \mathcal{Q}$, $P \in \mathcal{P}$, then Definition 2.3 implies that $R_Q \asymp R_P$. This can be used to prove that there exists $d_0 < \infty$ such that

$$\#A^{\mathcal{Q}}_{P} \le d_0, \;\; P \in \mathcal{P}.$$

Lemma 2.16 below then gives that $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$. By interchanging $\mathcal{Q}$ and $\mathcal{P}$, we also have that $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$. From **[4**, Theorem 1] it then follows that $M^{s,\alpha}_{p,q}(\mathbb{R}^2)$ only depends on the α-covering up to equivalence of the norms.

LEMMA 2.16.
Let $\mathcal{Q}$ and $\mathcal{P}$ be connected admissible coverings. Then $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$ if and only if there exists $d_0 < \infty$ such that

$$\#A_P^{\mathcal{Q}} \leq d_0, \ P \in \mathcal{P}. \tag{2.14}$$

Proof:
Let us first assume that $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$, and choose $P \in \mathcal{P}$. Then there exists $Q \in \mathcal{Q}$ such that $P \subseteq \widetilde{Q}^{(k)}$. One can easily prove that $\widetilde{\mathcal{Q}}^{(k)}$ is a connected admissible covering so it follows that

$$\#A_P^{\mathcal{Q}} \leq \#A_{\widetilde{Q}^{(k)}}^{\mathcal{Q}} \leq \#A_{\widetilde{Q}^{(k)}}^{\widetilde{\mathcal{Q}}^{(k)}} \leq d_0.$$

To prove the opposite way, let us assume that (2.14) is satisfied. Choose $P \in \mathcal{P}$ and $Q \in A_P^{\mathcal{Q}}$. If $A_P^{\mathcal{Q}} \backslash \{Q\} = \emptyset$, then $P \subseteq Q$, and we are done. If instead $A_P^{\mathcal{Q}} \backslash \{Q\} \neq \emptyset$ and $\bar{Q} \cap \bar{Q}' = \emptyset$ for all $Q' \in A_P^{\mathcal{Q}} \backslash \{Q\}$, then

$$P \backslash \bar{Q} = \bigcup_{Q' \in A_P^{\mathcal{Q}} \backslash \{Q\}} \bar{Q}' \cap P.$$

However, this proves that $P \backslash \bar{Q}$ is both open and closed on P which contradicts that P is a connected set. It follows that $Q' \subset \widetilde{Q}$ for some $Q' \in A_P^{\mathcal{Q}} \backslash \{Q\}$. Next, we use the same argument with $\widetilde{Q}$, and either $P \subseteq \widetilde{Q}$ or there exists $Q'' \in A_P^{\mathcal{Q}} \backslash \{\widetilde{Q}\}$ such that $\bar{Q}'' \cap \widetilde{Q} \neq \emptyset$. As $A_P^{\mathcal{Q}}$ contains at most d_0 elements, we can repeat the argument $d_0 - 1$ times to get

$$P \subseteq \bigcup_{Q' \in A_P^{\mathcal{Q}}} \bar{Q}' \subseteq \widetilde{Q}^{(d_0 - 1)}$$

which proves that $\mathcal{P}$ is almost subordinate to $\mathcal{Q}$. ■

CHAPTER 3

Compactly supported frames for decomposition spaces

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Abstract: In this paper we study a construction of compactly supported frame expansions for decomposition spaces of Triebel-Lizorkin type and for the associated modulation spaces. This is done by showing that finite linear combinations of shifts and dilates of a single function with sufficient decay in both the direct and the frequency space can constitute a frame for Triebel-Lizorkin type spaces and the associated modulation spaces. First, we extend the machinery of almost diagonal matrices to Triebel-Lizorkin type spaces and the associated modulation spaces. Next, we prove that two function systems which are sufficiently close have an almost diagonal "change of frame coefficient" matrix. Finally, we approximate to an arbitrary degree an already known frame for Triebel-Lizorkin type spaces and the associated modulation spaces with a single function with sufficient decay in both the direct and the frequency space.

3.1. Introduction

Smoothness spaces such as the Triebel-Lizorkin (T-L) and Besov spaces play an important role in approximation theory and harmonic analysis. Often they are characterized by (or at least imply) some decay or sparseness of an associated discrete expansion. For example, a certain sparseness of a wavelet expansion is equivalent to smoothness measured in a Besov space [**43**]. A consequence of this is that a sufficiently smooth function can be compressed by thresholding the expansion coefficients of a sparse representation of the function [**13**, **14**]. More generally in nonlinear approximation, the coefficient norm characterization leads to better understanding of the approximation spaces (see e.g. [**27**, **34**]).

The T-L and Besov spaces are special cases of T-L type spaces and the associated modulation spaces which again form a broad subclass of the decomposition spaces defined on $\mathbb{R}^d$. Decomposition spaces were introduced by Feichtinger and Gröbner [**19**] and Feichtinger [**17**], and are based on structured coverings of the frequency space $\mathbb{R}^d$. Here the classical T-L and Besov spaces correspond to dyadic coverings [**51**]. Many authors have used modulation spaces to study pseudodifferential operators see e.g. [**46**] and references therein.

In this paper we study a flexible method of generating frames for T-L type spaces and the associated modulation spaces. Frames are redundant decomposition systems with extra structure between the expansion coefficients and the function being represented which make them useful for nonlinear approximation. The advantage of redundant decomposition systems is that they provide extra flexibility compared to bases as we have more than one way of representing the function. Recently this has lead to sparser representations of certain natural images than with wavelets; two examples of this are curvelet frames [**48**] and bandlets [**38**].

Frames for T-L type spaces and the associated modulation spaces have been considered earlier: Banach frames for α-modulation spaces in [**10, 23**] and Banach frames for T-L type spaces and the associated modulation spaces in [**4,5**]. However, these frames were constructed using band-limited functions which rules out compact support in direct space.

The goal of this paper is to construct frames with compact support for inhomogeneous T-L type spaces and the associated modulation spaces. An obvious modification produces frames for homogeneous spaces as well. The idea we employ is a perturbation principle which was first introduced in [**45**], further generalized in [**34**] and refined for frames in [**35**]. With this perturbation principle, finite linear combinations of shifts and dilates of a single function with sufficient decay in both the direct and the frequency space can be used to construct frame expansions with a prescribed nature such as compact support. These frame expansions are constructed from the atomic decomposition in [**4,5**]; thereby, generating frame expansions which share the same sparseness properties as the already known representation.

Next, we discuss frames in more detail. Suppose that X is a quasi-Banach space and Y the associated sequence space. We say that a countable family of functions Ψ in the dual X^* of X is a frame for X if there exists constants $C_1, C_2 > 0$ such that for all $f \in X$,

$$C_1\|f\|_X \le \|\{\langle f,\psi\rangle\}_{\psi\in\Psi}\|_Y \le C_2\|f\|_X,$$

where $\langle f,\psi\rangle := \psi(f)$. In the $L_2(\mathbb{R}^d)$ case, frames have the expansion

$$f = \sum_{\psi\in\Psi}\langle f, S^{-1}\psi\rangle\psi, \tag{3.1}$$

where S is the frame operator $Sf = \sum_{\psi\in\Psi}\langle f,\psi\rangle\psi$, $f \in X$. In the general case, (3.1) is not a byproduct of the theory, but we show that the frame condition is the key to proving that (3.1) holds and that $\{S^{-1}\psi\}_{\psi\in\Psi}$ is a frame. This also proves that $\{\psi\}_{\psi\in\Psi}$ is an atomic decomposition.

As the general setup requires a great deal of notation, we give an example of what is proven for α-modulation spaces, $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$.

THEOREM 3.1.
Choose $s\in\mathbb{R}$, $0<p\leq\infty$, $0<q<\infty$, $0\leq\alpha<1$, and $\delta>0$. Let $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ be the α-modulation spaces, $r:=\min(1,p,q)$, and $1/\beta:=\alpha/(1-\alpha)$. If $g\in C^1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$, $\hat{g}(0)\neq 0$, satisfies

$$|g^{(\kappa)}(x)|\leq C(1+|x|)^{-2\left(\frac{d}{r}+\delta\right)-1},\ |\kappa|\leq 1,$$
$$|\hat{g}(\xi)|\leq C(1+|x|)^{-2\left(\frac{d}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2d}{r}+\frac{3\delta}{2}\right)-1},$$

then there exists $K\in\mathbb{N}$ and $\psi_{k,n}(x):=e^{ix\cdot d_k}\sum_{i=1}^{K}a_{k,i}g(c_kx+b_{k,n,i})$, $a_{k,i}\in\mathbb{C}$, $b_{k,n,i},d_k\in\mathbb{R}^d$, $c_k\in\mathbb{R}$, such that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constitutes a frame for $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ and

$$f=\sum_{k,n\in\mathbb{Z}^d}\langle f,S^{-1}\psi_{k,n}\rangle\psi_{k,n}$$

for all $f\in M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ with convergence in $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$.

$\square$

The outline of the paper is as follows. In Section 3.2 we introduce homogeneous type spaces on $\mathbb{R}^d$ which are used to generate admissible coverings of the frequency space. These coverings are then used to define T-L type spaces and to construct associated frames. In Section 3.3 almost diagonal matrices are introduced, and we derive conditions under which the "change of frame coefficient" matrix is almost diagonal. Next, we use the machinery of almost diagonal matrices to construct new frames from old frames in Section 3.4 by using function systems which are sufficiently close to the frame from Section 3.2. Finally, in Section 3.5 we show that a system which consists of finite linear combinations of shifts and dilates of a single function with sufficient decay in both the direct and the frequency space can approximate another system with similar decay to an arbitrary degree. Thereby, creating systems which are sufficiently close to the frame from Section 3.2 and by using Section 3.4 constituting frames themselves which is our main result. We end the paper with a small discussion in Section 3.6 of the possible functions which can be used to construct the frames.

Throughout the paper we will make use of some standard notation. We let $\hat{f}(\xi):=\mathcal{F}(f)(\xi):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}f(x)e^{-ix\cdot\xi}\,\mathrm{d}x, f\in L_1(\mathbb{R}^d)$, and by duality extend it uniquely from Schwartz functions, $\mathcal{S}:=\mathcal{S}(\mathbb{R}^d)$, to tempered distributions, $\mathcal{S}':=\mathcal{S}'(\mathbb{R}^d)$. Similarly, we use $\langle f,\eta\rangle$ for the standard inner product of two functions $\int f\bar{\eta}$, and the same notation is employed for the action of a distribution $f \in \mathcal{S}'$ on $\bar{\eta} \in \mathcal{S}$. By $F \asymp G$ we mean that there exists two constants $0 < C_1 \leq C_2 < \infty$, depending only on "allowable" parameters, such that $C_1 F \leq G \leq C_2 F$. In general the constants C, C_1 and C_2 will change throughout the paper. For the sake of convenience, we write $\|f_k\|$ instead of $\|\{f_k\}_{k\in K}\|$ when the index set is well-known. Finally, for $\kappa \in \mathbb{N}_0^d$ we let $|\kappa| := \kappa_1 + \cdots + \kappa_d$, and for suitably differentiable functions we define $f^{(\kappa)} := \frac{\partial^{|\kappa|} f}{\partial_{\xi_1}^{\kappa_1} \cdots \partial_{\xi_d}^{\kappa_d}}$.

3.2. Triebel-Lizorkin type spaces

In this section we give a brief description of T-L type spaces and the associated modulation spaces. To define T-L type spaces and the associated modulation spaces, we need a suitable resolution of the identity on $\mathbb{R}^d$ in the sense that we need a countable collection of functions $\{\varphi_k\}$ with $\sum_k \varphi_k = 1$. To construct the resolution of the identity, we use a suitable covering of the frequency space. For a much more detailed discussion of the T-L type spaces see **[5]**, and for the associated modulation spaces see **[4]**.

Homogeneous type spaces on $\mathbb{R}^d$. Here we define homogeneous type spaces on $\mathbb{R}^d$ which will be used later to construct a suitable covering of the frequency space. These spaces are created with a quasi-norm induced by a one-parameter group of dilations.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$ induced by the inner product $\langle \cdot, \cdot \rangle$. We assume that A is a real $d \times d$ matrix with eigenvalues having positive real parts. For $t > 0$ define the group of dilations $\delta_t : \mathbb{R}^d \to \mathbb{R}^d$ by $\delta_t := \exp(A \ln t)$ and let $\nu := \text{trace}(A)$. The matrix A will be kept fixed throughout the paper. Some well-known properties of δ_t are (see **[50]**),

- $\delta_{ts} = \delta_t \delta_s$.
- $\delta_1 = Id$ (identity on $\mathbb{R}^d$).
- $\delta_t \xi$ is jointly continuous in t and ξ, and $\delta_t \xi \to 0$ as $t \to 0^+$.
- $|\delta_t| := \det(\delta_t) = t^\nu$.

According to [**50**, Proposition 1.7] there exists a strictly positive symmetric matrix P such that for all $\xi \in \mathbb{R}^d$,

$$[\delta_t \xi]_P := \langle P \delta_t \xi, \delta_t \xi \rangle^{\frac{1}{2}}$$

is a strictly increasing function of t. This helps use introduce a quasi-norm $|\cdot|_A$ associated with A.

DEFINITION 3.2.
We define the function $|\cdot|_A : \mathbb{R}^d \to \mathbb{R}_+$ by $|0|_A := 0$ and for $\xi \in \mathbb{R}^d \backslash \{0\}$ by letting $|\xi|_A$ be the unique solution t to the equation $[\delta_{1/t}\xi]_P = 1$. $\diamond$

It can be shown that:

- $|\cdot|_A \in C^\infty(\mathbb{R}^d\backslash\{0\})$.
- There exists a constant $C_A > 0$ such that

$$|\xi+\zeta|_A \le C_A(|\xi|_A + |\zeta|_A),\ \xi,\zeta \in \mathbb{R}^d. \tag{3.2}$$

- $|\delta_t\xi|_A = t|\xi|_A$.
- There exists constants $C_1, C_2, \alpha_1, \alpha_2 > 0$ such that

$$C_1 \min(|\xi|_A^{\alpha_1}, |\xi|_A^{\alpha_2}) \le |\xi| \le C_2 \max(|\xi|_A^{\alpha_1}, |\xi|_A^{\alpha_2}),\ \xi \in \mathbb{R}^d. \tag{3.3}$$

Example 3.3.
For $A = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_d)$, $\beta_i > 0$, we have $\delta_t = \operatorname{diag}(t^{\beta_1}, t^{\beta_2}, \ldots, t^{\beta_d})$, and one can verify that

$$|\xi|_A \asymp \sum_{j=1}^{d} |\xi_j|^{\frac{1}{\beta_j}}, \xi \in \mathbb{R}^d.$$

⋆

Finally, we define the balls $\mathcal{B}_A(\xi, r) := \{\zeta \in \mathbb{R}^d : |\xi - \zeta|_A < r\}$. It can be verified that $|\mathcal{B}_A(\xi,r)| = r^\nu \omega_d^A$ where $\omega_d^A := |\mathcal{B}_A(0,1)|$ so $(\mathbb{R}^d, |\cdot|_A, \mathrm{d}\xi)$ is a space of homogeneous type with homogeneous dimension ν.

The transpose of A with respect to $\langle\cdot,\cdot\rangle$, $B := A^\top$, will be useful for generating coverings of the direct space $\mathbb{R}^d$. Since the eigenvalues of B have positive real parts we can repeat the above construction for the group $\delta_t^\top := \exp(B \ln t), t > 0$. We let $|\cdot|_B$ denote the quasi-norm induced by $\delta_t^\top$, $\mathcal{B}_B(x,r)$ the balls associated with $|\cdot|_B$, and C_B the equivalent of C_A in (3.2). Furthermore, we have that the constants α_1 and α_2 in (3.3) also hold with B and $\operatorname{trace}(B) = \nu$. Notice that if $g_m(x) := m^\nu g(\delta_m^\top x)$, $g \in L_2(\mathbb{R}^d)$, then $\hat{g}_m(\xi) = \hat{g}(\delta_{\frac{1}{m}}\xi)$. We use the convention that δ_t acts on the frequency space while $\delta_t^\top$ acts on the direct space.

The following adaption of the Fefferman-Stein maximal inequality to the quasi-norm $|\cdot|_B$ will be essential for showing the boundedness of almost diagonal matrices. For $0 < r < \infty$, the parabolic maximal function of Hardy-Littlewood type is defined by

$$M_r^B u(x) := \sup_{t>0} \left(\frac{1}{\omega_d^B \cdot t^\nu} \int_{\mathcal{B}_B(x,t)} |u(y)|^r \,\mathrm{d}y \right)^{\frac{1}{r}}, u \in L_{r,\mathrm{loc}}(\mathbb{R}^d), \tag{3.4}$$

where $\omega_d^B := |\mathcal{B}_B(0,1)|$. There exists $C > 0$ so that the following vector-valued Fefferman-Stein maximal inequality holds for $r < q \le \infty$ and $r < p < \infty$ (see [**49**, Chapters I&II]),

$$\Big\| \Big(\sum_{k\in\mathbb{Z}^d} |M_r^B f_k|^q \Big)^{1/q} \Big\|_{L_p} \le C \Big\| \Big(\sum_{k\in\mathbb{Z}^d} |f_k|^q \Big)^{1/q} \Big\|_{L_p}. \tag{3.5}$$

If $q = \infty$, then the inner l_q-norm is replaced by the l_∞-norm.

Construction of frames. Here we first introduce admissible coverings and how to generate them (see e.g. [**17**]). These coverings are then used to construct a suitable resolution of unity and next define the T-L type spaces and the associated modulation spaces. Finally, we construct a frame which will be used in the following sections to generate compactly supported frame expansions.

DEFINITION 3.4.
A set $\mathcal{Q} := \{Q_k\}_{k\in\mathbb{Z}^d}$ of measurable subsets $Q_k \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} Q_k$ and there exists $n_0 < \infty$ such that $\#\{j \in \mathbb{Z}^d : Q_k \cap Q_j \neq \emptyset\} \leq n_0$ for all $k \in \mathbb{Z}^d$. ◇

To generate an admissible covering we will use a suitable collection of $|\cdot|_A$-balls, where the radius of a given ball is a so-called moderate function of its center.

DEFINITION 3.5.
A function $h : \mathbb{R}^d \to [\varepsilon_0, \infty)$ for $\varepsilon_0 > 0$ is called moderate if there exists constants $\rho_0, R_0 > 0$ such that $|\xi - \zeta|_A \leq \rho_0 h(\xi)$ implies $R_0^{-1} \leq h(\zeta)/h(\xi) \leq R_0$. ◇

EXAMPLE 3.6.
Let $0 \leq \alpha \leq 1$. Then

$$h(\xi) := (1 + |\xi|_A)^\alpha$$

is moderate. ⋆

With a moderate function h it is then possible to construct an admissible covering by using balls (see [**17**, Lemma 4.7] and [**5**, Lemma 5]):

LEMMA 3.7.
Given a moderate function h with constants $\rho_0, R_0 > 0$, there exists a countable admissible covering $\mathcal{C} := \{\mathcal{B}_A(\xi_k, \rho h(\xi_k))\}_{k\in\mathbb{Z}^d}$ for $\rho < \rho_0/2$, and there exists a constant $0 < \rho' < \rho$ such that the sets in $\mathcal{C}$ are pairwise disjoint.

□

By using that $\mathcal{B}_A(\xi_k, \rho' h(\xi_k))$ are disjoint it can be shown that $\mathcal{B}_A(\xi_k, 2\rho h(\xi_k))$ also give an admissible covering. Notice that the covering $\mathcal{C}$ from Lemma 3.7 is generated by a family of invertible affine transformations applied to $\mathcal{B}_A(0, \rho)$ in the sense that

$$\mathcal{B}_A(\xi_k, \rho h(\xi_k)) = T_k \mathcal{B}_A(0, \rho), \ T_k := \delta_{h(\xi_k)} \cdot + \xi_k.$$

We are now in a position to generate a suitable resolution of unity which additionally due to technical reasons has to satisfy the following conditions.

DEFINITION 3.8.
Let $\mathcal{C} := \{T_k \mathcal{B}_A(0, \rho)\}_{k\in\mathbb{Z}^d}$ be an admissible covering of $\mathbb{R}^d$ from Lemma 3.7. A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\{\varphi_k\}_{k\in\mathbb{Z}^d} \subset \mathcal{S}$ satisfying:

- $\text{supp}(\varphi_k) \subseteq T_k \mathcal{B}_A(0, 2\rho)$, $k \in \mathbb{Z}^d$.
- $\sum_{k \in \mathbb{Z}^d} \varphi_k(\xi) = 1$, $\xi \in \mathbb{R}^d$.
- $\sup_{k \in \mathbb{Z}^d} \|\varphi_k(T_k \cdot)\|_{H_2^s} < \infty$, $s > 0$,

where $\|f\|_{H_2^s} := \left(\int |\mathcal{F}^{-1} f(x)|^2 (1 + |x|_B)^{2s} \, dx\right)^{1/2}$. ◇

A standard trick for generating a BAPU for $\mathcal{C}$ is to pick $\Phi \in C^\infty(\mathbb{R}^d)$ non-negative with $\text{supp}(\Phi) \subseteq \mathcal{B}_A(0, 2\rho)$ and $\Phi(\xi) = 1$ for $\xi \in \mathcal{B}_A(0, \rho)$. One can then show that

$$\varphi_k(\xi) := \frac{\Phi(T_k^{-1}\xi)}{\sum_{j \in \mathbb{Z}^d} \Phi(T_j^{-1}\xi)}$$

defines a BAPU for $\mathcal{C}$. For later use, we also introduce

$$\phi_k(\xi) := \frac{\Phi(T_k^{-1}\xi)}{\sqrt{\sum_{j \in \mathbb{Z}^d} \Phi(T_j^{-1}\xi)^2}}, \tag{3.6}$$

which in a sense defines a square root of the BAPU.

With a BAPU in hand we can now define the T-L type spaces and the associated modulation spaces.

DEFINITION 3.9.
Let h be a moderate function satisfying

$$C_1(1 + |\xi|_A)^{\gamma_1} \leq h(\xi) \leq C_2(1 + |\xi|_A)^{\gamma_2}, \ \xi \in \mathbb{R}^d, \tag{3.7}$$

for some $0 < \gamma_1 \leq \gamma_2 < \infty$. Let $\mathcal{C}$ be an admissible covering of $\mathbb{R}^d$ from Lemma 3.7, $\{\varphi_k\}_{k \in \mathbb{Z}^d}$ a corresponding BAPU and $\varphi_k(D) f := \mathcal{F}^{-1}(\varphi_k \mathcal{F} f)$.

- For $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$, we define $F_{p,q}^s(h)$ as the set of distributions $f \in \mathcal{S}'$ satisfying
$$\|f\|_{F_{p,q}^s(h)} := \left\| \Big(\sum_{k \in \mathbb{Z}^d} |h(\xi_k)^s \varphi_k(D) f|^q \Big)^{1/q} \right\|_{L_p} < \infty.$$
- For $s \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q < \infty$, we define $M_{p,q}^s(h)$ as the set of distributions $f \in \mathcal{S}'$ satisfying
$$\|f\|_{M_{p,q}^s(h)} := \Big(\sum_{k \in \mathbb{Z}^d} \|h(\xi_k)^s \varphi_k(D) f\|_{L_p}^q \Big)^{1/q} < \infty.$$

If $q = \infty$, then the l_q-norm is replaced by the l_∞-norm. ◇

It can be shown that $F_{p,q}^s(h)$ depends only on h up to equivalence of the norms (see [**5**, Proposition 5.3]), so the T-L type spaces are well-defined. Similar for the modulation spaces. Furthermore, they both constitute quasi-Banach spaces, and for $p, q < \infty$, $\mathcal{S}$ is dense in both (see [**5**, Proposition 5.2]).

Next, we construct a frame for the T-L type spaces and the associated modulation spaces. Consider the system $\{\phi_k\}_{k\in\mathbb{Z}^d}$ from (3.6) which in a sense is a square root of a BAPU. Let K_a be a cube in $\mathbb{R}^d$ which is aligned with the coordinate axes and has side-length $2a$ satisfying $\mathcal{B}_A(0,2\rho)\subseteq K_a$. For the sake of convenience, put

$$t_k := h(\xi_k). \tag{3.8}$$

We then define

$$e_{k,n}(\xi) := (2a)^{-\frac{d}{2}} t_k^{-\frac{\nu}{2}} \chi_{K_a}(T_k^{-1}\xi) e^{-i\frac{\pi}{a} n\cdot T_k^{-1}\xi},\ n,k\in\mathbb{Z}^d,$$

and

$$\hat{\eta}_{k,n} := \phi_k e_{k,n},\ n,k\in\mathbb{Z}^d. \tag{3.9}$$

One can verify that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a tight frame for $L_2(\mathbb{R}^d)$. By defining $\hat{\mu}_k(\xi) := \phi_k(T_k\xi)$, we get an explicit representation of $\eta_{k,n}$ in the direct space

$$\eta_{k,n}(x) = (2a)^{-\frac{d}{2}} t_k^{\frac{\nu}{2}} \mu_k(\delta_{t_k}^{\top} x - \frac{\pi}{a} n) e^{ix\cdot\xi_k}, \tag{3.10}$$

and for $\kappa\in\mathbb{N}_0^d$, $N\in\mathbb{N}$ there exists $C>0$ such that

$$|\mu_k(x)^{(\kappa)}| \le C(1+|x|_B)^{-N} \tag{3.11}$$

independent of $k\in\mathbb{Z}^d$. To show that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constitutes a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$, we need associated sequence spaces. The following point sets will be useful for that,

$$Q(k,n) = \Big\{ y\in\mathbb{R}^d : \delta_{t_k}^{\top} y - \frac{\pi}{a} n \in \mathcal{B}_B(0,1) \Big\}. \tag{3.12}$$

It can easily be verified that there exists $n_0<\infty$ such that uniformly in x and k, $\sum_{n\in\mathbb{Z}^d}\chi_{Q(k,n)}(x)\le n_0$. With this property in hand, we can define the associated sequence spaces.

Definition 3.10.
Let $s\in\mathbb{R}$, $0<p<\infty$, and $0<q\le\infty$. We then define the sequence space $f^s_{p,q}(h)$ as the set of sequences $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d}\subset\mathbb{C}$ satisfying

$$\|s_{k,n}\|_{f^s_{p,q}(h)} := \Big\| \Big(\sum_{k,n\in\mathbb{Z}^d} \big(t_k^{s+\frac{\nu}{2}} |s_{k,n}|\big)^q \chi_{Q(k,n)} \Big)^{1/q} \Big\|_{L_p} < \infty.$$

Let $s\in\mathbb{R}$, $0<p\le\infty$, and $0<q<\infty$. We then define the sequence space $m^s_{p,q}(h)$ as the set of sequences $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d}\subset\mathbb{C}$ satisfying

$$\|s_{k,n}\|_{m^s_{p,q}(h)} := \Big\| t_k^{s+\frac{\nu}{2}-\frac{\nu}{p}} \Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|^p \Big)^{1/p} \Big\|_{l_q} < \infty.$$

If $p=\infty$ or $q=\infty$, then the l_p-norm or l_q-norm, respectively, is replaced by the l_∞-norm. ◇

Finally, we have that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ (3.9) constitutes a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$ (see [**4**, Theorem 2] and [**5**, Theorem 6.4]):

PROPOSITION 3.11.
Assume that $s \in \mathbb{R}$, $0 < p, q \le \infty$, $p < \infty$ for $F^s_{p,q}(h)$, and $q < \infty$ for $M^s_{p,q}(h)$. For any finite sequence $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$, we have

$$\Big\| \sum_{k,n\in\mathbb{Z}^d} s_{k,n}\eta_{k,n} \Big\|_{F^s_{p,q}(h)} \le C \|s_{k,n}\|_{f^s_{p,q}(h)}.$$

Furthermore, $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$,

$$\|f\|_{F^s_{p,q}(h)} \asymp \|\langle f, \eta_{k,n}\rangle\|_{f^s_{p,q}(h)}, \; f \in F^s_{p,q}(h).$$

Similar results hold for $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

□

3.3. Almost diagonal matrices

To later generate new frame expansions for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$ from the already known frames, we introduce an associated notion of almost diagonal matrices in this section. The machinery of almost diagonal matrices was used in [**24**, **25**] for the Triebel-Lizorkin and Besov spaces respectively. The goal is to find a new definition for almost diagonal matrices for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$, and then show that they are bounded on the associated sequence spaces $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$, and closed under composition.

From here on we shall add some further restrictions to the moderate function h used to generate admissible coverings:

$$\begin{cases} \text{There exists } \beta, R_1, \rho_1 > 0 \text{ such that } h^{1+\beta} \text{ is moderate and} \\ \quad |\xi - \zeta|_A \le ah(\xi) \text{ for } a \ge \rho_1 \text{ implies } h(\zeta) \le R_1 a h(\xi). \end{cases} \tag{3.13}$$

An abundance of functions h satisfying these conditions can be generated by using functions $s : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy $s(2b) \le Cs(b)$, $b \in \mathbb{R}_+$, and

$$(1+b)^\gamma \le s(b) \le (1+b)^{\frac{1}{1+\beta}}$$

for some $\gamma > 0$. We assign $h = s(|\cdot|_A)$ and use that s is weakly sub-additive to get the results (see [**17**]). Notice that $s(b) = (1+b)^\alpha, 0 \le \alpha < 1$, gives Example 3.6 and fulfills the mentioned conditions.

To motivate the definition of almost diagonal matrices, we let $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ be the frame defined in (3.10). By using (3.11) it can be verified that for fixed $N, M, L > 0$, $\eta_{k,n}$ has the following decay in the direct and the frequency space,

$$|\eta_{k,n}(x)| \le Ct_k^{\frac{\nu}{2}}(1 + t_k|x_{k,n} - x|_B)^{-2N}, \tag{3.14}$$

$$|\hat{\eta}_{k,n}(\xi)| \le Ct_k^{-\frac{\nu}{2}}(1 + t_k^{-1}|\xi_k - \xi|_A)^{-2M-2\frac{L}{\beta}}, \tag{3.15}$$

where

$$x_{k,n} = \delta^{\top}_{t_k^{-1}} \frac{\pi}{a} n,\ k,n \in \mathbb{Z}^d \tag{3.16}$$

and t_k was defined in (3.8). Let $\{\psi_{k,n}\}_{k,n\in Z^d} \subset L_2(\mathbb{R}^d)$ be a system with similar decay,

$$|\psi_{j,m}(x)| \leq C t_j^{\frac{\nu}{2}} (1 + t_j |x_{j,m} - x|_B)^{-2N}, \tag{3.17}$$

$$|\hat{\psi}_{j,m}(\xi)| \leq C t_j^{-\frac{\nu}{2}} (1 + t_j^{-1} |\xi_j - \xi|_A)^{-2M-2\frac{L}{\beta}}. \tag{3.18}$$

By examining $\langle \eta_{k,n}, \psi_{j,m}\rangle$ we then get the following lemma.

Lemma 3.12.
Choose $N, M, L > 0$ such that $2N > \nu$ and $2M + 2\frac{L}{\beta} > \nu$. If $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.14) and (3.15), and $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}^d}$ satisfies (3.17) and (3.18), we have

$$\begin{aligned}|\langle \eta_{k,n}, \psi_{j,m}\rangle| \leq & C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}+L} (1 + \max(t_k, t_j)^{-1} |\xi_k - \xi_j|_A)^{-M} \\ & \times (1 + \min(t_k, t_j) |x_{k,n} - x_{j,m}|_B)^{-N}.\end{aligned}$$

Proof:
From Lemma 3.23 we have

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| \leq C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}} (1 + \min(t_k, t_j) |x_{k,n} - x_{j,m}|_B)^{-2N}. \tag{3.19}$$

Using Lemma 3.23 for $\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m}\rangle$ gives

$$|\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m}\rangle| \leq C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}} (1 + \max(t_k, t_j)^{-1} |\xi_k - \xi_j|_A)^{-2M-2\frac{L}{\beta}}. \tag{3.20}$$

Next we raise the power of the first term in (3.20) at the expense of the second term. Without loss of generality assume that $t_k \leq t_j$. We first consider the case $|\xi_k - \xi_j|_A \leq \rho_0 t_j^{1+\beta}$, and use that $h^{1+\beta}$ is moderate (3.13) to get

$$\frac{1}{1 + t_j^{-1} |\xi_k - \xi_j|_A} \leq 1 \leq R_0^{\frac{\beta}{1+\beta}} \left(\frac{t_k}{t_j}\right)^{\beta}.$$

In the other case, $|\xi_k - \xi_j|_A > \rho_0 t_j^{1+\beta}$, and it follows by using $t_k \geq \varepsilon_0$ that

$$\frac{1}{1 + t_j^{-1} |\xi_k - \xi_j|_A} \leq \frac{1}{\rho_0 \varepsilon_0^{\beta}} \left(\frac{t_k}{t_j}\right)^{\beta}.$$

Hence we have

$$|\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m}\rangle| \le C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}+2L} (1+\max(t_k,t_j)^{-1}|\xi_k - \xi_j|_A)^{-2M}. \tag{3.21}$$

The lemma follows by combining (3.19) and (3.21), and using

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| = |\langle \eta_{k,n}, \psi_{j,m}\rangle|^{\frac{1}{2}} |\langle \hat{\eta}_{k,n}, \hat{\psi}_{j,m}\rangle|^{\frac{1}{2}}.$$

■

We are now ready to define almost diagonal matrices for the T-L type spaces and show that they act boundedly on the T-L type spaces. A similar result also follows for the associated modulation spaces.

Definition 3.13.
Assume that $s \in \mathbb{R}$, $0 < p, q \le \infty$, $p < \infty$ for $f^s_{p,q}(h)$, and $q < \infty$ for $m^s_{p,q}(h)$. Let $r := \min(1, p, q)$. A matrix $\mathbf{A} := \{a_{(j,m)(k,n)}\}_{j,m,k,n\in\mathbb{Z}^d}$ is called almost diagonal on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$ if there exists $C, \delta > 0$ such that

$$\begin{aligned} |a_{(j,m)(k,n)}| \le & C\left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}}, \left(\frac{t_k}{t_j}\right)^{\frac{\delta}{2}}\right) c^{\delta}_{jk} \\ & \times (1+\min(t_k,t_j)|x_{k,n} - x_{j,m}|_B)^{-\frac{\nu}{r}-\delta}, \end{aligned}$$

where

$$c^{\delta}_{jk} := \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\delta}, \left(\frac{t_k}{t_j}\right)^{\delta}\right)(1+\max(t_k,t_j)^{-1}|\xi_k - \xi_j|_A)^{-\frac{\nu}{r}-\delta}$$

with t_k defined in (3.8) and $x_{k,n}$ in (3.16). We denote the set of almost diagonal matrices on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$ by $\mathrm{ad}^s_{p,q}(h)$. ◇

The fact that almost diagonal matrices are bounded will be essential for generating compactly supported frame expansions.

Proposition 3.14.
Suppose that $A \in \mathrm{ad}^s_{p,q}(h)$. Then A is bounded on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

Proof:
We only prove the result for $f^s_{p,q}(h)$ when $q < \infty$ as $q = \infty$ follows in a similar way with l_q replaced by l_∞, and the proof for $m^s_{p,q}(h)$ is similar to the one for $f^s_{p,q}(h)$. Let $s := \{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \in f^s_{p,q}(h)$ and assume for now that $p, q > 1$. We write $\mathbf{A} := \mathbf{A}_0 + \mathbf{A}_1$ such that

$$(\mathbf{A}_0 s)_{(j,m)} = \sum_{k:t_k\ge t_j} \sum_{n\in\mathbb{Z}^d} a_{(j,m)(k,n)} s_{k,n} \text{ and } (\mathbf{A}_1 s)_{(j,m)} = \sum_{k:t_k< t_j} \sum_{n\in\mathbb{Z}^d} a_{(j,m)(k,n)} s_{k,n}.$$

By using Lemma 3.24 we have

$$
\begin{aligned}
|(\mathbf{A}_0 s)_{(j,m)}| &\le C \sum_{k:t_k \ge t_j} \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}-\frac{\nu}{r}-\frac{\delta}{2}} c_{jk}^{\delta} \sum_{n\in\mathbb{Z}^d} \frac{|s_{k,n}|}{\left(1+t_j\left|x_{k,n}-x_{j,m}\right|_B\right)^{\frac{\nu}{r}+\delta}} \\
&\le C \sum_{k:t_k \ge t_j} \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}-\frac{\delta}{2}} c_{jk}^{\delta} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)(x),
\end{aligned}
$$

for $x \in Q(j,m)$. It then follows by Hölder's inequality and Lemma 3.25 that

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}^d} |(\mathbf{A}_0 s)_{(j,m)}\chi_{Q(j,m)}|^q &\le C\Big(\sum_{k:t_k \ge t_j} \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} c_{jk}^{\delta} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Big)^q \\
&\le C \sum_{k:t_k \ge t_j} c_{jk}^{\delta}\Big(\left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Big)^q \Big(\sum_{i:t_i \ge t_j} c_{ji}^{\delta}\Big)^{q-1} \\
&\le C \sum_{k:t_k \ge t_j} c_{jk}^{\delta}\Big(\left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Big)^q.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\|\mathbf{A}_0 s\|_{f^s_{p,q}(h)} &\le C\Big\|\Big(\sum_{j\in\mathbb{Z}^d}\sum_{k:t_k \ge t_j} c_{jk}^{\delta}\Big(t_k^{s+\frac{\nu}{2}} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Big)^q\Big)^{1/q}\Big\|_{L_p} \\
&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}^d}\Big(t_k^{s+\frac{\nu}{2}} M_r^B\Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|\chi_{Q(k,n)}\Big)\Big)^q\Big)^{1/q}\Big\|_{L_p}.
\end{aligned}
$$

Using the vector-valued Fefferman-Stein maximal inequality (3.5), we arrive at

$$
\|\mathbf{A}_0 s\|_{f^s_{p,q}(h)} \le C\Big\|\Big(\sum_{k,n\in\mathbb{Z}^d} (t_k^{s+\frac{\nu}{2}}|s_{k,n}|)^q \chi_{Q(k,n)}\Big)^{1/q}\Big\|_{L_p} = C\,\|s\|_{f^s_{p,q}(h)}.
$$

The corresponding estimate for $\mathbf{A}_1$ follows from the same type of arguments resulting in both $\mathbf{A}_0$ and $\mathbf{A}_1$ being bounded on $f^s_{p,q}(h)$ and thereby $\mathbf{A}$. For the cases $q = 1$ and $p \le 1, q > 1$ choose $0 < \tilde{r} < r$ and $0 < \tilde{\delta} < \delta$ such that $\nu/r + \delta/2 \ge \nu/\tilde{r} + \tilde{\delta}/2$ and repeat the argument with $r := \tilde{r}$, and $\delta := \tilde{\delta}$. The case $q < 1$ follows from first observing that

$$
\tilde{\mathbf{A}} := \{\tilde{a}_{(j,m)(k,n)}\} := \left\{ |a_{(j,m)(k,n)}|^q \left(\frac{t_k}{t_j}\right)^{\frac{\nu}{2}-\frac{\nu q}{2}} \right\}
$$

is almost diagonal on $f^{sq}_{\frac{p}{q},1}(h)$. Furthermore, if $v := \{v_{k,n}\} := \{|s_{k,n}|^q t_k^{\frac{\nu q}{2}-\frac{\nu}{2}}\}$ we have

$$\|v\|^{\frac{1}{q}}_{f^{sq}_{\frac{p}{q},1}(h)} = \Big\| \Big(\sum_{k,n\in\mathbb{Z}^d} \big(t_k^{s+\frac{\nu}{2}} |s_{k,n}|\big)^q \chi_{Q(k,n)} \Big)^{1/q} \Big\|_{L_p} = \|s\|_{f^s_{p,q}(h)}.$$

Before we can put these two observations into use we need that

$$|(\mathbf{A}s)_{(j,m)}|^q \leq \sum_k \sum_{n\in\mathbb{Z}^d} |a_{(j,m)(k,n)}|^q |s_{k,n}|^q = t_j^{\frac{\nu}{2}-\frac{\nu q}{2}} \sum_k \sum_{n\in\mathbb{Z}^d} \tilde{a}_{(j,m)(k,n)} v_{k,n}.$$

We then have

$$\|\mathbf{A}s\|_{f^s_{p,q}(h)} \leq \|\tilde{\mathbf{A}}v\|^{\frac{1}{q}}_{f^{sq}_{\frac{p}{q},1}(h)} \leq C\|v\|^{\frac{1}{q}}_{f^{sq}_{\frac{p}{q},1}(h)} = C\|s\|_{f^s_{p,q}(h)}.$$

■

The following shows that almost diagonal matrices are closed under composition. First, we simplify the notation by defining

$$w^{s,\delta}_{(j,m)(k,n)} := \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}}, \left(\frac{t_k}{t_j}\right)^{\frac{\delta}{2}} \right) c^{\delta}_{jk} \\ \times \left(1+\min(t_k,t_j)|x_{k,n}-x_{j,m}|_B\right)^{-\frac{\nu}{r}-\delta},$$

where we have used the notation from Definition 3.13.

Proposition 3.15.
Let $s \in \mathbb{R}$, $0 < r \leq 1$ and $\delta > 0$. We then have

$$\sum_{i,l\in\mathbb{Z}^d} w^{s,\delta}_{(j,m)(i,l)} w^{s,\delta}_{(i,l)(k,n)} \leq C w^{s,\delta/2}_{(j,m)(k,n)},$$

Proof:
Notice that the factors $t_i^{s+\frac{\nu}{2}}$ in the first terms of $w^{s,\delta}_{(j,m)(i,l)}$ and $w^{s,\delta}_{(i,l)(k,n)}$ cancel leaving $(t_k/t_j)^{s+\frac{\nu}{2}}$ which can be moved outside the sums. Therefore we only need to deal with the last three terms in $w^{s,\delta}_{(j,m)(i,l)}$ and $w^{s,\delta}_{(i,l)(k,n)}$. First we consider the case $t_j \leq t_k$ and split the sum over i into three parts,

$$\begin{aligned} \sum_{i,l\in\mathbb{Z}^d} w^{s,\delta}_{(j,m)(i,l)} w^{s,\delta}_{(i,l)(k,n)} &= \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \left(\sum_{i:t_i>t_k} + \sum_{i:t_j\leq t_i\leq t_k} + \sum_{i:t_i<t_j} \right) \sum_{l\in\mathbb{Z}^d} \cdots \\ &= \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} (\mathrm{I}+\mathrm{II}+\mathrm{III}). \end{aligned}$$

For I, by using Lemma 3.26 and Lemma 3.27, we have

$$\begin{aligned}
\mathrm{I} =& \sum_{i:t_i>t_k} \sum_{l\in\mathbb{Z}^d} \left(\frac{t_j}{t_i}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} \left(\frac{t_k}{t_i}\right)^{\frac{\delta}{2}} c_{ji}^{\delta} c_{ik}^{\delta} \\
&\times \frac{1}{(1+t_j|x_{j,m}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \frac{1}{(1+t_k|x_{k,n}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \\
\leq& \frac{C}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}} \sum_{i:t_i>t_k} \left(\frac{t_j}{t_i}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} \left(\frac{t_k}{t_i}\right)^{\frac{\delta}{2}-\nu} c_{ji}^{\delta} c_{ik}^{\delta} \\
\leq& C \left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{jk}^{\delta/2} \frac{1}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}}.
\end{aligned}$$

Similarly for II we get

$$\begin{aligned}
\mathrm{II} =& \sum_{i:t_j\leq t_i\leq t_k} \sum_{l\in\mathbb{Z}^d} \left(\frac{t_j}{t_i}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} \left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{ji}^{\delta} c_{ki}^{\delta} \\
&\times \frac{1}{(1+t_j|x_{j,m}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \frac{1}{(1+t_i|x_{k,n}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \\
\leq& C \left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{jk}^{\delta/2} \frac{1}{(1+t_j|x_{j,m}-x_{k,n}|)^{\frac{\nu}{r}+\delta}}.
\end{aligned}$$

For III we get

$$\begin{aligned}
\mathrm{III} =& \sum_{i:t_i<t_j} \sum_{l\in\mathbb{Z}^d} \left(\frac{t_i}{t_j}\right)^{\frac{\delta}{2}} \left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{ji}^{\delta} c_{ik}^{\delta} \\
&\times \frac{1}{(1+t_i|x_{j,m}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \frac{1}{(1+t_i|x_{k,n}-x_{i,l}|_B)^{\frac{\nu}{r}+\delta}} \\
\leq& \sum_{i:t_i<t_j} C \left(\frac{t_i}{t_j}\right)^{\frac{\delta}{2}} \left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{ji}^{\delta} c_{ik}^{\delta} \frac{1}{(1+t_i|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}} \\
\leq& \frac{C}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}} \sum_{i:t_i<t_j} C \left(\frac{t_i}{t_j}\right)^{\frac{\delta}{2}-\frac{\nu}{r}-\delta} \left(\frac{t_i}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{ji}^{\delta} c_{ik}^{\delta} \\
\leq& C \left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}} c_{jk}^{\delta/2} \frac{1}{(1+t_j|x_{j,m}-x_{k,n}|_B)^{\frac{\nu}{r}+\delta}}.
\end{aligned}$$

In the case $t_j > t_k$, we observe that $w^{s,\delta}_{(j,m)(k,n)} = w^{2\nu/r-s-\nu,\delta}_{(k,n)(j,m)}$ so applying the first case to $w^{2\nu/r-s-\nu,\delta}_{(k,n)(j,m)}$ proves the proposition for $t_j > t_k$.

■

It follows from Proposition 3.15 that for $\delta_1, \delta_2 > 0$ we have

$$\sum_{i,l\in\mathbb{Z}^d} w^{s,\delta_1}_{(j,m)(i,l)} w^{s,\delta_2}_{(i,l)(k,n)} \leq C w^{s,\min(\delta_1,\delta_2)/2}_{(j,m)(k,n)} \tag{3.22}$$

which proves that $\mathrm{ad}^s_{p,q}(h)$ is closed under composition.

3.4. New frames from old frames

In this section we study a system $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ which is a small perturbation of the frame $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constructed in (3.10). The goal is first to show that a system $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ which is close enough to $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is also a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$. Next, to get a frame expansion, we show that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is also a frame, where S is the frame operator

$$Sf = \sum_{k,n\in\mathbb{Z}^d} \langle f, \psi_{k,n}\rangle \psi_{k,n}.$$

The results are inspired by **[35]** where perturbations of frames were studied in classical Triebel-Lizorkin and Besov spaces.
Let $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset L_2(\mathbb{R}^d)$ be a system that is close to $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ in the sense that there exists $\varepsilon, \delta > 0$ such that

$$|\eta_{k,n}(x) - \psi_{k,n}(x)| \leq \varepsilon t_k^{\frac{\nu}{2}} (1 + t_k |x_{k,n} - x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \tag{3.23}$$

$$|\hat{\eta}_{k,n}(\xi) - \hat{\psi}_{k,n}(\xi)| \leq \varepsilon t_k^{-\frac{\nu}{2}} (1 + t_k^{-1} |\xi_k - \xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right) - \frac{2}{\beta}\left(|s| + \frac{2\nu}{r} + \frac{3\delta}{2}\right)}, \tag{3.24}$$

where we have used the notation from Definition 3.13. Motivated by the fact that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a tight frame for $L_2(\mathbb{R}^d)$, we formally define $\langle f, \psi_{j,m}\rangle$ as

$$\langle f, \psi_{j,m}\rangle := \sum_{k,n\in\mathbb{Z}^d} \langle \eta_{k,n}, \psi_{j,m}\rangle \langle f, \eta_{k,n}\rangle, \; f \in F^s_{p,q}(h).$$

It follows from Lemma 3.12 and Proposition 3.14 that $\langle \cdot, \psi_{j,m}\rangle$ is a bounded linear functional on $F^s_{p,q}(h)$; in fact we have

$$\begin{aligned} \sum_{k,n\in\mathbb{Z}^d} |\langle \eta_{k,n}, \psi_{j,m}\rangle| |\langle f, \eta_{k,n}\rangle| &\leq \Big\| \Big\{ \sum_{k,n\in\mathbb{Z}^d} |\langle \eta_{k,n}, \psi_{j,m}\rangle| |\langle f, \eta_{k,n}\rangle| \Big\}_{j,m\in\mathbb{Z}^d} \Big\|_{f^s_{p,q}(h)} \\ &\leq C \|\langle f, \eta_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\|f\|_{F^s_{p,q}(h)}. \end{aligned} \tag{3.25}$$

Furthermore, $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a norming family for $F^s_{p,q}(h)$ as it satisfies $\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\|f\|_{F^s_{p,q}(h)}$. This can be used to show that S is a bounded operator on $F^s_{p,q}(h)$, and for small enough ε, this will be the key to showing that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$.

THEOREM 3.16.
There exists $\varepsilon_0, C_1, C_2 > 0$ such that if $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.23) and (3.24) for some $0 < \varepsilon \leq \varepsilon_0$ and $f \in F^s_{p,q}(h)$, then we have

$$C_1\|f\|_{F^s_{p,q}(h)} \leq \|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C_2\|f\|_{F^s_{p,q}(h)}.$$

Similarly for $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

Proof:
The proof will only be given for $F^s_{p,q}(h)$ as it follows the same way for $M^s_{p,q}(h)$. That $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a norming family gives the upper bound, thus we only need to establish the lower bound. For this we notice that $\{\varepsilon^{-1}(\eta_{k,n} - \psi_{k,n})\}_{k,n\in\mathbb{Z}^d}$ is also a norming family so we have

$$\|\langle f, \eta_{k,n} - \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\varepsilon\|f\|_{F^s_{p,q}(h)}.$$

It then follows that

$$\begin{aligned}\|f\|_{F^s_{p,q}(h)} &\leq C\|\langle f, \eta_{k,n}\rangle\|_{f^s_{p,q}(h)}\\ &\leq C(\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} + \|\langle f, \eta_{k,n} - \psi_{k,n}\rangle\|_{f^s_{p,q}(h)})\\ &\leq C(\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} + \varepsilon\|f\|_{F^s_{p,q}(h)}).\end{aligned}$$

By choosing $\varepsilon < 1/C$ we get the lower bound.

■

As one might guess from Theorem 3.16, the boundedness of the matrix $\{\langle\eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ on $f^s_{p,q}(h)$ is the key to showing that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is also a frame.

PROPOSITION 3.17.
There exists $\varepsilon_0 > 0$ such that if $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^0_{22}(h) = L_2(\mathbb{R}^d)$ and satisfies (3.23) and (3.24) for some $0 < \varepsilon \leq \varepsilon_0$, then $\{\langle\eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

Proof:
The proof will only be given for $f^s_{p,q}(h)$ as it follows similarly for $m^s_{p,q}(h)$. The fact that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ ensures that S^{-1} is a bounded operator on $L_2(\mathbb{R}^d)$. We first show that S^{-1} is bounded on $F^s_{p,q}(h)$. This will follow from showing that

$$\|(I - S)f\|_{F^s_{p,q}(h)} \leq C\varepsilon\|f\|_{F^s_{p,q}(h)}, f \in F^s_{p,q}(h), \tag{3.26}$$

choosing ε small enough and using the Neumann series. Assume for the moment that $\mathcal{D} := \{d_{(j,m)(k,n)}\} := \{\langle (I-S)\eta_{k,n}, \eta_{j,m}\rangle\}$ satisfies

$$\|\mathcal{D}s\|_{f^s_{p,q}(h)} \le C\varepsilon \|s\|_{f^s_{p,q}(h)} \tag{3.27}$$

By using that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$, we have that S is self-adjoint which leads to

$$\begin{aligned}\|(I-S)f\|_{F^s_{p,q}(h)} &\le C\|\langle (I-S)f, \eta_{j,m}\rangle\|_{f^s_{p,q}(h)} = C\|\mathcal{D}\{\langle f, \eta_{k,n}\rangle\}_{k,n\in\mathbb{Z}^d}\|_{f^s_{p,q}(h)}\\ &\le C\varepsilon\|\langle f, \eta_{j,m}\rangle\|_{f^s_{p,q}(h)} \le C\varepsilon\|f\|_{F^s_{p,q}(h)}.\end{aligned}$$

So to show (3.26) it suffices to prove (3.27). Note that

$$\begin{aligned}\langle (I-S)\eta_{k,n}, \eta_{j,m}\rangle &= \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n}, \eta_{i,l}\rangle\langle \eta_{i,l}, \eta_{j,m}\rangle - \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n}, \psi_{i,l}\rangle\langle \psi_{i,l}, \eta_{j,m}\rangle\\ &= \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n}, \eta_{i,l}\rangle\langle \eta_{i,l}-\psi_{i,l}, \eta_{j,m}\rangle + \sum_{i,l\in\mathbb{Z}^d}\langle \eta_{k,n}, \eta_{i,l}-\psi_{i,l}\rangle\langle \psi_{i,l}, \eta_{j,m}\rangle.\end{aligned}$$

By setting

$$\begin{aligned}\mathcal{D}_1 &:= \{d_{1(j,m)(i,l)}\} := \{\langle \eta_{i,l}-\psi_{i,l}, \eta_{j,m}\rangle\},\\ \mathcal{D}_2 &:= \{d_{2(i,l)(k,n)}\} := \{\langle \eta_{k,n}, \eta_{i,l}\rangle\},\\ \mathcal{D}_3 &:= \{d_{3(j,m)(i,l)}\} := \{\langle \psi_{i,l}, \eta_{j,m}\rangle\},\\ \mathcal{D}_4 &:= \{d_{4(i,l)(k,n)}\} := \{\langle \eta_{k,n}, \eta_{i,l}-\psi_{i,l}\rangle\},\end{aligned}$$

we have the decomposition

$$\mathcal{D} = \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_3\mathcal{D}_4.$$

Since $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.23) and (3.24), we have from Lemma 3.12 that $\varepsilon^{-1}\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \varepsilon^{-1}\mathcal{D}_4 \in \mathrm{ad}^s_{p,q}(h)$. Next, we use that $\mathrm{ad}^s_{p,q}(h)$ is closed under composition (3.22), and by Proposition 3.14,

$$\|\mathcal{D}s\|_{f^s_{p,q}(h)} \le C\varepsilon\|s\|_{f^s_{p,q}(h)}.$$

Consequently, (3.26) holds, and for sufficiently small ε, the operator S^{-1} is bounded on $F^s_{p,q}(h)$. Finally, let $s := \{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \in f^s_{p,q}(h)$ and $g := \sum_{k,n\in\mathbb{Z}^d} s_{k,n}\eta_{k,n}$. By using Proposition 3.11 we have that $g \in F^s_{p,q}(h)$, and as $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$, we have that S^{-1} is self-adjoint which gives

$$\sum_{k,n\in\mathbb{Z}^d}\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle s_{k,n} = \sum_{k,n\in\mathbb{Z}^d}\langle S^{-1}\eta_{k,n}, \psi_{j,m}\rangle s_{k,n} = \langle S^{-1}g, \psi_{j,m}\rangle.$$

If we combine this with $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ being a norming family for $F^s_{p,q}(h)$, see (3.25), we get

$$\Big\|\sum_{k,n\in\mathbb{Z}^d}\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle s_{k,n}\Big\|_{f^s_{p,q}(h)} = \|\langle S^{-1}g, \psi_{j,m}\rangle\|_{f^s_{p,q}(h)} \le C\|S^{-1}g\|_{F^s_{p,q}(h)}$$

$$\leq C\|g\|_{F^s_{p,q}(h)} \leq C\|s\|_{f^s_{p,q}(h)}$$

which proves that $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$.

■

That $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$ now follows as a consequence of $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ being bounded on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$. We state the following results without proofs as they follow directly in the same way as in the classical Triebel-Lizorkin and Besov spaces. The proofs can be found in **[35]**. First, we have the frame expansion.

LEMMA 3.18.
Assume that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies

$$|\psi_{k,n}(x)| \leq Ct_k^{\frac{\nu}{2}}(1+t_k|x_{k,n}-x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \tag{3.28}$$

$$|\hat{\psi}_{k,n}(\xi)| \leq Ct_k^{-\frac{\nu}{2}}(1+t_k^{-1}|\xi_k-\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)}, \tag{3.29}$$

where we have used the notation from Definition 3.13. If $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$, then for $f\in F^s_{p,q}(h)$ we have

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, S^{-1}\psi_{k,n}\rangle \psi_{k,n}$$

in the sense of $\mathcal{S}'$. Similarly for $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

□

Moreover, we have that $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame.

THEOREM 3.19.
Assume that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies (3.28) and (3.29). Then $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $F^s_{p,q}(h)$ if and only if $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$. Similarly for $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

□

It is worth noting that Proposition 3.17, Lemma 3.18 and Theorem 3.19 imply that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is an atomic decomposition if it satisfies (3.23) and (3.24) with sufficiently small ε and $p,q\geq 1$. Furthermore, we also have a frame expansion with $\{S^{-1}\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$.

LEMMA 3.20.
Assume that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies (3.28) and (3.29). If the transpose of $\{\langle \eta_{k,n}, S^{-1}\psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ is bounded on $f^s_{p,q}(h)$, then for $f\in F^s_{p,q}(h)$ we have

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, \psi_{k,n}\rangle S^{-1}\psi_{k,n}$$

in the sense of $\mathcal{S}'$. Similarly for $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

$\square$

REMARK 3.21.
If we have that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is normalized in $L_2(\mathbb{R}^d)$, then $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is an orthonormal basis for $L_2(\mathbb{R}^d)$ as a consequence of $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ being a tight frame for $L_2(\mathbb{R}^d)$ with constant 1. With arguments similar to the ones used in the proof of Proposition 3.17, it can be shown that there exists ε_0 such that if $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.23) and (3.24) for some $\varepsilon \leq \varepsilon_0$, then $\{\langle \eta_{k,n}, \psi_{j,m}\rangle\}_{k,n,j,m\in\mathbb{Z}^d}$ has a bounded inverse on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$, and consequently $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is an unconditional basis for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$.

For example, by using the unconditional basis for anisotropic α-modulation spaces constructed in Chapter 2, one can generate a compactly supported basis for the anisotropic α-modulation spaces. $\circ$

3.5. Construction of new frames

In this section we generate compactly supported frame expansions for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$. More precisely, we show that finite linear combinations, $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$, of shifts and dilates of a function g with sufficient decay in both the direct and the frequency space can fulfill (3.23) and (3.24). As a consequence of the previous section, $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ will then constitute frames for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$. In particular, by using a generating function g with compact support one can construct a compactly supported frame expansion. This is, as far as the authors are aware, a new approach. Earlier work, as in [**34**], used finite linear combinations of a function with sufficient smoothness and decay in the direct space and vanishing moments.

It suffices to prove that there exists a system of functions $\{\tau_k\}_{k\in\mathbb{Z}^d} \subset L_2(\mathbb{R}^d)$ which is close enough to $\{\mu_k\}_{k\in\mathbb{Z}^d}$ (3.10):

$$|\mu_k(x) - \tau_k(x)| \leq \varepsilon(1+|x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)},$$
$$|\hat{\mu}_k(\xi) - \hat{\tau}_k(\xi)| \leq \varepsilon(1+|\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)}.$$

The system

$$\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d} := \left\{ t_k^{\nu/2}\tau_k\Big(\delta_{t_k}^\top x - \frac{\pi}{a}n\Big)e^{ix\cdot\xi_k}\right\}_{k,n\in\mathbb{Z}^d}$$

will then satisfy (3.23) and (3.24). First, we take $g \in C^1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$, $\hat{g}(0)\neq 0$, which for fixed $N, M > 0$ satisfies

$$|g^{(\kappa)}(x)| \leq C(1+|x|_B)^{-N-\alpha_1},\ |\kappa|\leq 1, \tag{3.30}$$

$$|\hat{g}(\xi)| \leq C(1+|\xi|_A)^{-M-\alpha_2}. \tag{3.31}$$

Next for $m \geq 1$, we define $g_m(x) := C_g m^\nu g(\delta_m^\top x)$, where $C_g := \hat{g}(0)^{-1}$. It then follows that

$$
\begin{aligned}
|g_m^{(\kappa)}(x)| &\leq C m^{\nu+\alpha_2|\kappa|}(1+m|x|_B)^{-N-\alpha_1}, \ |\kappa| \leq 1, \\
\int_{\mathbb{R}} g_m(x)\,\mathrm{d}x &= 1, \\
|\hat{g}_m(\xi)| &\leq C m^{M+\alpha_2}(1+|\xi|_A)^{-M-\alpha_2}.
\end{aligned}
\tag{3.32}
$$

To construct τ_k we also need a set of finite linear combinations,

$$\Theta_{K,m} = \{\psi : \psi(\cdot) = \sum_{i=1}^{K} a_i g_m(\cdot + b_i), a_i \in \mathbb{C}, b_i \in \mathbb{R}^d\}.$$

We are now ready to show that any function with sufficient decay in the direct and the frequency space can be approximated to an arbitrary degree by a finite linear combination of another function with similar decay.

Proposition 3.22.
Let $N' > N > \nu$ and $M' > M > \nu$. If $g \in C^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, $\hat{g}(0) \neq 0$, fulfills (3.30) and (3.31) and $\mu_k \in C^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ fulfills

$$
\begin{aligned}
|\mu_k(x)| &\leq C(1+|x|_B)^{-N'}, \\
|\mu_k^{(\kappa)}(x)| &\leq C, \ |\kappa| \leq 1, \\
|\hat{\mu}_k(\xi)| &\leq C(1+|\xi|_A)^{-M'},
\end{aligned}
$$

then for any $\varepsilon > 0$ there exists $K, m \geq 1$ and $\tau_k \in \Theta_{K,m}$ such that

$$|\mu_k(x) - \tau_k(x)| \leq \varepsilon(1+|x|_B)^{-N}, \tag{3.33}$$

$$|\hat{\mu}_k(\xi) - \hat{\tau}_k(\xi)| \leq \varepsilon(1+|\xi|_A)^{-M}. \tag{3.34}$$

Proof:
We construct the approximation of μ_k in the direct space in three steps. First, by a convolution operator $\omega_m = \mu_k * g_m$, then, by $\theta_{q,m}$ which is the integral in ω_m taken over a region Q, and finally, by a discretization over dyadic cubes $\tau_{l,q,m}$. From (3.32) we have

$$\mu_k(x) - \omega_m(x) = \int_{\mathbb{R}^d} (\mu_k(x) - \mu_k(x-y)) g_m(y)\,\mathrm{d}y. \tag{3.35}$$

Define $U := m^{\lambda/2N}$, where $\lambda := \min(\alpha_1, N'-N)$. For $|x|_B \leq U$, we use the mean value theorem to get

$$|\mu_k(x) - \mu_k(x-y)| \leq C \min(1, |y|).$$

Inserting this in (3.35) we have

$$|\mu_k(x) - \omega_m(x)| \leq C \int_{\mathbb{R}^d} \frac{\min(1, |y|_B^{\alpha_1}) m^\nu}{(1+m|y|_B)^{N+\alpha_1}}\,\mathrm{d}y$$

$$\leq Cm^{-\alpha_1} \leq \frac{Cm^{-\lambda/2}}{U^N} \leq \frac{Cm^{-\lambda/2}}{(1+|x|_B)^N}. \tag{3.36}$$

For $|x|_B > U$, we split the integral over $\Omega := \{y : |y|_B \leq |x|_B/2C_B\}$ and Ω^c. If $y \in \Omega$, then $|x-y|_B \geq |x|_B/2C_B$, and we have

$$\begin{aligned}\int_\Omega |\mu_k(x) - \mu_k(x-y)||g_m(y)|\,\mathrm{d}y &\leq C(1+|x|_B)^{-N'} \\ &\leq \frac{C}{(1+U)^\lambda(1+|x|_B)^N} \leq \frac{Cm^{-\lambda^2/2N}}{(1+|x|_B)^N}.\end{aligned} \tag{3.37}$$

Integrating over Ω^c with $|x|_B > U$ gives

$$\begin{aligned}&\int_{\Omega^c} |\mu_k(x) - \mu_k(x-y)||g_m(y)|\,\mathrm{d}y \\ &\qquad\leq \frac{C}{(1+|x|_B)^{N'}} + \int_{\Omega^c} \frac{Cm^\nu}{(1+|x-y|_B)^{N'}(1+m|y|_B)^{N+\alpha_1}}\,\mathrm{d}y \\ &\qquad\leq \frac{C}{(1+|x|_B)^{N'}} + \frac{Cm^{-\lambda}}{(1+|x|_B)^N} \leq \frac{C(m^{-\lambda^2/2N}+m^{-\lambda})}{(1+|x|_B)^N}.\end{aligned} \tag{3.38}$$

So by choosing m sufficiently large in (3.36)-(3.38), we get

$$|\mu_k(x) - \omega_m(x)| \leq \frac{\varepsilon}{3}(1+|x|_B)^{-N}. \tag{3.39}$$

For the next step we fix m and choose $q \in \mathbb{N}$. Let $H_{l,q}$ denote the smallest set of dyadic cubes aligned with the coordinate axes and sidelength 2^{-l}, $l \in \mathbb{N}$, that covers $\mathcal{B}_B(0,2^q)$. We then approximate ω_m with $\theta_{q,m}$ defined as

$$\theta_{q,m}(\cdot) = \int_Q \mu_k(y)g_m(\cdot - y)\,\mathrm{d}y,$$

where $Q = \cup_{I\in H_{l,q}} I$. In which case we have

$$\omega_m(x) - \theta_{q,m}(x) = \int_{Q^c} \mu_k(y)g_m(x-y)\,\mathrm{d}y,$$

and it follows that

$$|\omega_m(x) - \theta_{q,m}(x)| \leq \int_{|y|_B\geq 2^q} \frac{Cm^\nu}{(1+|y|_B)^{N'}(1+m|x-y|_B)^{N+\alpha_1}}\,\mathrm{d}y := L.$$

We first estimate the integral for $|x|_B \leq 2^{q-1}/C_B$ which gives $|y|_B \geq |x|_B$ and $|x-y|_B \geq 2^{q-1}/C_B$. Hence we obtain

$$L \leq \frac{C}{(1+|x|_B)^{N'}} \int_{|u|\geq \frac{2^{q-1}}{C_B}} \frac{m^\nu}{(1+m|u|_B)^{N+\alpha_1}}\,\mathrm{d}u \leq \frac{Cm^{-\lambda}2^{-\lambda q}}{(1+|x|_B)^{N'}}. \tag{3.40}$$

For $|x|_B > 2^{q-1}/C_B$, we split the integral over $\Omega := \{y : |y|_B \geq 2^q\} \cap \{y : |y|_B \leq |x|_B/2C_B\}$ and $\Omega' := \{y : |y|_B \geq 2^q\} \cap \{y : |y|_B > |x|_B/2C_B\}$. If $y \in \Omega$, then $|x-y|_B \geq |x|_B/2C_B$, and we get

$$\begin{aligned}\int_\Omega \frac{m^\nu}{(1+|y|_B)^{N'}(1+m|x-y|_B)^{N+\alpha_1}}\,\mathrm{d}y &\leq \frac{Cm^\nu}{(1+m|x|_B)^{N+\alpha_1}}\int_{|y|_B\geq 2^q}\frac{1}{(1+|y|_B)^{N'}}\,\mathrm{d}y\\ &\leq \frac{Cm^{-\lambda}2^{-\lambda q}}{(1+|x|_B)^N}.\end{aligned} \tag{3.41}$$

Similar for Ω' we have

$$\begin{aligned}\int_{\Omega'} \frac{m^\nu}{(1+|y|_B)^{N'}(1+m|x-y|_B)^{N+\alpha_1}}\,\mathrm{d}y &\leq \frac{C}{(1+|x|_B)^{N'}}\int_{\mathbb{R}^d}\frac{m^\nu}{(1+m|x-y|_B)^{N+\alpha_1}}\,\mathrm{d}y\\ &\leq \frac{C}{(1+|x|_B)^{N'}} \leq \frac{C2^{-\lambda q}}{(1+|x|_B)^N}.\end{aligned} \tag{3.42}$$

By choosing q sufficiently large in (3.40)-(3.42), we obtain

$$|\omega_m(x) - \theta_{q,m}(x)| \leq \frac{\varepsilon}{3}(1+|x|_B)^{-N}. \tag{3.43}$$

For the final step, we fix q and approximate $\theta_{q,m}$ by a discretization $\tau_{l,q,m}$,

$$\tau_{l,q,m}(\cdot) = \sum_{I\in H_{l,q}} |I|\mu_k(x_I)g_m(\cdot - x_I),$$

where x_I is the center of the dyadic cube I. Now choose $q' > q$ such that $Q \subset \mathcal{B}_B(0, 2^{q'})$, and note that $\tau_{l,q,m} \in \Theta_{K,m}$, $K < 2^{dl+\nu q'}$. We introduce $F(\cdot) := \mu_k(\cdot)g_m(x-\cdot)$ which gives

$$\begin{aligned}|\theta_{q,m}(x) - \tau_{l,q,m}(x)| &\leq \sum_{I\in H_{l,q}}\int_I |\mu_k(y)g_m(x-y) - \mu_k(x_I)g_m(x-x_I)|\,\mathrm{d}y\\ &\leq \sum_{I\in H_{l,q}}\int_I |F(y)-F(x_I)|\,\mathrm{d}y.\end{aligned}$$

By using the mean value theorem, we then get

$$\begin{aligned}|\theta_{q,m}(x) - \tau_{l,q,m}(x)| &\leq \sum_{I\in H_{l,q}}\int_I |y-x_I| \max_{\substack{z\in l(x_I,y)\\ |\kappa|\leq 1}} |F^{(\kappa)}(z)|\,\mathrm{d}y\\ &\leq C2^{\nu q'-l}\max_{\substack{z\in\mathcal{B}_B(0,2^{q'})\\ |\kappa|\leq 1}} |g_m^{(\kappa)}(x-z)|,\end{aligned} \tag{3.44}$$

where $l(x_I, y)$ is the line-segment between x_I and y. If $|x|_B \le 2^{q'+1}C_B$ and $|\kappa| \le 1$, then we have

$$|g_m^{(\kappa)}(x-z)| \le Cm^{\nu+\alpha_2} \le \frac{Cm^{\nu+\alpha_2}2^{q'N}}{(1+|x|_B)^N}. \tag{3.45}$$

For $|x|_B > 2^{q'+1}C_B$ and $z \in \mathcal{B}_B(0, 2^{q'})$, we have $|x-z|_B \ge |x|_B/2C_B$, and hence for $|\kappa| \le 1$, it follows that

$$|g_m^{(\kappa)}(x-z)| \le \frac{Cm^{\nu+\alpha_2}}{(1+m|x|_B)^{N+\alpha_1}} \le \frac{Cm^{\alpha_2-\alpha_1}}{(1+|x|_B)^{N+\alpha_1}}. \tag{3.46}$$

By choosing l sufficiently large, and combining (3.44)-(3.46), we have

$$|\theta_{q,m}(x) - \tau_{l,q,m}(x)| \le \frac{\varepsilon}{3}(1+|x|_B)^{-N}. \tag{3.47}$$

Finally by combining (3.39), (3.43) and (3.47), we get

$$|\mu_k(x) - \tau_{l,q,m}(x)| \le \varepsilon(1+|x|_B)^{-N}. \tag{3.48}$$

To approximate μ_k in the frequency space we use three steps similar to the approximation in the direct space. Note that $\tau_{l,q,m}$ still fulfills (3.48) if we choose l, q, m even larger. First, we use $\hat{\omega}_m$ to approximate $\hat{\mu}_k$ in which case we have

$$\begin{aligned} |\hat{\mu}_k(\xi) - \hat{\omega}_m(\xi)| &= |\hat{\mu}_k(\xi)(1 - C_g\hat{g}(\delta_{\frac{1}{m}}\xi))| \\ &\le C(1+|\xi|_A)^{-M}(1+|\xi|_A)^{M-M'}|1 - C_g\hat{g}(\delta_{\frac{1}{m}}\xi)|. \end{aligned}$$

By choosing $a > 0$ such that $C(1+a)^{M-M'}|1 - C_g\hat{g}(\delta_{\frac{1}{m}}\xi)| \le \varepsilon/3$ and m such that $C|1 - C_g\hat{g}(\delta_{\frac{1}{m}}\xi)| \le \varepsilon/3$ for $|\xi|_A < a$, we get

$$|\hat{\mu}_k(\xi) - \hat{\omega}_m(\xi)| \le \frac{\varepsilon}{3}(1+|\xi|_A)^{-M}. \tag{3.49}$$

Next, we fix m, choose q and limit the Fourier integral of μ_k to Q from the approximation in the direct space,

$$\theta'_{q,m}(\xi) = \hat{g}_m(\xi)\int_Q \mu_k(x)e^{ix\cdot\xi}\,\mathrm{d}x.$$

This gives

$$|\hat{\omega}_m(\xi) - \theta'_{q,m}(\xi)| \le |\hat{g}_m(\xi)|\int_{|x|_B \ge 2^q} |\mu_k(x)e^{ix\cdot\xi}|\,\mathrm{d}x \le \frac{Cm^M 2^{-\lambda q}}{(1+|\xi|_A)^M}. \tag{3.50}$$

In the last step, we fix q and approximate $\theta'_{q,m}$ by $\hat{\tau}_{l,q,m}$. We introduce $G(x) := \mu_k(x)e^{ix\cdot\xi}$ and reuse q' from the approximation in the direct space to get

$$|\theta'_{q,m}(\xi) - \hat{\tau}_{l,q,m}(\xi)| \le |\hat{g}_m(\xi)|\left|\int_Q \mu_k(x)e^{ix\cdot\xi}\,\mathrm{d}x - \sum_{I\in H_{l,q}} |I|\mu_k(x_I)e^{ix_I\cdot\xi}\right|$$

$$
\begin{aligned}
&\leq |\hat{g}_m(\xi)| \sum_{I \in H_{l,q}} \int_I |G(x) - G(x_I)| \, \mathrm{d}x \\
&\leq \frac{Cm^{M+\alpha_2} 2^{\nu q' - l}}{(1+|\xi|_A)^{M+\alpha_2}} \max_{\substack{x \in \mathbb{R}^d \\ |\kappa| \leq 1}} |G^{(\kappa)}(x)| \leq \frac{Cm^{M+\alpha_2} 2^{\nu q' - l}}{(1+|\xi|_A)^{M}}. \qquad (3.51)
\end{aligned}
$$

By combining (3.49)-(3.51) with sufficiently large l, q, m, we get

$$|\hat{\mu}_k(\xi) - \hat{\tau}_{l,q,m}(\xi)| \leq \varepsilon (1 + |\xi|_A)^{-M}.$$

It follows that by choosing l, q, m large enough, $\tau_{l,q,m}$ fulfills both (3.33) and (3.34). Furthermore, we have $\tau_{l,q,m} \in \Theta_{K,m}$, $K < 2^{dl + \nu q'}$.

■

3.6. Discussion and further examples

In this paper we studied a flexible method of generating frames for T-L type spaces and the associated modulation spaces. With Proposition 3.17, Lemma 3.18 and Theorem 3.19, we proved that a system, which is sufficiently close to a frame for certain types of T-L type spaces and the associated modulation spaces, also constitutes a frame for these spaces. Furthermore, with Proposition 3.22 we construct such a system from finite linear combinations of shifts and dilates of a single function with sufficient decay in both the direct and the frequency space.

Examples of functions with sufficient decay in both the direct and the frequency space are $e^{-|\cdot|_B}$ and $(1 + |\cdot|_B)^{-N}$ with N sufficiently large. By using (3.3), we can simplify this even further and use the exponential function $e^{-|\cdot|^2}$ or the rational functions $(1 + |\cdot|^2)^{-N/2\alpha_1}$. An example with compact support can be constructed by using a spline with compact support. Furthermore, as the system is constructed using finite linear combinations of splines, we get a system consisting of modulated compactly supported splines.

As a last remark, we draw attention to the fact that the methods used in Section 3.4 and 3.5 do not depend on the assumptions made on the function h in the beginning of Section 3.3. These assumptions are only needed to prove that the "change of frame coefficient" matrices are bounded and closed under compositions. For the anisotropic α-modulation spaces, spaces with $0 \leq \alpha < 1$ satisfy the assumptions in Section 3.3, but the case $\alpha = 1$ does not. To deal with the case $\alpha = 1$, we mention that one can use a definition of almost diagonal matrices closer to the one for the classical Triebel-Lizorkin and Besov spaces which does not require these assumptions. These almost diagonal matrices were introduced in **[6]** and proven to be bounded. Furthermore, they can be used to show that the "change of frame coefficient" matrices are also bounded and closed under compositions in the case $\alpha = 1$. It follows that the methods in Sections 3.4 and 3.5 can be used to construct frames for a variety of decomposition spaces given the right definition of almost diagonal matrices.

Appendix

In this appendix we prove five technical lemmas which we used in Section 3.3. We use the same notation as in Sections 3.2 and 3.3. First, we used the following lemma to prove Lemma 3.12.

LEMMA 3.23.
Let $N > \nu$ and suppose $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ satisfies (3.14), and $\{\psi_{k,n}\}_{k,n\mathbb{Z}^d}$ satisfies (3.17). We then have

$$|\langle \eta_{k,n}, \psi_{j,m}\rangle| \leq C \min\left(\frac{t_k}{t_j}, \frac{t_j}{t_k}\right)^{\frac{\nu}{2}} (1+\min(t_k,t_j)|x_{k,n}-x_{j,m}|_B)^{-N}, \tag{3.52}$$

with t_k defined in (3.8) and $x_{k,n}$ in (3.16).

Proof:
Without loss of generality assume that $t_k \leq t_j$. First we consider the case $t_k|x_{k,m}-x_{j,m}|_B \leq 1$. It then follows that

$$\frac{t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x|_B)^N} \leq t_k^{\frac{\nu}{2}} \leq \frac{2^N t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N}, \tag{3.53}$$

and we have

$$\begin{aligned}
|\langle \eta_{k,n}, \psi_{j,m}\rangle| &\leq \frac{C t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N} \int_{\mathbb{R}^d} \frac{t_j^{\frac{\nu}{2}}}{(1+t_j|x_{j,m}-x|_B)^N}\,\mathrm{d}x \\
&= \frac{C t_k^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N} \int_{\mathbb{R}^d} \frac{t_j^{-\frac{\nu}{2}}}{(1+|x|_B)^N}\,\mathrm{d}x \\
&\leq C\left(\frac{t_k}{t_j}\right)^{\frac{\nu}{2}} (1+t_k|x_{k,n}-x_{j,m}|_B)^{-N} \qquad (3.54)
\end{aligned}$$

since the space associated with $|\cdot|_B$ has homogeneous dimension ν. For the other case, $t_k|x_{k,m}-x_{j,m}|_B > 1$, we consider two additional cases. In the first case, we assume that $|x_{k,n}-x|_B \geq \frac{1}{2C_B}|x_{k,n}-x_{j,m}|_B$. Similar to above we then get (3.53) which leads to (3.54). In the last case, we have $|x_{k,n}-x|_B < \frac{1}{2C_B}|x_{k,n}-x_{j,m}|_B$ which gives $|x_{j,m}-x|_B > \frac{1}{2C_B}|x_{k,n}-x_{j,m}|_B$. It then follows that

$$\frac{1}{(1+t_j|x_{j,m}-x|_B)^N} \leq \frac{C_1}{(1+t_j|x_{k,n}-x_{j,m}|_B)^N} \leq \frac{C_2(t_k/t_j)^N}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N},$$

and we have

$$\begin{aligned}|\langle \eta_{k,n}, \psi_{j,m}\rangle| &\leq \frac{C(t_k/t_j)^{\frac{\nu}{2}}}{(1+t_k|x_{k,n}-x_{j,m}|_B)^N}\int_{\mathbb{R}^d}\frac{t_k^\nu}{(1+t_k|x_{k,n}-x|_B)^N}\,\mathrm{d}x\\ &\leq C\left(\frac{t_k}{t_j}\right)^{\frac{\nu}{2}}(1+t_k|x_{k,n}-x_{j,m}|_B)^{-N}.\end{aligned}$$

■

The following estimate in the direct space was used to prove Proposition 3.14.

LEMMA 3.24.
Suppose that $0 < r \leq 1$ and $N > \nu/r$. Then for any sequence $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$, and for $x \in Q(j,m)$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}\frac{|s_{k,n}|}{(1+\min(t_k,t_j)|x_{k,n}-x_{j,m}|_B)^N} \leq &C\max\left(\frac{t_k}{t_j},1\right)^{\frac{\nu}{r}}\\ &\times M_r^B\Big(\sum_{n\in\mathbb{Z}^d}|s_{k,n}|\chi_{Q(k,n)}\Big)(x),\end{aligned}\tag{3.55}$$

with t_k defined in (3.8), $x_{k,n}$ in (3.16), $Q(j,m)$ in (3.12) and M_r^B in (3.4).

Proof:
Without loss of generality we may assume $x_{j,m} = 0$ and begin by considering the case $t_k \leq t_j$. We define the sets,

$$\begin{aligned}A_0 &= \{n\in\mathbb{Z}^d : t_k|x_{k,n}|_B \leq 1\},\\ A_i &= \{n\in\mathbb{Z}^d : 2^{i-1} < t_k|x_{k,n}|_B \leq 2^i\},\;\; i\geq 1.\end{aligned}$$

Choose $x \in Q(j,m)$. There exists $C_1 > 0$ such that $\cup_{n\in A_i}Q(k,n) \subset \mathcal{B}_B(x, C_1 2^i t_k^{-1})$, and by using $\int \chi_{Q(k,n)} = \omega_d^B t_k^{-\nu}$, we get

$$\begin{aligned}\sum_{n\in A_i}\frac{|s_{k,n}|}{(1+t_k|x_{k,n}|)^N} &\leq C2^{-iN}\sum_{n\in A_i}|s_{k,n}| \leq C2^{-iN}\Big(\sum_{n\in A_i}|s_{k,n}|^r\Big)^{\frac{1}{r}}\\ &\leq C2^{-iN}\left(\frac{t_k^\nu}{\omega_d^B}\int_{\mathcal{B}_B(x,C_1 2^i t_k^{-1})}\sum_{n\in A_i}|s_{k,n}|^r\chi_{Q(k,n)}\right)^{\frac{1}{r}}.\end{aligned}$$

Hence by the definition of the maximal operator (3.4) we have

$$\begin{aligned}\sum_{n\in A_i}\frac{|s_{k,n}|}{(1+t_k|x_{k,n}|)^N} &\leq C2^{i(\frac{\nu}{r}-N)}\left(\frac{t_k^\nu}{2^{i\nu}\omega_d^B}\int_{\mathcal{B}_B(x,C_1 2^i t_k^{-1})}\sum_{n\in A_i}|s_{k,n}|^r\chi_{Q(k,n)}\right)^{\frac{1}{r}}\\ &\leq C2^{i(\frac{\nu}{r}-N)}M_r^B\Big(\sum_{n\in\mathbb{Z}^d}|s_{k,n}|\chi_{Q(k,n)}\Big)(x)\end{aligned}$$

by using $\sum_{n\in\mathbb{Z}^d}\chi_{Q(k,n)} \leq n_0$. Summing over $i \geq 0$ and using $N > \nu/r$ gives (3.55). For the second case, $t_k > t_j$, we redefine the sets,

$$A_0 = \{n \in \mathbb{Z}^d : t_j|x_{k,n}|_B \leq 1\}$$
$$A_i = \{n \in \mathbb{Z}^d : 2^{i-1} < t_j|x_{k,n}|_B \leq 2^i\}, i \geq 1.$$

As before we have

$$\begin{aligned}\sum_{n\in A_i}\frac{|s_{k,n}|}{(1+t_j|x_{k,n}|)^M} \leq & C2^{-iN}\left(\frac{t_k^\nu}{\omega_d^B}\int_{\mathcal{B}_B(x,C_1 2^i t_j^{-1})}\sum_{n\in A_i}|s_{k,n}|^r\chi_{Q(k,n)}\right)^{\frac{1}{r}} \\ \leq & C2^{i(\frac{\nu}{r}-N)}\left(\frac{t_k}{t_j}\right)^{\frac{\nu}{r}} M_r^B\Big(\sum_{n\in\mathbb{Z}^d}|s_{k,n}|\chi_{Q(k,n)}\Big)(x).\end{aligned}$$

Summing over $i \geq 0$ again gives (3.55).

■

To prove Proposition 3.14 we also used the following estimate in the frequency space.

Lemma 3.25.
Let $\delta > 0$. There exists $C > 0$ independent of k such that

$$\sum_{j\in\mathbb{Z}^d}\min\left(\left(\frac{t_j}{t_k}\right)^\nu, \left(\frac{t_k}{t_j}\right)^\delta\right)(1+\max(t_k,t_j)^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta} \leq C,$$

with t_k defined in (3.8) and ξ_k in Lemma 3.7.

Proof:
We begin by dividing the indices into sets,

$$A_0 = \{j \in \mathbb{Z}^d : |\xi_j - \xi_k|_A \leq \rho_1 t_k\}$$
$$A_i = \{j \in \mathbb{Z}^d : 2^{i-1}\rho_1 t_k < |\xi_j - \xi_k|_A \leq 2^i\rho_1 t_k\},\ \ i \geq 1,$$

with ρ_1 defined in (3.13). For $j \in A_i$, we have $\mathcal{B}_A(\xi_j, t_j) \subset \mathcal{B}_A(\xi_k, C_1 2^i t_k)$ which follows from using (3.13):

$$\begin{aligned}|\xi_k - \xi|_A \leq C_A(|\xi_k - \xi_j|_A + |\xi_j - \xi|_A) \leq & C_A(2^i\rho_1 t_k + t_j) \\ \leq & C_A(2^i\rho_1 t_k + R_1 2^i t_k) \\ = & C_1 2^i t_k,\end{aligned}$$

for $\xi \in \mathcal{B}_A(\xi_j, t_j)$. Next, we divide the sum even further by first looking at $t_k \geq t_j$, and by using that the covering $\{\mathcal{B}_A(\xi_j, t_j)\}_j$ is admissible, we get

$$\begin{aligned}
&\sum_{\substack{j \in A_i \\ j : t_j \leq t_k}} \left(\frac{t_j}{t_k}\right)^{\nu} (1 + t_k^{-1}|\xi_j - \xi_k|_A)^{-\nu-\delta} \\
&\qquad \leq C 2^{-i(\nu+\delta)} \sum_{\substack{j \in A_i \\ j : t_j \leq t_k}} \left(\frac{t_j}{t_k}\right)^{\nu} \frac{1}{\omega_d^A t_j^{\nu}} \int_{\mathcal{B}_A(\xi_j, t_j)} \chi_{\mathcal{B}_A(\xi_j, t_j)}(\xi) \, \mathrm{d}\xi \\
&\qquad \leq C 2^{-i(\nu+\delta)} \frac{1}{\omega_d^A t_k^{\nu}} \int_{\mathcal{B}_A(\xi_k, C_1 2^i t_k)} \sum_{\substack{j \in A_i \\ j : t_j \leq t_k}} \chi_{\mathcal{B}_A(\xi_j, t_j)}(\xi) \, \mathrm{d}\xi \\
&\qquad \leq C 2^{-i\delta}.
\end{aligned}$$

Summing over i gives the lemma for the $t_k \geq t_j$ part of the sum. In a similar way, the result for $t_k < t_j$ follows by using

$$\sum_{\substack{j \in A_i \\ j : t_j > t_k}} \left(\frac{t_k}{t_j}\right)^{\delta} (1 + t_j^{-1}|\xi_j - \xi_k|_A)^{-\nu-\delta} \leq \sum_{\substack{j \in A_i \\ j : t_j > t_k}} \left(\frac{t_j}{t_k}\right)^{\nu} (1 + t_k^{-1}|\xi_j - \xi_k|_A)^{-\nu-\delta}.$$

■

The following estimate in the direct space was used to prove Proposition 3.15.

Lemma 3.26.
Assume that $t_j \leq t_k$, $N > \nu$ and

$$g := \sum_{l \in \mathbb{Z}^d} \frac{1}{(1 + \min(t_j, t_i)|x_{j,m} - x_{i,l}|_B)^N} \frac{1}{(1 + \min(t_k, t_i)|x_{k,n} - x_{i,l}|_B)^N},$$

with t_k defined in (3.8) and $x_{k,n}$ in (3.16). We then have

$$g \leq \frac{C}{(1 + \min(t_j, t_i)|x_{j,m} - x_{k,n}|_B)^N} \max\left(\frac{t_i}{t_k}, 1\right)^{\nu}.$$

Proof:
Note that from Lemma 3.24 with $r = 1$ and $s_{k,n} = 1$, it follows that

$$\sum_{l \in \mathbb{Z}^d} \frac{1}{(1 + \min(t_k, t_i)|x_{k,n} - x_{i,l}|_B)^N} \leq C \max\left(\frac{t_i}{t_k}, 1\right)^{\nu}. \tag{3.56}$$

We first consider the case $\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B \le 1$ which gives

$$\begin{aligned} g &\le \sum_{l\in\mathbb{Z}^d} \frac{1}{(1+\min(t_k,t_i)|x_{k,n}-x_{i,l}|_B)^N} \\ &\le C\max\left(\frac{t_i}{t_k},1\right)^\nu \\ &\le \frac{C}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N}\max\left(\frac{t_i}{t_k},1\right)^\nu . \end{aligned}$$

For the case $\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B > 1$ we split the sum into

$$A = \left\{l\in\mathbb{Z}^d : |x_{j,m}-x_{i,l}|_B < \tfrac{1}{2C_B}|x_{j,m}-x_{k,n}|_B\right\}$$

and its complement. For A^c we have

$$\frac{1}{(1+\min(t_j,t_i)|x_{j,m}-x_{i,l}|_B)^N} \le \frac{(2C_B)^N}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N},$$

and by using (3.56), the desired estimate follows. For $l\in A$, we notice that $|x_{k,n}-x_{i,l}|_B > \frac{1}{2C_B}|x_{j,m}-x_{k,n}|_B$ and get

$$\begin{aligned} &(1+\min(t_k,t_i)|x_{k,n}-x_{i,l}|_B)^{-N} \\ &\qquad\le \left(1+\tfrac{1}{2C_B}\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B\frac{\min(t_k,t_i)}{\min(t_j,t_i)}\right)^{-N} \\ &\qquad\le \frac{C}{(1+\min(t_j,t_i)|x_{j,m}-x_{k,n}|_B)^N}\left(\frac{\min(t_j,t_i)}{\min(t_k,t_i)}\right)^\nu . \end{aligned} \tag{3.57}$$

Next, by using (3.56) with j instead of k we get

$$\sum_{l\in\mathbb{Z}^d}\frac{1}{(1+\min(t_j,t_i)|x_{j,m}-x_{i,l}|_B)^N} \le C\max\left(\frac{t_i}{t_j},1\right)^\nu . \tag{3.58}$$

The lemma follows by combining (3.57) and (3.58).

■

Finally, we also used the following estimate in the frequency space to prove Proposition 3.15.

LEMMA 3.27.
Let $\delta>0$ and $0<r\le 1$. We then have

$$h := \sum_{i\in\mathbb{Z}^d} c_{ji}^\delta c_{ik}^\delta \le C c_{jk}^{\delta/2},$$

where

$$c_{jk}^{\delta} := \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\delta}, \left(\frac{t_k}{t_j}\right)^{\delta}\right)(1+\max(t_k,t_j)^{-1}|\xi_k-\xi_j|_A)^{-\frac{\nu}{r}-\delta},$$

with t_k defined in (3.8) and ξ_k in Lemma 3.7.

Proof:
Without loss of generality assume that $r = 1$. We will begin with assuming that $t_j \le t_k$. Furthermore, if $t_k^{-1}|\xi_j - \xi_k|_A \le \rho_0$ we have $t_k/t_j \le R_0$ by using that h is moderate (see Definition 3.5). Combining this with Lemma 3.25 gives

$$h \le \sum_{i\in\mathbb{Z}^d} c_{ik}^{\delta} \le C_1 \le C_2 c_{jk}^{\delta}.$$

In the other case, $t_k^{-1}|\xi_j - \xi_k|_A > \rho_0$, we split the sum into

$$A = \{i : |\xi_j - \xi_i|_A < \tfrac{1}{2C_A}|\xi_j - \xi_k|_A\}$$

and its complement. For $i \in A^c$ and $t_i \ge t_k \ge t_j$ we have

$$\begin{aligned}
h \le& C \sum_{\substack{i\in A^c\\ i:t_i\ge t_k}} \left(\frac{t_j}{t_i}\right)^{\nu+\delta}(1+t_i^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}c_{ik}^{\delta}\\
\le& C\left(\frac{t_j}{t_k}\right)^{\nu+\delta}(1+t_k^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}\sum_{\substack{i\in A^c\\ i:t_i\ge t_k}} c_{ik}^{\delta}\\
\le& C c_{jk}^{\delta}
\end{aligned}$$

and similarly for $t_k > t_i \ge t_j$. For $t_k \ge t_j > t_i$ we get

$$\begin{aligned}
h \le& C \sum_{\substack{i\in A^c\\ i:t_i< t_j}} \left(\frac{t_i}{t_j}\right)^{\delta}(1+t_j^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}c_{ik}^{\delta}\\
\le& C\left(\frac{t_j}{t_k}\right)^{\nu+\delta}(1+t_k^{-1}|\xi_j-\xi_k|_A)^{-\nu-\delta}\sum_{\substack{i\in A^c\\ i:t_i< t_j}} c_{ik}^{\delta}\\
\le& C c_{jk}^{\delta}.
\end{aligned}$$

Finally, when $i \in A$ we have $|\xi_i - \xi_k|_A > \frac{1}{2C_A}|\xi_j - \xi_k|_A$ which for $t_i \ge t_k \ge t_j$ gives

$$h \le C \sum_{\substack{i\in A\\ i:t_i\ge t_k}} \left(\frac{t_k}{t_i}\right)^{\delta}\left(\frac{t_j}{t_i}\right)^{\nu+\delta}(1+t_i^{-1}|\xi_j-\xi_i|_A)^{-\nu-\frac{\delta}{2}}(1+t_i^{-1}|\xi_j-\xi_k|_A)^{-\nu-\frac{\delta}{2}}$$

$$\leq C\left(\frac{t_j}{t_k}\right)^{\nu+\frac{\delta}{2}}(1+t_k^{-1}|\xi_j-\xi_k|_A)^{-\nu-\frac{\delta}{2}}\sum_{\substack{i\in A\\ i:t_i\geq t_k}}\left(\frac{t_j}{t_i}\right)^{\frac{\delta}{2}}(1+t_i^{-1}|\xi_j-\xi_i|_A)^{-\nu-\frac{\delta}{2}}$$

$$\leq Cc_{jk}^{\delta/2}.$$

For $t_k > t_i \geq t_j$ and $t_k \geq t_j > t_i$ the argument can be repeated in a similar way which proves the lemma when $t_k \geq t_j$. For $t_k < t_j$, it suffices to use that $c_{jk}^{\delta} = (t_j/t_k)^{\nu} c_{kj}^{\delta}$, and we get

$$h = \sum_{i\in\mathbb{Z}^d}\left(\frac{t_j}{t_i}\right)^{\nu} c_{ij}^{\delta}\left(\frac{t_i}{t_k}\right)^{\nu} c_{ki}^{\delta} \leq C\left(\frac{t_j}{t_k}\right)^{\nu} c_{kj}^{\delta/2} = c_{jk}^{\delta/2}.$$

■

CHAPTER 4

Compactly supported curvelet-type systems

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Abstract: In this paper we study a flexible method for constructing curvelet type frames. These curvelet type systems have the same sparse representation properties as curvelets for appropriate classes of smooth functions, and the flexibility of the method allows us to give a constructive description of how to construct curvelet type systems with a prescribed nature such as compact support in the direct space. The method consists of using the machinery of almost diagonal matrices to show that a system of curvelet molecules which is sufficiently close to curvelets constitutes a frame for curvelet type spaces. Such a system of curvelet molecules can then be constructed using finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay.

4.1. Introduction

Second generation curvelets were introduced by Candès and Donoho, who also proved that curvelets give an essentially optimal sparse representation of images (functions) that are C^2 except for discontinuities along piecewise C^2-curves [**9**]. It follows that efficient compression of such images can be achieved by thresholding their curvelet expansions. Curvelets form a multiscale system with effective support that follows a parabolic scaling relation $\textit{width} \approx \textit{length}^2$. Moreover, they also provide an essentially optimal sparse representation of Fourier integral operators [**7**] and an optimal sparse and well organized solution operator for a wide class of linear hyperbolic differential equations [**8**]. However, curvelets are band-limited, and contrary to wavelets it is an open question whether compactly supported curvelet type systems exist.

In this paper we study a flexible method for generating curvelet type systems with the same sparse representation properties as curvelets (when sparseness is measured in curvelet type sequence spaces). The method uses a perturbation principle which was first introduced in [**45**], further generalized in [**34**] and refined for frames in [**35**]. We give a constructive description of how to construct curvelet type systems consisting of finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay. This gives the flexibility to construct curvelet type systems with a prescribed nature (see Section 4.6) such as compact support in the direct space. For the sake of convenience the construction will only be done in $\mathbb{R}^2$, but it can easily be extended to $\mathbb{R}^d$. The main results can be found in Sections 4.4 and 4.5.

The curvelet type sequence spaces we use are associated with curvelet type spaces $G^s_{p,q}$ which were introduced in [**4**]. Here $G^s_{p,q}$ was constructed by applying a curvelet type splitting of the frequency space to a general construction of decomposition spaces; thereby obtaining a natural family of smoothness spaces for which curvelets constitute frames (see Section 4.2). Originally, this construction of decomposition spaces based on structured coverings of the frequency space was introduced by Feichtinger and Gröbner [**19**] and Feichtinger [**17**]. For example, the classical Triebel-Lizorkin and Besov spaces correspond to dyadic coverings of the frequency space [**51**].

The outline of the paper is as follows. In Section 4.2 we define second generation curvelets and curvelet type spaces. Furthermore, we introduce curvelet molecules which will be the building blocks for our compactly supported curvelet type frames. Next, in Section 4.3 we use the properties of curvelet molecules to show that the "change of frame coefficient" matrix is almost diagonal if the curvelet molecules have sufficient regularity. With the machinery of almost diagonal matrices, we can then in Section 4.4 show that curvelet molecules which are close enough to curvelets constitute frames for the curvelet type spaces. Finally, in Section 4.5 we give a constructive description of how to construct these curvelet molecules from finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay. We conclude the paper with a short discussion in Section 4.6 of the possible functions which can used to construct the curvelet molecules.

4.2. Second generation curvelets

We begin this section with a brief definition of curvelets and curvelet molecules which will later be used to construct curvelet type frames. Furthermore, we define the curvelet type spaces for which curvelets constitute frames. For a much more detailed discussion of the curvelet construction, we refer the reader to [**8**, **9**], and for decomposition spaces, of which the curvelet type spaces are a subclass, we refer to [**4**, **19**].

Let ν be an even $C^\infty(\mathbb{R})$ window that is supported on $[-\pi, \pi]$ such that its 2π-periodic extension obeys

$$|\nu(\theta)|^2 + |\nu(\theta - \pi)|^2 = 1, \ \theta \in [0, 2\pi). \tag{4.1}$$

Define $\nu_{j,l}(\theta) := \nu(2^{\lfloor j/2 \rfloor}\theta - \pi l)$ for $j \geq 2$ and $l = 0, 1, \ldots, 2^{\lfloor j/2 \rfloor} - 1$. Next, with the angular window in place, let $w \in C_c^\infty(\mathbb{R})$ obey

$$|w_0(t)|^2 + \sum_{j \geq 2} |w(2^{-j}t)|^2 = 1, \; t \in \mathbb{R}, \tag{4.2}$$

with $w_0 \in C_c^\infty(\mathbb{R})$ supported in a neighborhood of the origin. We then define

$$\phi_{j,l}(\xi) := w(2^{-j}|\xi|)(\nu_{j,l}(\theta) + \nu_{j,l}(\theta + \pi)), \; \xi = |\xi|(\cos\theta, \sin\theta) \in \mathbb{R}^2. \tag{4.3}$$

Notice that the support of $w(2^{-j}|\xi|)\nu_{j,0}(\theta)$ is contained in a rectangle $R_j = I_{1j} \times I_{2j}$ given by

$$I_{1j} := \{\xi_1, t_j \leq \xi_1 \leq t_j + L_j\}, \qquad I_{2j} := \{\xi_2, 2|\xi_2| \leq l_j\},$$

where t_j is determined uniquely for a minimal L_j, $L_j := \delta_1 \pi 2^j$ and $l_j := \delta_2 2\pi 2^{j/2}$ (δ_1 depends weakly on j, see [**9**, Section 2.2]). With $\tilde{I}_{1j} := \pm I_{1j}$ and $\tilde{R}_j = \tilde{I}_{1j} \times I_{2j}$ the system

$$e_{j,k}(\xi) := \frac{2^{-3j/4}}{2\pi\sqrt{\delta_1\delta_2}} e^{i\frac{(k_1+1/2)2^{-j}\xi_1}{\delta_1}} e^{i\frac{k_2 2^{-j/2}\xi_2}{\delta_2}}, \; k \in \mathbb{Z}^2,$$

is an orthonormal basis for $L_2(\tilde{R}_j)$.
We let $\hat{f}(\xi) := \mathcal{F}(f)(\xi) := (2\pi)^{-1} \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi}\, dx$, $f \in L_1(\mathbb{R}^2)$, and by duality extend it uniquely from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$. Finally, we define

$$\hat{\eta}_\mu(\xi) := \phi_{j,l}(\xi) e_{j,k}(R_{\theta_\mu}^\top \xi), \; \mu = (j, l, k), \tag{4.4}$$

where R_{θ_μ} is rotation by the angle $\theta_\mu := \pi 2^{-\lfloor j/2 \rfloor} l$, and as coarse-scale elements we define $\hat{\eta}_{1,0,k}(\xi) := \delta_0^{-1}\phi_{1,0}(\xi)e^{ik\cdot\xi/\delta_0}$, where $\phi_{1,0}(\xi) := \omega_0(|\xi|)$ and $\delta_0 > 0$ is sufficiently small. The system $\{\eta_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ is called *curvelets*, $\mathcal{J} := \{(j,l) | j \geq 1, l = 0, 1, \ldots, 2^{\lfloor j/2 \rfloor} - 1\}$. It can be shown that curvelets constitute a tight frame for $L_2(\mathbb{R}^2)$ (see [**9**, Section 2.2]).

To later construct curvelet type frames, we need a system of functions which share the essential properties of curvelets. As we shall see, curvelet molecules, which were introduced in [**8**] and used there to study hyperbolic differential equations, have all the properties we need. For $\kappa \in \mathbb{N}_0^2$, we define $|\kappa| := \kappa_1 + \kappa_2$, and for suitably differentiable functions we define $f^{(\kappa)} := \frac{\partial^{|\kappa|} f}{\partial_{\xi_1}^{\kappa_1} \partial_{\xi_2}^{\kappa_2}}$.

Definition 4.1.
A family of functions $\{\psi_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ is said to be a *family of curvelet molecules with regularity* R, $R \in \mathbb{N}$, if for $j \geq 2$ they may be expressed as

$$\psi_\mu(x) = 2^{\frac{3j}{4}} a_\mu(D_{2^{-j}} R_{\theta_\mu} x - (k_1/\delta_1, k_2/\delta_2)),$$

where $D_{2^{-j}}x = (2^j x_1, 2^{j/2} x_2)$, $\delta_1, \delta_2 > 0$ and all functions a_μ satisfy the following:

- For $|\kappa| \leq R$ there exists constants $C > 0$ independent of μ such that

$$|a_\mu^{(\kappa)}(x)| \leq C(1 + |x|)^{-2R}. \tag{4.5}$$

- There exists constants $C > 0$ independent of μ such that

$$|\hat{a}_\mu(\xi)| \leq C \min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^R. \tag{4.6}$$

The coarse-scale molecules, $j = 1$, must take the form $\psi_\mu(x) = a_\mu(x - k/\delta_0)$, $\delta_0 > 0$, where a_μ satisfies (4.5). ◇

It can be shown that curvelets constitute a family of curvelet molecules with regularity R for any $R \in \mathbb{N}$ (see [**8**, p. 1489]).

To define the curvelet type spaces which together with the associated sequence spaces will characterize the sparse representation properties of curvelets we need a suitable partition of unity.

Definition 4.2.
Let $Q_{j,l} := \operatorname{supp}(\phi_{j,l})$ for $(j,l) \in \mathcal{J}$, where $\phi_{j,l}$ was defined in (4.3). A bounded admissible partition of unity (BAPU) is a family of functions $\{\Psi_{j,l}\}_{(j,l)\in\mathcal{J}} \subset \mathcal{S} := \mathcal{S}(\mathbb{R}^2)$ satisfying:

- $\operatorname{supp}(\Psi_{j,l}) \subseteq Q_{j,l}$, $(j,l) \in \mathcal{J}$.
- $\sum_{(j,l)\in\mathcal{J}} \Psi_{j,l}(\xi) = 1$, $\xi \in \mathbb{R}^2$.
- $\sup_{(j,l)\in\mathcal{J}} |Q_{j,l}|^{1/p-1} \|\mathcal{F}^{-1}\Psi_{j,l}\|_{L_p(\mathbb{R}^2)} < \infty$, $p \in (0,1]$.

◇

An example of a BAPU is $\{|\phi_{j,l}|^2\}_{(j,l)\in\mathcal{J}}$ which follows from the construction of $\phi_{j,l}$ (see (4.1) and (4.2)) and curvelets being curvelet molecules with regularity R for any $R \in \mathbb{N}$. We are now ready to define curvelet type spaces.

Definition 4.3.
Let $\{\Psi_{j,l}\}_{(j,l)\in\mathcal{J}}$ be a BAPU and $\Psi_{j,l}(D)f := \mathcal{F}^{-1}(\Psi_{j,l}\mathcal{F}f)$. For $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$, we define $G^s_{p,q} := G^s_{p,q}(\mathbb{R}^2)$ as the set of distributions $f \in \mathcal{S}' := \mathcal{S}'(\mathbb{R}^2)$ satisfying

$$\|f\|_{G^s_{p,q}} := \Big(\sum_{(j,l)\in\mathcal{J}} \|2^{js}\Psi_{j,l}(D)f\|^q_{L_p} \Big)^{1/q} < \infty.$$

◇

It can be shown that $G^s_{p,q}$ is a quasi-Banach space (Banach space for $p, q \geq 1$), and $\mathcal{S}$ is dense in $G^s_{p,q}$ [**4, 19**]. Furthermore, $G^s_{p,q}$ is independent of the choice of BAPU.

We also need the sequence spaces associated with the curvelet type spaces. For the sake of convenience, we write $\|f_k\|$ instead of $\|\{f_k\}_{k\in K}\|$ when the index set is clear from the context.

DEFINITION 4.4.
For $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$, we define the sequence space $g^s_{p,q}$ as the set of sequences $\{z_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \subset \mathbb{C}$ satisfying

$$\|z_\mu\|_{g^s_{p,q}} := \left\|2^{j\left(s+\frac{3}{2}(\frac{1}{2}-\frac{1}{p})\right)}\Big(\sum_{k\in\mathbb{Z}^2}|z_\mu|^p\Big)^{1/p}\right\|_{l_q} < \infty,$$

where the l_p-norm is replaced with the l_∞-norm if $p = \infty$. $\diamond$

Notice that the sequence spaces l_q are special cases of $g^s_{p,q}$ as we have $g_{q,q}^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})} = l_q$.

Next, we introduce frames for $G^s_{p,q}$ and use the notation $F \asymp G$ when there exists two constants $0 < C_1 \leq C_2 < \infty$, depending only on "allowable" parameters, such that $C_1F \leq G \leq C_2F$.

DEFINITION 4.5.
We say that a family of functions $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ in the dual of $G^s_{p,q}$ is a frame for $G^s_{p,q}$ if for all $f \in G^s_{p,q}$ we have

$$\|f\|_{G^s_{p,q}} \asymp \|\langle f, \psi_\mu\rangle\|_{g^s_{p,q}}.$$

The following is called the frame expansion of $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ when it exists,

$$f = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f, S^{-1}\psi_\mu\rangle\psi_\mu \tag{4.7}$$

in the sense of $\mathcal{S}'$, where S is the frame operator $Sf = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f, \psi_\mu\rangle\psi_\mu$, $f \in G^s_{p,q}$. $\diamond$

From [**4**, Lemma 4 and Section 7.3] we have that curvelets (4.4) constitute a frame for the curvelet type spaces with a frame operator S that is equal to the identity, $S = I$:

PROPOSITION 4.6.
Assume that $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$. For any finite sequence $\{z_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \subset \mathbb{C}$, we have

$$\Big\|\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} z_\mu\eta_\mu\Big\|_{G^s_{p,q}} \leq C\|z_\mu\|_{g^s_{p,q}}.$$

Furthermore, $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$ with frame operator $S = I$,

$$\|f\|_{G^s_{p,q}} \asymp \|\langle f, \eta_\mu\rangle\|_{g^s_{p,q}},\ f \in G^s_{p,q}.$$

$\square$

Notice that frame expansions for two frames $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ and $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ have the same sparseness when measured in the associated sequence space $g^s_{p,q}$ if $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ and $\{S^{-1}\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ also constitute frames for $G^s_{p,q}$,

$$\|\langle f, S^{-1}\psi_\mu\rangle\|_{g^s_{p,q}} \asymp \|f\|_{G^s_{p,q}(\mathbb{R}^2)} \asymp \|\langle f, S^{-1}\eta_\mu\rangle\|_{g^s_{p,q}}.$$

Hence, to get a curvelet type system $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ with the same sparse representation properties as curvelets $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$, it suffices to prove that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ constitutes a frame for $G^s_{p,q}$.

4.3. Almost diagonal matrices

To generate curvelet type frames in the following sections we introduce the machinery of almost diagonal matrices in this section. Almost diagonal matrices where used in **[24]** on Besov spaces, and here we find an associated notion of almost diagonal matrices on $g^s_{p,q}$. The goal is to find a definition so that the composition of two almost diagonal matrices gives a new almost diagonal matrix and almost diagonal matrices are bounded on $g^s_{p,q}$.

To help us define almost diagonal matrices we use a slight variation of the pseudodistance introduced in **[47]** which was constructed in **[8]**. For this we need the center of η_μ in the direct space, $x_\mu := R_{\theta_\mu}(k_1 2^{-j}/\delta_1, k_2 2^{-j/2}\delta_2)$, and the "direction" of η_μ, $\rho_\mu := (\cos\theta_\mu, \sin\theta_\mu)$.

DEFINITION 4.7.
Given a pair of indices $\mu = (j,l,k)$ and $\mu' = (j',l',k')$, we define the *dyadic-parabolic pseudodistance* as

$$\omega(\mu,\mu') := 2^{|j-j'|}(1+\min(2^j,2^{j'})d(\mu,\mu')),$$

where

$$d(\mu,\mu') := |\theta_\mu-\theta_{\mu'}|^2 + |x_\mu - x_{\mu'}|^2 + |\langle\rho_\mu, x_\mu - x_{\mu'}\rangle|.$$

◇

The dyadic-parabolic pseudodistance was studied in detail in **[8]**, and from there we can deduce the following properties:

- For $\delta > 0$ there exists $C > 0$ such that

$$\sum_{k\in\mathbb{Z}^2}\omega(\mu,\mu')^{-\frac{3}{2}-\delta} \le C. \tag{4.8}$$

- For $\delta > 0$ there exists $C > 0$ such that

$$\sum_{(j,l)\in\mathcal{J}}\omega(\mu,\mu')^{-\frac{1}{2}-\delta} \le C. \tag{4.9}$$

- For $N \geq 2$ and $\delta > 0$ there exists $C > 0$ such that

$$\sum_{\mu'' \in \mathcal{J} \times \mathbb{Z}^2} \omega(\mu, \mu'')^{-N-\delta} \omega(\mu'', \mu')^{-N-\delta} \leq C \omega(\mu, \mu')^{-N-\frac{\delta}{2}}. \tag{4.10}$$

- Let $\{\psi_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ and $\{\eta_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ be two families of curvelet molecules with regularity $4R$, $R \in \mathbb{N}$. Then there exists $C > 0$ such that

$$|\langle \psi_\mu, \eta_{\mu'} \rangle| \leq C \omega(\mu, \mu')^{-R}. \tag{4.11}$$

These properties lead us to the following definition of almost diagonal matrices on $g^s_{p,q}$.

Definition 4.8.
Assume that $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$. Let $r := \min(1, p, q)$ and $t := s + \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$. A matrix $\mathbf{A} = \{a_{\mu\mu'}\}_{\mu,\mu' \in \mathcal{J} \times \mathbb{Z}^2}$ is called *almost diagonal on* $g^s_{p,q}$ if there exists $C, \delta > 0$ such that

$$|a_{\mu\mu'}| \leq C 2^{(j'-j)t} \omega(\mu, \mu')^{-\frac{2}{r}-\delta}.$$

$\diamond$

Remark 4.9.
Note that by using (4.10), we get that the composition of two almost diagonal matrices on $g^s_{p,q}$ gives a new almost diagonal matrix on $g^s_{p,q}$. $\circ$

We are now ready to show the most important property of almost diagonal matrices; they act boundedly on the curvelet type spaces.

Proposition 4.10.
If $\mathbf{A}$ is almost diagonal on $g^s_{p,q}$, then $\mathbf{A}$ is bounded on $g^s_{p,q}$.

Proof:
We only prove the result for $p < \infty$ as the result for $p = \infty$ follows in a similar way with l_p replaced by l_∞. Let $\omega_0(\mu, \mu') := \omega(j, l, 0, j', l', 0)$, $z := \{z_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2} \in g^s_{p,q}$, and assume for now that $p \geq 1$. We begin with looking at the l_p-norm of $\|\mathbf{A}z\|_{g^s_{p,q}}$. By using Minkowski's inequality, Hölder's inequality and (4.8) we get

$$\begin{aligned}
&\Big(\sum_{k \in \mathbb{Z}^2} |(\mathbf{A}z)_\mu|^p \Big)^{1/p} \\
&\quad \leq C \bigg(\sum_{k \in \mathbb{Z}^2} \Big(\sum_{(j',l') \in \mathcal{J}} 2^{(j'-j)t} \omega_0(\mu, \mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \sum_{k' \in \mathbb{Z}^2} |z_{\mu'}| \omega(\mu, \mu')^{-\frac{3}{2r}-\frac{\delta}{2}} \Big)^p \bigg)^{1/p} \\
&\quad \leq C \sum_{(j',l') \in \mathcal{J}} 2^{(j'-j)t} \omega_0(\mu, \mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \bigg(\sum_{k \in \mathbb{Z}^2} \Big(\sum_{k' \in \mathbb{Z}^2} |z_{\mu'}| \omega(\mu, \mu')^{-\frac{3}{2r}-\frac{\delta}{2}} \Big)^p \bigg)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{(j',l')\in\mathcal{J}} 2^{(j'-j)t}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \\
&\qquad\times\Bigg(\sum_{k\in\mathbb{Z}^2}\sum_{k'\in\mathbb{Z}^2}|z_{\mu'}|^p\omega(\mu,\mu')^{-\frac{3}{2r}-\frac{\delta}{2}}\Big(\sum_{k'\in\mathbb{Z}^2}\omega(\mu,\mu')^{-\frac{3}{2r}-\frac{\delta}{2}}\Big)^{p-1}\Bigg)^{1/p} \\
&\leq C \sum_{(j',l')\in\mathcal{J}} 2^{(j'-j)t}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}}\Big(\sum_{k'\in\mathbb{Z}^2}|z_{\mu'}|^p\Big)^{1/p}.
\end{aligned}$$

We then have

$$\|\mathbf{A}z\|_{g^s_{p,q}} \leq C\Bigg(\sum_{(j,l)\in\mathcal{J}}\Bigg(\sum_{(j',l')\in\mathcal{J}} 2^{j't}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}}\Big(\sum_{k'\in\mathbb{Z}^2}|z_{\mu'}|^p\Big)^{1/p}\Bigg)^q\Bigg)^{1/q}.$$

For $q \geq 1$ we use Hölder's inequality and (4.9) to get

$$\begin{aligned}
\|\mathbf{A}z\|_{g^s_{p,q}} \leq& C\Bigg(\sum_{(j,l)\in\mathcal{J}}\sum_{(j',l')\in\mathcal{J}} 2^{j'qt}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \\
&\qquad\times\Big(\sum_{k'\in\mathbb{Z}^2}|z_{\mu'}|^p\Big)^{q/p}\Big(\sum_{(j',l')\in\mathcal{J}}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}}\Big)^{q-1}\Bigg)^{1/q} \\
&\leq C\|z\|_{g^s_{p,q}}.
\end{aligned}$$

For $q < 1$ the result follows by a direct estimate. The case $p < 1$ remains, and here we first observe that

$$\tilde{\mathbf{A}} := \{\tilde{a}_{\mu\mu'}\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2} = \big\{|a_{\mu\mu'}|^p 2^{(j'-j)(t-tp)}\big\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$$

is almost diagonal on $g^s_{1,\frac{q}{p}}$. Furthermore, if we let
$v := \{v_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} := \{|z_\mu|^p 2^{-j(t-tp)}\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ we have

$$\|v\|^{1/p}_{g^s_{1,\frac{q}{p}}} = \Bigg(\sum_{(j,l)\in\mathcal{J}}\Bigg(\sum_{k\in\mathbb{Z}^2}2^{jtp}|z_\mu|^p\Bigg)^{q/p}\Bigg)^{1/q} = \|z\|_{g^s_{p,q}}.$$

Before we can put these two observations into use, we need that

$$|(\mathbf{A}z)_\mu|^p \leq \sum_{(j',l')\in\mathcal{J}}\sum_{k'\in\mathbb{Z}^2}|a_{\mu\mu'}|^p|z_{\mu'}|^p = 2^{j(t-tp)}\sum_{(j',l')\in\mathcal{J}}\sum_{k'\in\mathbb{Z}^2}\tilde{a}_{\mu\mu'}v_\mu.$$

We then have

$$\|\mathbf{A}z\|_{g^s_{p,q}} \leq \|\tilde{\mathbf{A}}v\|^{1/p}_{g^s_{1,\frac{q}{p}}} \leq C\|v\|^{1/p}_{g^s_{1,\frac{q}{p}}} = C\|z\|_{g^s_{p,q}}.$$

■

4.4. Curvelet type frames

In this section we study a family of curvelet molecules $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ which is a small perturbation of curvelets $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$. The goal is first to show that if $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is close enough to $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$, then $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$. Next to get a frame expansion, we show that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is also a frame. The results are inspired by **[35]** where perturbations of frames were studied in Triebel-Lizorkin and Besov spaces.

Let $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\subset L_2(\mathbb{R}^2)$ be a system that is close to $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ in the sense that there exists $\varepsilon,\delta>0$ such that for $j\geq 2$

$$\eta_\mu(x)-\psi_\mu(x)=2^{\frac{3j}{4}}c_\mu(D_{2^{-j}}R_{\theta_\mu}x-(k_1/\delta_1,k_2/\delta_2)), \tag{4.12}$$

where $D_{2^{-j}}x=(2^jx_1,2^{j/2}x_2)$, $\delta_1,\delta_2>0$ and all functions c_μ satisfy the following:

- For $|\kappa|\leq 4\lceil |t|+\frac{2}{r}+\delta\rceil$ we need

$$|c^{(\kappa)}_\mu(x)|\leq \varepsilon(1+|x|)^{-8\lceil |t|+\frac{2}{r}+\delta\rceil}. \tag{4.13}$$

- Furthermore we need

$$|\hat{c}_\mu(\xi)|\leq \varepsilon\min(1,2^{-j}+|\xi_1|+2^{-\frac{j}{2}}|\xi_2|)^{4\lceil |t|+\frac{2}{r}+\delta\rceil}. \tag{4.14}$$

We have used the notation from Definition 4.8. The coarse-scale molecules, $j=1$, must take the form $\eta_\mu(x)-\psi_\mu(x)=c_\mu(x-k/\delta_0)$, $\delta_0>0$, where c_μ satisfies (4.13).

Then $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t|+\frac{2}{r}+\delta\rceil$, and motivated by $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ being a tight frame for $L_2(\mathbb{R}^2)$, we formally define $\langle f,\psi_{\mu'}\rangle$ as

$$\langle f,\psi_{\mu'}\rangle := \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle \eta_\mu,\psi_{\mu'}\rangle\langle f,\eta_\mu\rangle,\ f\in G^s_{p,q}.$$

It follows from (4.11) and Proposition 4.10 that $\langle\cdot,\psi_{\mu'}\rangle$ is a bounded linear functional on $G^s_{p,q}$; in fact we have

$$\begin{aligned}\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}|\langle \eta_\mu,\psi_{\mu'}\rangle||\langle f,\eta_\mu\rangle| &\leq \Big\|\Big\{\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}|\langle \eta_\mu,\psi_{\mu'}\rangle||\langle f,\eta_\mu\rangle|\Big\}_{\mu'\in\mathcal{J}\times\mathbb{Z}^2}\Big\|_{g^s_{p,q}}\\ &\leq C\|\langle f,\eta_\mu\rangle\|_{g^s_{p,q}}\leq C\|f\|_{G^s_{p,q}}.\end{aligned} \tag{4.15}$$

Furthermore, $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a norming family for $G^s_{p,q}$ as it satisfies $\|\langle f,\psi_\mu\rangle\|_{g^s_{p,q}}\leq C\|f\|_{G^s_{p,q}}$. This can be used to show that S is a bounded operator on $G^s_{p,q}$, and for small enough ε this will be the key to showing that $\{\psi_\mu\}$ is a frame for $G^s_{p,q}$.

THEOREM 4.11.
There exists $\varepsilon_0, C_1, C_2 > 0$ such that if $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ satisfies (4.12) for some $0 < \varepsilon \le \varepsilon_0$, then we have

$$C_1\|f\|_{G^s_{p,q}} \le \|\langle f, \psi_\mu\rangle\|_{g^s_{p,q}} \le C_2\|f\|_{G^s_{p,q}}\ f \in G^s_{p,q}.$$

Proof:
That $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a norming family gives the upper bound, thus we only need to establish the lower bound. For this we use that $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is also a norming family so we have

$$\|\langle f, \eta_\mu - \psi_\mu\rangle\|_{g^s_{p,q}} \le C\varepsilon\|f\|_{G^s_{p,q}}.$$

It then follows that

$$\begin{aligned}\|f\|_{G^s_{p,q}} &\le C\|\langle f, \eta_\mu\rangle\|_{g^s_{p,q}}\\ &\le C\big(\|\langle f, \psi_\mu\rangle\|_{g^s_{\mu}} + \|\langle f, \eta_\mu - \psi_\mu\rangle\|_{g^s_{p,q}}\big)\\ &\le C\big(\|\langle f, \psi_\mu\rangle\|_{g^s_{p,q}} + \varepsilon\|f\|_{G^s_{p,q}}\big).\end{aligned}$$

By choosing $\varepsilon < 1/C$ we get the lower bound.

■

As one might guess from Theorem 4.11, the boundedness of the matrix $\{\langle\eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ on $g^s_{p,q}$ is the key to showing that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is also a frame for $G^s_{p,q}$.

PROPOSITION 4.12.
There exists $\varepsilon_0 > 0$ such that if $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ satisfies (4.12) for some $0 < \varepsilon \le \varepsilon_0$ and furthermore is a frame for $G^0_{22} = L_2(\mathbb{R}^2)$, then $\{\langle\eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu\,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$.

Proof:
The fact that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $L_2(\mathbb{R}^2)$ ensures that S^{-1} is a bounded operator on $L_2(\mathbb{R}^2)$. We first show that S^{-1} is bounded on $G^s_{p,q}$. This will follow from showing that

$$\|(I-S)f\|_{G^s_{p,q}} \le C\varepsilon\|f\|_{G^s_{p,q}}, f \in G^s_{p,q}, \tag{4.16}$$

choosing ε small enough and using the Neumann series. Assume for a moment that
$\mathcal{D} := \{d_{\mu'\mu}\}_{\mu',\mu\in\mathcal{J}\times\mathbb{Z}^2} := \{\langle(I-S)\eta_\mu, \eta_{\mu'}\rangle\}_{\mu',\mu\in\mathcal{J}\times\mathbb{Z}^2}$ satisfies

$$\|\mathcal{D}z\|_{g^s_{p,q}} \le C\varepsilon\|z\|_{g^s_{p,q}}. \tag{4.17}$$

By using that S is self-adjoint, we then have

$$\|(I-S)f\|_{G^s_{p,q}} \le C\|\{\langle(I-S)f, \eta_{\mu'}\rangle\}\|_{g^s_{p,q}} = C\|\mathcal{D}\{\langle f, \eta_\mu\rangle\}\|_{g^s_{p,q}}$$

$$\leq C\varepsilon \|\{\langle f, \eta_\mu \rangle\}\|_{g^s_{p,q}} \leq C\varepsilon \|f\|_{G^s_{p,q}}.$$

So to show (4.16) it suffices to prove (4.17). Note that

$$\langle (I-S)\eta_\mu, \eta_{\mu'} \rangle = \sum_{\mu'' \in \mathcal{J} \times \mathbb{Z}^2} \langle \eta_\mu, \eta_{\mu''} \rangle \langle \eta_{\mu''}, \eta_{\mu'} \rangle - \sum_{\mu'' \in \mathcal{J} \times \mathbb{Z}^2} \langle \eta_\mu, \psi_{\mu''} \rangle \langle \psi_{\mu''}, \eta_{\mu'} \rangle$$

$$= \sum_{\mu'' \in \mathcal{J} \times \mathbb{Z}^2} \langle \eta_\mu, \eta_{\mu''} \rangle \langle \eta_{\mu''} - \psi_{\mu''}, \eta_{\mu'} \rangle + \sum_{\mu'' \in \mathcal{J} \times \mathbb{Z}^2} \langle \eta_\mu, \eta_{\mu''} - \psi_{\mu''} \rangle \langle \psi_{\mu''}, \eta_{\mu'} \rangle.$$

By setting

$$\begin{aligned}
\mathcal{D}_1 &:= \{d_{1(\mu')(\mu'')}\} := \{\langle \eta_{\mu''} - \psi_{\mu''}, \eta_{\mu'} \rangle\}, \\
\mathcal{D}_2 &:= \{d_{2(\mu'')(\mu)}\} := \{\langle \eta_\mu, \eta_{\mu''} \rangle\}, \\
\mathcal{D}_3 &:= \{d_{3(\mu')(\mu'')}\} := \{\langle \psi_{\mu''}, \eta_{\mu'} \rangle\}, \\
\mathcal{D}_4 &:= \{d_{4(\mu'')(\mu)}\} := \{\langle \eta_\mu, \eta_{\mu''} - \psi_{\mu''} \rangle\},
\end{aligned}$$

we have the decomposition

$$\mathcal{D} = \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_3\mathcal{D}_4.$$

Since $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta \rceil$, we have from (4.11) that $\varepsilon^{-1}\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \varepsilon^{-1}\mathcal{D}_4$ are almost diagonal on $g^s_{p,q}$. Next, we use Remark 4.9, and by Proposition 4.10,

$$\|\mathcal{D}z\|_{g^s_{p,q}} \leq C\varepsilon \|z\|_{g^s_{p,q}}.$$

Consequently, (4.16) holds, and for sufficiently small ε the operator S^{-1} is bounded on $G^s_{p,q}$. Finally, let $z := \{z_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2} \in g^s_{p,q}$ and $h =: \sum_\mu z_\mu \eta_\mu$. By using Proposition 4.6 we have that $h \in G^s_{p,q}$, and as $\{\psi_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ is a frame for $L_2(\mathbb{R}^2)$, we have that S^{-1} is self-adjoint which gives

$$\sum_{\mu \in \mathcal{J} \times \mathbb{Z}^2} \langle \eta_\mu, S^{-1}\psi_{\mu'} \rangle z_\mu = \sum_{\mu \in \mathcal{J} \times \mathbb{Z}^2} \langle S^{-1}\eta_\mu, \psi_{\mu'} \rangle z_\mu = \langle S^{-1}h, \psi_{\mu'} \rangle.$$

If we combine this with $\{\psi_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ being a norming family (4.15), we get

$$\begin{aligned}
\Big\| \sum_{\mu \in \mathcal{J} \times \mathbb{Z}^2} \langle \eta_\mu, S^{-1}\psi_{\mu'} \rangle z_\mu \Big\|_{g^s_{p,q}} &= \|\langle S^{-1}h, \psi_{\mu'} \rangle\|_{g^s_{p,q}} \leq C\|S^{-1}h\|_{G^s_{p,q}} \\
&\leq C\|h\|_{G^s_{p,q}} \leq C\|z\|_{g^s_{p,q}}
\end{aligned}$$

which proves that $\{\langle \eta_\mu, S^{-1}\psi_{\mu'} \rangle\}_{\mu,\mu' \in \mathcal{J} \times \mathbb{Z}^2}$ is bounded on $g^s_{p,q}$.

■

That $\{S^{-1}\psi_\mu\}_{\mu \in \mathcal{J} \times \mathbb{Z}^2}$ is a frame for $G^s_{p,q}$ now follows as a consequence of $\{\langle \eta_\mu, S^{-1}\psi_{\mu'} \rangle\}_{\mu,\mu' \in \mathcal{J} \times \mathbb{Z}^2}$ being bounded on $g^s_{p,q}$. We state the following results without proofs as they follow directly in the same way as in the Besov space case. The proofs can be found in [**35**]. First, we have the frame expansion.

LEMMA 4.13.
Assume that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ and a frame for $L_2(\mathbb{R}^2)$. If $\{\langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$, then for $f\in G^s_{p,q}$ we have

$$f = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle f, S^{-1}\psi_\mu\rangle \psi_\mu$$

in the sense of $\mathcal{S}'$.

□

Next, we have that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$

THEOREM 4.14.
Assume that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ and a frame for $L_2(\mathbb{R}^2)$. Then $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$ if and only if $\{\langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$.

□

It follows from Proposition 4.12, Lemma 4.13 and Theorem 4.14 that if $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules which is close enough to curvelets, then the representation $\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f, S^{-1}\psi_\mu\rangle\psi_\mu$, $f\in G^s_{p,q}$, has the same sparse representation properties as curvelets when measured in $g^s_{p,q}$. As a final result we also have a frame expansion with $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$.

LEMMA 4.15.
Assume that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ and a frame for $L_2(\mathbb{R}^2)$. If the transpose of $\{\langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$, then for $f\in G^s_{p,q}$ we have

$$f = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle f, \psi_\mu\rangle S^{-1}\psi_\mu$$

in the sense of $\mathcal{S}'$.

□

All that remains now is to construct a flexible family of curvelet molecules which is close enough to curvelets in the sense of (4.12).

4.5. Construction of curvelet type systems

In this section we construct a flexible curvelet type system. We do this by showing that finite linear combinations of shifts and dilates of a function g with sufficient smoothness and decay can be used to construct a system $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$

that satisfies (4.12). From the previous section, we then have that the representation $\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f, S^{-1}\psi_\mu\rangle\psi_\mu$, $f\in G^s_{p,q}$, has the same sparse representation properties as curvelets when measured in $g^s_{p,q}$.

First we take $g\in C^{M+1}(\mathbb{R}^2)$, $\hat{g}(0)\neq 0$, which for fixed $N'>2, M>0$ satisfies

$$|g^{(\kappa)}(x)|\leq C(1+|x|)^{-N'},\ |\kappa|\leq M+1, \tag{4.18}$$

Next, for $m\geq 1$ we define $g_m(x):=C_g m^2 g(mx)$, where $C_g=:\hat{g}(0)^{-1}$. It then follows that

$$\begin{aligned}|g_m^{(\kappa)}(x)| &\leq Cm^{2+|\kappa|}(1+m|x|)^{-N'},\ |\kappa|\leq M+1,\\ \int_{\mathbb{R}^2} g_m(x)\,\mathrm{d}x &= 1.\end{aligned} \tag{4.19}$$

We recall that curvelets (4.4) are a family of curvelet molecules for any regularity $R\in\mathbb{N}$. From the definition of a family of curvelet molecules (Definition 4.1), we have that for $j\geq 2$ curvelet molecules can be expressed as

$$\eta_\mu(x)=2^{\frac{3j}{4}}a_\mu(D_{2^{-j}}R_{\theta_\mu}x-(k_1/\delta_1,k_2/\delta_2)),$$

where a_μ must satisfy (4.5) and (4.6). So to construct a family of curvelet molecules $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ which satisfy (4.12), we need to construct a family of functions $\{b_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ such that $a_\mu-b_\mu$ satisfy (4.13) and (4.14). We define $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ as

$$\psi_\mu(x):=2^{\frac{3j}{4}}b_\mu(D_{2^{-j}}R_{\theta_\mu}x-(k_1/\delta_1,k_2/\delta_2))$$

for $j\geq 2$ and to construct $\{b_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ we also need the following set of finite linear combinations,

$$\Theta_{K,m}:=\{b_\mu : b_\mu(\cdot)=\sum_{i=1}^{K}c_i g_m(\cdot+d_i), c_i\in\mathbb{C}, d_i\in\mathbb{R}^2\}.$$

We have omitted the construction of ψ_μ for $j=1$ as it follow in a similar way.

Proposition 4.16.
Let $N'>N>2$, $M>0$ and $j>0$. If $g\in C^{M+1}(\mathbb{R}^2)$, $\hat{g}(0)\neq 0$, fulfills (4.18) and $a_\mu\in L_2(\mathbb{R}^2)\cap C^{M+1}(\mathbb{R}^2)$ fulfills

$$\begin{aligned}|a_\mu^{(\kappa)}(x)| &\leq C(1+|x|)^{-N'},\ |\kappa|\leq M+1\\ |\hat{a}_\mu(\xi)| &\leq C\min(1,2^{-j}+|\xi_1|+2^{-\frac{j}{2}}|\xi_2|)^{M+1},\end{aligned}$$

then for any $\varepsilon > 0$ there exists $K, m \geq 1$ (m independent of j) and $b_\mu \in \Theta_{K,m}$ such that

$$|a_\mu^{(\kappa)}(x) - b_\mu^{(\kappa)}(x)| \leq \varepsilon(1+|x|)^{-N}, \ |\kappa| \leq M \tag{4.20}$$

$$|\hat{a}_\mu(\xi) - \hat{b}_\mu(\xi)| \leq \varepsilon \min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M. \tag{4.21}$$

Proof:
Let $\varepsilon > 0$ and κ, $|\kappa| \leq M$, be given. We construct the approximation of a_μ in the direct space in three steps. First by a convolution operator $\omega_m = a_\mu * g_m$, then by $\theta_{q,m}$ which is the integral in ω_m taken over a dyadic cube Q, and finally by a discretization over smaller dyadic cubes $b_{l,q,m}$. From (4.19) we have

$$a_\mu^{(\kappa)}(x) - \omega_m^{(\kappa)}(x) = \int_{\mathbb{R}^2} \big(a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)\big) g_m(y)\, dy. \tag{4.22}$$

Define $U := m^{\lambda/2N}$, where $\lambda := \min(1, N' - N)$. For $|x| \leq U$, we use the mean value theorem to get

$$|a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)| \leq C \min(1, |y|).$$

Inserting this in (4.22) we have

$$\begin{aligned} |a_\mu^{(\kappa)}(x) - \omega_m^{(\kappa)}(x)| &\leq C \int_{\mathbb{R}^2} \frac{\min(1,|y|)m^2}{(1+m|y|)^{N'}}\, dy \\ &\leq Cm^{-\lambda} \leq \frac{Cm^{-\lambda/2}}{U^N} \leq \frac{Cm^{-\lambda/2}}{(1+|x|)^N}. \end{aligned} \tag{4.23}$$

For $|x| > U$, we split the integral over $\Omega := \{y : |y| \leq |x|/2\}$ and Ω^c. If $y \in \Omega$, then $|x-y| \geq |x|/2$, and we have

$$\begin{aligned} \int_\Omega |a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)||g_m(y)|\, dy &\leq C(1+|x|)^{-N'} \\ &\leq \frac{C}{(1+U)^\lambda (1+|x|)^N} \leq \frac{Cm^{-\lambda^2/2N}}{(1+|x|)^N}. \end{aligned} \tag{4.24}$$

Integrating over Ω^c with $|x| > U$ gives

$$\begin{aligned} &\int_{\Omega^c} |a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)||g_m(y)|\, dy \\ &\leq \frac{C}{(1+|x|)^{N'}} + \int_{\Omega^c} \frac{Cm^2}{(1+|x-y|)^{N'}(1+m|y|)^{N'}}\, dy \\ &\leq \frac{C}{(1+|x|)^{N'}} + \frac{Cm^{-\lambda}}{(1+|x|)^N} \leq \frac{C(m^{-\lambda^2/2N} + m^{-\lambda})}{(1+|x|)^N}. \end{aligned} \tag{4.25}$$

So by choosing m sufficiently large in (4.23)-(4.25), we get

$$|a_\mu^{(\kappa)}(x) - \omega_m^{(\kappa)}(x)| \le \frac{\varepsilon}{3}(1+|x|)^{-N}. \tag{4.26}$$

For the next step we fix m and choose $q \in \mathbb{N}$. Let Q denote the dyadic cube with sidelength 2^{q+1}, sides parallel with the axes and centered at the origin. We then approximate ω_m with $\theta_{q,m}$ defined as

$$\theta_{q,m}(\cdot) = \int_Q a_\mu(y) g_m(\cdot - y)\, \mathrm{d}y.$$

In which case we have

$$\omega_m^{(\kappa)}(x) - \theta_{q,m}^{(\kappa)}(x) = \int_{Q^c} a_\mu(y) g_m^{(\kappa)}(x-y)\, \mathrm{d}y,$$

and it follows that,

$$|\omega_m^{(\kappa)}(x) - \theta_{q,m}^{(\kappa)}(x)| \le \int_{|y|\ge 2^q} \frac{Cm^{2+|\kappa|}}{(1+|y|)^{N'}(1+m|x-y|_B)^{N'}}\, \mathrm{d}y := L.$$

We first estimate the integral for $|x| \le 2^{q-1}$ which gives $|y| > |x|$ and $|x-y| \ge 2^{q-1}$. Hence we obtain

$$L \le \frac{Cm^{2+|\kappa|}}{(1+|x|)^{N'}} \int_{|u|\ge 2^{q-1}} \frac{1}{(1+m|u|)^{N'}}\, \mathrm{d}u \le \frac{Cm^{|\kappa|-\lambda}2^{-\lambda q}}{(1+|x|)^{N'}}. \tag{4.27}$$

For $|x| > 2^{q-1}$, we split the integral over $\Omega := \{y : |y| \ge 2^q\} \cap \{y : |y| \le |x|/2\}$ and $\Omega' := \{y : |y| \ge 2^q\} \cap \{y : |y| > |x|/2\}$. If $y \in \Omega$, then $|x-y| \ge |x|/2$, and we get

$$\begin{aligned}\int_\Omega \frac{m^{2+|\kappa|}}{(1+|y|)^{N'}(1+m|x-y|)^{N'}}\, \mathrm{d}y &\le \frac{Cm^{2+|\kappa|}}{(1+m|x|)^{N'}} \int_{|y|\ge 2^q} \frac{1}{(1+|y|)^{N'}}\, \mathrm{d}y \\ &\le \frac{Cm^{|\kappa|-\lambda}2^{-\lambda q}}{(1+|x|)^{N}}.\end{aligned} \tag{4.28}$$

Similar for Ω' we have

$$\begin{aligned}\int_{\Omega'} \frac{m^{2+|\kappa|}}{(1+|y|)^{N'}(1+m|x-y|)^{N'}}\, \mathrm{d}y &\le \frac{C}{(1+|x|)^{N'}} \int_{\mathbb{R}^2} \frac{m^{2+|\kappa|}}{(1+m|x-y|)^{N'}}\, \mathrm{d}y \\ &\le \frac{Cm^{|\kappa|}}{(1+|x|)^{N'}} \le \frac{m^{|\kappa|}2^{-\lambda q}}{(1+|x|)^{N}}.\end{aligned} \tag{4.29}$$

By choosing q sufficiently large in (4.27)-(4.29), we obtain

$$|\omega_m^{(\kappa)}(x) - \theta_{q,m}^{(\kappa)}(x)| \le \frac{\varepsilon}{3}(1+|x|)^{-N}. \tag{4.30}$$

For the final step we fix q, choose $l \in \mathbb{N}$ and approximate $\theta_{q,m}$ by a discretization

$$b_{l,q,m}(\cdot) = \sum_{I \in H_{l,q}} |I| a_\mu(x_I) g_m(\cdot - x_I),$$

where x_I is the center of the dyadic cube I and $H_{l,q}$ is the set of dyadic cubes with sidelength 2^{-l} which together give Q. Note that $b_{l,q,m} \in \Theta_{K,m}$, $K = 2^{q+l+1}$. We introduce $F(\cdot) := a_\mu(\cdot) g_m^{(\kappa)}(x - \cdot)$ which gives

$$\begin{aligned} |\theta_{q,m}^{(\kappa)}(x) - b_{l,q,m}^{(\kappa)}(x)| &\leq \sum_{I \in H_{l,q}} \int_I |a_\mu(y) g_m^{(\kappa)}(x-y) - a_\mu(x_I) g_m^{(\kappa)}(x - x_I)| \,\mathrm{d}y \\ &\leq \sum_{I \in H_{l,q}} \int_I |F(y) - F(x_I)| \,\mathrm{d}y. \end{aligned}$$

By using the mean value theorem, we then get

$$\begin{aligned} |\theta_{q,m}^{(\kappa)}(x) - b_{l,q,m}^{(\kappa)}(x)| &\leq \sum_{I \in H_{l,q}} \int_I |y - x_I| \max_{\substack{z \in l(x_I, y) \\ |\kappa'| \leq 1}} |F^{(\kappa')}(z)| \,\mathrm{d}y \\ &\leq C 2^{2q-l} \max_{\substack{|z| \leq 2^{q+1} \\ |\kappa'| \leq |\kappa|+1}} |g_m^{(\kappa')}(x - z)|, \end{aligned} \tag{4.31}$$

where $l(x_I, y)$ is the line-segment between x_I and y. If $|x| \leq 2^{q+2}$ and $|\kappa'| \leq |\kappa| + 1$, then we have

$$|g_m^{(\kappa')}(x - z)| \leq C m^{3+|\kappa|} \leq \frac{C m^{3+|\kappa|} 2^{qN}}{(1 + |x|)^N}. \tag{4.32}$$

For $|x| > 2^{q+2}$ and $|z| \leq 2^{q+1}$, we have $|x - z| \geq |x|/2$, and hence for $|\kappa'| \leq |\kappa| + 1$, it follows that

$$|g_m^{(\kappa')}(x - z)| \leq \frac{C m^{3+|\kappa|}}{(1 + m|x|)^{N'}} \leq \frac{C m^{3+|\kappa|}}{(1 + |x|)^{N'}}. \tag{4.33}$$

By choosing l sufficiently large, we obtain by combining (4.31)-(4.33) that

$$|\theta_{q,m}^{(\kappa)}(x) - b_{l,q,m}^{(\kappa)}(x)| \leq \frac{\varepsilon}{3} (1 + |x|)^{-N}. \tag{4.34}$$

Finally by combining (4.26), (4.30) and (4.34), we get

$$|a_\mu^{(\kappa)}(x) - b_{l,q,m}^{(\kappa)}(x)| \leq \varepsilon (1 + |x|)^{-N}. \tag{4.35}$$

To approximate a_μ in the frequency space we use three steps similar to the approximation in the direct space. Note that $b_{l,q,m}$ still fulfills (4.35) if we choose l, q, m even larger. First we use $\hat{\omega}_m$ to approximate $\hat{a}_\mu$ in which case we have

$$|\hat{a}_\mu(\xi) - \hat{\omega}_m(\xi)| = |\hat{a}_\mu(\xi)^{\frac{M}{1+M}}\hat{a}_\mu(\xi)^{\frac{1}{1+M}}(1 - C_g\hat{g}(\xi/m))|$$
$$\leq C\min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M(1+|\xi|)^{-1}|1 - C_g\hat{g}(\xi/m)|.$$

By choosing $\xi_g > 0$ such that $C(1+\xi_g)^{-1}|1 - C_g\hat{g}(\xi/m)| \leq \varepsilon/3$ and m such that $C|1 - C_g\hat{g}(\xi/m)| \leq \varepsilon/3$ for $|\xi| < \xi_g$, we get

$$|\hat{a}_\mu(\xi) - \hat{\omega_m}(\xi)| \leq \frac{\varepsilon}{3}\min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M. \tag{4.36}$$

Next, we fix m, choose q and limit the Fourier integral of a_μ to Q from the approximation in the direct space,

$$\theta'_{q,m}(\xi) = \hat{g}_m(\xi)\int_Q a_\mu(x)e^{ix\cdot\xi}\,\mathrm{d}x.$$

This gives

$$|\hat{\omega}_m(\xi) - \theta'_{q,m}(\xi)| \leq |\hat{g}_m(\xi)|\int_{|x|>2^q}|a_\mu(x)e^{ix\cdot\xi}|\,\mathrm{d}x \leq C2^{-\lambda q}. \tag{4.37}$$

In the last step, we fix q and approximate $\theta'_{q,m}$ by $\hat{b}_{l,q,m}$. We introduce $G(x) := a_\mu(x)e^{ix\cdot\xi}$ which gives

$$|\theta'_{q,m}(\xi) - \hat{b}_{l,q,m}(\xi)| \leq |\hat{g}_m(\xi)|\Big|\int_Q a_\mu(x)e^{ix\cdot\xi}\,\mathrm{d}x - \sum_{I\in H_{l,q}}|I|a_\mu(x_I)e^{ix_I\cdot\xi}\Big|$$
$$\leq |\hat{g}_m(\xi)|\sum_{I\in H_{l,q}}\int_I|G(x) - G(x_I)|\,\mathrm{d}x$$
$$\leq \frac{C2^{2q-l}}{1+|\xi/m|}\max_{\substack{x\in\mathbb{R}^2\\ |\kappa'|\leq 1}}|G^{(\kappa')}(x)| \leq Cm2^{2q-l}. \tag{4.38}$$

By combining (4.36)-(4.38) for sufficiently large l, q, m, we get

$$|\hat{a}_\mu(\xi) - \hat{b}_{l,q,m}(\xi)| \leq \varepsilon\min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M.$$

It follows that by choosing l, q, m large enough $b_{l,q,m}$ fulfills both (4.20) and (4.21). Furthermore, we have $b_{l,q,m} \in \Theta_{K,m}$, $K = 2^{q+l+1}$.

■

4.6. Discussion

In this paper we studied a flexible method for generation curvelet type systems with the same sparse representation properties as curvelets when measured in $g^s_{p,q}$. With Proposition 4.12, Lemma 4.13 and Theorem 4.14 we proved that a system of curvelet molecules which is close enough to curvelets has these sparse representation properties. Furthermore, with Proposition 4.16 we gave a constructive description of how such a system of curvelet molecules can be constructed from finite linear combinations of shifts and dilates for a single function with sufficient smoothness and decay.

Examples of functions with sufficient smoothness and decay are the exponential function $e^{-|\cdot|^2}$ and the rational functions $(1+|\cdot|^2)^{-N}$ with N sufficiently large. An example with compact support can be constructed by using a spline with compact support. Furthermore as the system is constructed using finite linear combinations of splines, we get a system consisting of modulated compactly supported splines.

CHAPTER 5

Epilogue

The case for my life, then, or for that of any one else who has been a mathematician in the same sense which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.

G.H. Hardy, A Mathematician's Apology

In this chapter we discuss some of the open problems which present themselves in extension of the papers in Chapters 2-4.

Curvelet type bases. Curvelets belong to a collection of directional representation systems of which Shearlets [**36**] and Contourlets [**16**] are the other prominent members. As far as the author is aware a basis which shares the properties of these systems has yet to be constructed. In the same way as in Chapter 2 it is possible to construct a brushlet basis for $L_2(\mathbb{R}^2)$ with a curvelet type decomposition of the frequency space. However, the norm characterization of curvelet type spaces proves difficult especially estimates like (2.13). So the construction of a curvelet type basis is an interesting problem which would give a full characterization of curvelet type spaces in terms of nonlinear approximation. More generally, there is the open challenge of constructing an unconditional basis for T-L type spaces, the associated modulation spaces and beyond.

Group representations. Coorbit spaces were introduced by Feichtinger and Gröchenig in [**20–22**] and are a more structured class of smoothness spaces than decomposition spaces. They are build on group representations and here Besov spaces are associated with the Weyl-Heisenberg group and the classical modulation spaces are associated with the affine group. The additional structure gives an atomic decomposition for all coorbit spaces, but also excludes α-modulation spaces. This was remedied by Dahlke, Fornasier, Rauhut, Steidl and Teschke who generalized coorbit spaces to also include α-modulation spaces by using group representations modulo quotients [**10**]. The construction of a basis for α-modulation spaces with a group representation is an open question which would help the construction of more efficient algorithms for

α-modulation spaces. However, there is no obvious modification of the basis in Chapter 2 which would solve this problem or at least give a tight frame.

Flexible frame construction. We saw in Chapter 1 that Daubechies wavelets can provide better compressed images than Meyer wavelets. Similarly one would expect that a discrete curvelet transform built on the curvelet type frame in Chapter 4 would preform better than the classical curvelet transform. The open question is how to make such a transform feasible in a practical setting especially computation of the coefficients in the frame expansion. The general frame construction in Chapters 3 and 4 could also prove useful in the study of integral and pseudodifferential operators with the right choice of generating function g.

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