Southern Illinois University Carbondale
OpenSIUC

3-2000

Two Rank Order Tests for M-ary Detection

Viswanath Annampedu
Lucent Technologies

Vladimir V. Roganov
Southern Illinois University Carbondale

Ramanarayanan Viswanathan
Southern Illinois University Carbondale, viswa@engr.siu.edu

Recommended Citation

Two Rank Order Tests for M-ary Detection

Viswanath Annampedu, Vladimir V. Roganov, and Ramanarayanan Viswanathan, *Senior Member, IEEE*

Abstract—We consider a general M-ary detection problem where, given M groups of L samples each, the problem is to identify which unique group of L samples have come from the signal hypothesis. The optimal likelihood ratio test is unrealizable, when the joint distribution of ML samples is not completely known. In this paper we consider two rank order types of tests termed as the modified rank test (MRT) and the modified rank test with row sort (MRTRS). We examine through simulation, the small sample probability of error performances of MRT and MRTRS for detecting a signal among contaminants. Numerically computable closed –form error expressions are derived for some special cases. Asymptotic (large sample) error rate of MRT is also derived. The results indicate that MRTRS provides improved performance over other previously known rank tests.

Index Terms—Asymptotic error, decision fusion, M-ary communication, rank tests, signal detection.

I. INTRODUCTION

LET $\{X_{ij}, i = 1, 2, \cdots, M$ and $j = 1, 2, \cdots, L\}$ denote the set of observations such that the samples $\{X_{kj}, j = 1, 2, \cdots, L\}$ are all independent and identically distributed (i.i.d.) with the signal density $f_1(.)$,[1] whereas the rest of the samples $\{X_{ij}, j = 1, 2, \cdots, L$ and $i = 1, 2, \cdots, M(i \neq k)\}$ are i.i.d. with the noise density $f_0(.)$. It is not known that the kth group of L samples are from the signal distribution (hypothesis H_k) and the problem is to decide, based on the observations, which unique group of L samples have come from $f_1(.)$. By arranging the observations in a matrix with M rows and L columns, the detection problem can be stated as follows: identify the unique row of samples that belong to the density $f_1(.)$. Rank tests, which are nonparametric in nature, are natural candidates when the two densities $f_1(.)$ and $f_0(.)$ are stochasti-

cally ordered. In this paper it is assumed that a signal sample is stochastically larger than a noise sample

$$F_1(x) \leq F_0(x), \text{ for all } x,$$
$$\text{where } F_i(x) \text{ is the CDF of the densiy } f_i(x), i = 0, 1.$$

The corresponding testing problem under stochastic ordering of the two samples is characterized as the slippage problem in statistical literature [1].

In many communication problems, the L samples in a row are the result of processing L diversity paths or L pieces of information [2]. As a first approximation, it is usually assumed that all the L samples are statistically independent of each other. The L samples arise in different ways corresponding to different situations. For example, L could be the number of hops per symbol in a frequency-hopped multilevel frequency-shift keying (FH-MFSK) communication system. In a multiuser mobile radio system, an FH-MFSK scheme is used to combat interference on the desired user's signal from the other users' signals (multiple-access interference) [3]. In military applications, FH-MFSK modulation scheme is used to improve performance against partial band noise (PBN) jamming and tone jamming [4]. In mobile-radio environment, where multipath propagation is present, multiple copies of transmitted signal arrive at the receiver with different amplitudes and at different times. In IS-95 DS/CDMA systems, a proposed 2D-RAKE receiver, in addition to exploiting the spatial structure, takes advantage of multipath signaling to realize a form of time diversity [5]. A recent study shows that the rank type tests can provide robust performance for these code-division multiple access (CDMA) systems [6], [7]. In all these problems, even if the densities of the observations can be assumed known, the parameters of these densities are usually unknown, and hence, a likelihood ratio test (LRT) cannot be implemented. Also, the strengths of different diversity paths may be different, thereby implying that the signal densities in different paths are nonidentically distributed. In this paper we assume the simpler model of the observations mentioned in the previous paragraph.

A rank-based test for the M-ary signal detection problem can be formulated as follows. A rank matrix is first created by rank ordering all the observations $\{X_{ij}, i = 1, 2, \cdots, M$ and $j = 1, 2, \cdots, L\}$ and then replacing the samples with their corresponding ranks. Then a rank sum test (RST) declares the row with the maximum rank sum as the row corresponding to the signal hypothesis [1]. A reduced rank sum test (RRST) rank orders the samples in each column separately into values of 1 through M, and then picks the row with the maximum rank sum. The reduced ranking is appealing when the densities of the signal observations corresponding to different diversity paths are nonidentical. For a frequency-hopped multilevel

[1]Typically in a communication problem, $f_1(.)$ is due to the processing of a signal plus noise process. For convenience, we call $f_1(.)$ simply as the signal density.

frequency-shift keying (FH-MFSK) system, the RST and the RRST are found to be nearly identical in performance and they serve as a competing alternative to the parametric receivers [3]. List rank sum diversity combining based on RST was devised in [8]. In this method, a rank-list value table is to be created based on the channel condition and the interference threat. In general, the list table can be chosen only in an *ad hoc* manner. The receiver picks the row with the maximum list rank sum. A few other nonparametric tests are also discussed in [8].

In this paper, we consider two specific rank order tests termed as the modified rank test (MRT) and the modified rank test with row sort (MRTRS). MRT is a variation of RRST. Let the rank of $\{x_{ij}\}$ in the ordering (reduced rank ordering) of the samples $\{x_{ij}, i = 1, 2, \cdots, M\}$ be denoted as r_{ij}. A variation of the RRST is to create a value matrix where the (i, j) element of the value matrix is given by

$$v_{ij} = \begin{cases} r_{ij}, & \text{if } r_{ij} \geq M - p + 1, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq p \leq M. \quad (1)$$

The value sums are then computed as

$$s_i = \sum_{j=1}^{L} v_{ij}. \quad (2)$$

The MRT then decides l as the signal row where $l = \arg\max_{i \in \{1, \ldots, M\}} s_i$. If $p = 1$, the MRT retains only the maximum rank of M in each column and assigns zero values to the others. In other words, independently for each column, the row with the largest rank is decided as the signal row. Therefore, for $p = 1$, MRT can be thought of as a majority logic combining (MLC) of the decisions made in each column. For other values of p, MRT can be thought of as combining decisions, when decisions are presented with confidence weights. Decision fusion has been discussed extensively in several recent publications [9]–[11]. Since the value sum of each row is integer-valued, it is possible that more than one row may be tied as having the largest value sum. If a tie occurs among several rows, then the MRT picks at random one of the tied rows as the signal row.

Modified rank test with row sort is implemented by first sorting the elements in each row of $\{x_{ij}\}$ in increasing order of values and then applying the MRT procedure to the elements of the row sorted matrix. That is, if the row sorted matrix is denoted as $\{y_{ij}, i = 1, 2, \cdots, M, j = 1, 2, \cdots, L\}$, then y_{ij} is the jth largest among $\{x_{ik}, k = 1, 2, \cdots, L\}$, for $i = 1, 2, \cdots, M$. The rest of the operations of reduced ranking, clipping, and summing are carried out exactly as in MRT, but these are applied to $\{y_{ij}\}$ and not to $\{x_{ij}\}$. Thus MRTRS compares the strongest signal against each of the $M - 1$ strongest noise components, the next strongest signal against each of the $M - 1$ next strongest noise, and so on. It is certainly difficult to predict whether such comparisons would yield more accurate identification of the signal than that provided by L comparisons of independent signal and noise samples, as in MRT. Only a probability of error assessment provides the definitive answer.

In Section II we discuss finite sample probability of error performance of MRT and MRTRS for three example (f_1, f_0)

pairs. The first is the result of noncoherent processing of a Rayleigh fading signal received in additive white Gaussian noise (AWGN), whereas the next two are based on constant signal-plus-noise models. The signal and the noise models for different application areas mentioned earlier differ from these simpler models because, in the former, multiple-access interference or jammer interference may be present. Here we show that the MRTRS with $p = 1$ performs very well in all three example cases. Another aim of this study is to determine how the choice of the parameter p affects the performances of MRT and MRTRS. Large sample asymptotic performance of MRT is evaluated in Section III. An Appendix provides closed-form probability of error expressions for the special case of $p = 1$, $L = 3, 5$ for MRT and $p = 1$, $L = 3$ for MRTRS. We conclude this paper in Section IV.

II. FINITE SAMPLE PROBABILITY OF ERROR PERFORMANCE

The performances of MRT and MRTRS are evaluated by finding the probabilities of error in identifying the signal row. We consider three cases of (f_1, f_0) pairs:

Case i) Exponential

$$f_1(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \lambda < 1$$
$$f_0(x) = e^{-x}, \qquad x \geq 0. \quad (3)$$

Case ii) Gaussian

$$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-((x-\theta)^2/2)}, \qquad \theta > 0$$
$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}. \quad (4)$$

Case iii) Double Exponential (or Laplace)

$$f_1(x) = \frac{1}{2} e^{-|x-\theta|}, \qquad \theta > 0$$
$$f_0(x) = \frac{1}{2} e^{-|x|}. \quad (5)$$

The finite sample performances of MRT and MRTRS are evaluated through simulation. Simulation of the required samples in (3)–(5) is obtained from IMSL random number generation routines. In each case, the number of runs is sufficiently large to ensure an error count of 30 or more. Observing an error count of 10 implies that, with a confidence level of 95%, the actual error is within a factor of 2 of the estimated error [12], [13]. Thus our error estimates are obtained with an even greater accuracy. In the Appendix we provide the closed-form probability of error expressions for the special case of $p = 1$, $L = 3, 5$ for MRT and $p = 1$, $L = 3$ for MRTRS. As shown there, the derivations require some ingenious steps in order to arrive at numerically computable expressions for the probability of error. Error rates determined through these analytical derivations are compared against the results from simulation studies. In all the cases we have excellent agreement between the simulation and the theoretical results.

In Figs. 1–4 we show probabilities of error of various tests for exponential distribution. The error rates for the normal and the Laplace are shown in Figs. 5–8 and 9–12, respectively. Observe that the probability of error is plotted on a logarithmic scale in all
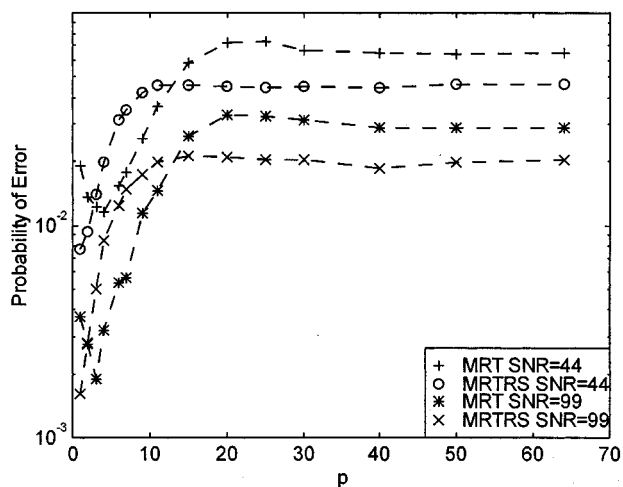
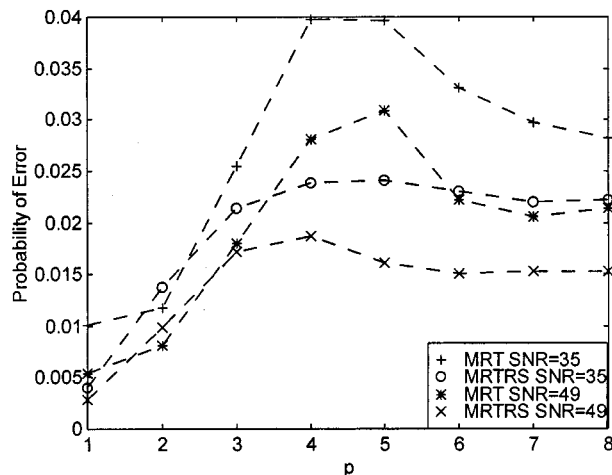| SNR	RST	LRT
44	6.78E-2	1.07E-3
99	2.93E-2	1.07E-4

Fig. 1.   Probability of error for exponential distribution $L = 3, M = 64$.
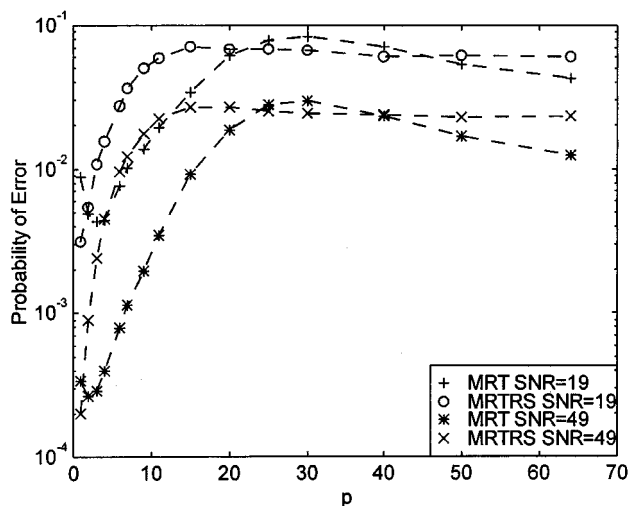
SNR	RST	LRT
19	4.13E-2	4.29E-4
49	1.30E-2	6.10E-6

Fig. 2.   Probability of error for exponential distribution $L = 5, M = 64$.

SNR	RST	LRT
35	2.37E-2	7.21E-4
49	1.55E-2	2.81E-4

Fig. 3.   Probability of error for exponential distribution $L = 3, M = 8$.

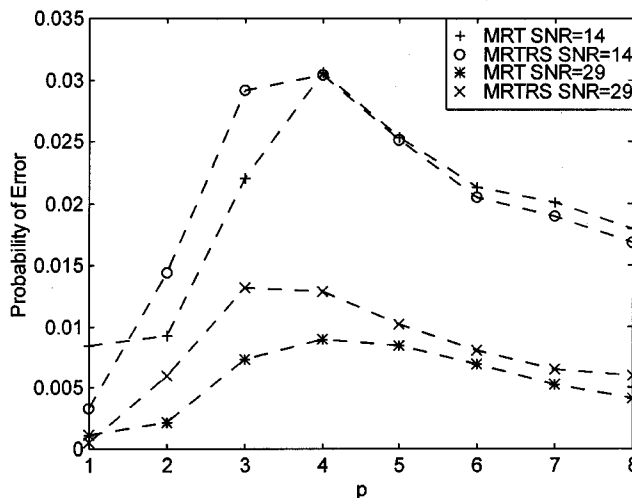| SNR	RST	LRT
14	1.27E-2	4.35E-4
29	2.90E-3	1.84E-5

Fig. 4.   Probability of error for exponential distribution $L = 5, M = 8$.

the figures except in Figs. 3 and 4, where it is on a linear scale. The parameters for the signal densities under the three cases are chosen so that the error rates of MRT and MRTRS are in the easily estimable range (i.e., greater than 10^{-5}). This assures a reasonable simulation time requirement. In Figs. 1–8, we also provide the error rates of the LRT for comparison purposes [LRT is the uniformly most powerful test (UMP) for these two cases]. For exponential and normal cases, the LRT error rates are obtained through standard analytical expressions. For the double exponential case, no uniformly most powerful test for one-sided alternative exists. Implementation of a LRT requires a complete knowledge of the signal density, which is unavailable in many

situations. Even though the error rate of LRT for the double exponential case is obtainable through simulation studies, we have not presented it for the simple reason that the LRT is not realizable in many situations. The following observations from the first four tables pertain to the exponential case. The error rate of the reduced rank sum test (RRST), which is MRT with $p = M$, is about the same as the error rate of the rank sum test (RST). The MRTRS $(p = 1)$ performs very well providing about 1/10 of the error rate of RST at low signal-to-noise ratios $(\mathrm{SNR} = 1/\lambda - 1)$. For $M = 64$, $L = 5$, and high SNR, even a higher reduction of error rate is achieved (Fig. 2). At high SNR values, the simple MRT also achieves error rates close to that of MRTRS for low
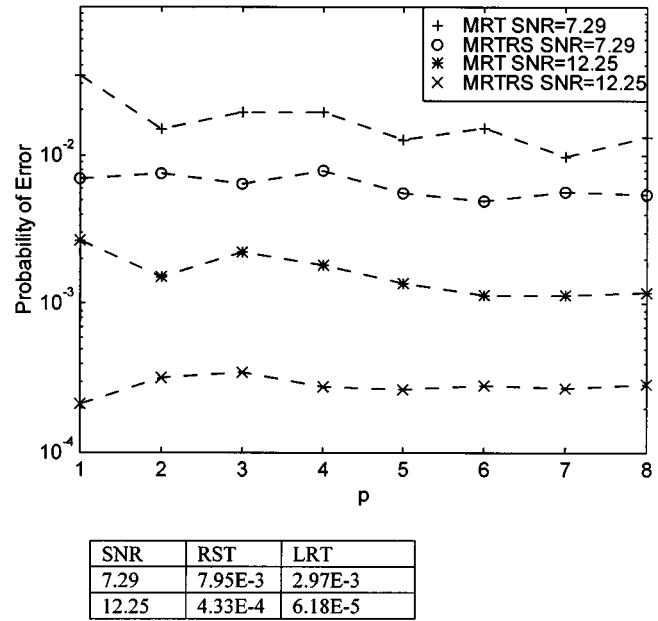
SNR	RST	LRT
10.24	1.48E-2	2.13E-3
16.00	1.03E-3	2.82E-5

Fig. 5.   Probability of error for normal distribution $L = 3$, $M = 64$.

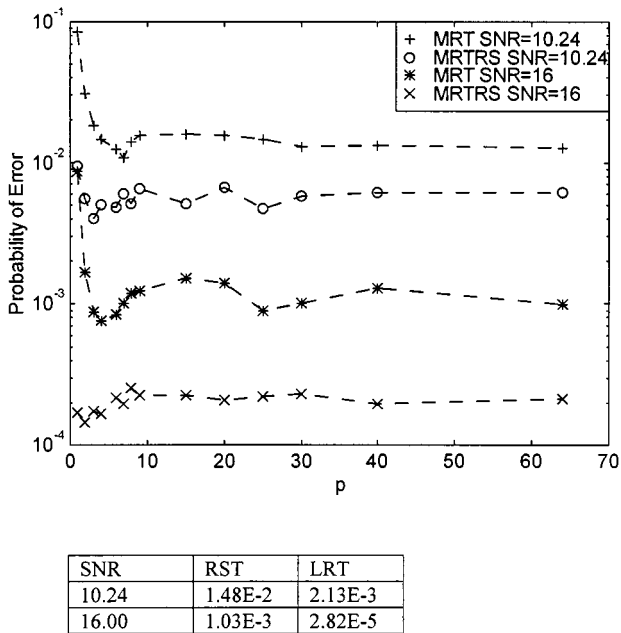| SNR	RST	LRT
7.29	7.95E-3	2.97E-3
12.25	4.33E-4	6.18E-5

Fig. 7.   Probability of error for normal distribution $L = 3$, $M = 8$.

SNR	RST	LRT
4.0	5.59E-2	2.71E-2
9.0	9.60E-4	6.04E-5

Fig. 6.   Probability of error for normal distribution $L = 5$, $M = 64$.

SNR	RST	LRT
4.0	9.60E-3	4.83E-3
7.84	1.39E-4	3.27E-5
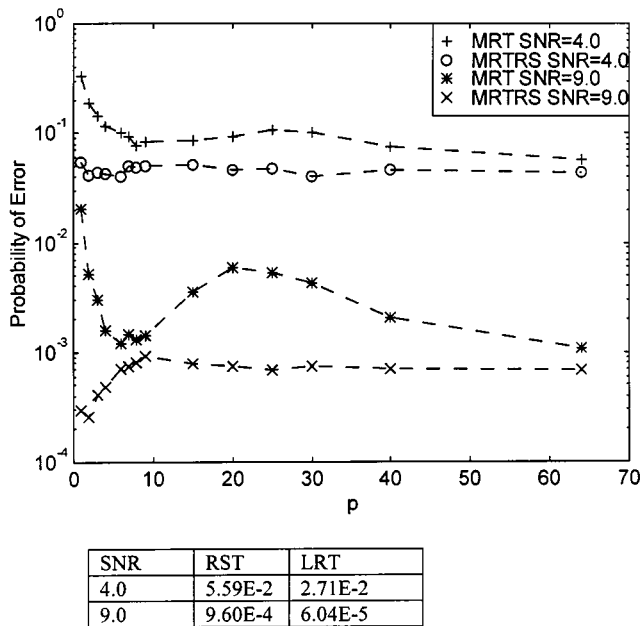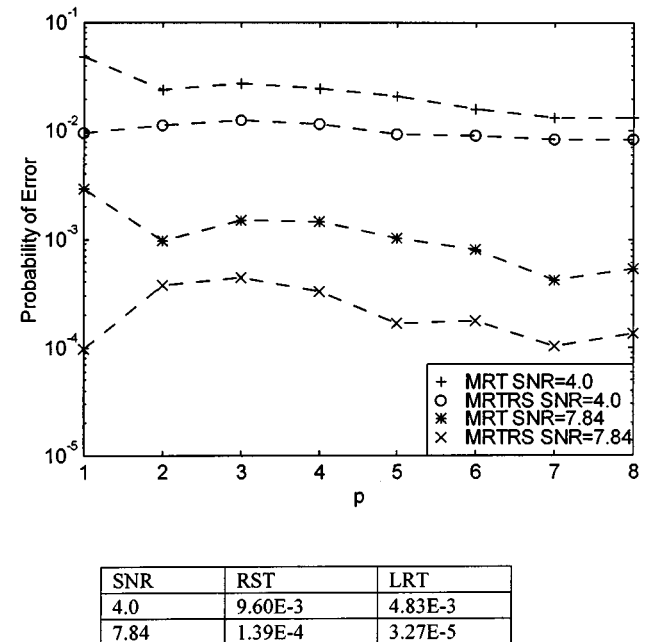
Fig. 8.   Probability of error for normal distribution $L = 5$, $M = 8$.

values of p. For $M = 8$, $p = 1$ gives the smallest probability of error for both MRT and MRTRS. The error rate of LRT ranges from 1/6 through 1/30 of the error rate of MRTRS ($p = 1$).

In the Gaussian case (Figs. 5–8), the error rate of MRTRS ($p = 1$) is closer to that of RST, except for large alphabet sizes and large $\mathrm{SNR}\,(=\theta^2)$ [see $M = 64$, $L = 3$, $\mathrm{SNR} = 16$, and $L = 5$, $\mathrm{SNR} = 9$ in Figs. 5 and 6, in which case the error rate of MRTRS ($p = 1$) is about 1/10 and 1/3 of that of RST, respectively]. Hence, the additional row sorting is beneficial only for large alphabet sizes and high SNR's. Moreover, the MRTRS error rate seems relatively insensitive to p. The LRT has an error rate that ranges from about 1/5 of that of RST for low SNR

to about 1/35 of that of RST for high SNR. The error rates of RRST (MRT with $p = M$) and RST are comparable at low SNR values. At high SNR's and $M = 8$, the error rate of RST is somewhat smaller than that of RRST. At high SNR, the MRT with $p = 1$ shows a considerably higher error rate than RST or MRTRS. Hence, $p = 1$ is not recommended for MRT in the normal case.

For double exponential distribution, while considering the variation of error of MRTRS with respect to p, MRTRS with $p = 1$ provides the best performance in almost all the cases considered (Figs. 9–12). As compared to the RST, the error rate is reduced by a factor ranging from 2 through 5 in low $\mathrm{SNR}\,(=\theta^2)$
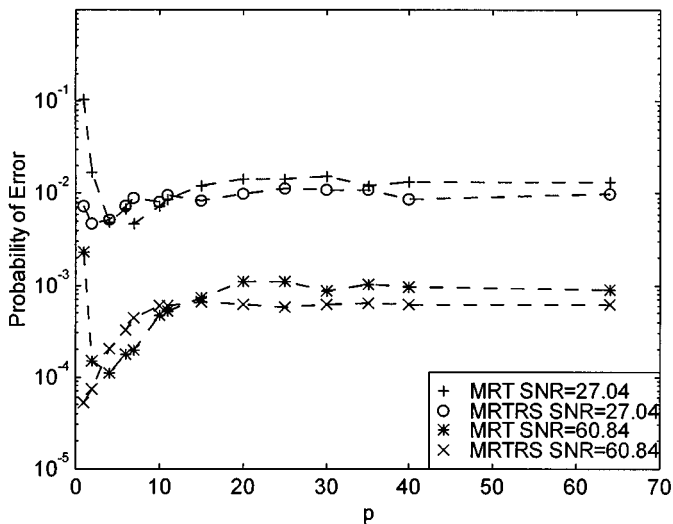
SNR	RST
27.04	1.30E-2
60.84	1.03E-3

Fig. 9. Probability of error for double exponential distribution $L = 3$, $M = 64$.
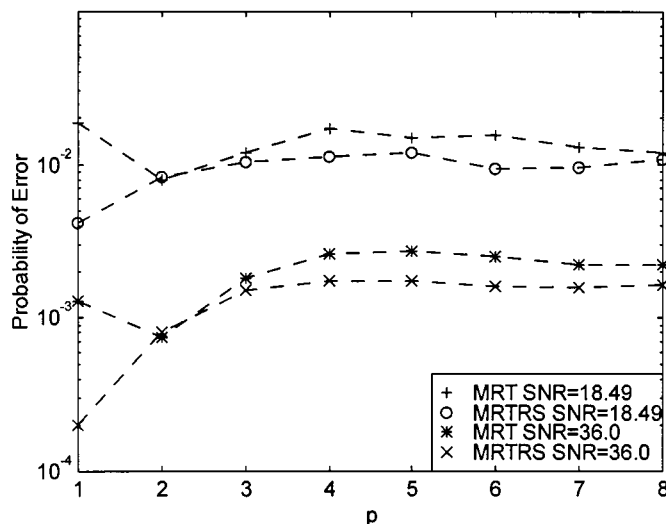
SNR	RST
18.49	9.0E-3
36.0	1.41E-3

Fig. 11. Probability of error for double exponential distribution $L = 3$, $M = 8$.

SNR	RST
16	1.11E-2
28.09	2.05E-3

Fig. 10. Probability of error for double exponential distribution $L = 5$, $M = 64$.
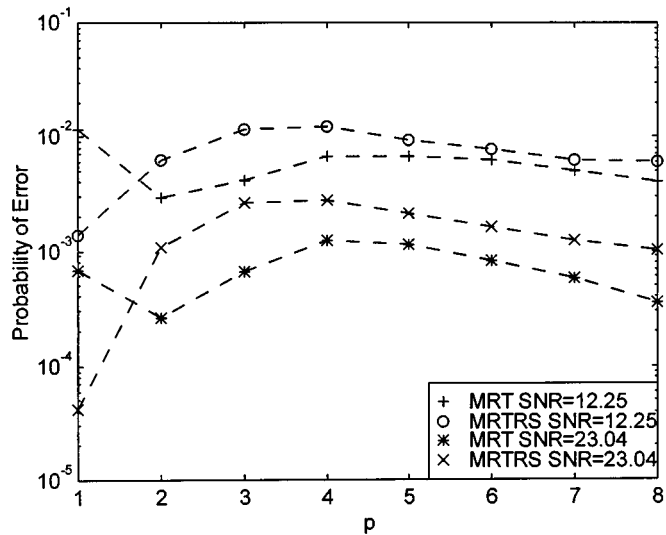
SNR	RST
12.25	2.45E-3
23.04	1.95E-4

Fig. 12. Probability of error for double exponential distribution $L = 5$, $M = 8$.

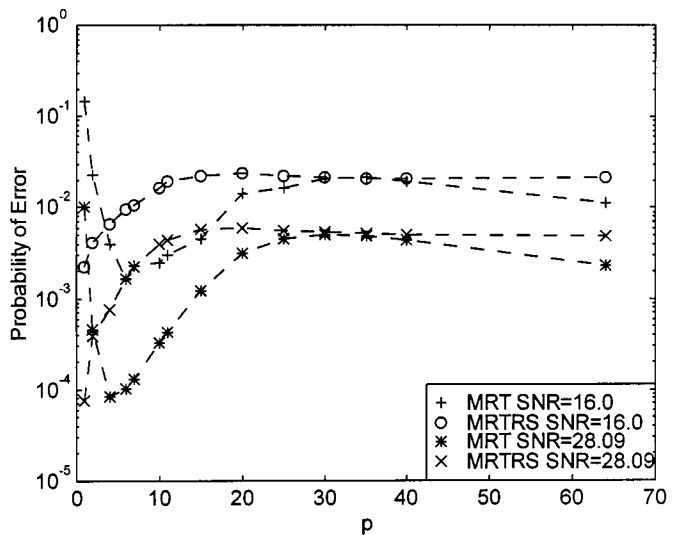and from 5 through 20 for high SNR. When $p = 1$, the row sorting helps to reduce the error rate of MRT significantly by more than an order of magnitude.

The above results show the following. The simple MRT is highly sensitive to the choice of p, with the best choice dependent on the distribution, M, and L. For Gaussian and Laplace densities, there is a reduction in error as p is changed from 1 to 2. Over certain range of p values, it is possible that MRT performs better than MRTRS (see exponential and double exponential Figs. 1, 2, and 9, 10). However, if p is appropriately chosen,

then MRTRS does perform well. In fact, MRTRS ($p = 1$) provides robust and very good performance in heavy tail densities such as exponential or double exponential. Even in the case of normal, MRTRS ($p = 1$) outperforms RST for large alphabet sizes and high SNR's. Only for $M = 8$, the performances of the two are comparable. With regard to the choice of p, for MRT, $p = 1$ is near optimal only for the exponential case. For MRTRS, the choice of $p = 1$ provides nearly optimal performance in all the three cases studied in this paper.

III. ASYMPTOTIC PERFORMANCE OF MRT

In this section, we provide a procedure for arriving at an expression for the probability of error of MRT for very large values of L. The row-sorting operation in MRTRS introduces statistical dependency among samples, thereby complicating any derivation for asymptotic performance. Without any loss of generality assume that the first row of samples $\{x_{1j}, j = 1, 2, \cdots, L\}$ come from the signal density $f_1(.)$. By a multivariate central limit theorem [14, p. 74], as L increases without any bound, $S_i, i = 1, 2, \cdots, M$ are jointly Gaussian with mean and covariances denoted by the following notations:

$$
\begin{aligned}
E(S_1) &= m_s \\
E(S_i) &= m_0, & i &= 2, 3, \cdots, M \\
\sigma_{s_1}^2 &= \sigma_s^2 \\
\sigma_{s_i}^2 &= \sigma_0^2, & i &= 2, 3, \cdots, M \\
\mathrm{COV}(S_1, S_i) &= \rho_{10}\sigma_0\sigma_s, & i &= 2, 3, \cdots, M \\
\mathrm{COV}(S_i, S_j) &= \rho_0\sigma_0^2, & i &\neq j, (i, j) \in (2, 3, \cdots, M).
\end{aligned}
\tag{6}
$$

If we let $S_i^* = (S_i - m_s)/\sigma_s$, then the conditional density of $\{S_i^*, i = 2, 3, \cdots, M\}$ given $S_1^* = s$ is also a multivariate normal density $N_{M-1}(\mu_c^*, \Sigma_c^*)$ with mean and covariances given by [15, eq. (8a2.11), p. 522]

$$\mu_c^* = m_c^* \cdot \mathbf{1}^T$$

(i, i)th element of $\Sigma_c^* = \Sigma_{cii}^* = \sigma_c^{*2}$,

$$i = 1, 2, \cdots, M - 1$$

(i, j)th element of $\Sigma_c^* = \Sigma_{cij}^* = \sigma_c^{*2}\rho_c^*$,

$$i \neq j \in (1, 2, \cdots M - 1). \tag{7}$$

Now, the probability of error $P(e)$ is given by

$$
\begin{aligned}
P(e) &= 1 - P(c) \\
&= 1 - \int_{-\infty}^{\infty} P\left(\frac{S_2^* - m_c^*}{\sigma_c^*} < h, \frac{S_3^* - m_c^*}{\sigma_c^*} < h, \right. \\
&\quad \left. \cdots, \frac{S_M^* - m_c^*}{\sigma_c^*} < h \middle| S_1^* = s\right)\phi(s)\,ds
\end{aligned}
\tag{8}
$$

where

$$h = \frac{s - m_c^*}{\sigma_c^*} \tag{9}$$

and $\phi(\ )$ denotes the standard Gaussian density. Because of equal correlation among any pair of variables from $(S_2^*, S_3^*, \cdots, S_M^*)$, the $(M-1)$–fold integral embedded in the integrand of (8) can be reduced to a single integral involving a normal CDF and a normal density function [16]. Using this reduction, and using large L, (8) can be approximated in a straightforward fashion to yield the asymptotic probability of error

$$P(e) \approx \frac{(M-1)}{\sqrt{2\pi}}\frac{1}{\sqrt{\frac{c_h^2}{(1+m_h^2)}}}\exp\left(-\frac{c_h^2}{2(1+m_h^2)}\right) \tag{10}$$
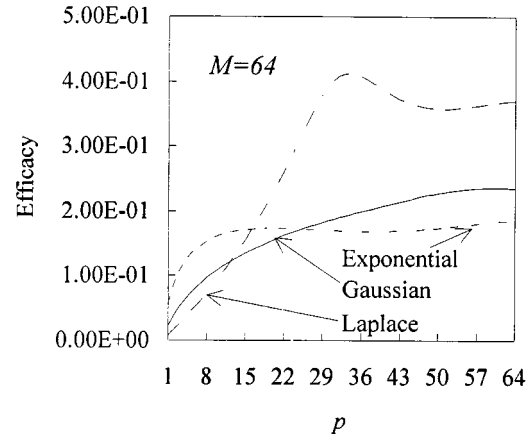

Fig. 13. Weak signal efficacy of MRT for $M = 64$.

where

$$c_h = \frac{m_S - m_0}{\sigma_0\sqrt{1-\rho_{10}^2}} \qquad m_h = \frac{\sigma_S - \rho_{10}\sigma_0}{\sigma_0\sqrt{1-\rho_{10}^2}}.$$

If necessary, a reader can consult [17] for more details on the derivation of (10).

Now

$$
\begin{aligned}
\frac{c_h^2}{(1+m_h^2)} &= \frac{(m_s - m_o)^2}{(\sigma_0^2 + \sigma_s^2 - 2\rho_{10}\sigma_0\sigma_s)} \\
&= \frac{[E(S_1 - S_2)]^2}{(\mathrm{Var}(S_1 - S_2))}.
\end{aligned}
\tag{11}
$$

The last term on the right-hand side of (11) can be termed as the signal-to-noise ratio (SNR). Equation (10) shows that the error approaches zero exponentially with the factor (SNR/2). This result is similar to the one obtained for the detection of M-ary orthogonal signals in AWGN ([18, eq. (65), p. 264], notice that the union bound is asymptotically tight). We define the weak signal asymptotic efficacy η_p of MRT with a specific p as

$$\eta_p = \frac{\mathrm{SNR}}{2L\varepsilon^2} \tag{12}$$

where ε is the weak signal level.

For the weak signal condition (i.e., $f_1(\ ) \to f_0(\ )$), (12) was computed corresponding to the three cases (3)–(5) (as can be seen from [17], the calculations are straightforward, but tedious) and the results are shown in Figs. 13 ($M=64$) and 14 ($M=8$). Fig. 13 shows that for the exponential density, RRST (MRT with $p = M$) gives the best performance and that a similar performance can be achieved at a considerably lower value of $p = 20$. For the Gaussian case, η_p is monotonic for p values from 1 through 60, with the best performance achieved at $p = 60$. Efficacy decreases by a very small amount after $p = 60$. For the double exponential density, the results show that the efficacy peaks at $p = 34$ such that its peak value is somewhat larger than the efficacy at $p = M$. That is, not retaining all the rank values yields a slightly better performance than retaining all the rank values. We have computed all the expressions for efficacy with high numerical accuracy. Therefore, we are confident that this behavior as a function of p is really exhibited by MRT.
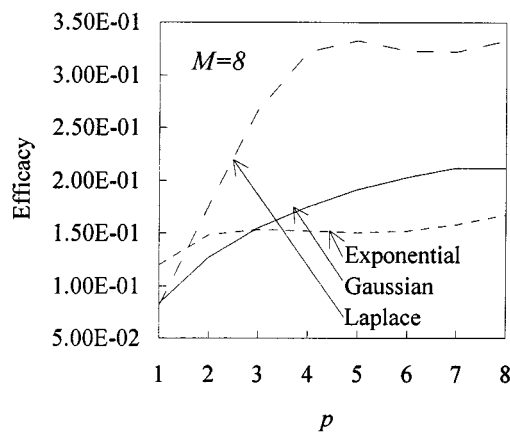
Fig. 14. Weak signal efficacy of MRT for $M = 8$.

Fig. 14 shows that for the exponential density, RRST $(p = 8)$ gives the best performance and that an efficacy comparable to that of RRST can be achieved at a lower p value of 3. For Gaussian density, η_p increases monotonically up to $p = 7$ and then decreases slightly at $p = 8$. Results for the double exponential density show that RRST achieves the highest efficacy and that the efficacy at $p = 5$ is almost equal to that of RRST.

From the results on the asymptotic performance of MRT, we can note that, in all the cases that we have considered, there is a gain in efficacy by going from $p = 1$ to $p = 2$. That is, retaining the first two largest rank values in each column and then picking the maximum value sum yields a better η_p than the majority logic combining scheme. Results also show that there is a difference between the small-sample and large-sample performance. In the cases of exponential and double exponential, whereas very low p values are preferred for small samples, large p values are preferred for large samples. We also computed the weak signal asymptotic efficacy of the linear detector. The linear detector obtains the sum of observations in each row and declares the row corresponding to the largest sum as the signal row. Using the information that all of the observations are statistically independent, it can be seen from (12) that the asymptotic efficacy of linear detector is 0.25, for the three (f_1, f_0) pairs (3)–(5). It is well known that linear detector is optimal (i.e., LRT) for cases i) and ii). Even in these cases the MRT performs close to the optimal test, for large p and M values.

IV. CONCLUSION

In this paper we consider two specific rank order type of tests termed as the modified rank test (MRT) and the modified rank test with row sort (MRTRS). By introducing a simple row-sorting operation in the MRT we obtain the MRTRS. Both these tests require a choice for the parameter p to be made. For three specific signal and contaminant models, the probability of error performances of these receivers and that of the traditional rank sum test are obtained through simulation studies. Numerically computable probability of error expressions for some special cases allow us to verify the results from the simulation studies. Whereas with MRT, $p = 1$ is near-optimal only for the exponential case, with MRTRS, it is near-optimal for all the three cases considered. MRTRS with $p = 1$ provides robust

and competitive performance. In heavy tail densities, the performance of MRTRS $(p = 1)$ is significantly better than that of the rank sum test.

It has been shown recently that both MRT and MRTRS with $p = 1$ perform better than an equal-gain combiner in a DS-CDMA context [6], [7]. In this case, the RST performs slightly better than the MRTRS. It will be of interest to evaluate the performances of rank tests and determine their behavior in combating partial band jamming noise in FH-MFSK systems and in other M-ary detection problems.

APPENDIX
PROBABILITY OF ERROR OF MRT FOR $p = 1$, $L = 3$, 5 AND MRTRS FOR $p = 1$, $L = 3$

In this appendix we derive probability of error expressions for MRT and MRTRS corresponding to the special case of $p = 1$. Our aim is to derive analytically tractable and computationally feasible expressions for the error probabilities. It seems impossible to arrive at a general expression for arbitrary p and L. While evaluating MRTRS, it is realized that the computational complexity limits our consideration to the case of $L = 3$. Without any loss of generality it is assumed that the first row contains the signal samples. Therefore, the MRT (MRTRS) makes an error in decision if it chooses any row other than the first row as the signal row. Consider a tie event A that corresponds to the situation of m noise rows and the signal row having the largest value sum. The probability of correct decision of MRT (MRTRS) under this situation is then given by $P(A)/(m + 1)$. Such tie events are taken into consideration while deriving the probability of error expression. It is assumed in the following that $M \geq L$ and that L is odd.

A. MRT

When $p = 1$, only one of the elements in any column of the value matrix is M, whereas the rest of the elements in that column assume zero value. The correct decision event consists of two components, namely, the nontie and the tie situations. In the former case, the signal row gets maximum value of M in l or more columns, any other row gets a value of M in less than l columns, where $l \geq (M + 1)/2$. Let the corresponding probability be denoted as

$$P_{c1} = \sum_l P(\text{Signal row value sum} \geq Ml, \\ \text{all other row value sums} < Ml). \quad \text{(A.1)}$$

In the latter case, the signal row gets maximum value in

$$k = \left(\frac{L + 1}{2} - i \right) \qquad i \in \left\{ 1, 2, \cdots, \frac{L - 1}{2} \right\}$$

columns, no other row gets maximum value in more than k columns, and exactly m other rows have their maximum in k columns. The corresponding probability is denoted as

$$P_{c2} = \sum_{k, m} \frac{1}{m + 1} P(\text{Signal row value sum} = Mk,$$

$$\text{exactly } m \text{ other rows have value sum} = Mk,$$

$$\text{rest of the rows have value sum} < Mk).$$

$$\text{(A.2)}$$

The probability of correct decision and the probability of error of MRT are given by

$$P_c = P_{c1} + P_{c2} \tag{A.3}$$

$$P_e = 1 - P_c. \tag{A.4}$$

The probability that the signal sample X_{1j} is the largest among the combined ordering of $(M-1)$ noise samples and the signal sample in the jth column is given by

$$q = P(R_{1j} = M) = \int_{-\infty}^{\infty} F_0^{M-1}(x) f_1(x)\, dx,$$
$$j = 1, 2, \cdots, L \tag{A.5}$$

where $F_0(.)$ is the CDF corresponding to the density $f_0(.)$.

$$P(R_{kj} = M) = \frac{1-q}{(M-1)}, \qquad k = 2, 3, \cdots, M$$

because the noise samples are i.i.d.

i) $L = 3$:

Using the definition of q it can be easily seen that

$$P_{c1} = q^3 + \binom{3}{2} q^2 (1 - q). \tag{A.6}$$

A tie situation occurs when, in a particular column, the signal sample has maximum rank M and the maximum ranks in other columns occur in different noise rows. Hence

$$P_{c2} = \frac{1}{3} \left(\binom{3}{1} q \left(\frac{1-q}{M-1} \right)^2 (M-1)(M-2) \right). \tag{A.7}$$

ii) $L = 5$:

$$P_{c1} = q^5 + \binom{5}{4} q^4 (1-q) + \binom{5}{3} q^3 (1-q)^2 + \binom{5}{2}$$
$$\cdot q^2 \left(\frac{1-q}{M-1} \right)^3 (M-1)(M-2)(M-3). \tag{A.8}$$

A tie event can happen in two ways, i) as in $L = 3$ case and ii) when, in two columns, the signal samples have maximum ranks and in exactly two out of the three remaining columns, a particular noise row has the maximum ranks. Hence,

$$P_{c2} = \frac{1}{5} \left(\binom{5}{1} q \left(\frac{1-q}{M-1} \right)^4 \prod_{i=1}^{4} (M-i) \right)$$
$$+ \frac{1}{2} \left(\binom{5}{2} \binom{3}{2} q^2 \left(\frac{1-q}{M-1} \right)^3 (M-1)(M-2) \right). \tag{A.9}$$

B. MRTRS

Let $L = 3$. As in MRT, the correct decision event has two components, the nontie and the tie situations. The probability of correct decision is given by

$$P_c = P_{c1} + P_{c2} \tag{A.10}$$

where the first probability on the right-hand side corresponds to the nontie event and the second one corresponds to the tie event. The nontie is a union of two events specified by ($\bigcap_{k=1}^{3} y_{1k}$ is maximum among y_{jk}, $j = 1, 2, \cdots, M$) and (y_{1j} is maximum in any two out of three columns, $j = 1, 2, 3$). Let us denote the probability of the above first event as P_{123} and the probabilities

corresponding to three distinct subsets of the second event as $P_{12\bar{3}}$, $P_{13\bar{2}}$, $P_{23\bar{1}}$. Hence

$$P_{c1} = P_{123} + P_{12\bar{3}} + P_{13\bar{2}} + P_{23\bar{1}}. \tag{A.11}$$

The tie event is the union of three mutually exclusive events specified as $\bigcup_{j=1}^{3}$ ($V_{1j} = M$, exactly two noise rows have their value sums equal M, all other noise rows have their value sums equal 0). By denoting the corresponding probabilities as P_{1T}, P_{2T}, P_{3T}, we have

$$P_{c2} = \frac{1}{3} (P_{1T} + P_{2T} + P_{3T}). \tag{A.12}$$

Using standard procedures involving random variables, we can obtain expressions for various probabilities appearing in (A.11) and (A.12). Procedures for obtaining some of these are discussed below. The expressions for the rest are obtained through similar steps.

Let

$$Z_j = \max(Y_{ij}, i = 2, 3, \cdots, M), \qquad j = 1, 2, 3 \tag{A.13}$$

$$P_{123} = P(Y_{11} > Z_1, Y_{12} > Z_2, Y_{13} > Z_3)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y_{13}} \int_{-\infty}^{y_{12}} F_{Z_1, Z_2, Z_3}(y_{11}, y_{12}, y_{13})$$
$$\cdot f_{Y_{11}, Y_{12}, Y_{13}}(y_{11}, y_{12}, y_{13})\, dy_{11}\, dy_{12}\, dy_{13} \tag{A.14}$$

where

$$f_{Y_{11}, Y_{12}, Y_{13}}(y_{11}, y_{12}, y_{13})$$
$$= \begin{cases} 3! f_1(y_{11}) f_1(y_{12}) f_1(y_{13}), & y_{11} < y_{12} < y_{13} \\ 0, & \text{elsewhere} \end{cases} \tag{A.15}$$

$$F_{Z_1, Z_2, Z_3}(z_1, z_2, z_3) = P \left(\bigcap_{j=1}^{3} \bigcap_{i=2}^{M} Y_{ij} < z_j \right)$$
$$= P \left(\bigcap_{i=2}^{M} \bigcap_{j=1}^{3} Y_{ij} < z_j \right)$$
$$= P^{M-1}(Y_{21} < z_1, Y_{22} < z_2, Y_{23} < z_3)$$
$$= F_{Y_{21}, Y_{22}, Y_{23}}^{M-1}(z_1, z_2, z_3) \tag{A.16}$$

$$F_{Y_{21}, Y_{22}, Y_{23}}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$$
$$= \sum_{s=2}^{3} \sum_{r=1}^{s} \frac{3!}{r!(s-r)!(3-s)!} F_0^r(\varepsilon_1)(F_0(\varepsilon_2) - F_0(\varepsilon_1))^{s-r}$$
$$\cdot (F_0(\varepsilon_3) - F_0(\varepsilon_2))^{3-s}, \qquad \varepsilon_1 < \varepsilon_2 < \varepsilon_3. \tag{A.17}$$

Next

$$P_{12\bar{3}} = P(Y_{11} > Z_1, Y_{12} > Z_2, Y_{13} < Z_3)$$
$$= P(Y_{11} > Z_1, Y_{12} > Z_2) - P_{123}$$
$$= P_{12} - P_{123} \tag{A.18}$$
$$P_{12} = \int_{-\infty}^{\infty} \int_{-\infty}^{y_{12}} F_{Z_1, Z_2}(y_{11}, y_{12})$$
$$\cdot f_{Y_{11}, Y_{12}}(y_{11}, y_{12})\, dy_{11}\, dy_{12} \tag{A.19}$$

$$f_{Y_{11}, Y_{12}}(y_{11}, y_{12})$$
$$= \begin{cases} 3!(1 - F_1(y_{12}))f_1(y_{11})f_1(y_{12}), & y_{11} < y_{12} \\ 0, & \text{otherwise} \end{cases}$$
$$(A.20)$$

$$F_{Z_1, Z_2}(z_1, z_2)$$
$$= \left[\sum_{s=2}^{3} \sum_{r=1}^{s} \frac{3!}{r!(s-r)!(3-s)!} F_0^r(z_1) \right.$$
$$\left. \cdot (F_0(z_2 - z_1))^{s-r}(1 - F_0(z_2))^{3-s} \right]^{M-1}, \quad z_1 < z_2.$$
$$(A.21)$$

Similar expressions can be derived for $P_{1\bar{3}\bar{2}}$ and $P_{2\bar{3}\bar{1}}$. Next, an expression for P_{1T} is derived

$$P_{1T} = P\left(Y_{11} \text{ is max in 1st column, } \bigcap_{j=2}^{3} \right.$$

$$\left. j\text{th column max not in 1st row} \right)$$

$$- P\left(\bigcup_{j=2}^{M} Y_{11} \text{ is max in 1st column, max in} \right.$$

$$\left. \text{2nd and 3rd column both in } j\text{th row} \right)$$

$$= P_{1\bar{2}\bar{3}} - (M - 1)P_{01} \qquad (A.22)$$

where

$$P_{01} = P\left(Y_{11} \text{ is max in 1st column, } \bigcap_{j=2}^{3} Y_{2j} \right.$$

$$\left. \text{is max in } j\text{th column} \right) \qquad (A.23)$$

$$P_{1\bar{2}\bar{3}} = P(Y_{11} > Z_1, Y_{12} < Z_2, Y_{13} < Z_3)$$
$$= P(Y_{11} > Z_1) - P_{12} - P_{13} + P_{123}. \qquad (A.24)$$

By denoting the first probability on the right-hand side of the above expression as P_1, we get

$$P_1 = P(Y_{11} > Z_1) = \int_{-\infty}^{\infty} F_{Z_1}(y_{11})f_{Y_{11}}(y_{11})\, dy_{11} \qquad (A.25)$$

where

$$f_{Y_{11}}(y_{11}) = 3(1 - F_1(y_{11}))^2 f_1(y_{11}) \qquad (A.26)$$

$$F_{Z_1}(z) = \left(\sum_{r=1}^{3} \binom{3}{r} F_0^r(z)(1 - F_0(z))^{3-r} \right)^{M-1}. \qquad (A.27)$$

In order to derive an expression for P_{01}, we define new variables

$$R_j = \max(Y_{ij}, i = 3, 4, \cdots, M), \qquad j = 1, 2, 3 \quad (A.28)$$

$$P_{01} = \int F_{R_1, R_2, R_3}(y_{11}, y_{22}, y_{23})$$
$$\cdot f_{Y_{11}, Y_{12}, Y_{13}}(y_{11}, y_{12}, y_{13})$$
$$\cdot f_{Y_{21}, Y_{22}, Y_{23}}(y_{21}, y_{22}, y_{23})$$
$$\cdot dy_{21}\, dy_{13}\, dy_{12}\, dy_{23}\, dy_{22}\, dy_{11} \qquad (A.29)$$

where the sixfold integral is carried out over the region

$$-\infty < y_{21} < y_{11}, \quad y_{12} < y_{13} < y_{23}, \quad y_{11} < y_{12} < y_{22},$$
$$y_{22} < y_{23} < \infty, \quad y_{11} < y_{22} < \infty, \quad -\infty < y_{11} < \infty$$

and

$$F_{R_1, R_2, R_3}(r_1, r_2, r_3)$$
$$= \left(\sum_{s=2}^{3} \sum_{r=1}^{s} \frac{3!}{r!(s-r)!(3-s)!} F_0^r(r_1) \right.$$
$$\left. \cdot (F_0(r_2) - F_0(r_1))^{s-r}(F_0(r_3) - F_0(r_2))^{3-s} \right)^{M-2},$$
$$r_1 < r_2 < r_3. \qquad (A.30)$$

Equation (A.29) can be simplified because the three inner integrals can be evaluated analytically. Hence

$$P_{01} = \int_{-\infty}^{\infty} \int_{y_{11}}^{\infty} \int_{y_{22}}^{\infty} F_{R_1, R_2, R_3}(y_{11}, y_{22}, y_{33})$$
$$\cdot I f_0(y_{23})f_0(y_{22})f_0(y_{11})\, dy_{23}\, dy_{22}\, dy_{11} \quad (A.31)$$

where

$$I = 36 F_0(y_{11}) \left\{ F_1(y_{23})(F_1(y_{22}) - F_1(y_{11})) \right.$$
$$\left. - \frac{F_1^2(y_{22})}{2} + \frac{F_1^2(y_{11})}{2} \right\}. \qquad (A.32)$$

Equations such as (A.14), (A.19), (A.25), and (A.31) can be numerically integrated using routines such as the IMSL (International Mathematical and Statistical Library) routines. By proceeding along similar lines we can get expressions for P_{2T} and P_{3T}.

ACKNOWLEDGMENT

REFERENCES

[1] K. Alam and J. R. Thompson, "A selection procedure based on ranks," *Ann. Inst. Statist. Math.*, pp. 253–262, 1971.

[2]   J. G. Proakis, *Digital Communications*, 3rd ed.   New York: McGraw-Hill, 1995.

[3]   R. Viswanathan and S. C. Gupta, "Nonparametric receiver for FH-MFSK mobile radio," *IEEE Trans. Commun.*, vol. COM-33, pp. 178–184, Feb. 1985.

[4]   R. Viswanathan and K. Taghizadeh, "Diversity combining in FH/BFSK systems to combat partial band jamming," *IEEE Trans. Commun.*, vol. 36, pp. 1062–1069, Sept. 1988.

[5]   A. F. Naguib and A. Paulraj, "Performance of DS/CDMA with M-ary orthogonal modulation cell site antenna arrays," *IEEE J. Select. Areas. Commun.*, pp. 1770–1783, Dec. 1996.

[6]   C. H. Gowda, V. Annampedu, and R. Viswanathan, "Diversity combining in antenna base station receiver for DS/CDMA system," *IEEE Commun. Lett.*, vol. 2, pp. 180–182, July 1998.

[7]   C. H. Gowda, "Order statistics and rank tests for radar detection and wireless CDMA," Ph.D. dissertation, Dept. Elec. Eng., Southern Illinois Univ., Carbondale, IL 62901, Aug. 1998.

[8]   T. A. Gulliver and C. Felstead, "Nonparametric diversity combining for fast frequency hopping," in *Conf. Proc., IEEE MILCOM*, 1995, pp. 60–64.

[9]   R. Viswanathan and P. K. Varshney, "Distributed detection with multiple sensors—Part I: Fundamentals," *Proc. IEEE*, pp. 54–63, Jan. 1997.

[10]   J. N. Tsitsiklis, "Decentralized detection," in *Advances in Statistical Signal Processing, Vol. 2 of Signal Detection*, H. V. Poor and J. B. Thomas, Eds.   Greenwich, CT: JAI, 1990.

[11]   P. K. Varshney, *Distributed Detection and Data Fusion*.   New York: Springer-Verlag, 1996.

[12]   M. C. Jeruchim, "Techniques for estimating the bit error rates in the simulation of digital communication systems," *IEEE J. Select. Areas Commun.*, pp. 153–170, Jan. 1984.

[13]   F. M. Gardner and J. D. Baker, *Simulation Techniques—Models of Communication Signals and Processes*.   New York: Wiley, 1997, ch. 13.

[14]   T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*.   New York: Wiley, 1958.

[15]   C. R. Rao, *Linear Statistical Inference and its Applications*, 2nd ed.   New York: Wiley Eastern, 1989.

[16]   S. S. Gupta, "Probability integral of multivariate normal and multivariate t," *Ann. Inst. Statist. Math.*, pp. 792–838, 1963.

[17]   V. Annampedu, "Decision combining in M-ary communication problems: A rank based test," M.S. thesis, Dept. Elec. Eng., Southern Illinois Univ. at Carbondale, Carbondale, IL 62901, Feb. 1997.

[18]   H. L. Van Trees, *Detection, Estimation, and Modulation Theory*.   New York: Wiley, 1968, pt. I.