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John P. McSorley

*Southern Illinois University Carbondale*, [mcsorley60@hotmail.com](mailto:mcsorley60@hotmail.com)

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# Double Arrays, Triple Arrays, and Balanced Grids with $v = r + c - 1$

John P. McSorley,  
Department of Mathematics,  
Southern Illinois University,  
Carbondale. IL 62901-4408.  
mcsorley60@hotmail.com

## Abstract

In Theorem 6.1 of [3] it was shown that, when  $v = r + c - 1$ , every triple array  $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$  is a balanced grid  $BG(v, k, k : r \times c)$ . Here we prove the converse of this Theorem. Our final result is: Let  $v = r + c - 1$ . Then every triple array is a  $TA(v, k, c - k, r - k, k : r \times c)$  and every balanced grid is a  $BG(v, k, k : r \times c)$ , and they are equivalent.

*Keywords:* arrays, double arrays, triple arrays, balanced grids, designs

# 1 Introduction, Main Result

We briefly introduce the main players: arrays, double arrays, triple arrays, and balanced grids. See [3] for more details.

Consider a rectangle with  $r$  rows and  $c$  columns, in which each cell contains exactly one element from the set  $V = \{1, 2, \dots, v\}$ . Suppose that the rectangle is *binary*, *i.e.*, every row contains distinct elements and every column contains distinct elements. Further, suppose that the rectangle is *equireplicate*, *i.e.*, every element of  $V$  occurs exactly  $k$  times in the rectangle for some  $k \geq 1$ . Call such a rectangle a  $r \times c$  *array* based on the set  $V$ , and denote it by  $\mathcal{A} = A(v, k : r \times c)$ .

An array  $\mathcal{A}$  is a *double array* if it satisfies the following two properties:

- (P1) any two distinct rows have the same number,  $\lambda_{rr}$ , of common elements;
- (P2) any two distinct columns have the same number,  $\lambda_{cc}$ , of common elements.

Such an array is denoted by  $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ . Suppose further that  $\mathcal{A}$  satisfies the third property:

- (P3) any row and any column have the same number,  $\lambda_{rc}$ , of common elements,

then  $\mathcal{A}$  is called a *triple array*, a  $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ .

Now consider a pair of distinct elements  $x \in V$  and  $y \in V$ . If both occur in the same row of  $\mathcal{A}$  then we say that the pair  $\{x, y\}$  occurs in this row, similarly for columns. Suppose that  $\{x, y\}$  occurs in  $r_1$  rows of  $\mathcal{A}$  and in  $c_1$  columns of  $\mathcal{A}$ , then we say that it occurs  $\mu_{\{x,y\}} = r_1 + c_1$  times in the *grid*  $\mathcal{A}$ . We call  $\mathcal{A}$  a *balanced grid* if there is a constant  $\mu$  such that  $\mu = \mu_{\{x,y\}}$  for every  $x$  and  $y$ . We denote such a balanced grid by  $BG(v, k, \mu : r \times c)$ .

In Theorem 6.1 of [3] it was shown that, when  $v = r + c - 1$ , every triple array  $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$  is a balanced grid  $BG(v, k, k : r \times c)$ . It was then stated that examples to the converse of this Theorem had been found. In Theorem 2.5 of this paper we prove the converse of Theorem 6.1 of [3]. Our main result (Theorem 2.6) is: Let  $v = r + c - 1$ . Then every triple array is a  $TA(v, k, c - k, r - k, k : r \times c)$  and every balanced grid is a  $BG(v, k, k : r \times c)$ , and they are equivalent.

Finally, we restate a conjecture of Agrawal [1] concerning symmetric balanced incomplete block designs and triple arrays.

## 2 For $v=r+c-1$ , TA and BG are equivalent

We work mainly with the variables  $r$ ,  $c$ , and  $k$ ; writing other variables in terms of these three variables, see Theorems 2.2, 3.1, and 4.1 of [3].

$$v = \frac{rc}{k}, \quad \lambda_{rr} = \frac{c(k-1)}{r-1}, \quad \lambda_{cc} = \frac{r(k-1)}{c-1}, \quad \lambda_{rc} = k, \quad \mu = \frac{k^2(r+c-2)}{rc-k}. \quad (1)$$

When  $v = r + c - 1$  if values of the two parameters  $r$  and  $c$  are given then all parameters in (1) can be expressed in terms of them, and so are ‘forced’. But we prefer to keep  $k$  in our formulae:

### Lemma 2.1

- (i) In a triple array  $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$  the following are equivalent:  
 $v = r + c - 1$  and  $\lambda_{rr} = c - k$  and  $\lambda_{cc} = r - k$ .
- (ii) In a balanced grid  $BG(v, k, \mu : r \times c)$  we have  $v = r + c - 1$  if and only if  $\mu = k$ .

*Proof.* (i) If  $v = r + c - 1$  then  $c = v - r + 1$ . Then  $ck = vk - rk + k = rc - rk + k$ , and so  $ck - c = rc - rk - c + k = (r-1)(c-k)$ . But, from (1),  $\lambda_{rr} = \frac{c(k-1)}{r-1}$ , and so  $\lambda_{rr} = c - k$ . The converse is given by working backwards. Hence  $v = r + c - 1$  if and only if  $\lambda_{rr} = c - k$ . Similarly we can prove that  $v = r + c - 1$  if and only if  $\lambda_{cc} = r - k$ .

(ii) Suppose that  $v = r + c - 1$ . Then, from (1),  $v = \frac{rc}{k} = r + c - 1$ . So  $\frac{rc}{k} - 1 = \frac{rc-k}{k} = r + c - 2$ . Now (1) gives  $\mu = k$ . The converse is given by working backwards. ■

The following Corollary was not explicitly stated in [3].

**Corollary 2.2** When  $v = r + c - 1$  every triple array is a  $TA(v, k, c - k, r - k, k : r \times c)$ , and every balanced grid is a  $BG(v, k, k : r \times c)$ . ■

### Matching BIBD's

Let  $\mathcal{D}_1$  be a  $(v_1, b, r_1, \kappa, \lambda_1) - BIBD$  based on a  $v_1$ -set  $V_1$ , and  $\mathcal{D}_2$  a  $(v_2, b, r_2, \kappa, \lambda_2) - BIBD$  based on a  $v_2$ -set  $V_2$ , with  $v_1 v_2 = b\kappa$ . Let the  $b$  blocks of  $\mathcal{D}_1$  be arranged in any fixed order, and let the  $\kappa$  elements in each block be arranged in any fixed order. Then  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are *matching* if the  $b$  blocks of  $\mathcal{D}_2$ , and the  $\kappa$  elements within each block, can be arranged so that when  $\mathcal{D}_2$  is superimposed onto  $\mathcal{D}_1$  then each of the  $v_1 v_2$  pairs from  $V_1 \times V_2$  appears exactly once amongst the  $b\kappa$  pairs covered. See Preece [4] Section 6, definition (b), for an equivalent definition of matching *BIBD's*. Such superimpositions are generally known as Graeco-Latin designs.

**Example 1** Two matching *BIBD's*: a  $(5, 10, 6, 3, 3) - BIBD$  based on  $\{R_1, R_2, R_3, R_4, R_5\}$  and a  $(6, 10, 5, 3, 2) - BIBD$  based on  $\{C_1, C_2, C_3, C_4, C_5, C_6\}$ , and their superimposition.

$R_1$	$R_2$	$R_3$	$C_1$	$C_4$	$C_5$	$R_1 C_1$	$R_2 C_4$	$R_3 C_5$
$R_1$	$R_3$	$R_5$	$C_2$	$C_3$	$C_5$	$R_1 C_2$	$R_3 C_3$	$R_5 C_5$
$R_1$	$R_3$	$R_4$	$C_3$	$C_5$	$C_6$	$R_1 C_3$	$R_3 C_6$	$R_4 C_5$
$R_1$	$R_4$	$R_5$	$C_1$	$C_3$	$C_4$	$R_1 C_4$	$R_4 C_3$	$R_5 C_1$
$R_1$	$R_2$	$R_5$	$C_1$	$C_5$	$C_6$	$R_1 C_5$	$R_2 C_1$	$R_5 C_6$
$R_1$	$R_2$	$R_4$	$C_2$	$C_4$	$C_6$	$R_1 C_6$	$R_2 C_2$	$R_4 C_4$
$R_2$	$R_4$	$R_5$	$C_3$	$C_4$	$C_6$	$R_2 C_3$	$R_4 C_6$	$R_5 C_4$
$R_2$	$R_3$	$R_4$	$C_2$	$C_4$	$C_5$	$R_2 C_5$	$R_3 C_4$	$R_4 C_2$
$R_2$	$R_3$	$R_5$	$C_1$	$C_2$	$C_6$	$R_2 C_6$	$R_3 C_1$	$R_5 C_2$
$R_3$	$R_4$	$R_5$	$C_1$	$C_2$	$C_3$	$R_3 C_2$	$R_4 C_1$	$R_5 C_3$

### Block structures $\mathcal{R}^\perp$ , $\mathcal{C}^\perp$ , and $\mathcal{S}$

Let  $\mathcal{A}$  be an arbitrary array  $A(v, k : r \times c)$ . Label the  $r$  rows of  $\mathcal{A}$  with  $R_1, R_2, \dots, R_r$ , and the  $c$  columns with  $C_1, C_2, \dots, C_c$ .

Let  $\mathcal{R} = \{R_1, R_2, \dots, R_r\}$  be the block structure composed of the  $r$  rows of  $\mathcal{A}$ . Similarly, let  $\mathcal{C} = \{C_1, C_2, \dots, C_c\}$  be the block structure composed of the  $c$  columns of  $\mathcal{A}$ .

For any  $x \in V$  let  $R_x^\perp = \{R_i \mid x \in R_i\}$ . Then  $\mathcal{R}^\perp = \{R_x^\perp \mid x \in V\}$  is the dual of  $\mathcal{R}$  and is a block structure based on the set  $\{R_1, R_2, \dots, R_r\}$  with  $v$  blocks each of size  $k$ . Similarly, for any  $x \in V$  let  $C_x^\perp = \{C_j \mid x \in C_j\}$ . Then

$\mathcal{C}^\perp = \{C_x^\perp \mid x \in V\}$  is the dual of  $\mathcal{C}$  and is a block structure based on the set  $\{C_1, C_2, \dots, C_c\}$  with  $v$  blocks each of size  $k$ .

Define  $S_x = R_x^\perp \cup C_x^\perp$  for every  $x \in V$ , and let  $\mathcal{S}$  be the block structure  $\{S_x \mid x \in V\}$ .

By definition of a double array and matching *BIBD*'s we have (compare Lemma 2.1 of [3]):

**Lemma 2.3** *Let  $\mathcal{A}$  be an arbitrary array  $A(v, k : r \times c)$ . Then  $\mathcal{A}$  is a double array  $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$  if and only if  $\mathcal{R}^\perp$  is a  $(r, v, c, k, \lambda_{rr})$ -*BIBD* and  $\mathcal{C}^\perp$  is a  $(c, v, r, k, \lambda_{cc})$ -*BIBD*, and  $\mathcal{R}^\perp$  and  $\mathcal{C}^\perp$  are matching. ■*

When  $\mathcal{A}$  is a double array we call  $\mathcal{R}^\perp$  its *BIBD<sub>R</sub>* and  $\mathcal{C}^\perp$  its *BIBD<sub>C</sub>*.

**Example 2** A double array  $DA(10, 3, 3, 2 : 5 \times 6)$  whose matching *BIBD<sub>R</sub>* and *BIBD<sub>C</sub>* were given above in Example 1.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$
$R_1$	1	2	3	4	5	6
$R_2$	5	6	7	1	8	9
$R_3$	9	10	2	8	1	3
$R_4$	10	8	4	6	3	7
$R_5$	4	9	10	7	2	5

Before the next Theorem, we need the following result of Ryser [6], Chapter 8, Theorem 2.2:

*Let  $\mathcal{B}$  be an incidence structure based on a  $v$ -set with  $v$  blocks each of size  $k$ , in which any two distinct blocks intersect in the same number  $\lambda$  of elements. Then  $\mathcal{B}$  is a  $(v, k, \lambda)$ -*SBIBD*.*

Compare the following Theorem with Theorem 5.2 of [3].

**Theorem 2.4** *Let  $\mathcal{G}$  be a  $BG(v, k, \mu : r \times c)$  with  $v = r + c - 1$ . Then there exists a  $(v + 1, r, r - k)$ -*SBIBD*.*

*Proof.* Recall the definitions of the block structures  $\mathcal{R}^\perp$  and  $\mathcal{C}^\perp$  above. Let  $B_0 = \{R_1, R_2, \dots, R_r\}$ . For each  $x \in V$  put  $\overline{R}_x^\perp = B_0 \setminus R_x^\perp$ , then  $|\overline{R}_x^\perp| = r - k$ .

Let  $B_x = \overline{R}_x^\perp \cup C_x^\perp$  for each  $x \in V$ . Then  $|B_x| = (r - k) + k = r$ . Now consider the block structure  $\mathcal{B} = \{B_x \mid x \in V\} \cup \{B_0\}$ . It is based on

the  $r + c = v + 1$  elements from  $\mathcal{R} \cup \mathcal{C} = \{R_1, R_2, \dots, R_r, C_1, C_2, \dots, C_c\}$  and has  $v + 1$  blocks each of size  $r$ . We now show that  $\mathcal{B}$  is the required  $(v + 1, r, r - k) - SBIBD$ .

Now  $\mathcal{G}$  is a  $BG$  in which every pair  $\{x, y\}$  occurs  $\mu = k$  (Lemma 2.1(ii)) times, so  $|S_x^\perp \cap S_y^\perp| = |R_x^\perp \cap R_y^\perp| + |C_x^\perp \cap C_y^\perp| = k$ . We have:

$$\begin{aligned}
|B_x \cap B_y| &= |\overline{R}_x^\perp \cap \overline{R}_y^\perp| + |C_x^\perp \cap C_y^\perp| \\
&= |\overline{R}_x^\perp| + |\overline{R}_y^\perp| - |\overline{R}_x^\perp \cup \overline{R}_y^\perp| + |C_x^\perp \cap C_y^\perp| \\
&= (r - k) + (r - k) - |\overline{R_x^\perp \cap R_y^\perp}| + |C_x^\perp \cap C_y^\perp| \\
&= 2r - 2k - (r - |R_x^\perp \cap R_y^\perp|) + |C_x^\perp \cap C_y^\perp| \\
&= r - 2k + (|R_x^\perp \cap R_y^\perp| + |C_x^\perp \cap C_y^\perp|) \\
&= r - 2k + k = r - k.
\end{aligned}$$

Also, for all  $x \in V$ , we have  $|B_x \cap B_0| = r - k$ . Thus any two distinct blocks of  $\mathcal{B}$  intersect in  $r - k$  elements. So, from Ryser's result above,  $\mathcal{B}$  is a  $(v + 1, r, r - k) - SBIBD$ .  $\blacksquare$

Next is the converse to Theorem 6.1 of [3]:

**Theorem 2.5** *Let  $v = r + c - 1$ . Every  $BG(v, k, k : r \times c)$  is a  $TA(v, k, c - k, r - k, k : r \times c)$ .*

*Proof.* Let  $\mathcal{G}$  be a  $BG(v, k, k : r \times c)$ . Recall from Theorem 2.4 above that  $\mathcal{B}$  is a  $(v + 1, r, r - k) - SBIBD$ . The construction of  $\mathcal{B}$  from  $\mathcal{R}^\perp$  and  $\mathcal{C}^\perp$  gives: Firstly,  $\mathcal{R}^\perp$  is the complement of the derived design of  $\mathcal{B}$  with respect to block  $B_0$ , hence  $\mathcal{R}^\perp$  is a  $(r, v, c, k, c - k) - BIBD$ . Secondly,  $\mathcal{C}^\perp$  is the residual design of  $\mathcal{B}$  with respect to  $B_0$ , hence  $\mathcal{C}^\perp$  is a  $(c, v, r, k, r - k) - BIBD$ . Since  $\mathcal{R}^\perp$  and  $\mathcal{C}^\perp$  are also constructed from an array, they are matching. Hence, via Lemma 2.3,  $\mathcal{G}$  is a double array, a  $DA(v, k, c - k, r - k : r \times c)$ .

Consider any pair  $\{R_i, C_j\}$ . Then  $C_j$  occurs  $r$  times in the first  $v$  blocks of  $\mathcal{B}$ , and pair  $\{R_i, C_j\}$  occurs  $r - k$  times in these blocks. So, amongst the first  $v$  blocks of  $\mathcal{B}$ , there are  $r - (r - k) = k$  blocks which do not contain  $R_i$  but do contain  $C_j$ . Hence, in  $\mathcal{S}$ , there are  $k$  blocks containing pair  $\{R_i, C_j\}$ . Thus  $|R_i \cap C_j| = k$  for every  $i$  and  $j$ , and so  $\mathcal{G}$  is a triple array, a  $TA(v, k, c - k, r - k, k : r \times c)$ .  $\blacksquare$

Using Theorem 6.1 from [3] and Corollary 2.2 above, we have:

**Theorem 2.6** *Let  $v = r + c - 1$ . Then every triple array is a  $TA(v, k, c - k, r - k, k : r \times c)$  and every balanced grid is a  $BG(v, k, k : r \times c)$ , and they are equivalent.  $\blacksquare$*

**Example 3** An array  $\mathcal{A}$ , a  $A(10, 3 : 5 \times 6)$ , which is both a balanced grid  $BG(10, 3, 3 : 5 \times 6)$  and a triple array  $TA(10, 3, 3, 2, 3 : 5 \times 6)$ . The three block structures shown are its  $BIBD_R$ , a  $(5, 10, 6, 3, 3) - BIBD$ ; its  $BIBD_C$ , a  $(6, 10, 5, 3, 2) - BIBD$ ; and  $\mathcal{B}$ , a  $(11, 5, 2) - SBIBD$ .

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$
$R_1$	1	2	3	4	5	6
$R_2$	4	7	1	3	8	9
$R_3$	2	5	10	8	9	3
$R_4$	10	8	7	6	1	2
$R_5$	9	4	5	10	6	7

$R_1$	$R_2$	$R_4$	$C_1$	$C_3$	$C_5$	$R_3$	$R_5$	$C_1$	$C_3$	$C_5$
$R_1$	$R_3$	$R_4$	$C_1$	$C_2$	$C_6$	$R_2$	$R_5$	$C_1$	$C_2$	$C_6$
$R_1$	$R_2$	$R_3$	$C_3$	$C_4$	$C_6$	$R_4$	$R_5$	$C_3$	$C_4$	$C_6$
$R_1$	$R_2$	$R_5$	$C_1$	$C_2$	$C_4$	$R_3$	$R_4$	$C_1$	$C_2$	$C_4$
$R_1$	$R_3$	$R_5$	$C_2$	$C_3$	$C_5$	$R_2$	$R_4$	$C_2$	$C_3$	$C_5$
$R_1$	$R_4$	$R_5$	$C_4$	$C_5$	$C_6$	$R_2$	$R_3$	$C_4$	$C_5$	$C_6$
$R_2$	$R_4$	$R_5$	$C_2$	$C_3$	$C_6$	$R_1$	$R_3$	$C_2$	$C_3$	$C_6$
$R_2$	$R_3$	$R_4$	$C_2$	$C_4$	$C_5$	$R_1$	$R_5$	$C_2$	$C_4$	$C_5$
$R_2$	$R_3$	$R_5$	$C_1$	$C_5$	$C_6$	$R_1$	$R_4$	$C_1$	$C_5$	$C_6$
$R_3$	$R_4$	$R_5$	$C_1$	$C_3$	$C_4$	$R_1$	$R_2$	$C_1$	$C_3$	$C_4$
						$R_1$	$R_2$	$R_3$	$R_4$	$R_5$

### Agrawal's Conjecture

The second paragraph in the proof of Theorem 2.5 above is essentially Agrawal's method of constructing a triple array  $TA(v, k, c - k, r - k, k : r \times c)$  with  $v = r + c - 1$  from a  $(v + 1, r, r - k) - SBIBD$  with  $k > 2$ , see Agrawal [1]. It seems worthwhile to restate his conjecture in terms of matching BIBD's:

*Let  $\mathcal{S}$  be a  $(v_s, k_s, \lambda_s) - SBIBD$  with  $k_s - \lambda_s > 2$ . For any fixed block  $S_0$  let  $\mathcal{S}_{\text{der}}$  denote the derived design of  $\mathcal{S}$  with respect to  $S_0$ , and let  $\mathcal{S}_{\text{res}}$  denote the residual design of  $\mathcal{S}$  with respect to  $S_0$ .*

*Then the complement of  $\mathcal{S}_{\text{der}}$  and  $\mathcal{S}_{\text{res}}$  are matching.*



An incorrect proof of this conjecture appeared in Raghavarao and Nageswararao [5], as was pointed out in Bailey and Heidtmann [2], and Wallis and Yucas [7]. It appears that this conjecture is still open.

If Agrawal's conjecture is correct then any  $(v_s, k_s, \lambda_s) - SBIBD$  with  $k_s - \lambda_s > 2$  gives rise to a  $TA(v_s - 1, k_s - \lambda_s, v_s - 2k_s + \lambda_s, \lambda_s, k_s - \lambda_s : k_s \times (v_s - k_s))$ , a triple array with ' $v = r + c - 1$ '.

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