Performance Comparison of Likelihood, Hard-Limited, and Linear Combining Receivers for FH-MFSK Mobile Radio--Base-to-Mobile Transmission

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where q and P_c are the number and power, respectively, of the clipped samples. Note that we have equality in (2.21) if and only if the absolute values of all the clipped samples are equal.

Now, we prove that if a nominal signal contains nonzero unclipped samples, then it is not optimum. Suppose that we distribute a fixed amount of power P_b between the clipped part of the signal and an unclipped sample s_{oj}, and furthermore, assume that the allocation of the power in the clipped part is optimum. Then the corresponding contribution to the signal-to-noise ratio is given by

$$g(s_{oj}^2) = \left([(q(P_b - s_{oj}^2))^{1/2} - \Delta]^2 \Big/ \sum_i^* \lambda_i \right) + \lambda_j^{-1} s_{oj}^2 \tag{2.22}$$

which is a convex function of s_{oj}^2. Therefore, the optimal choice of $|s_{oj}|$ must lie on the boundary of the set of its possible values; i.e., it must be either zero or clipped.

We have now shown that all nonzero samples of an optimum nominal signal have constant absolute value. Thus, we need only to specify the optimum number and locations of the nonzero samples. From (2.21) it is readily seen that the nonzero samples must correspond to the q lowest eigenvalues of the noise, and that the optimum q maximizes the right side of (2.21) with $P_c = \|s_o\|^2$ or, equivalently, $\Gamma(q)$. □

An interesting result which follows from Proposition 3 is that, if the first (lowest) m eigenvalues are equal, then we have that

$$\Gamma(n) = [1 - n^{-1/2}\Delta/\|s_o\|]^2/\lambda_o, \quad 1 \leq n \leq m \tag{2.23}$$

and therefore $m \leq q$, i.e., the first m samples are assumed to be nonzero. Applying this fact when the least favorable noise is white, we deduce that the optimum nominal signal is constant in absolute value for all its samples. Also, if $\Delta = 0$ it is easy to check that Proposition 3 results in the classical minimum-eigenvalue eigenvector solution. An important aspect in which Proposition 3 differs from Propositions 1 and 2 is that it gives an optimum signal that is dependent on the degree of distortion (through the ratio $r = \Delta/P^{1/2}$). Note from (2.19) that the solutions of Propositions 1–3 coincide when $\lambda_1/\lambda_0 > H(r)$, where $H(r)$ is an increasing function defined on $[0, 1)$ by

$$H(r) = (\sqrt{2} - r)^2/(1 - r)^2 - 1.$$

III. CONCLUDING REMARKS

The classical solution to the problem of optimum signal selection for matched filtering has been generalized in this paper to admit the existence of uncertainties in the received signal and in the noise covariance matrix. Following the minimax approach to the design of finite-length discrete-time robust matched filters, the goal of the selection (under a power constraint) of the transmitted signal is the optimization of the lower bound of performance guaranteed by the robust matched filter design.

The discussion has emphasized the presence of signal uncertainties due to channel distortion and the noise covariance uncertainty class has been restricted to contain a maximal element, or, equivalently, a signal-independent least favorable matrix (see [8]). Three types of distortion uncertainty models that cover a wide area of practical application have been studied and different results for the signal selection problem have been shown to hold. By use of weighted l_1, l_2, and l_∞ norms, these uncertainty models can be further generalized to accommodate for different degrees of distortion in the directions of the signal space. In such cases the results related to minimax matched filtering and optimum signal design can be extended straightforwardly.

With respect to the mean-square distortion model, a threefold justification for the classical signal design using the minimum-eigenvalue eigenvector of the covariance matrix has been found: it optimizes the signal-to-noise ratio when the received signal coincides with the transmitted one, its associated matched filter is minimax robust for any degree of mean-square distortion and it optimizes the worst case signal-to-noise ratio. However, for the other types of distortion considered here, the set of optimum transmitted signal under distortion no longer coincides with the minimum-eigenvalue eigenspace. The maximum and mean absolute distortion models lend themselves to an analytical solution of the signal design problem under a mean-square power constraint in the case of uncorrelated (not necessarily stationary) least favorable noise. For these models, the corresponding results indicate the advisability of avoiding comparatively small nominal signal samples and of allocating, in some cases, signal power to nonminimum-eigenvalue samples. Note finally that, for a given covariance matrix with a one-dimensional minimum-eigenvalue eigenspace, and with a sufficiently large allowable power (relative to the degree of distortion), the optimum signals for the three types of distortion classes coincide.

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Performance Comparison of Likelihood, Hard-Limited, and Linear Combining Receivers for FH-MFSK Mobile Radio—Base-to-Mobile Transmission

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Abstract—**The performances of three receivers, namely, the hard-limited, the linear, and the maximum-likelihood combiners for the detection of frequency-hopped multilevel frequency-shift keyed sig-**

Paper approved by the Editor for Communication Theory of the IEEE Communications Society for publication after presentation in part at the International Telemetering Conference, San Diego, CA, September 1982. Manuscript received June 3, 1982; revised November 7, 1982. This work was supported by the Air Force Office of Scientific Research under Grant 77-3277C.

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0090-6778/83/0500-0670$01.00 © 1983 IEEE

nals transmitted from mobiles to base have been reported earlier [11]. Only the hard-limited combiner has been analyzed with respect to base-to-mobile link [1]. Here, we give new results on the performance of the likelihood and the linear combining receivers operating at the mobiles. Whereas it is possible to find exactly the union bound on the probability of bit error for a linear combiner, for a likelihood receiver, bounding and approximation techniques such as simple Chernoff bound and saddle point integration were employed. We also observe the asymptotic (SNR → ∞) equivalence of the hard-limited and the likelihood receivers. This, together with the approximate error estimates at finite SNR, leads us to believe that the likelihood receiver is only marginally superior to a hard-limited combiner. As expected, the linear combiner performs poorly.

INTRODUCTION

Recently, performance analysis of the likelihood, linear, and hard-limited receivers, for the mobile-to-base transmission, has been reported [11]. The above paper discusses few interference models and continues the analysis, based on one of the models. None of these models is applicable to base-to-mobile transmission. Whereas it is possible to arrive at the various receiver structures with a suitable model applicable to base-to-mobile transmission, instead, we present a much simpler and unified approach. In all the receivers, the attempt is to discriminate spurious rows of the decoded matrix of a user, which consists of samples from an exponential mixture, from the correct row, which consists of samples from a simple exponential density. To assess the performance, we use Chernoff bounding and saddle point integration techniques for evaluating the probability of bit error for likelihood receiver and use an exact method for the linear combiner.

In Section I we briefly discuss various receiver structures for the FH-MFSK modulation scheme. In Section II we make an approximate estimate of the likelihood receiver performance. In Section II we also observe the asymptotic equivalence of the likelihood and hard-limited receivers. The exact probability of bit error is calculated for a linear combiner in Section III.

I. RECEIVER STRUCTURE FOR FH-MFSK MODULATION

Fig. 1 shows a section of the noncoherent envelope analyzer. As in [1], let τ be the chip duration, K be the number of bits of information transmitted every $L\tau$ seconds, $W = 20$ MHz be the one-way bandwidth, and R be the bit rate. Then we have 2^K such sections in operation corresponding to different orthogonal tones. Let ϵ_{ij} denote the envelope squared output at the ith envelope analyzer after the jth chip. Corresponding to either the signal-plus-noise or the noise-only case, we have ϵ_{ij} to be either exponentially distributed with mean value $(1/\lambda_1)$ or exponentially distributed with mean value $(1/\lambda_0)$, respectively.

A mobile user u receives the signals from the base and creates a decoded matrix every $L\tau$ seconds. The values ϵ_{kj} become the entries X_{ij} in the decoded matrix (the decoding is done on the received matrix with the address of user u). In general, a receiver chooses a row as the row corresponding to the transmitted word, based on some decision criterion. In [1], where hard-limited combining is employed, corresponding to each entry (i, j) in the matrix, a number n_{ij} is assigned such that

$$n_{ij} = \begin{cases} 1 & \text{iff } X_{ij} \geqslant T \\ 0 & \text{otherwise.} \end{cases}$$

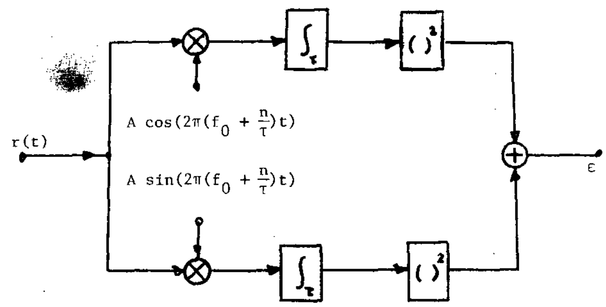

Fig. 1. Envelope analyzer.

A row k is declared as the correct row if

$$\sum_{j=1}^{L} n_{kj} > \sum_{j=1}^{L} n_{ij} \qquad i \neq k.$$

In case two or more rows have the same maximum sum Σ, then any row among these rows is chosen at random as the correct row. In [2], a linear combiner based on choosing the kth row as the correct row such that

$$\max_i \left(\sum_{j=1}^{L} X_{ij} \right) = \sum_{j=1}^{L} X_{kj}$$

was analyzed, for mobile-to-base transmission, using some approximate techniques.

Likelihood Receiver: We shall assume that the minimum frequency spacing between the hops in the transmitted waves is larger than the coherent bandwidth of the Rayleigh fading channel. This, then, implies that X_{ij} are independent and exponentially distributed. Among the 2^K rows in the decoded matrix, only one row is the correct row, wherein all the X_{ij}'s have a mean value $(1/\lambda_1)$. In each of the rest of the $(2^K - 1)$ spurious rows, some elements have a mean value of $(1/\lambda_0)$ and the rest have a mean value of $(1/\lambda_1)$. A spurious row has contributions partly from the interfering users plus noise and partly from the receiver noise. On an average, each spurious row will have a proportion p of X_{ij}'s created due to interference, where p is given by

$$p = 1 - (1 - 2^{-K})^{M-1} \tag{1}$$

and M equals the number of users operating in the cell.

Since each row can be a spurious row (hypothesis H_0) or not (hypothesis H_1), we have the following testing problem applied to an lth row:

$$H_0: X_{lj} \sim p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_0 e^{-\lambda_0 x} \tag{2}$$

versus

$$H_1: X_{lj} \sim \lambda_1 e^{-\lambda_1 x}$$

where

$$j = 1, 2, \cdots, L$$

$$l = 1, \cdots, 2^K.$$

It can be noticed that the proportion p is known once the number of users operating in the cell is known. In statistical literature, whereas a lot of attention has been paid to estimate

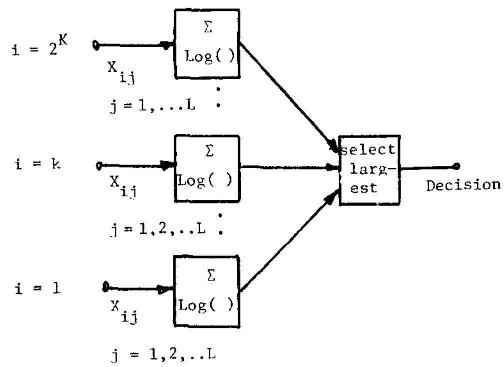

Fig. 2. Likelihood receiver.

the parameters of a mixture distribution [3], [4], there have been no significant results concerning the testing of whether a sample originates from a mixture family or from a member of the family and not the mixture.

Normalizing X_{lj}'s with respect to the mean value of signal plus noise energy, we have

$$Y_{lj} = \lambda_1 X_{lj}. \tag{2a}$$

Therefore, (2) gets modified as

$$H_0\colon Y_{lj} \sim pe^{-y} + (1-p)be^{-by}$$

versus

$$H_1\colon Y_{lj} \sim e^{-y} \tag{3}$$

where $b = \lambda_0/\lambda_1$, signal-plus-noise-to-noise power ratio (SNNR). Forming the likelihood ratio [5], we have

$$S_l = -\sum_{j=1}^{L} \ln\left(p + (1-p)be^{-(b-1)y_{lj}}\right). \tag{4}$$

Then the likelihood receiver chooses the row having $\max_l \{S_l\}$ as the correct row. The block diagram of the likelihood receiver is shown in Fig. 2.

It can be shown that the same receiver can be arrived at by using the approach in [11] with an interference model applicable to base-to-mobile transmission, viz.

$$p(\Gamma_{nl} \mid H_0) = p\delta(\Gamma - 1) + (1-p)\delta(\Gamma)$$

$$p(\Gamma_{nl} \mid H_1) = \delta(\Gamma).$$

II. THE LIKELIHOOD RECEIVER PERFORMANCE

As discussed earlier, we have the decision rule of the likelihood receiver. Decide the row having the $\max_l \{S_l\}$ as the correct row. The statistic S_l behaves differently depending on whether y_{lj}'s belong to H_0 or H_1 (3). However, it is neither possible to find the distribution of S_l exactly under either hypothesis, nor it is possible to calculate exactly the probability of bit error rate.

Let

$$S_l = \sum_{j=1}^{L} Z_{lj}. \tag{5}$$

Without loss of generality, let us assume that the kth row is the correct row corresponding to the word transmitted to the user. In other words, all the rows other than the kth row are spurious. Upon finding the distribution of Z_{lj} under hypotheses H_0 and H_1, we have

$$p_{Z_{ij}}(z) = \begin{cases} cd^{-c}e^{-2z}(e^{-z}-p)^{c-1} & -\ln(p+d) < z < -\ln p \\ 0 & \text{elsewhere} \end{cases} \tag{6}$$

$$p_{Z_{kj}}(z) = \begin{cases} cd^{-c}e^{-z}(e^{-z}-p)^{c-1} & -\ln(p+d) < z < -\ln p \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$

where

$$c = 1/(b-1)$$

$$d = (1-p)b. \tag{8}$$

Simple Chernoff Bound

Here we evaluate an upper bound on the probability of bit error using the Chernoff method [6].

For identically and independently distributed random variables X_i, we have

$$\text{Prob}\left[\frac{1}{L}\sum_{i=1}^{L} X_i \geq 0\right] \leq \{E[e^{r_0 X_i}]\}^L \tag{9}$$

where $r_0 > 0$ is the Chernoff parameter.

We are interested in the union bound on the probability of bit error P_b given by

$$P_b \leq 2^{K-1} P_0 \tag{10}$$

where

$$\begin{aligned} P_0 &= \text{Prob}\,[S_i - S_k > 0] \\ &= \text{Prob}\left[\sum_{j=1}^{L} Z_{ij} - Z_{kj} > 0\right]. \end{aligned} \tag{11}$$

Therefore, by upper bounding P_0 using (9), we can upper bound P_b.

Let

$$y_j = Z_{ij} - Z_{kj}. \tag{12}$$

Then

$$P_0 = \text{Prob}\left[\sum_{j=1}^{L} y_j > 0\right] \leq \{E(e^{r_0 y_j})\}^L \tag{13}$$

and the parameter r_0 is found as the solution to

$$E(y_j e^{r_0 y_j}) = 0. \tag{14}$$

The above equation implies that

$$E(Z_{ij}e^{r_0 Z_{ij}})E(e^{-r_0 Z_{kj}}) - E(Z_{kj}e^{-r_0 Z_{kj}})E(e^{r_0 Z_{ij}}) = 0. \tag{15}$$

Also,

$$E(e^{r_0 y_j}) = E(e^{r_0 Z_{ij}})E(e^{-r_0 Z_{kj}}). \tag{16}$$

Using the density function given in (6) and (7), it can be shown that $r_0 = 1/2$ is the only solution to (15).

We also observe that

$$E(e^{(1/2)Z_{ij}}) = E(e^{-(1/2)Z_{kj}}). \tag{17}$$

Therefore, P_b can be upper bounded by numerically evaluating the right-hand side (RHS) of (13), using (16) and (17). Doing this, we arrive at the curves shown as A in Fig. 4.

Chernoff Bound with Central Limit Theorem

We obtain another approximation to the probability P_0 defined in (13) by using the results in [7]. The idea is to derive a tilted density from the density of y_j [see (12)] and express the probability P_0 in terms of the tilted density variable obtained from y_j's. We define

$$p_{T_j}(t) = p_{y_j}(t)e^{r_0 t}/E(e^{r_0 y_j}) \tag{18}$$

and

$$T = \sum_{j=1}^{L} T_j.$$

It can be shown that [7]

$$P_0 = [E(e^{r_0 y_j})]^L \int_0^\infty p_T(\alpha)e^{-r_0\alpha}\, d\alpha \tag{19}$$

and r_0 is chosen so as to make $E(T) = 0$. T is the sum of L identically and independently distributed variables, and hence, p_T is approximately Gaussian, especially in the vicinity of $E(T) = 0$. The condition $E(T) = 0$ implies that $E(y_j e^{r_0 y_j}) = 0$ and, hence, by the results in the previous subsection, $r_0 = 1/2$. We approximate $p_T(\alpha)$ as

$$p_T(\alpha) \approx \frac{1}{\sqrt{2\pi}\sigma_T} e^{-\alpha^2/2\sigma_T{}^2} \tag{20}$$

where

$$\begin{aligned}\sigma_T{}^2 &= L \operatorname{var}[T_j] \\ &= L\left\{\frac{d^2}{dr_0{}^2}[E(e^{r_0 y_j})] \Big/ E(e^{r_0 y_j})\right\}_{r_0=1/2}\end{aligned} \tag{21}$$

It can be shown that the above bracketed term reduces to

$$\operatorname{var}(T_j) = 2\,\frac{\left[\int e^{\frac{-3S}{2}}(e^{-S}-p)^{c-1}\,dS\right]\left[\int S^2 e^{\frac{-3S}{2}}(e^{-S}-p)^{c-1}\,dS\right] - \left[\int S e^{\frac{-3S}{2}}(e^{-S}-p)^{c-1}\,dS\right]^2}{\left[\int e^{\frac{-3S}{2}}(e^{-S}-p)^{c-1}\,dS\right]^2} \tag{22}$$

where all the integrals are between the limits $(-\ln(p+d), -\ln p)$. Therefore,

$$P_0 \approx (E(e^{r_0 y_j}))^L \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_T} e^{-\alpha^2/2\sigma_T{}^2} e^{-r_0\alpha}\, d\alpha, \quad \text{at } r_0 = 1/2. \tag{23}$$

Evaluating the RHS of (22) numerically, we can evaluate the approximate value of P_b using (10), (16), (17), and (23). The resulting P_b is plotted against the number of users M in Fig. 4. Comparing this approximate P_b curve against [1, Fig. 8], we see tnat the likelihood receiver is marginally better than the hard-limited combiner. Although not shown, it was observed that the effect of variation of K on the receiver performance is similar to the one encountered in the hard-limited combiner, suggesting an optimum K for a given set of W and R. For example, with $W = 20$ MHz and $R = 32$ kbits/s, we have the optimum K to be 8.

Asymptotic Equivalence of Hard-Limited and Likelihood Receivers

In Fig. 3 we show the plot of the nonlinearity

$$F(y) = -\ln(p + (1-p)be^{-(b-1)y}) \tag{24}$$

which is nothing but the likelihood ratio. The plot is for fixed $p = 0.5$ and for various values of b. Several observations can be made by looking at the figure. First of all, the function $F(y)$ is nonlinear and, therefore, a receiver based on Σy_{ij} would not be optimum. Second, as b increases, the curve shifts towards the origin, simultaneously making the transition sharper. Ultimately, as $b \to \infty$, the nonlinearity becomes degenerate with $F(y) = -\ln p$, $y \neq 0$ and an infinite jump discontinuity at the origin. Therefore, $F(y)$ has a resemblance toward hard limiter characteristics, as $b \to \infty$. Moreover, its asymptotic performance is identical to a hard limiter, as will be shown below.

As $b \to \infty$, (3) is modified as

$$\begin{aligned} H_0&: y_{lj} \sim pe^{-y} + (1-p)\delta(y) \\ H_1&: y_{lj} \sim e^{-y}. \end{aligned} \tag{25}$$

Therefore, in the correct row, the random variables $Z_{kj} = F(y_{kj})$ are all degenerate, taking on values $-\ln p$ with probability 1. However, in a spurious row, the random variables Z_{ij} are all identically distributed Bernoulli, taking values $-\ln p$ with probability p and $-\infty$ with probability $(1-p)$. Therefore, an error occurs in our decision only when

$$S_k = S_l \qquad \text{for some } l \neq k. \tag{26}$$

This can happen only when all the L Bernoulli variables take on the value $-\ln p$ and therefore, the probability of this event equals p^L.

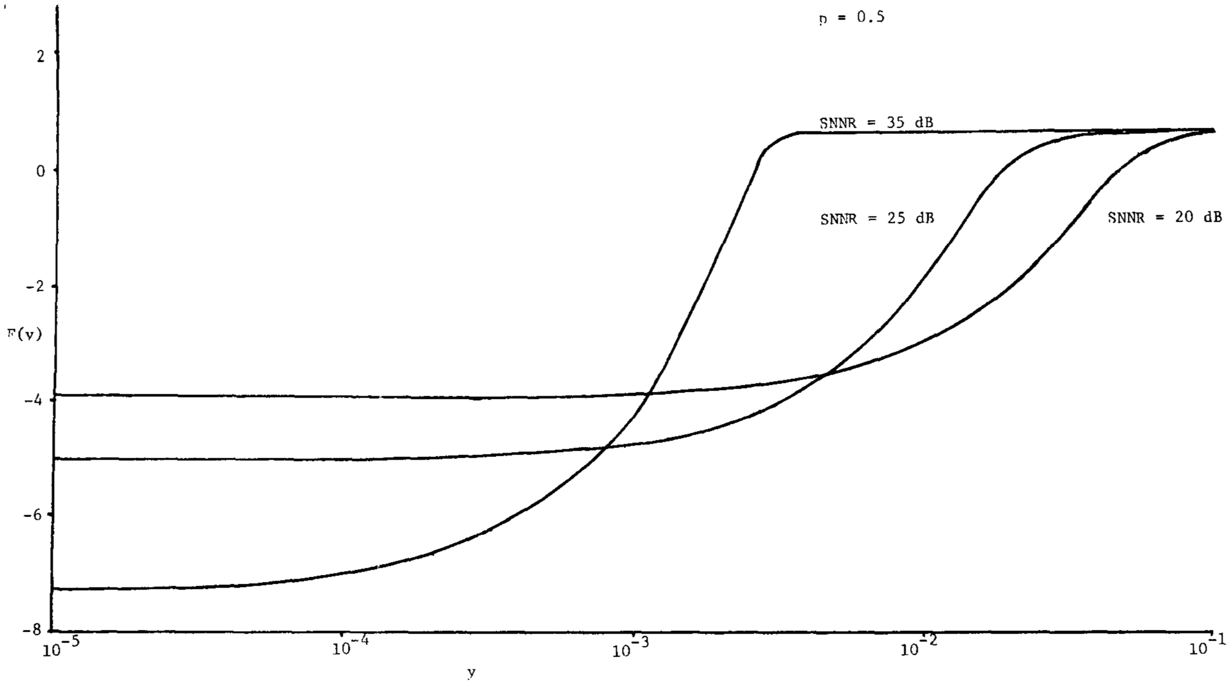

Fig. 3. Plot of nonlinearity.

Hence,

$$\begin{aligned} P_1 &= \text{probability of correctly identifying a row as either spurious or not} \\ &= 1 - p^L. \end{aligned} \tag{27}$$

The above discussion suggests the possibility of more than one row competing for the correct row, although only one exists truly. As in the hard-limited receiver, we resort to random choice of a row among these as the correct row. The probability of correct word decision becomes

$$P_c = \sum_{i=0}^{2^K-1} \binom{2^K-1}{i} P_1{}^{2^K-1-i}(1-P_1)^i \frac{1}{(i+1)}. \tag{28}$$

Only the first few terms in the above equation have significant contribution. Finally the probability of bit error P_b is given by

$$P_b = (1-P_c)\frac{2^{K-1}}{(2^K-1)}. \tag{29}$$

The asymptotic bit error rate is plotted as curve D in Fig. 4.

Now, the hard-limited receiver has a nonlinearity of the form

$$F(y) = \begin{cases} 1 & y \geqslant T \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, as $b \to \infty$, (25) implies that P_1 of (27) also holds good for a hard-limited receiver, provided $T \ll 1$. (In [1], we need the receiver threshold $\beta \gg 1$, since in this case the normalization of the envelope is done with respect to the receiver noise.) Thus, we have established the asymptotic (SNR $\to \infty$) equivalence of likelihood and hard-limited receivers. However, for mobile-to-base transmission, the saddle point approximation [11] predicts uniformly better performance of the likelihood receiver (approximated as a soft limiter) over the hard-limited combiner. One reason for this difference could be that the interferers in mobile-to-base transmission could contribute energy to some elements of the correct row of the decoded matrix of a user under consideration. Such is not the case with the base-to-mobile transmission, where the interferers create only the spurious rows.

Saddle Point Integration

In the previous subsection on the Chernoff bound, we bounded the probability P_0 that the sum of L random variables exceeds zero value. Denoting $\phi(u)$ as the characteristic function of the random variable y_j defined in (12), we have

$$\Pr\left(\sum_{j=1}^{L} y_j > 0\right) = P_0 = \frac{1}{2\pi i}\int_c \frac{\phi^L(u)}{u}\,du \tag{30}$$

where c is a contour whose real part goes from $-\infty$ to $+\infty$ and whose imaginary part lies in the lower half of the complex u plane. Here, $i = \sqrt{-1}$. The above equation can be rewritten as

$$P_0 = \frac{1}{2\pi i}\int_c \frac{e^{L\ln\phi(u)}}{u}\,du. \tag{31}$$

When L is large, the contour c can be deformed into another contour c', such that only a portion of the contour c', around the saddle point, has a dominant contribution to the integral [8], [9]. In fact, it turns out that the first term approximation of the asymptotic expansion of (31) is equivalent to the result achieved with the Chernoff bound and the central limit theorem.[1]

With

$$G(u) = \ln\phi(u) \tag{32}$$

[1] We thank the reviewer for pointing out this equivalence.

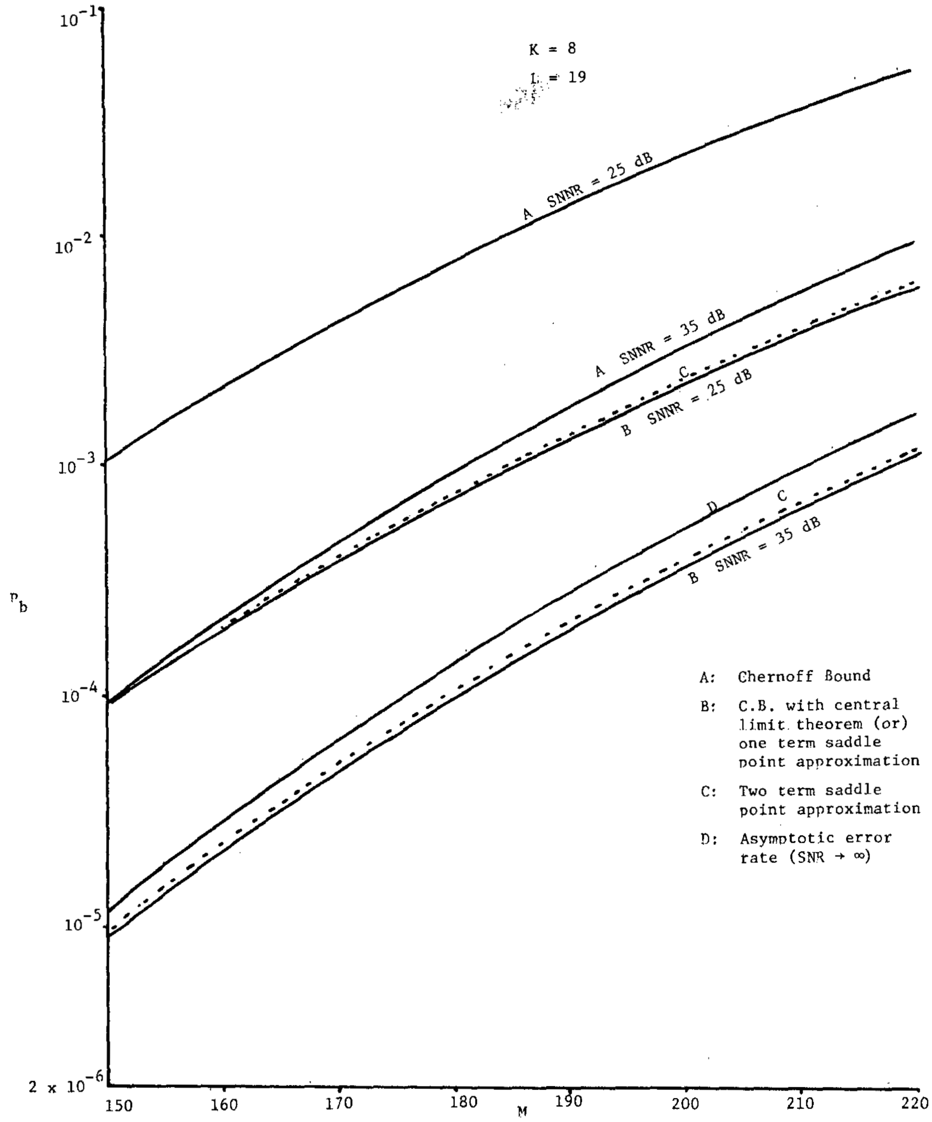

Fig. 4. Probability of bit error versus number of users for likelihood receiver.

the saddle points are the solutions to

$$G'(u) = \frac{dG(u)}{du} = \frac{\phi'(u)}{\phi(u)} = 0. \tag{33}$$

Using our results on the Chernoff bound, it can be observed that there is a unique saddle point on the imaginary axis at $u = -i/2$. The deformed contour c' is then the line parallel to the real axis and going through the point $-i/2$. On the contour c', Im $[G(u)]$ is constant and Re $[G(u)]$ reaches a maximum at $-i/2$. Therefore, by using the standard saddle point expansion, we can write

$$P_0 \cong a_0{}^L \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-L\left(\frac{a_2}{a_0}\right)\frac{x^2}{2}}}{(x - i/2)}\, dx + \frac{1}{2\pi i} \frac{LG^{IV}}{24} \int_{-\infty}^{\infty} \frac{e^{-L\left(\frac{a_2}{a_0}\right)\frac{x^2}{2}}}{(x - i/2)}\, x^4\, dx \right) \tag{34}$$

where

$$a_0 = \phi(u)\,|_{u=-i/2}$$

$$a_2 = -\left.\frac{d^2\phi(u)}{du^2}\right|_{u=-i/2}$$

$$G^{IV} = \left.\frac{d^4 G(u)}{du^4}\right|_{u=-i/2}.$$

In Fig. 4 we show as curve C the probability of bit error P_b of (10), when P_0 is computed using (34). We notice that the inclusion of the second term, as in the RHS of (34), resulted only in very little change in P_b (observe the closeness of curves B and C). Also, the curve C at 35 dB SNNR lies slightly below the theoretical infinite SNNR curve D. This discrepancy can be attributed only to the saddle point approximation technique. Since L is not really large ($L = 19$), the interaction between the pole at the origin of the integrand in (31) and the not-too-far saddle point at $-i/2$ must be considered. Therefore, a more refined saddle point technique is needed for a better estimate [10]. Assuming that the optimism of curve C at 35 dB SNNR is retained at 25 dB SNNR, and with $P_b < 10^{-3}$ we observe that only about 15 users more than possible with a hard-limited receiver could be accommodated (see Fig. 4; also, recall that the hard-limited receiver can accommodate about 170 users under the same conditions). This suggests that the likelihood receiver is only marginally superior to a hard-limited receiver.

III. LINEAR COMBINING RECEIVER

Based on our exponential mixture model, we can evaluate the probability of bit error for the linear combiner, without invoking any approximations.

Let

$$S_l = \sum_{j=1}^{L} Y_{lj} \tag{35}$$

where Y_{lj} are distributed as in (3). The receiver chooses the row m as the correct row such that $S_m = \max_l (S_l)$. It is of interest to find the distribution of S_l under H_0 and H_1. As before, assume that the kth row is the correct row and all $i \neq k$ are spurious. Then

$$S_k \sim \text{gamma}\,(L, 1)$$

i.e.,

$$p_{S_k}(S) = \begin{cases} \dfrac{S^{L-1}}{\Gamma(L)} e^{-S} & S \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

We find the distribution of S_i through the use of characteristic functions. Precisely,

$$p_{S_i}(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\left(\frac{p}{1-it} \right) + \frac{(1-p)b}{b-it} \right)^L e^{-itS}\, dt. \tag{37}$$

The above equation can be rewritten as

$$p_{S_i}(S) = \sum_{m=0}^{L} \binom{L}{m} p^m ((1-p)b)^{L-m} \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itS}\, dt}{(1-it)^m (b-it)^{L-m}} \right]. \tag{38}$$

Evaluating the integral in the above equation using the residue theorem [12], we have

$$\begin{aligned} p_{S_i}(S) &= \left[\frac{[(1-p)b]^L (-1)^{L-1}}{(L-1)!} S^{L-1} e^{-bS} \right. \\ &+ \sum_{m=1}^{L-1} \binom{L}{m} p^m ((1-p)b)^{L-m} \left\{ \frac{(-1)^{m-1}}{(m-1)!} \right. \\ &\cdot \sum_{k=0}^{m-1} \binom{m-1}{k} [(L-m) \cdots (L-m+k-1)] \\ &\cdot (1-b)^{-L+m-k} S^{m-k-1} e^{-S} + \frac{(-1)^{L-m-1}}{(L-m-1)!} \\ &\cdot \sum_{k=0}^{L-m-1} \binom{L-m-1}{k} [m(m+1) \cdots (m+k-1)] \\ &\left. \cdot (b-1)^{-m-k} S^{L-m-1-k} e^{-bS} \right\} \\ &\left. + \frac{p^L (-1)^{L-1}}{(L-1)!} S^{L-1} e^{-S} \right] (-1)^{L+1} \qquad S \geqslant 0 \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{39}$$

Above, the terms $[(L-M) \cdots (L-m+k-1)]$ and $[m(m+1) \cdots (m+k-1)]$ equal 1 when $k = 0$. Therefore, the probability of bit error P_b can be calculated as

$$P_b \leqslant 2^{K-1}(1 - P_1) \tag{40}$$

where

$$\begin{aligned} P_1 &= \text{Prob}\,[S_k > S_i] \\ &= \int_0^{\infty} F_{S_i}(s) p_{S_k}(s)\, ds \end{aligned} \tag{41}$$

and $F_{S_i}(s)$ is the distribution function of the density function given in (39). P_b was evaluated numerically using a computer and the results are shown in Fig. 5. It is seen that the linear combiner performs very poorly. Similar dismal performance of the linear combiner with respect to mobile-to-base transmission was established in [2].

CONCLUSION

Considering the base-to-mobile transmission, we compared the performance of the likelihood, hard-limited, and linear combining receivers. The linear combiner performs the worst, agreeing with the expectations, after observing the performance in the mobile-to-base link. A simple Chernoff bound technique gives an upper bound on the probability of bit error for the likelihood receiver. The bound is not very tight, but assures a minimum performance. As SNR $\to \infty$, it is shown that the theoretical considerations imply the equivalence of hard-limited and likelihood receivers. This is slightly different from the mobile-to-base link, where the likelihood receiver seems to have a slightly better performance than the hard-limited receiver [11].

We observed, by employing saddle point integration, that the likelihood receiver is only marginally superior to a hard-limited receiver at finite SNNR and hence, because of the simplicity of implementation, the latter is to be preferred. Also, it is noticed that a refined saddle point integration technique is required for a better error estimate.

ACKNOWLEDGMENT

The authors wish to thank the reviewers for their many useful comments.

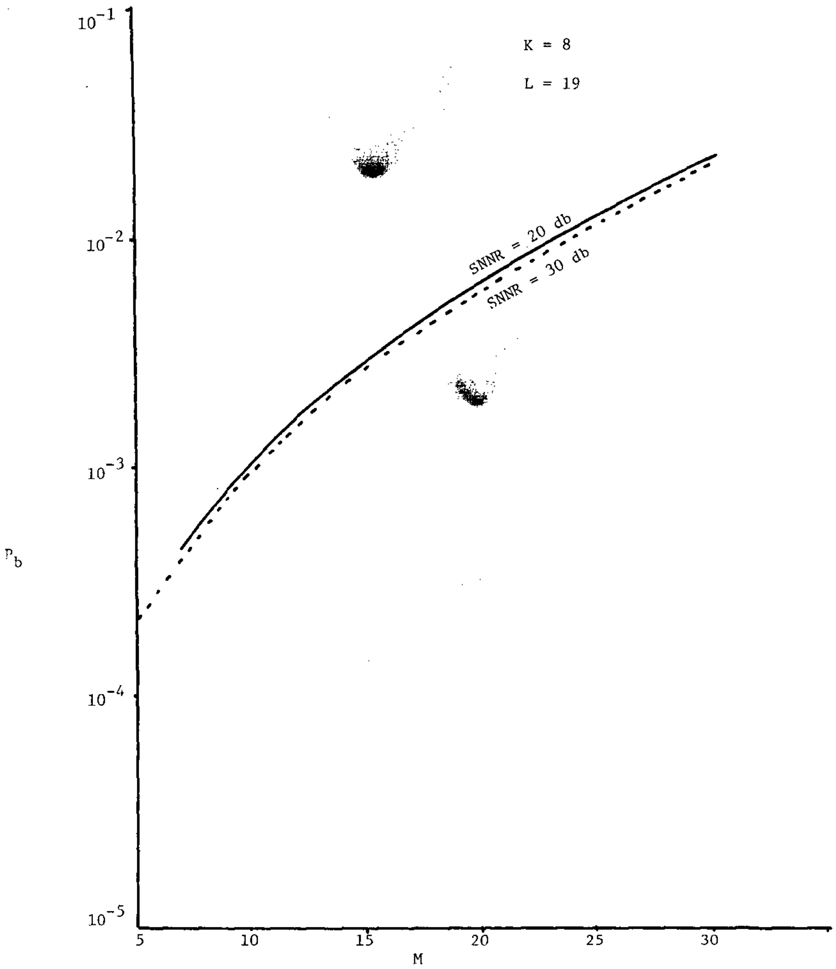

Fig. 5. Probability of bit error versus number of users for linear combiner.

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The Use of Moment Space Bounds for Evaluating the Performance of a Nonlinear Digital Communication System

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Abstract—**Certain moments of the output of a bandpass nonlinear system whose input is a stationary Gaussian random process will be used to evaluate the performance (i.e., the average probability of error) of the system. The main application of this procedure is the analysis of a frequency translating saturating satellite link. The basic**

Paper approved by the Editor for Communication Theory of the IEEE Communications Society for publication after presentation at the International Conference on Communications, Denver, CO, June 1981. Manuscript received May 18, 1982; revised January 10, 1983. This work was supported in part by the Electronics Program of the Office of Naval Research.

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0090-6778/83/0500-0677$01.00 © 1983 IEEE