12-2010

Tight Double-Change Covering Designs

Kristen A. Lindbloom
Southern Illinois University Carbondale, kwisten621@yahoo.com

TIGHT DOUBLE-CHANGE COVERING DESIGNS

by

Kristen A. Lindbloom

B.S., Illinois State University, 2004

A Research Paper
Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
May, 2011

RESEARCH PAPER APPROVAL

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Approved by:

Dr. John McSorley, Chair

Dr. Thomas Porter

Dr. Joseph Yucas

Graduate School
Southern Illinois University Carbondale
November 30, 2010

ACKNOWLEDGMENTS

My sincere thanks goes to Dr. McSorley [8] [9] for his assistance, patience, and encouragement. Without his guidance and insights, I would not have been able to grow and learn so much as a graduate student. I would also like to thank all of the members of the Mathematics Department at SIUC for the many opportunities you have given me, and providing such a wonderful environment for my studies. A special thanks goes to the members of my graduate committee, Dr. Porter and Dr. Yucas [9] [11], for their support and cooperation.

I would especially like to thank my family and friends for their encouragement and support. None of this would have been possible if I did not have the confidence in myself, and belief that I can accomplish anything I set my mind to. A special thanks to Russ for . . . everything.

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INTRODUCTION

Combinatorics is a rapidly growing area of mathematics due to the many useful applications to other fields in the sciences including algebra, probability and statistics, computer science, industrial and electrical engineering, biology, and chemistry, etc. This paper will explore two branches of combinatorics: Design Theory and Graph Theory, having roots in some of the oldest discoveries in combinatorics including Euler's Latin Squares dating from the 18th century, and Kirkman's Schoolgirl Problem proposed by Thomas Kirkman in 1850.

In the first chapter, general definitions and properties of double-change covering designs are introduced, as well as examples and useful applications. We will also give main results that lead into the next two chapters.

Chapters 2 and 3 explore for what v tight, circular double-change designs exist when $k = 2$ and $k = 3$. We find that when $k = 2$, non-trivial tight, circular designs exist for $v \geq 6$ using a direct construction. Using constructions for Steiner triple systems, we find that tight, circular, double-change designs are possible for $v \equiv 1$ or $3 \pmod 6$ when $k = 3$. However, economical designs can be constructed for $v \equiv 0, 2, 4, 5 \pmod 6$ (not discussed in this paper).

CHAPTER 1

DOUBLE-CHANGE COVERING DESIGNS

1.1 INTRODUCTION TO COMBINATORIAL COVERING DESIGNS AND BLOCK DESIGNS

A *combinatorial covering design* is a selection and/or construction of subsets from a finite set that satisfy certain properties, mainly intersection conditions. For instance, a chef at a restaurant has nine different specialty entrées, and prepares three of them each day for twelve consecutive days. Is it possible to schedule the menu in a way such that: (1) each entrée must be made at least once; (2) two entrées made on the same day cannot be made together on the same day again, i.e. each pair of entrées is made together on the same day exactly once; (3) and each day differs from the day previous by exactly two entrées, i.e. exactly one of the entrees is made two days in a row? Answers to questions such as this one will be revealed in this paper.

A specific type of covering design called a *block design* will provide the backbone for this paper. We reference Brualdi [2] for the following example: Suppose a company wishes to ask a random pool of its consumers to compare a certain number, say v, of varieties of a product that are being tested for acceptability in the marketplace. It would be expensive and time consuming if each consumer compared all v varieties, so the company decides to have each consumer compare a portion of the varieties. In this scenario, the number of varieties being tested must equal the number of consumers making comparisons, in this case v. So the company forms subsets of the v varieties into *blocks*, say B_1, B_2, $\ldots$, B_v, to distribute to the v consumers. Thus a consumer can make comparisons of each pair of varieties in the given block; but in order to be efficient, we wish that each pair of varieties is tested exactly once. Hence each pair of varieties must appear in only one block.

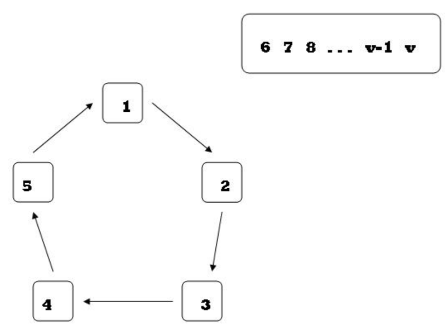
"Let X be any set of v elements, called *varieties*, and let B be a collection B_1, $B_2, \ldots, B_b$ of k-element subsets of X called *blocks*. Then B is a *balanced block design* on X, provided that each pair of elements of X occurs together in exactly λ blocks." [2] Here we assume $k, \lambda, v \in \mathbb{Z}^+$ and $2 \leq k \leq v$. When a balanced block design is complete, $k = v$, and all varieties appear in each block; or in other words, every consumer compares all v varieties. However, when $k < v$, we say B is a *balanced incomplete block design (BIBD).*

1.2 APPLICATIONS

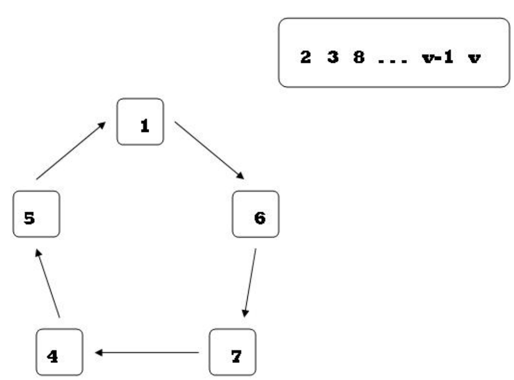
A valuable "real-world" application of combinatorial designs was seen as far back as the 1930's as discussed by F. Yates and R.A. Fisher, who were largely interested in agricultural field trials. Suppose a new crop is introduced to a certain region of a country. "What is the right way to set about determining the best varieties and the appropriate manuring and cultivations?" When there are a large number of varieties to be compared, the problem of physical arrangement becomes apparent. When plots contain all varieties, it is difficult to efficiently determine which environmental factors are affecting the crops, such as soil fertility, different fertilizers, or even interaction between varieties. So they selected a certain number of varieties to be controls, and divided the remaining varieties into sets, "each set being arranged with the controls in a number of randomised blocks", so that randomly assigned treatments (for example, nitrogen fertilizer versus phosphate fertilizer) could be tested. Thus when comparing a certain selection of plots with fairly consistent environments, "accuracy of the treatment comparisons is considerably enhanced..." "The process of random arrangement within the blocks ensures that no treatment shall be unduly favoured, and, moreover, enables an unbiased estimate of experimental error to be obtained, which is itself the basis of valid tests of significance."

Since all varieties do not appear in the same plot, and the probabilities of seeing certain treatments are fairly consistent, we see that this construction is actually a balanced incomplete block design. [12]

We see another useful application in combatibility testing of electrical components. Suppose there are v different electrical components that need to be tested, i.e. {1,2,. . . ,v-1,v}, such that each component must be tested with every other component. A number of components, say for example {1,2,3,4,5}, are loaded into a testing device and tested concurrently; hence, the components in the device together form a block.



At the completion of the test, a certain quantity, say two, of the components are removed and replaced with two other components, thus forming the second block.



Here we see the blocks are ordered, so we call this construction an *ordered covering* *design*. We continue the process until all v electrical components have been tested against each other. In this example, we switched out two components at a time, with the blocks forming a *double-change* design. Because the removal and insertion of the components requires the employment of either a human or computerized operator, unnecessary and redundant testing can be costly. Thus an optimized, minimal cost design is desired for the testing.

1.3 DOUBLE-CHANGE DESIGNS

A *double-change covering design (dccd)* is an ordered set of blocks, each block of size k consisting of unordered elements from the set $[v] = \{1, 2, ..., v\}$, and which follow the rules: (1) each block after the first differs from the previous block by changing exactly two elements, and (2) every pair in $[v]$ appears in at least one block. Note that within each block the elements are unordered, but the blocks themselves are ordered.

We say an element is *introduced* in a block if it is one of the two new elements changed in the block. We say that a pair is *covered* if it appears in at least one block of the dccd. We write blocks horizontally as rows, and will leave unchanged elements in their original positions. We define:

- v as the number of varieties tested, taken from $[v] = \{1, 2, ..., v\}$, and $v \geq k$;
- b as the number of blocks in the design;
- k as the block size, i.e. the number of elements in each block, and $k \leq v$;
- λ as the number of times each pair (i, j) appears, $i \in \{1, 2, \ldots, v-1\}, j \in \{2, 3, \ldots, v\}$

If $\lambda = 1$ for all pairs, then each pair is covered exactly once, and we call the dccd *tight*. However, if repeated pairs are necessary, then we call the dccd *economical* if we can construct the design with a minimal number of blocks. If we can change exactly two elements of the last block of a dccd to return to the first block, producing a loop, we call this dccd *circular*.

For example, a dccd with $(v, k) = (5, 2)$ is

3	4
1	2
4	5
1	3
2	5
1	4
2	3
1	5
2	4
3	5

This example is tight, since there are no repeated pairs.

Now consider the following double change covering design with $(v, k) = (7, 5)$:

1	2	3	4	5
1	2	3	6	7
1	4	5	6	7

Note that it is economical and circular, but not tight since the pairs 12, 13, 23, 14, 15 and 45 are repeated.

Finally consider the dccd with $(v, k) = (7, 3)$. We can see that it is both tight and circular:

1	2	3
1	4	6
1	5	7
3	5	4
2	5	6
3	7	6
2	7	4

1.4 MAIN RESULTS

Theorem 1.4.1. *The number of blocks, b, in a tight, double-change covering design is*

$$b = \frac{\binom{v}{2} - \binom{k}{2}}{2k-3} + 1 \tag{1.1}$$

Proof. In a double-change covering design, there are $\binom{v}{2}$ pairs to be covered, and every pair is covered at least once. Let b be the number of blocks in a dccd. The first block of a $dccd(v, k)$ has $\binom{k}{2}$ pairs, and each of the subsequent $(b-1)$ blocks has $2(k-2)+1$ pairs. We see this is true since each new element introduced forms pairs with the rest of the elements in the block except itself and the other new element, so $(k-2)$, likewise for the other new element introduced, $2(k-2)$. Also, the two new elements form a pair, thus we have $2(k-2)+1$ pairs. Hence,

$$\binom{v}{2} = \binom{k}{2} + (b-1)[2(k-2)+1]$$

Therefore we see,

$$b = \frac{\binom{v}{2} - \binom{k}{2}}{2k-3} + 1$$

□

If the number of blocks in a dccd equal (1.1), then the dccd is tight. If we can find a design that meets the following bound:

$$b = \left\lceil \frac{\binom{v}{2} - \binom{k}{2}}{2k-3} + 1 \right\rceil \tag{1.2}$$

then the design is economical.

Note that it is necessary that either $v = k$ or $v \geq k+2$. The first is the trivial complete case where the $dccd(v = k, k)$ is $1 \quad 2 \quad \ldots \quad v = k$. To illustrate why $v \geq k+2$, suppose we wish to construct a $dccd(v, k)$ where $k = v - 1$. Suppose the first block is $(1,\ 2,\ \ldots,\ v-2,\ v-1)$. However, there is just one more element to introduce in the next block, and introducing only it would violate the definition of a dccd.

This brings us to permissible values of v and k. Using (1.1), if $k = 2$, we see that $\binom{v}{2} = b$, so $v(v-1) = 2b$. Since the product of any two consecutive integers is always even, we see that as long as $v \geq 4$ since v must be greater than or equal to $k+2$, the permissible values of v are 4, 5, 6, 7, ... Again using (1.1) for $k = 3$, we see $v \geq 5$ and $\binom{v}{2}$ must be divisible by 3; thus v is restricted to the values 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, ... For $k = 4$, we have v restricted to 7, 9, 12, 14, 17, 19, 22, 24, 27, 29, ...; for $k = 5$, we have v restricted to 10, 12, 17, 19, 24, 26, 31, ...; and for $k = 6$, we have v restricted to 13, 15, 22, 24, 31, ...

CHAPTER 2

DOUBLE-CHANGE DESIGNS WITH $K = 2$

2.1 DIRECT CONSTRUCTION

Each pair, (i, j), where $i = 1, 2, \ldots, v-1$ and $j = 2, 3, \ldots, v$, is a block in a $dccd(v, 2)$, and any given two consecutive blocks are disjoint. Note that each variety appears $v - 1$ times in the design, since each $x \in \{1, 2, \ldots, v\}$ is paired with every variety but itself.

The trivial case when $v = 2$ is $dccd(2, 2) = 1\ \ 2$. The $dccd(3, 2)$ does not exist by the above argument that $3 \not\geq k + 2 = 2 + 2 = 4$. Nevertheless, double change designs do not exist for $v = 4$ when $k = 2$. For example, choose 2 elements from $V = \{1, 2, 3, 4\}$ for the first block, leaving the two remaining elements of V for the second block. However, the only possibility for the third block will be identical to the first block. Thus it is possible for only two out of the $\binom{4}{2} = 6$ pairs to be covered.

Tight double-change designs exist for $v = 5$, but none of which are circular. The construction for $dccd(5, 2)$ will be given at the end of this section.

Tight, circular designs exist for $v \geq 6$. How can we construct a $dccd(v, 2)$ when $v \geq 6$ in general? Consider all pairs of $V = \{1, 2, \ldots v\}$, and some arbitrary pair (i, j) in a $dccd(v, 2)$, where i, $i = 1, 2, \ldots, v - 1$, represents the element from an arbitrary row and j, $j = 2, 3, \ldots v$, represents the element from an arbitrary column. Write each pair of V, lining up identical $j's$ by column and identical $i's$ by row, as illustrated on the next page.

$$
\begin{array}{cccccc}
12 & 13 & 14 & 15 & \dots & 1v \\
 & 23 & 24 & 25 & \dots & 2v \\
 & & 34 & 35 & \dots & 3v \\
 & & & 45 & \dots & 4v \\
 & & & & \dots & 5v \\
 & & & & \ddots & \vdots \\
 & & & & & (v-1)v
\end{array} \tag{2.1}
$$

Note that for any odd or even v, we always have $i < j$ and $i+1 \leq j \leq v$. Notice that if we choose an arbitrary pair (i, j) for the first block; then for the second block, we cannot take from:

Columns:	Rows:
If $i = 1$, from column i	If $j = v$, from row $j - 1$
If $i > 1$, from column $i - 1$	If $j < v$, from row j
And from column $j - 1$	And from row i

Here we have four possible cases, but we see there are $\binom{v-2}{2}$ choices for the second block in every case. To illustrate, take all $\binom{v}{2}$ pairs and subtract the pairs that it cannot be adjacent to (noting again that each variety appears $v - 1$ times in the design), and add one for itself. Thus

$$\binom{v}{2} - 2(v-1) + 1 \;\Leftrightarrow\; \binom{v-2}{2}$$

The question arises: Can we construct a dccd using this triangular structure?

Let us call the farthest left diagonal the *main diagonal*. Examining a diagonal that starts from the top left to the bottom right, *except* for the main diagonal, e.g.

(1,3), (2,4), (3,5), ..., $(v-2, v)$, we see that each pair is distinct from the previous. Or in other words, choosing some arbitrary pair (i, j), the pair $(i+1, j+1)$ is its subsequent block, and the two are always distinct, that is $i \neq i+1 \neq j \neq j+1$ for any i, j since:

1. $i+1 \neq i$, for if they were equal, these two blocks would be in the same row, and hence could not be on a diagonal;

2. $j+1 \neq i$, since $j > i$ implies $j+1 > i+1$, $\forall\ i, j$;

3. $i+1 \neq j$, for if they were equal, the pair $(j, j+1)$ is produced, so that we have the blocks (i, j) then $(j, j+1)$, which are always on the main diagonal;

4. $j+1 \neq j$, for if they were equal, these two blocks would be in the same column, and hence could not be on a diagonal.

Now examining the main diagonal, i.e. (1,2), (2,3), (3,4), ..., $(v-1, v)$, notice that for any arbitrary pair (i, j), its subsequent pair is $(j, j+1)$. Also note that every other pair, i.e. (i, j) and $(i+2, j+2)$, are always distinct by a similar argument as above.

Since odd v and even v must be constructed slightly differently, we break our direct constructions into the two cases.

2.1.1 Construction of $dccd(v, 2)$ with odd v

1. To construct a dccd when v is odd, start on the main diagonal at the pair $(1, 2)$.

2. Each subsequent block should be every other pair from left to right along the main diagonal. So for any block (i, j) along the main diagonal, its consecutive block will be the pair $(i+2, j+2)$.

3. When we reach the pair $(v-2, v-1)$, this will be the last pair from the main diagonal used in the dccd for the time being; and the next block will be the pair $(1, \lfloor \frac{v}{2} \rfloor)$, or in words, go back up to the beginning of the diagonal above the first diagonal.

4. Now we go straight down each diagonal from left to right so that for any block (i, j) on the diagonal, its subsequent block will be the pair $(i+1, j+1)$.

5. When we come to some arbitrary (i, v), such that $i \neq v-1$ and $i \neq 2$, the next block will always be $(1, v-i+2)$. In other words, taking pairs diagonally from left to right, when we come to the bottom of one, we go up to the top of the one above it.

6. When we come to $(2, v)$, where $v \geq 6$, the next block cannot be $(1, v)$ since this would violate the definition of a dccd. Thus we go back to the main diagonal, but skipping over (2,3), which is also not permissible. Thus starting with $(4, 5)$ is permissible, since $v \geq 6$.

7. We are now on the main diagonal again, so we take every other pair (the remaining pairs along the diagonal) until we reach $(v-1, v)$.

8. Looping back to the top of the main diagonal, the next block should be $(2, 3)$; and finally the last block in the design is $(1, v)$.

This procedure will construct a linear dccd every time for any odd $v \geq 7$, and to get a circular dccd, just interchange the first two blocks.

For example, let $v = 7$. We first create our triangular structure, and then following the above procedure for odd v, we construct the $dccd(7, 2)$.

12	13	14	15	16	17	1	2	4	7
	23	24	25	26	27	3	4	1	5
		34	35	36	37	5	6	2	6
			45	46	47	1	3	3	7
				56	57	2	4	1	6
					67	3	5	2	7
						4	6	4	5
						5	7	6	7
						1	4	2	3
						2	5	1	7
						3	6		

This tight dccd is linear, and we could interchange the first two pairs to get a circular dccd.

2.1.2 Construction of $dccd(v, 2)$ with even v

1. To construct a dccd when v is even, start on the main diagonal at the pair $(1, 2)$.

2. Each subsequent block should be every other pair from left to right along the main diagonal. So for any block (i, j) along the main diagonal, its consecutive block will be the pair $(i + 2, j + 2)$.

3. When we reach the pair $(v - 1, v)$, this will be the last pair from the main diagonal used in the dccd for the time being; and the next block will be the pair $(1, v - i + 2)$, or in words, go back up to the beginning of the diagonal

above the first diagonal.

4. Now similar to the odd case, we take pairs diagonally from left to right, so that for any block (i, j) on the diagonal, its subsequent block will be the pair $(i + 1, j + 1)$.

5. When we come to the bottom of one diagonal, we go up to the top of the one above it (except in one instance). Usually when we come to some arbitrary (i, v), such that $i \neq 2$, the next block will be $(1, v - i + 1)$ since we see that for all but one arbitrary (i, v), $i \neq v - i + 2$; however, when $i = \frac{v}{2} + 1$, $i = v - i + 2$ so that these two pairs are not distinct. Hence, when we get to $(\frac{v}{2} + 1, v)$, we must skip over $(1, \frac{v}{2} + 1)$. We continue the process of taking pairs along the diagonals until we reach $(2, v)$.

6. When we come to $(2, v)$ the next block cannot be $(1, v)$ since this would violate the definition of a dccd, so we now place the pair we originally skipped over, $(1, \frac{v}{2} + 1)$, since $2 \neq v \neq 1 \neq \frac{v}{2} + 1$ for any $v \geq 6$.

7. Notice that for any $v \geq 6$, $2 < \frac{v}{2} + 1$, so we now go back to the main diagonal starting from $(2, 3)$, since the two pairs are indeed distinct. We are now on the main diagonal again, so we take every other pair (the remaining pairs), the last one being $(v - 2, v - 1)$.

8. Finally, the last block in the design is $(1, v)$.

This procedure produces a tight linear dccd for any $v \geq 6$, and we can interchange the first two pairs to get a circular dccd. To illustrate, let $v = 8$.

12	13	14	15	16	17	18		1	2		1	4	3	8
	23	24	25	26	27	28		3	4		2	5	1	7
		34	35	36	37	38		5	6		3	6	2	8
			45	46	47	48		7	8		4	7	1	5
				56	57	58		1	3		5	8	2	3
					67	68		2	4		2	6	4	5
						78		3	5		3	7	6	7
								4	6		4	8	1	8
								5	7		1	6		
								6	8		2	7		

Again, we started with our triangular structure, and followed the above procedure for even v to construct the dccd. This tight dccd is linear, and we could interchange the first two pairs to get a circular dccd.

2.1.3 Construction of $dccd(5, 2)$

Lastly, let us construct a double-change design when $v = 5$. We will again use the triangular structure, but instead of beginning on the main diagonal, let us start on the diagonal above it, with the first block being (1,3). We go down this diagonal, then up to the one above it, and then down that one until we get to (2,5). Then we go back to the main diagonal starting with (3,4) and then taking every other pair from the bottom up (although the only other pair will be (1,2)). Next, go back to the bottom of the main diagonal. The next block after (1,2) will be (4,5), and then again we take every other pair (although the only pair left will be (2,3)). Lastly, the last block in the design is (1,5).

12	13	14	15	1	3
	23	24	25	2	4
		34	35	3	5
			45	1	4
				2	5
				3	4
				1	2
				4	5
				2	3
				1	5

2.2 HAMILTONIAN CYCLES IN DOUBLE-CHANGE GRAPHS

We reference Chartrand's Corollary 6.7 (Dirac's Theorem): *"Let G be a graph of order $n \geq 3$. If deg $v \geq n/2$ for each vertex v of G, then G is Hamiltonian."* [4] A graph contains a Hamiltonian cycle if there exists a cycle in G in which every vertex is visited exactly once (except for the first and last), and starts and finishes at the same vertex. If we can visit each vertex once, but cannot start and end at the same vertex, then the graph has a Hamiltonian path. We wish to explore for what double-change graphs when $k = 2$, $DCG(v, 2)$, does a Hamiltonian cycle exist?

If the dccd is circular, then it potentially has a Hamiltonian cycle, but if the dccd is linear, it potentially has a Hamiltonian path. For $DCG(v, 2)$, i.e. when $k = 2$, each vertex is a pair, and we form edges between pairs that are disjoint. The order of the graph is $\binom{v}{2}$, and we see the degree of each vertex is $\binom{v-2}{2}$. To illustrate, take all $\binom{v}{2}$ pairs and subtract the pairs that it cannot be adjacent to, and add one for itself. Thus we see the degree of each vertex is

$$\binom{v}{2} - 2(v-1) + 1 \;\Leftrightarrow\; \binom{v-2}{2} \tag{2.2}$$

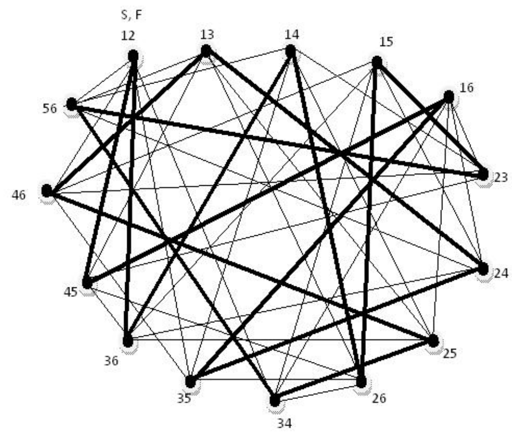
Theorem 2.2.1. *Hamiltonian cycles exist for $DCG(v,2)$ where $v \geq 6$.*

Proof. Using Dirac's Theorem, if $\binom{v-2}{2} \geq \frac{\binom{v}{2}}{2}$, or equivalently, if $\binom{v}{2} - 2(v-1) + 1 \geq \frac{\binom{v}{2}}{2}$, then $DCG(v,2)$ has a Hamiltonian cycle.

$$\begin{aligned}
\frac{v(v-1)}{2} - 2(v-1) + 1 &\geq \frac{\binom{v}{2}}{2} \\
v(v-1) - 4(v-1) + 2 &\geq \frac{v(v-1)}{2} \\
2v(v-1) - 8(v-1) + 4 &\geq v(v-1) \\
2v - 8 + \frac{4}{v-1} &\geq v \\
v - 8 + \frac{4}{v-1} &\geq 0
\end{aligned}$$

The inequality is true for $v \geq 8$. Thus, Dirac's Theorem implies a circular dccd exists, which implies that a linear dccd exists. Dirac's Theorem is a sufficient condition for the presence of a Hamiltonian cycle, but not necessary. For example, if $v = 6$ or $v = 7$, we see $\binom{v-2}{2} \ngeq \frac{\binom{v}{2}}{2}$, but $DCG(6,2)$ and $DCG(7,2)$ still have Hamiltonian cycles. □

Here we see $DCG(6,2)$ and $DCG(7,2)$, with Hamiltonian cycles starting and finishing at vertex 12: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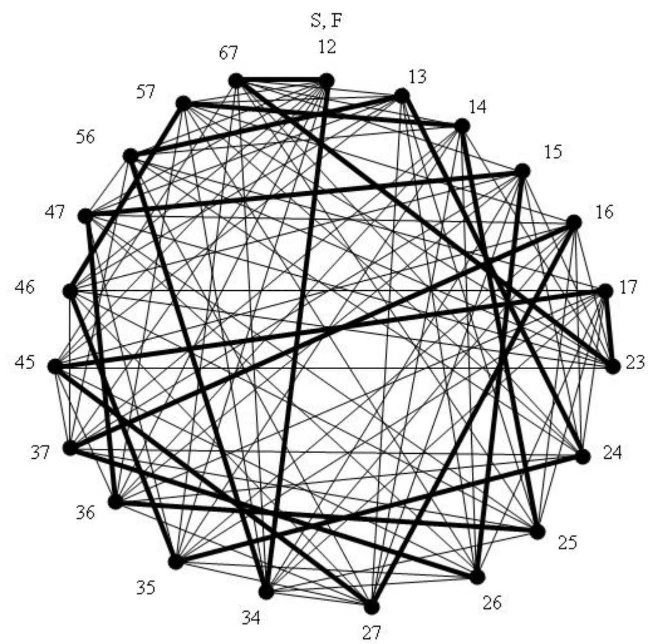
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CHAPTER 3

DOUBLE-CHANGE DESIGNS WITH $K = 3$

3.1 STEINER TRIPLE SYSTEMS

Steiner triple systems originate from Reverand Thomas Kirkman's question posed in 1850, which is now known as "Kirkman's schoolgirl problem": "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast." [10] In other words, is it possible to arrange the fifteen girls in five rows, three girls in each row, so that no two girls walk in the same triple more than once? This problem asks for a Steiner triple system with $\lambda = 1$ and $v = 15$.

A *Steiner triple system STS(V)* based on $V = \{1, 2, ..., v\}$ is a collection of b 3-sets or triples from V such that $\lambda = 1$, and every pair from V occurs exactly once in some triple. A triple, say xyz, covers three pairs: xy, xz, and yz. So $3b = \lambda\binom{v}{2} = 1 \cdot \binom{v}{2}$ implies that $b = \frac{\binom{v}{2}}{3} = \frac{v(v-1)}{6}$. Hence this last term must be an integer in order for a $STS(V)$ to be possible.

For instance, let $v = 3$, then $V = \{1, 2, 3\}$. We see $b = \frac{3 \cdot 2}{6} = 1 \in \mathbb{Z}$, so we should be able to construct $STS(3)$ in one block. Indeed it is possible: $STS(3) =$ 1 2 3. If $v = 4$, then $V = \{1, 2, 3, 4\}$, and $b = \frac{4 \cdot 3}{6} = 2 \in \mathbb{Z}$, but exhaustive analysis shows that we are unable to construct a $STS(4)$ in two blocks since one pair is never covered. For example, $STS(4) \neq (123)(423)$ because the pair (1,4) is missing. Therefore, $STS(4)$ does not exist. We use similar arguments that $STS(5)$ and $STS(6)$ do not exist.

We saw that it is necessary that $\frac{v(v-1)}{6} \in \mathbb{Z}$, but we also must have $\frac{v-1}{2} \in \mathbb{Z}$ since in a $STS(V)$ each $x \in V$ lies in $\frac{n-1}{2}$ blocks. To illustrate, there are $v - 1$ elements in $V \setminus \{x\}$ and x must appear in a triple exactly once with each of these $n - 1$ elements. There are two other elements in each block with x, so it follows that twice the number of blocks containing each variety is $v-1$. Thus we see each variety must lie in exactly $\frac{v-1}{2}$ triples. Note that this necessary condition also proves that $STS(4)$ does not exist since $\frac{4-1}{2} \notin \mathbb{Z}$.

For which v does a $STS(v)$ exist? We use a well known theorem:

$$\textit{A } STS(V) \textit{ exists if and only if } v \equiv 1 \textit{ or } 3 \pmod{6}.$$

The proof of this theorem follows from the implications of both of the above two necessary requirements. If $\frac{v-1}{2}$ must be an integer, then v has to be odd, i.e. $v = 2s+1$, where $s = 0, 1, 2, \ldots$; thus we have $v \equiv 1, 3$, or $5 \pmod{6}$. But suppose $v \equiv 5 \pmod{6}$, then $v = 6s+5$. This implies that $b = \frac{n(n-1)}{6} = \frac{(6s+5)(6s+4)}{3 \cdot 2} = \frac{(6s+5)(3s+2)}{3} \notin \mathbb{Z}$ since 3 cannot divide 5. Thus both necessities imply $v \equiv 1$ or $3 \pmod{6}$. It is true that for a $STS(V)$ to exist, it is necessary that $v \equiv 1$ or $3 \pmod{6}$, but if we have a v such that $v \equiv 1$ or $3 \pmod{6}$, then it is a sufficient condition for the existence of a $STS(V)$.

The $STS(V)$ for $v \equiv 1 \pmod{6}$ and $v \equiv 3 \pmod{6}$ must be constructed differently, so let us look at each case separately.

3.2 THE KIRKMAN/STEINER CONSTRUCTION OF A $STS(V \equiv 3 \pmod{6})$

Consider $v \equiv 3 \pmod{6}$ and $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. For a fixed $k \in \mathbb{Z}_6$, the equation $i + j = k$ has six solutions, e.g. $k = 4$, $i + j = 4 : 4 + 0 = 4; 3 + 1 = 4; 2 + 2 = 4; 1 + 3 = 4; 0 + 4 = 4; 5 + 5 = 4$ in modulo 6. If you are given two of the i, j, or k, is the third uniquely determined?

Consider now $i + j = 2k$ in $\mathbb{Z}_6$. If $i = 0$ and $j = 4$, $0 + 4 = 2k$, which implies $k = 2, 5$. So given two of i, j, k the third is *not* uniquely determined. Thus the statement is not true for this equation in $\mathbb{Z}_6$. However, let us consider $i + j = 2k$ in $\mathbb{Z}_5$. Again, if $i = 0$ and $j = 4$, $0 + 4 = 2k$ implies $k = 2$. The statement is true here. Hence we see that if given two of i, j, k in $\mathbb{Z}_m$, m odd, the third *is* uniquely determined. In general, $i + j = t \cdot k$ in $\mathbb{Z}_m$ has this property when t and m have *no* common factors (in the even case, t and m have 2 as a common factor).

So if $v \equiv 3 \pmod 6$, then let $m = 2s + 1$, $s = 0, 1, 2, \ldots$, so that $v = 6s + 3 = 3(2s + 1) = 3m$, where m is of course odd. Let us reference Cameron [3] for the following construction. We consider $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$. Define $V = \{a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_{m-1}, c_0, c_1, \ldots, c_{m-1}\}$. The triples (blocks) in the construction will be of two types:

1. (a) $a_i a_j b_k, \quad i \neq j, \quad i + j \equiv 2k$ in $\mathbb{Z}_m$

   (b) $b_i b_j c_k, \quad i \neq j, \quad i + j \equiv 2k$ in $\mathbb{Z}_m$

   (c) $c_i c_j a_k, \quad i \neq j, \quad i + j \equiv 2k$ in $\mathbb{Z}_m$

2. $a_i b_i c_i, i \in \mathbb{Z}$.

So as an example, let us construct a $STS(9)$, where $v = 9$, and $m = 3$. Here $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2\}$. The first block will be $a_0 a_1 b_k$, and we solve the linear equation $0 + 1 = 2k$ to find k. We see that $k = 2$, so our first block is $a_0 a_1 b_2$. Similarly, the second block is $a_0 a_2 b_k$, and we solve $0+2 = 2k$ for k. We get $k = 1$, so our block becomes $a_0 a_2 b_1$. Our third block is $a_1 a_2 b_k$, and again solving the equation for k, we get $a_0 a_1 b_0$. These 3 blocks form Type 1a. We form blocks similarly for Types 1b, 1c, and 2, producing the following Steiner triple system. Then we substitute for each variable.

a_0	a_1	b_2		1	2	6
a_0	a_2	b_1		1	3	5
a_1	a_2	b_0		2	3	4
b_0	b_1	c_2		4	5	9
b_0	b_2	c_1		4	6	8
b_1	b_2	c_0		5	6	7
c_0	c_1	a_2		7	8	3
c_0	c_2	a_1		7	9	2
c_1	c_2	a_0		8	9	1
a_0	b_0	c_0		1	4	7
a_1	b_1	c_1		2	5	8
a_2	b_2	c_2		3	6	9

Let us now verify that each pair only appears once by checking the number of blocks. There are $\binom{m}{2}$ ways to choose an arbitrary pair a_ib_j, and k will be uniquely determined. There are three of these types, and m of Type 2. So we have:

$$\binom{m}{2} \cdot 1 \cdot 3 + m = \frac{3m(m-1)}{2} + m = \frac{3m(3m-1)}{6} = \frac{n(n-1)}{6}$$

Next, let us check that every pair occurs at least once in a triple:

1. (i) a_ia_j where $i \neq j$ occurs in $a_ia_jb_k$ where k is unique;

   (ii) b_ib_j is similar;

   (iii) c_ic_j is similar;

2. (i) a_ib_i occurs in $a_ib_ic_i$, but also could possibly occur in Type 1a if $i = k$. If $a_ib_i \subset a_ia_jb_k$, then $i = k$, but $i + j = 2k = 2i$ implies that $j = i$. Contradiction. So a_ib_i occurs only in $a_ib_ic_i$. For example, the block $a_3a_4b_3$ is impossible: if $3 + 4 = 2 \cdot 3$, then $3 + 4 = 3 + 3$ which implies $4 = 3$. Contradiction;

   (ii) a_ic_i similar;

   (iii) b_ic_i similar;

3. (i) a_ib_k where $i \neq k$ (and also $i \neq j$), occurs in $a_ia_jb_k$ where $i + j = 2k$;

   (ii) a_ic_k similar;

   (iii) b_ic_k similar.

Thus we have constructed a Steiner triple system in which every pair appears exactly once. Can we construct a double-change design from this $STS(V = 3 \pmod 6)$?

3.2.1 Construction of a $dccd(v, 3)$ with $v \equiv 3 \pmod 6$

We can construct the double-change design by rearranging the blocks of the $STS(V \equiv 3 \pmod 6)$ and using the triangular structure we used for constructing $dccd(v, 2)$. Looking at just the first two columns of the $STS(V)$ we have constructed, we notice that we can construct a single-change design. We will first arrange the blocks in such a way that the first two columns form a single-change design (while ignoring the third column, but not removing it). We use the triangular structure:

$$\begin{array}{cccccc} 01 & 02 & 03 & 04 & \ldots & 0(m\text{-}1) \\ & 12 & 13 & 14 & \ldots & 1(m\text{-}1) \\ & & 23 & 24 & \ldots & 2(m\text{-}1) \\ & & & 34 & \ldots & 3(m\text{-}1) \\ & & & & \ldots & 4(m\text{-}1) \\ & & & & \ddots & \vdots \\ & & & & & (m\text{-}2)(m\text{-}1) \end{array}$$

This procedure will take care of the blocks in the form 1a; and we construct 1b and 1c in the same manner.

1. Starting at the top left corner with (0,1), $a_0a_1b_{k_1}$, we move horizontally along the first row, stopping at $(0, m-1)$, $a_0a_{m-1}b_{k_2}$.

2. Next we go down the last column, starting with $(, m-1)$, $a_1a_{m-1}b_{k_3}$, and ending with $(m-2, m-1)$, $a_{m-2}a_{m-1}b_{k_4}$.

3. Next we go up to the second entry of the column to the left of the last column, $(1, m-2)$, $a_1a_{m-2}b_{k_5}$, and go straight down this column until we get to the bottom. Then we go up to the next column.

4. We continue this process until we finally reach (1, 2), $a_1a_2b_{k_r}$.

Thus we have covered every pair in {0, 1, ..., m-1}. For example, with $v = 15, m = 5$, the blocks in the form 1a will be:

$$
\begin{array}{ccc}
a_0 & a_1 & b_3 \\
a_0 & a_2 & b_1 \\
a_0 & a_3 & b_4 \\
a_0 & a_4 & b_2 \\
a_1 & a_4 & b_0 \\
a_2 & a_4 & b_3 \\
a_3 & a_4 & b_1 \\
a_1 & a_3 & b_2 \\
a_2 & a_3 & b_0 \\
a_1 & a_2 & b_4
\end{array} \tag{3.1}
$$

The blocks in the form 1b and 1c will have the same subscripts.

By constructing this "single-change" design, we have in fact created a double-change design *within* each of the three forms. Notice in (3.1), that as we go from block to block, each b_k is distinct from the previous, thus creating a double-change design. This is true in general, for if we have two blocks, $a_i a_j b_k$ and $a_i a_{j'} b_{k'}$ say, where $j \neq j'$; if $k = k'$, then $2k = 2k'$, which implies that $i + j = i + j'$ and hence $j = j'$. Contradiction. Thus each b is distinct. This is also true for the $c's$ in 1b and the $a's$ in 1c.

Now we join up our constructions for 1b, 1c, and the three blocks of type 2, $a_i b_i c_i$, $a_j b_j c_j$, $a_k b_k c_k$, at the end of 1a. After doing this, you can see that we do not quite have a complete double-change design; however, we can again rearrange blocks to produce one.

1. First insert the block $a_1 b_1 c_1$ in between 1a and 1b. This can always be done

since the last block in 1a is $a_1a_2b_{k'}$ and the first block in 1b is $b_0b_1c_{k''}$.

2. Now if we interchange the last two blocks of 1b, i.e. $b_2b_3c_{k'''}$ and $b_1b_2c_{k^{iv}}$, we can insert a block $a_ib_ic_i$ where $i = 0, 3, 4, ..., m-1$, but $i \neq 1, 2$, after $b_2b_3c_{k'''}$. Since $m \geq 5$, k must be less than 1, i.e. 0; or greater than 2. If $k = 1$, then $2 + 3 = 2k$ gives us that 5=2. Likewise for $k = 2$, we get 5=4. But if $m \geq 5$, 2 will always equal 2, and 4 will always equal 4 in modulo $m \geq 5$.

3. Next insert the block $a_2b_2c_2$ after 1c. This can always be done since we end with the block $c_1c_2a_{k^v}$.

4. Lastly we can insert the remaining $m - 3$ blocks in the form $a_ib_ic_i$ into the design where they fit. The $m - 3$ blocks will fit actually in just 1a since there are $\binom{m}{2} - 1 = \binom{m-2}{2}$ slots available, and $\frac{(m-2)(m-1)}{2} \geq m - 3$ for $m \geq 1$. Remember we constructed these blocks using the triangular structure by going across the top row and then down each column from right to left. If the block that needs to be inserted is $a_0b_0c_0$ then place it into the design where we were going along the top row. Place the block $a_jb_jc_j$ into the design where the respective pair in the triangular structure is (i, j).

5. Now to get a circular design, we can interchange the very first two blocks in 1a.

Thus for $m = 5$, we have the tight, circular double-change design:

a_0	a_2	b_1	b_0	b_1	c_3	c_0	c_1	a_3
a_0	a_1	b_3	b_0	b_2	c_1	c_0	c_2	a_1
a_0	a_3	b_4	b_0	b_3	c_4	c_0	c_3	a_4
a_0	a_4	b_2	b_0	b_4	c_2	c_0	c_4	a_2
a_4	b_4	c_4	b_1	b_4	c_0	c_1	c_4	a_0
a_1	a_4	b_0	b_2	b_4	c_3	c_2	c_4	a_3
a_2	a_4	b_3	b_3	b_4	c_1	c_3	c_4	a_1
a_3	a_4	b_1	b_1	b_3	c_2	c_1	c_3	a_2
a_3	b_3	c_3	b_1	b_2	c_4	c_2	c_3	a_0
a_1	a_3	b_2	b_2	b_3	c_0	c_1	c_2	a_4
a_2	a_3	b_0	a_0	b_0	c_0	a_2	b_2	c_2
a_1	a_2	b_4						
a_1	b_1	c_1						

Eulerian Circuit within a $dccd(v, 3)$ with $v \equiv 3 \pmod 6$

Quoting Wallis, Yucas, and Zhang, "When $k = 2$, we may interpret the elements as vertices and the blocks as edges of a complete graph on v vertices, and a single-change covering design is provided by a walk through the graph which covers every edge." [11] We can thus always find an Eulerian Circuit within our "single-change" design described for our example (3.1) using the triangular structure described. This procedure is useful since it allows us to keep track of edges covered.

1. First, let every pair in the triangle represent an edge between two vertices. Starting with (0,1) at the top left corner of the main diagonal, we connect

vertices 0 and 1 in our graph. Now we take each pair all the way down to the last pair on the main diagonal, $(m-2, m-1)$, connecting respective edges.

2. Now we go up to $(0, m-1)$ at the top of this column, i.e. the farthest right column, and start a *zig-zag* pattern. In words, this pattern will be: left, down, right, down, left, down, right, down, etc. So beginning with $(0, m-1)$, we go left to $(0, m-2)$, then down to $(1, m-2)$, right to $(1, m-1)$, down to $(2, m-1)$, left to $(2, m-2)$, down, right, $\ldots$, until we get down to the bottom to $(m-3, m-1)$. This zig-zag pattern never covers more than two columns at a time and assures us that we have a single-change from pair to pair. For instance, take the four pairs from the triangular structure

(i, j)	(i, j+1)
(i+1, j)	(i+1, j+1)

   There is always a single-change between two blocks in the same row; and there is always a single change between two blocks in the same column. Thus for any arbitrary pair of columns in this zig-zag pattern, we always have a single-change design.

3. Now we go up to the top of the next column to the left, beginning with $(0, m-3)$, and repeat the same zig-zag pattern.

4. Continue the pattern until we reach $(0, 2)$, in which case we are finished.

Hence we began our walk through the edges of the graph at vertex 0, and ended at vertex 0, thus indeed producing an Eulerian Circuit. For example, when $m = 7$, we begin at vertex (0,1) and draw edges in the following order:

01	25
12	35
23	36
34	46
45	04
56	03
06	13
05	14
15	24
16	02
26	

3.3 THE SKOLEM CONSTRUCTION OF A $STS(V \equiv 1 \pmod{6})$

We reference Lindner [7] for the Skolem Construction of a $STS(V)$ when $v \equiv 1 \pmod{6}$. Let $v = 6n+1$, and construct a half-idempotent commutative quasigroup of order $2n$. Recall that a latin square is an $n \times n$ array in which each element appears exactly once in each column and exactly once in each row. "A quasigroup of order n is a pair $(Q, \circ)$, where Q is a set of size n and "$\circ$" is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have *unique* solutions." [7] An idempotent quasigroup is one in which for $1 \leq i \leq n$ entry (i, i) contains i, and the quasi-group is commutative if for all $i \geq 1$, $j \leq n$, entries (i, j) and (j, i) contain the same element. So a half-idempotent commutative quasigroup is one in which for all $i \geq 1$, $j \leq n$, entries (i, j) and (j, i) contain the same element, and the entries (i, i) and $(n+i, n+i)$ for $1 \leq i \leq n$ contain the same element.

Let the elements of the STS be $\{\infty, a_1, a_2, ..., a_{2n}, b_1, b_2, ..., b_{2n}\}$. The triples (blocks) of the STS will be of three types:

Type 1: for $1 \leq i \leq n$,

$a_i b_i c_i$

Type 2: for $1 \leq i \leq n$,

$\infty a_{n+i} b_i$ We shall call this Type 2 form 1

$\infty b_{n+i} c_i$ Type 2 form 2

$\infty c_{n+i} a_i$ Type 2 form 3

Type 3: for $1 \leq i < j \leq 2n$,

$a_i a_j b_{i \circ j}$ Type 3 form 1

$b_i b_j c_{i \circ j}$ Type 3 form 2

$c_i c_j a_{i \circ j}$ Type 3 form 3

To illustrate the construction, let us construct a $STS(13)$ where $n = 2$. First we construct a half-idempotent commutative quasigroup of order 4, and then we construct the three types of blocks.

$\circ$	1	2	3	4
1	1	3	2	4
2	3	2	4	1
3	2	4	1	3
4	4	1	3	2

a_1	b_1	c_1	a_1	a_2	b_3	b_1	b_2	c_3	c_1	c_2	a_3
a_2	b_2	c_2	a_1	a_3	b_2	b_1	b_3	c_2	c_1	c_3	a_2
∞	a_3	b_1	a_1	a_4	b_4	b_1	b_4	c_4	c_1	c_4	a_4
∞	a_4	b_2	a_2	a_3	b_4	b_2	b_3	c_4	c_2	c_3	a_4
∞	b_3	c_1	a_2	a_4	b_1	b_2	b_4	c_1	c_2	c_4	a_1
∞	b_4	c_2	a_3	a_4	b_3	b_3	b_4	c_3	c_3	c_4	a_3
∞	c_3	a_1									
∞	c_4	a_2									

Let us now check that every pair occurs at least once in a triple:

1. (i) a_ib_i occurs in $a_ib_ic_i$;

   (ii) a_ic_i is similar;

   (iii) b_ic_i is similar;

2. (i) ∞a_i occurs in $\infty c_{n+i}ai$

   ∞a_{n+i} occurs in $\infty a_{n+i}b_i$

   (ii) ∞b_i is similar;

   ∞b_{n+i} is similar;

   (iii) ∞c_i is similar;

   ∞c_{n+i} is similar;

3. (i) a_ia_j where $i \neq j$ occurs in $a_ia_jb_{i\circ j}$;

   (ii) b_ib_j is similar;

   (iii) c_ic_j is similar;

4. (i) $a_ib_{i\circ j}$ occurs in $a_ia_jb_{i\circ j}$;

(ii) $b_i c_{i \circ j}$ is similar;

(iii) $c_i a_{i \circ j}$ is similar;

5. (i) $a_j b_{i \circ j}$ occurs in $a_i a_j b_{i \circ j}$;

(ii) $b_j c_{i \circ j}$ is similar;

(iii) $c_j a_{i \circ j}$ is similar.

Next, let us verify that each pair only appears once by checking the number of blocks. There are n blocks of Type 1, and $3n$ blocks of Type 2. Also, there are three forms of Type 3 blocks, in which there are $\binom{2n}{2}$ of each form. So we have:

$$n + 3n + 3\binom{2n}{2} = \frac{(6n+1)(6n)}{6} = \frac{v(v-1)}{6}$$

Thus we conclude that we have constructed a Steiner triple system in which each pair occurs exactly once in the design. Can we construct a double-change design from this construction?

3.3.1 Construction of a $dccd(v, 3)$ with $v \equiv 1 \pmod 6$

We break the construction into three cases, since the larger n is, the more blocks we have to work with. For $n = 1$, the construction described above already gives a circular double-change design:

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ \infty & a_2 & b_1 \\ \infty & b_2 & c_1 \\ \infty & c_2 & a_1 \\ a_1 & a_2 & b_2 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & a_2 \end{array}$$

For $n = 2$:

1. The first block will be $a_1b_1c_1$ from the Type 1 blocks.

2. Next, we place the Type 2 blocks since the first block in Type 2 will be $\infty a_{n+1}b_1$. Notice that the blocks of Type 2 are already in a double-change design, with ∞ being a constant element. It is true in general that for two blocks of this type, without loss of generality say $\infty a_{n+i}b_i$ and $\infty a_{n+i'}b_{i'}$, that $i \neq n+i \neq i' \neq n+i'$. It is clear that with $n \geq 1$, $i \neq n+i$ and $i' \neq n+i'$. In our two blocks we assume $n+i \neq n+i'$, but suppose that $i = i'$, then this implies that $n+i = n+i'$. Contradiction. Hence, there is a double-change within our two arbitrary blocks.

3. Next we will begin to place the first of the Type 3 blocks, but let us first rearrange each of the three forms of Type 3 blocks in the same fashion we did for $dccd(v \equiv 3 \pmod 6, 3)$. Using the triangular structure, we go across the first row, and then down each column from right to left, forming a "single-change" design within the first two columns of the blocks, and thus creating a double-change design within each of the three forms.

4. Since we end the Type 2 blocks with $\infty c_{2n}a_n$, i.e. ∞c_4a_2, we can now place the blocks of the first form of Type 3 after Type 2, since the first form of Type 3 begins with $a_1a_2b_3$.

5. Next, we see that we end this first form of Type 3 with $a_2a_3b_4$, and the second form begins with $b_1b_2c_3$, so we may place $a_2b_2c_2$ in between the two forms.

6. We do not need to do anything to Type 3 form 2, but we do need to arrange Type 3 form 3. Move the block $c_2c_4a_1$ to the top of the form 3 blocks, and move $c_1c_4a_4$ to the very bottom to get a circular design.

a_1	b_1	c_1	b_1	b_2	c_3
∞	a_3	b_1	b_1	b_3	c_2
∞	a_4	b_2	b_1	b_4	c_4
∞	b_3	c_1	b_2	b_4	c_1
∞	b_4	c_2	b_3	b_4	c_3
∞	c_3	a_1	b_2	b_3	c_4
∞	c_4	a_2	c_2	c_4	a_1
a_1	a_2	b_3	c_1	c_2	a_3
a_1	a_3	b_2	c_1	c_3	a_2
a_1	a_4	b_4	c_3	c_4	a_3
a_2	a_4	b_1	c_2	c_3	a_4
a_3	a_4	b_3	c_1	c_4	a_4
a_2	a_3	b_4			
a_2	b_2	c_2			

For $n \geq 3$:

1. The first block will be $a_1b_1c_1$ from the Type 1 blocks.

2. Next, we place the Type 2 blocks since the first block in Type 2 will be $\infty a_{n+1}b_1$. Notice that the blocks of Type 2 are already in a double-change design, with ∞ being a constant element. It is true in general that for two blocks of this type, without loss of generality say $\infty a_{n+i}b_i$ and $\infty a_{n+i'}b_{i'}$, that $i \neq n+i \neq i' \neq n+i'$. It is clear that with $n \geq 1$, $i \neq n+i$ and $i' \neq n+i'$. In our two blocks we assume $n+i \neq n+i'$, but suppose that $i = i'$, then this implies that $n+i = n+i'$. Contradiction. Hence, there is a double-change

within our two arbitrary blocks.

3. Next we will begin to place the first of the Type 3 blocks, but let us first rearrange the three forms of Type 3 blocks in the same fashion we did for $dccd(v \equiv 3 \pmod 6, 3)$. Using the triangular structure, we go across the first row, and then down each column from right to left, forming a "single-change" design within the first two columns of the blocks, and thus creating a double-change design within each of the three forms.

4. We now place the blocks of the first form of Type 3 after Type 2, but first place the block $a_1a_nb_k$ at the top of Type 3 form 1. Since we end the Type 2 blocks with $\infty c_{2n}a_n$, this guarantees that we will have a smooth double-change design from one type to the next.

5. Next, we see that we end this first form of Type 3 with $a_2a_3b_{k'}$, and the second form begins with $b_1b_2c_{k''}$, where $k \neq 2$, so we may place $a_2b_2c_2$ in between the two forms.

6. Similarly, we end the second form of Type 3 with $b_2b_3c_{k'''}$ and begin the third form with $c_1c_2a_{k^{iv}}$, so let us place $c_1c_3a_{k^v}$ before $c_1c_2a_{k^{iv}}$ so that we may insert $a_3b_3c_3$ in between.

7. For the remaining $n-3$ blocks of Type 1, we may place them where they fit in the design, particularly in the same fashion we did for $dccd(v \equiv 3 \pmod 6, 3)$. These $n-3$ blocks will fit into the Type 3 form 1 blocks since there are $\binom{2n}{2}-1$ possible slots, and $\binom{2n}{2}-1 \geq n-3$ for all $n \geq 1$.

8. To get a circular design, move the block $c_1c_2a_{k^{iv}}$ to the very bottom.

a_1	b_1	c_1	a_1	a_6	b_6	b_1	b_4	c_5	c_1	c_3	a_2
∞	a_4	b_1	a_2	a_6	b_1	b_1	b_5	c_3	c_1	c_4	a_5
∞	a_5	b_2	a_3	a_6	b_4	b_1	b_6	c_6	c_1	c_5	a_3
∞	a_6	b_3	a_4	a_6	b_2	b_2	b_6	c_1	c_1	c_6	a_6
∞	b_4	c_1	a_5	a_6	b_5	b_3	b_6	c_4	c_2	c_6	a_1
∞	b_5	c_2	a_2	a_5	b_6	b_4	b_6	c_2	c_3	c_6	a_4
∞	b_6	c_3	a_3	a_5	b_1	b_5	b_6	c_5	c_4	c_6	a_2
∞	c_4	a_1	a_4	a_5	b_4	b_2	b_5	c_6	c_5	c_6	a_5
∞	c_5	a_2	a_2	a_4	b_3	b_3	b_5	c_1	c_2	c_5	a_6
∞	c_6	a_3	a_3	a_4	b_6	b_5	b_4	c_4	c_3	c_5	a_1
a_1	a_3	b_2	a_2	a_3	b_5	b_2	b_4	c_3	c_4	c_5	a_4
a_1	a_2	b_4	a_2	b_2	c_2	b_3	b_4	c_6	c_2	c_4	a_3
a_1	a_4	b_5	b_1	b_2	c_4	b_2	b_3	c_5	c_3	c_4	a_6
a_1	a_5	b_3	b_1	b_3	c_2	a_3	b_3	c_3	c_2	c_3	a_5
									c_1	c_2	a_4

3.4 CONCLUSION

To answer the question given in the first section of the paper about the chef and his menu, we see that his proposed design *is* possible, since his problem asks to construct a $dccd(9, 3)$. We see that $9 \equiv 3 \pmod 6$, so we use the Kirkman/Steiner construction for $STS(9)$, and then arrange the blocks using the method described to get a double-change design. Also using (1.1), we see the number of blocks, b, in this design should be 12. So if the chef has nine entrées {1, 2, 3, 4, 5, 6, 7, 8, 9}, we get the following double-change design with twelve blocks:

1	3	5
1	2	6
2	3	4
1	4	7
4	5	9
4	6	8
5	6	7
2	5	8
7	8	3
8	9	1
7	9	2
3	6	9

Since this design is circular, the chef could seamlessly and efficiently start the designed menu over again on the 13th day while still fulfilling his requirements.

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VITA

Graduate School
Southern Illinois University

Kristen A. Lindbloom Date of Birth: June 21, 1982

608 N. Springer Street, Carbondale, Illinois 62901

11665 W. 195th Street, Mokena, Illinois, 60448

Illinois State University
Bachelor of Science, Mathematics, December 2004

Research Paper Title:
Tight Double-Change Covering Designs

Major Professor: Dr. J. McSorley