SINGLE-CHANGE CIRCULAR COVERING DESIGNS

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Abstract

A single-change circular covering design (scccd) based on the set $[v] = \{1, \ldots, v\}$ with block size k is an ordered collection of b blocks, $\mathcal{B} = \{B_1, \ldots, B_b\}$, each $B_i \subset [v]$, which obey: (1) each block differs from the previous block by a single element, as does the last from the first, and, (2) every pair of $[v]$ is covered by some B_i. The object is to minimize b for a fixed v and k. We present some minimal constructions of scccds for arbitrary v when $k = 2$ and 3, and for arbitrary k when $k+1 \leq v \leq 2k$. Tight designs are those in which each pair is covered exactly once. Start-Finish arrays are used to construct tight designs when $v > 2k$; there are 2 non-isomorphic tight designs with $(v, k) = (9, 4)$, and 12 with $(v, k) = (10, 4)$. Some non-existence results for tight designs, and standardized, element-regular, perfect, and column-regular designs are also considered.

1. Definitions; notation; examples

A *single-change circular covering design* based on the set $[v] = \{1, \ldots, v\}$ with block size k is an ordered collection of b blocks, $\mathcal{B} = \{B_1, \ldots, B_b\}$, each an unordered subset of k distinct elements from $[v]$, which obey:

(1) each block differs from the previous block by a single element, i.e., $|B_{i-1} \cap B_i| = k - 1$ for $i = 2, \ldots, b$; and the last block, B_b, differs from the first, B_1, by a single element, i.e., $|B_b \cap B_1| = k - 1$; and

(2) every (unordered) pair $\{x, y\}$ of $[v]$, with $x \neq y$, can be written as $\{e_i, z\}$ where $e_i \in B_i \backslash B_{i-1}$ and $z \in B_i$ for some $i = 2, \ldots, b$, or as $\{e_1, z\}$ where $e_1 \in B_1 \backslash B_b$ and $z \in B_1$.

For $i = 2, \ldots, b$ we say that element e_i is *introduced* in block B_i, and the pairs $\{e_i, z\}$ where $z \in B_i$ are *covered* by B_i. Similarly, e_1 is introduced in B_1 and pairs $\{e_1, z\}$ where $z \in B_1$ are covered by B_1. We also say that a pair is *covered* by $\mathcal{B}$ if it is covered by some block in $\mathcal{B}$.

A single-change circular covering design is simply a *single-change covering design* (see Wallis, Yucas, and Zhang [4], and Preece, Constable, Zhang, Yucas, Wallis, McSorley, and Phillips [2]) in which a 'single-change' is also required between B_b and B_1.

We denote a single-change circular covering design by scccd; and a scccd based on $[v]$ with block size k by scccd(v,k), or by scccd(v,k,b) if we wish to mention that it contains b blocks. For a fixed v and k, where $k \geq 2$ and $v \geq k+1$, we denote by $b_*(v,k)$ the smallest b for which there exists a scccd(v,k,b), and we call such a scccd$(v,k,b_*(v,k))$ *minimal*. We write our designs vertically as in [4].

Our first example is a scccd$(6,3,8)$ labelled $\mathcal{E}$ and shown in Fig. 1(a). In $\mathcal{E}$ each pair, except $\{4,1\}$, is covered once; $\{4,1\}$ is covered twice, in B_5 and in B_8. This is an *economical* design, see §5.

blocks B_i		*element introduced* e_i	*pairs covered* $\{e_i,z\},\ z \in B_i$	
B_1	6 4 2	6	$\{6,4\}$	$\{6,2\}$
B_2	6 3 2	3	$\{3,6\}$	$\{3,2\}$
B_3	6 3 5	5	$\{5,6\}$	$\{5,3\}$
B_4	6 3 1	1	$\{1,6\}$	$\{1,3\}$
B_5	4 3 1	4	$\{4,3\}$	$\{4,1\}$
B_6	4 5 1	5	$\{5,4\}$	$\{5,1\}$
B_7	2 5 1	2	$\{2,5\}$	$\{2,1\}$
B_8	2 4 1	4	$\{4,2\}$	$\{4,1\}$

$\mathcal{E}$, an economical scccd$(6,3,8)$

(a)

1 2 3
2 3 4
3 4 5
4 5 1
5 1 2

$\mathcal{Y}_3$, a tight scccd$(5,3,5)$

(b)

Fig. 1: Examples: $\mathcal{E}$ and $\mathcal{Y}_3$.

As the ordering of the elements in a block is immaterial, we often (but not always) leave a block's unchanged elements in the same columns as in the previous block, see $\mathcal{E}$ in Fig. 1(a); such a representation of $\mathcal{E}$ is called *column-strict*. In block B_4 element 1 is introduced because $1 \notin B_3$ and element 6 is *changed* because $6 \notin B_5$.

Our second example is a scccd$(5,3,5)$, see Fig. 1(b); we call this design $\mathcal{Y}_3$, see §6. Here the 10 pairs from the set $\{1,2,3,4,5\}$ are each covered exactly once, 2 per block; we call such a design *tight*, see §5.

We generally use the notation $\mathcal{B} = \{B_1,\ldots,B_b\}$ for an arbitrary scccd(v,k,b) and $\mathcal{C}$ for an arbitrary tight scccd(v,k,b), often with $v > 2k$.

The main object of this paper is to study scccds, with special interest in the function $b_*(v,k)$ and in tight designs. We will see that the structure and construction of these designs are somewhat different from those of the single-change covering designs of [4] and [2].

2. Requirements for $\mathcal{B}$ to form a single-change circular covering design

Let $\mathcal{B} = \{B_1, \ldots, B_b\}$ be an ordered collection of b blocks; each B_i contains k distinct elements from $[v]$.

We say that $\mathcal{B}$ has the *single-change circular* property if $|B_{i-1} \cap B_i| = k-1$ for $i = 2, \ldots, b$, and $|B_b \cap B_1| = k-1$.

Lemma 2.1. $\mathcal{B}$ is a scccd(v,k) if and only if
(i) $\mathcal{B}$ has the single-change circular property, and
(ii) every pair of $[v]$ is in some block of $\mathcal{B}$, and
(iii) no pair of $[v]$ is in every block of $\mathcal{B}$.

Proof. First suppose that $\mathcal{B}$ is a scccd(v,k). Then (i) and (ii) are true by definition of a scccd. Now suppose that (iii) is false, and that the pair $\{x,y\}$ is in every block of $\mathcal{B}$. Then neither x nor y is introduced in any block, hence $\{x,y\}$ is not covered by $\mathcal{B}$, a contradiction because $\mathcal{B}$ covers every pair.

Now suppose that $\mathcal{B}$ satisfies (i), (ii), and (iii). As $\mathcal{B}$ satisfies (i) we need only show that it covers every pair of $[v]$ to conclude that it is a scccd(v,k). Now, by (ii), every pair $\{x,y\}$ lies in some block of $\mathcal{B}$ and, by (iii), the pair $\{x,y\}$ is not in every block. So let $B_{i_1}, B_{i_2}, \ldots, B_{i_t}$, where $t \le b-1$, be a sequence of consecutive blocks each containing $\{x,y\}$; the block immediately 'before' B_{i_1}, say B, does not contain $\{x,y\}$. So exactly one of x or y lies in B and the other does not. Hence, either y or x (respectively) is introduced in B_{i_1}, and so $\{x,y\}$ is covered there. Hence $\mathcal{B}$ covers every pair and is a scccd(v,k). ■

3. Designs with k=2 and 3

k=2 If a tight scccd$(v,2,b)$ exists, then $b = v(v-1)/2$, see §5. Now, given a tight scccd$(v,2)$ with first block $B_1 = (1,2)$ and last block $B_b = (v,1)$, we may add on the v blocks as shown in Fig. 2(a) to obtain a tight scccd$(v+1, 2, v(v+1)/2)$ based on $[v+1]$ with last block $(v+1, 1)$.

Beginning with the tight scccd$(3,2,3)$ shown in Fig. 2(b) we can use this construction repeatedly to obtain a tight scccd$(v,2)$ for any $v \ge 3$.

k=3 If a tight scccd$(v,3,b)$ exists, then $b = v(v-1)/4$, see §5; and so $v \equiv 0$ or $1 \pmod 4$. Let $v \equiv 0 \pmod 4$, and suppose that we have a tight scccd$(v,3)$ with $B_1 = (1,2,3)$ and

$B_b = (v, 1, 2)$ in which element 1 is introduced in B_b. We can then construct a tight scccd$(v+4, 3, (v+3)(v+4)/4)$ based on $[v+4]$ by altering B_b to $(v, v+1, 2)$ and adding on the $2v+3$ blocks as shown in Fig. 2(c). This new design has last block $(v+4, 1, 2)$ in which 1 is introduced; so we can use this construction repeatedly to obtain a tight scccd$(v, 3)$ for any $v \equiv 0 \pmod 4$ beginning with the tight scccd$(4, 3, 3)$ shown in Fig. 2(d).

(a)
$1,\ 2$
$\cdot\ \cdot$
$\cdot\ \cdot$
$\cdot\ \cdot$
$\underline{v},\ \underline{1}$
$v,\ v+1$
$v-1, v+1$
$\cdot\ \cdot$
$\cdot\ \cdot$
$\cdot\ \cdot$
$1,\ v+1$

(a)

(b)
1 2
3 2
3 1

a tight scccd$(3, 2, 3)$

(b)

(c)
$1,\ 2,\ 3$
$\cdot\ \cdot\ \cdot$
$\cdot\ \cdot\ \cdot$
$\cdot\ \cdot\ \cdot$
$\underline{v},\ \underline{v+1},\ \underline{2}$
$v,\ v+2,\ 2$
$v,\ v+2,\ v+4$
$v,\ v+2,\ v+3$
$v,\ 1,\ v+3$
$v+1,\ 1,\ v+3$
$v+1,\ 1,\ v+2$
$v+1,\ v-1,\ v+2$
$\cdot\ \cdot\ \cdot$
$\cdot\ \cdot\ \cdot$
$\cdot\ \cdot\ \cdot$
$v+1,\ 3,\ v+2$
$v+1,\ 3,\ v+4$
$v+3,\ 3,\ v+4$
$\cdot\ \cdot\ \cdot$
$\cdot\ \cdot\ \cdot$
$\cdot\ \cdot\ \cdot$
$v+3,\ v-1,\ v+4$
$v+3,\ 2,\ v+4$
$v+4,\ 1,\ 2$

(c)

(d)
1 2 3
4 2 3
4 1 2

a tight scccd$(4, 3, 3)$

(d)

(e)
1 2 3
1 4 3
5 4 3
5 4 2
5 1 2

a tight scccd$(5, 3, 5)$

(e)

(f)
1 2 3
4 2 3
4 5 3
4 6 3
4 6 1
5 6 1
5 6 2
6 1 2

an economical scccd$(6, 3, 8)$

(f)

(g)
1 2 3
1 7 3
1 7 5
1 4 5
1 4 6
3 4 6
3 5 6
2 5 6
2 7 6
2 7 4
7 1 2

an economical scccd$(7, 3, 11)$

(g)

Fig. 2: Starter designs and additions for $k = 2$ and 3.

We can also construct a tight scccd$(v, 3)$ when $v \equiv 1 \pmod 4$ for any $v \geq 5$ starting with the tight scccd$(5, 3, 5)$ shown in Fig. 2(e); and an economical scccd$(v, 3, \lceil v(v-1)/4 \rceil)$ for $v \equiv 2$ or $3 \pmod 4$ for any $v \geq 6$ starting with the economical scccds in (f) or (g) respectively.

Theorem 3.1.

(i) A tight scccd$(v, 2)$ exists for all $v \geq 3$;
(ii) a tight scccd$(v, 3)$ exists for all $v \equiv 0$ or $1 \pmod 4$, $v \geq 4$;
(iii) an economical scccd$(v, 3)$ exists for all $v \equiv 2$ or $3 \pmod 4$, $v \geq 6$.

■

4. Standardized forms; isomorphisms; reverses

A scccd(v, k, b) is *standardized* or in *standardized form* (see §1 of [2]) if:

(1) the elements of the first block are $1, 2, \ldots, k$ in that order;
(2) the other elements are introduced initially in the order $k+1, k+2, \ldots, v$;
(3) the elements of the first block are changed initially in the order $k, k-1, \ldots, 2, 1$ (if our scccd(v, k, b) has one element, say element 1, in every block, then the elements of the first block are changed initially in the order $k, k-1, \ldots, 2$);
(4) beginning at the second block, a block's unchanged elements are in the same columns as in the previous block (i.e., it is column-strict).

Given any scccd(v, k, b), $\mathcal{B}$, that satisfies (4) above, in order to change it to its standardized form we need to apply a permutation of $[v]$ to it, followed by a permutation of its columns. For example, if we apply the permutation $(1, 6)(3, 4)$ to $\mathcal{E} = \{B_1, B_2, \ldots, B_8\}$ shown in Fig. 3(a), and then permute its 2nd and 3rd columns, we arrive at its standardized form shown in (b), with blocks labelled L_i.

A *cyclic shift* of the ordered blocks $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is one of the following rearrangements:

$$\begin{aligned}
\mathcal{B} = \mathcal{B}_1 &= \{B_1, B_2, \ldots, B_b\}, \\
\mathcal{B}_2 &= \{B_2, B_3, \ldots, B_b, B_1\}, \\
&\;\;\vdots \\
\mathcal{B}_i &= \{B_i, \ldots, B_b, B_1, \ldots, B_{i-1}\}, \\
&\;\;\vdots \\
\mathcal{B}_b &= \{B_b, B_1, \ldots, B_{b-1}\}.
\end{aligned}$$

The block arrangement $\mathcal{B}_i$ is called the *i-th cyclic shift* of $\mathcal{B}$. For each i the first block in $\mathcal{B}_i$ is B_i.

				in	*ch*			*in*	*ch*	
				e_i	e'_i			e_i	e'_i	
B_1	6 4 2	L_1	1 2 3	1	$\vert\overline{3}\vert$	L_1	1 2 3	2	$\vert\overline{3}\vert$	2 4 1
B_2	6 3 2	L_2	1 2 4	$\vert\overline{4}\vert$	$\vert\underline{2}\vert$	L_2	1 2 4	$\vert\overline{4}\vert$	$\vert 2\vert$	2 5 1
B_3	6 3 5	L_3	1 5 4	$\vert 5\vert$	5	L_3	1 5 4	$\vert 5\vert$	$\vert\underline{1}\vert$	4 5 1
B_4	6 3 1	L_4	1 6 4	$\vert\underline{6}\vert$	$\vert\overline{\underline{1}}\vert$	L_4	6 5 4	$\vert\underline{6}\vert$	6	4 3 1
B_5	4 3 1	L_5	3 6 4	3	4	L_5	3 5 4	3	4	6 3 1
B_6	4 5 1	L_6	3 6 5	5	3	L_6	3 5 2	2	5	6 3 5
B_7	2 5 1	L_7	2 6 5	2	5	L_7	3 6 2	6	2	6 3 2
B_8	2 4 1	L_8	2 6 3	3	6	L_8	3 6 1	1	6	6 4 2
$\mathcal{E}$		$\mathrm{sf}(\mathcal{E}) = \mathrm{sf}(\mathcal{E}_1)$				$\mathrm{rsf}(\mathcal{E}) = \mathrm{sf}(\mathcal{E}_8)$				$\mathrm{rev}(\mathcal{E})$
(a)		(b)				(c)				(d)

Fig. 3: The economical scccd$(6, 3, 8)$, $\mathcal{E}$, its standardized form, its representative standardized form, and its reverse.

Each scccd $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ has b standardized forms, one for each cyclic shift $\mathcal{B}_i$; let $\mathrm{sf}(\mathcal{B}_i)$ denote the standardized form of $\mathcal{B}_i$ for each $i = 1, \ldots, b$.

The scccd$(5, 3, 5)$, $\mathcal{Y}_3$, of Fig. 1(b) has each of its 5 standardized forms identical (shown in Fig. 15(d)); but $\mathcal{E}$ in Fig. 3(a) has each of its 8 standardized forms different. If a design $\mathcal{B}$ does not have all of its standardized forms identical, one of these forms can usefully be chosen as the *representative standardized form*, $\mathrm{rsf}(\mathcal{B})$. In order to do this for an arbitrary $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$, we will presently define, for each i, four finite sequences associated with $\mathrm{sf}(\mathcal{B}_i)$, namely S_1, S_2, S_3, and S_4. (In general, the sequences S_1, S_2, S_3, and S_4 will be different for each i, but we choose this notation for simplicity.) We then consider the ordered set $\{S_1, S_2, S_3, S_4\}$. So each $\mathrm{sf}(\mathcal{B}_i)$ gives us an ordered set of four sequences. Next, we order these ordered sets of four sequences according to the criteria below, and choose the 'least' in this ordering. Suppose this least ordered set comes from $\mathrm{sf}(\mathcal{B}_\ell)$, then $\mathrm{sf}(\mathcal{B}_\ell)$ is taken as $\mathrm{rsf}(\mathcal{B})$.

Again for simplicity, we use $L_1, L_2, \ldots, L_b$, to denote the b blocks of $\mathrm{sf}(\mathcal{B}_i)$, for every $i = 1, \ldots, b$, even though generally the blocks differ for each i. So $\mathrm{sf}(\mathcal{B}_i) = \{L_1, L_2, \ldots, L_b\}$ where $L_1 = (1, 2, \ldots, k)$ and $L_2 = (1, 2, \ldots, k + 1)$.

The sequences S_1 and S_2 are sequences of distinct blocks from $\{L_1, L_2, \ldots, L_b\}$; and S_3 and S_4 are sequences of not necessarily distinct elements from $[v]$. For a fixed $\mathrm{sf}(\mathcal{B}_i)$ we define S_1 with reference to (2) above. For $t = 1, \ldots, v - k$, the t-th member of S_1 is the block into which element $k+t$ is initially introduced. Thus S_1 begins with L_2. Sequence S_2 (see (3) above) is a sequence of blocks beginning at L_1. For $t = 1, \ldots, k$, the t-th member of S_2 is the first block from which element $k-t+1$ is initially changed (t stops at $k-1$ if $\mathcal{B}$ has element 1 in every block).

Just as for $i = 1, \ldots, b$, the element e_i is introduced into block L_i, let e'_i be the element changed from L_i. So we have $e'_i \in L_i \backslash L_{i+1}$ for $i = 1, \ldots, b-1$, and $e'_b \in L_b \backslash L_1$. Now define sequence $S_3 = \{e_i : L_i \notin S_1\}$, arranged with increasing i; so, e_1 is its first member. Similarly, we define $S_4 = \{e'_i : L_i \notin S_2\}$, arranged with increasing i.

For an example see sf($\mathcal{E}_1$) in Fig. 3(b), where the column of introduced elements, e_i, is labelled 'in', and the column of changed elements, e'_i, is labelled 'ch'. We have $S_1 = \{L_2, L_3, L_4\}$, $S_2 = \{L_1, L_2, L_4\}$, $S_3 = \{1, 3, 5, 2, 3\}$, and $S_4 = \{5, 4, 3, 5, 6\}$.

Thus each sf($\mathcal{B}_i$) gives us an ordered set of four sequences. We now order these ordered sets of four sequences by, first of all, lexicographically ordering their first elements, the S_1 sequences, according to the rule: $L_r < L_s$ if and only if $r < s$, and choosing the set(s) whose S_1 sequence is the first (i.e., the least) in this list. If two (or more) sets have identical S_1 sequences, we choose the one with the least S_2 sequence, using the same ordering. If two (or more) sets have identical S_1 and S_2 sequences we then compare their S_3 sequences and order them lexicographically using the natural $<$ ordering on $[v]$, and choose the least. If still identical, we compare their S_4 sequences, with the $<$ ordering, and choose the least. Two such sets with identical S_1, S_2, S_3, and S_4 sequences can easily be shown to correspond to standardized forms that are identical.

By this process we arrive at the particular sf($\mathcal{B}_\ell$) with the least set of sequences according to our lexicographic orderings; we take this particular standardized form as the representative standardized form of $\mathcal{B}$, rsf($\mathcal{B}$). For example, rsf($\mathcal{E}$), shown in Fig. 3(c), is sf($\mathcal{E}_8$); and rsf($\mathcal{Y}_3$) is shown in Fig. 15(d).

Two scccd(v, k, b)s $\mathcal{B}$ and $\mathcal{B}'$ are *isomorphic*, ($\mathcal{B} \cong \mathcal{B}'$), if we can apply a permutation of $[v]$ combined with a cyclic shift of the blocks of $\mathcal{B}$ to obtain $\mathcal{B}'$. Similarly, an *automorphism* of a scccd(v, k, b) $\mathcal{B}$ is a permutation of $[v]$ which, when applied to $\mathcal{B}$, produces a cyclic shift of $\mathcal{B}$. For example, the permutation $(1, 2, 3, 4, 5)$ is an automorphism of $\mathcal{Y}_3$.

Theorem 4.1. Let $\mathcal{B}$ and $\mathcal{B}'$ be two scccd(v, k, b)s.
Then $\mathcal{B} \cong \mathcal{B}'$ if and only if rsf($\mathcal{B}$) = rsf($\mathcal{B}'$).

Proof. Suppose $\mathcal{B} \cong \mathcal{B}'$, then, for any $i \in \{1, \ldots, b\}$, there exists a $j \in \{1, \ldots, b\}$ such that $\mathcal{B}_i$, the i-th cyclic shift of $\mathcal{B}$, can be changed into $\mathcal{B}'_j$ using only a permutation on v, i.e., with no cyclic shift of the blocks of $\mathcal{B}_i$. So sf($\mathcal{B}_i$) = sf($\mathcal{B}'_j$), and so {sf($\mathcal{B}_1$), ... ,sf($\mathcal{B}_b$)} $\subseteq$ {sf($\mathcal{B}'_1$), ... ,sf($\mathcal{B}'_b$)}. Similarly, {sf($\mathcal{B}'_1$), ... ,sf($\mathcal{B}'_b$)} $\subseteq$ {sf($\mathcal{B}_1$), ... ,sf($\mathcal{B}_b$)}. Finally we see that {sf($\mathcal{B}_1$), ... ,sf($\mathcal{B}_b$)} = {sf($\mathcal{B}'_1$), ... ,sf($\mathcal{B}'_b$)}, and so rsf($\mathcal{B}$) = rsf($\mathcal{B}'$).

Now suppose that rsf($\mathcal{B}$) = rsf($\mathcal{B}'$), then $\mathcal{B} \cong$ rsf($\mathcal{B}$) = rsf($\mathcal{B}'$) $\cong \mathcal{B}'$, as required.

■

The *reverse* of the scccd $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is the scccd obtained by reversing the order of the blocks of $\mathcal{B}$; we denote this design by $\text{rev}(\mathcal{B}) = \{B_b, B_{b-1}, \ldots, B_1\}$. The reverse of $\mathcal{E}$ is shown in Fig. 3(d). If $\mathcal{B} \cong \text{rev}(\mathcal{B})$ we say that $\mathcal{B}$ is *self-reverse*.

5. Lower bounds on $\mathbf{b_*(v,k)}$; constructions of designs for $\mathbf{k+1 \leq v \leq 2k}$

Lemma 5.1. For $v \geq 4$ and $k \geq 3$, the value of $b_*(v,k)$, the minimum number of blocks in a scccd(v,k), satisfies

$$b_*(v,k) \geq max\left\{v-1, \left\lceil \frac{v(v-1)}{2(k-1)} \right\rceil\right\}.$$

Proof. In a scccd(v,k) exactly one element is introduced per block, so, if $b_*(v,k) < v-1$, then at most $v-2$ distinct elements are introduced. Hence, at least 2 distinct elements are not introduced and the pair containing them is not covered, a contradiction. So $b_*(v,k) \geq v-1$.

A scccd(v,k) must cover all $v(v-1)/2$ pairs of $[v]$, and $k-1$ pairs are covered per block. Thus $b_*(v,k) \cdot (k-1) \geq v(v-1)/2$, and so the result.

■

Corollary 5.2. For $v \geq 4$ and $k \geq 3$ we have

$$b_*(v,k) \geq \begin{cases} v-1, & \text{for } k+1 \leq v \leq 2k-2; \\ \left\lceil \dfrac{v(v-1)}{2(k-1)} \right\rceil, & \text{for } v > 2k-2. \end{cases} \tag{1}$$

■

If $b_*(v,k)$ satisfies equation (1) with equality, then the corresponding scccd(v,k) is *economical*; except that if $b_*(v,k) = \dfrac{v(v-1)}{2(k-1)}$ then the corresponding scccd(v,k) is *tight*. An economical and a tight design are shown in Fig. 1.

Consider the first case of Corollary 5.2. If a scccd(v,k) with $k+1 \leq v \leq 2k-2$ and with the minimal number of $v-1$ blocks exists, then a total of $v-1$ elements are introduced in the design; from the proof of Lemma 5.1 these $v-1$ elements are distinct. Hence, without loss of generality, in such a design the elements $1, \ldots, v-1$ are each introduced exactly once; the element v is not introduced, and so appears in every block. The following construction satisfies these requirements:

Arrange the elements of $[v-1]$ in a circle and call this arrangement the *circular* $[v-1]$. Now, for $1 \leq i \leq v-1$, let A_i be the block beginning at i and containing $k-1$ consecutive elements taken clockwise from the circle; i.e., $A_i = (i, i+1, \ldots, i+k-2)$, where addition is taken modulo $v-1$ with $v-1$ replacing 0. Now let $B_i = A_i \cup \{v\}$ and $\mathcal{B} = \{B_1, \ldots, B_{v-1}\}$.

Theorem 5.3. For $k \geq 3$ and $k+1 \leq v \leq 2k-2$ the blocks $\mathcal{B} = \{B_1, \ldots, B_{v-1}\}$ where $B_i = A_i \cup \{v\}$ form an economical scccd$(v, k, v-1)$.

Proof. We need only show that $\mathcal{B}$ is a scccd(v, k). It then follows, since $|\mathcal{B}| = v - 1$, that it is an economical scccd$(v, k, v-1)$. We show that $\mathcal{B}$ satisfies (i), (ii), and (iii) of Lemma 2.1.

(i) This is clear from the definition of $\mathcal{B}$.

(ii) Now $B_i = (i, i+1, \ldots, i+k-2) \cup \{v\}$, where addition is taken modulo $v-1$ with $v-1$ replacing 0. So element $i+k-2$ is introduced in B_i, i.e., element i is introduced in B_{i-k+2}. Hence, for $1 \leq i \leq v-1$, pair $\{i, v\}$ is covered by B_{i-k+2}. This deals with pairs that contain v.

Now consider the pair $\{i, j\}$ where $1 \leq i < j \leq v-1$ and let v be even. The pair $\{i, j\}$ can be covered by a 'run' of $v/2$ consecutive elements of the circular $[v-1]$ starting either at i or at j. But $v \leq 2k-2$ and so $v/2 \leq k-1$, i.e., such a run is contained in A_i or A_j, so in B_i or B_j. Hence $\{i, j\}$ is covered by $\mathcal{B}$. A similar argument works when v is odd.

(iii) Let $\{i, j\}$ be in every block; then, without loss of generality, we have $1 \leq i \leq v-1$. But i is introduced in B_{i-k+2}, and so cannot be in the previous block, a contradiction. Hence no pair is in every block.

Thus $\mathcal{B}$ is an economical scccd$(v, k, v-1)$. ∎

An economical scccd$(5, 4, 4)$ constructed using Theorem 5.3 is shown in Fig. 4(a).

		B_1 1 2 3 4
		B_2 2 3 4 5
	B_1 1 2 3 6	B_3 3 4 5 6
B_1 1 2 3 5	B_2 2 3 4 6	B_4 4 5 6 7
B_2 2 3 4 5	B_3 3 4 5 6	B_5 5 6 7 1
B_3 3 4 1 5	B_4 4 5 1 6	B_6 6 7 1 2
B_4 4 1 2 5	B_5 5 1 2 6	B_7 7 1 2 3
an economical scccd$(5, 4, 4)$	a tight scccd$(6, 4, 5)$	a tight scccd$(7, 4, 7)$
(a)	(b)	(c)

Fig. 4: Designs with $k = 4$ constructed using Theorems 5.3, 5.4, and 5.5 respectively.

For a fixed $k \geq 3$ the only tight designs amongst the economical scccd$(v, k, v-1)$s with $k+1 \leq v \leq 2k-2$ occur when $v(v-1)/2 = (k-1)(v-1)$, i.e., when $v = 2k-2$. So Theorem 5.3 with $v = 2k-2$ yields tight designs:

Theorem 5.4. For $k \geq 3$ and $v = 2k-2$ the blocks $\mathcal{B} = \{B_1, \ldots, B_{2k-3}\}$ where $B_i = A_i \cup \{2k-2\}$ form a tight scccd$(2k-2, k, 2k-3)$.

■

See Fig. 4(b) for a tight scccd$(6, 4, 5)$ constructed using Theorem 5.4.

So far, for a fixed $k \geq 3$, we have constructed economical scccds when $k+1 \leq v \leq 2k-2$. We now consider $v = 2k-1$ and $v = 2k$, so we are in the second case of Corollary 5.2 in which a scccd has $b \geq \left\lceil \dfrac{v(v-1)}{2(k-1)} \right\rceil$ blocks.

v=2k-1 Here $b \geq 2k-1$. Consider the circular $[2k-1]$. For $1 \leq i \leq 2k-1$ let $B_i = (i, i+1, \ldots, i+k-1)$, where addition is taken modulo $2k-1$ with $2k-1$ replacing 0. Then we have the following result. This construction and the next also work for $k = 2$.

Theorem 5.5. For $k \geq 2$ and $v = 2k-1$ the blocks $\mathcal{B} = \{B_1, \ldots, B_{2k-1}\}$ where $B_i = (i, i+1, \ldots, i+k-1)$ form a tight scccd$(2k-1, k, 2k-1)$.

■

See Fig. 4(c) for a tight scccd$(7, 4, 7)$ constructed using Theorem 5.5.

v=2k Here $b \geq 2k+2$. To construct an economical scccd$(2k, k, 2k+2)$ based on $[2k]$ we take the blocks $\{B_1, \ldots, B_{2k-1}\}$ of the scccd$(2k-1, k, 2k-1)$ in Theorem 5.5 above and add on 3 new blocks: C between B_{k-1} and B_k, then C' after B_{2k-1}, and finally C'' after C', i.e., between C' and B_1; see Fig. 5(a).

In Fig. 5(a) the single-change circular property between the blocks is preserved. Further, as the elements introduced in B_1 and B_k are unchanged, the pairs covered by these blocks are unchanged; and pairs containing the new element $2k$ are covered in the 3 new blocks, thus all pairs are covered. Finally, no pair is in every block. Thus , from Lemma 2.1, this is a scccd$(2k, k, 2k+2)$, which is economical; it is tight only when $2k+2 = \dfrac{2k(2k-1)}{2(k-1)}$, i.e., when $k = 2$.

Theorem 5.6.
For $k \geq 2$ and $v = 2k$ the blocks $\mathcal{B} = \{B_1, B_2, \ldots, B_{k-1}, C, B_k, \ldots, B_{2k-1}, C', C''\}$ form an economical scccd$(2k, k, 2k+2)$, which is tight when $k = 2$.

■

See Fig. 5(b) for an economical scccd$(8, 4, 10)$ constructed using Theorem 5.6.

B_1	1,	2,	3,	…,	k
B_2	2,	3,	4,	…,	$k+1$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
B_{k-1}	$k-1$,	k,	$k+1$,	…,	$2k-2$
C	$2k$,	k,	$k+1$,	…,	$2k-2$
B_k	k,	$k+1$,	$k+2$,	…,	$2k-1$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
B_{2k-1}	$2k-1$,	1,	2,	…,	$k-1$
C'	$2k-1$,	$2k$,	2,	…,	$k-1$
C''	$2k$,	1,	2,	…,	$k-1$

scccd$(2k, k, 2k+2)$

(a)

B_1	1 2 3 4
B_2	2 3 4 5
B_3	3 4 5 6
C	8 4 5 6
B_4	4 5 6 7
B_5	5 6 7 1
B_6	6 7 1 2
B_7	7 1 2 3
C'	7 8 2 3
C''	8 1 2 3

scccd$(8, 4, 10)$

(b)

Fig. 5: The economical scccd$(2k, k, 2k+2)$ of Theorem 5.6 (tight only for $k = 2$), this design for $k = 4$.

The theorem below summarizes this section, cf., Corollary 5.1 of [4].

Theorem 5.7. For $k \geq 3$ we have

$$b_*(v,k) = \begin{cases} v-1, & \text{for } k+1 \leq v \leq 2k-2; \\ v, & \text{for } v = 2k-1; \\ v+2, & \text{for } v = 2k. \end{cases}$$

■

So, in this section, we have constructed scccds with a minimal number of blocks for all $k \geq 3$ and $k+1 \leq v \leq 2k$, and have given examples for $k = 4$.

From now on we concentrate mainly on tight designs.

6. Some families of tight designs

Tight scccds are of special interest; they are analogous to *tight* single-change covering designs, see [2].

So far we have three infinite families of tight scccds with a fixed k, see Theorem 3.1:

(i) scccd$(v, 2)$ for all $v \geq 3$,

(ii) scccd$(v, 3)$ for all $v \equiv 0 \pmod 4$, $v \geq 4$, and

(iii) scccd$(v, 3)$ for all $v \equiv 1 \pmod 4$, $v \geq 5$.

For the tight designs of [2] infinite families are not known to exist with k variable, except in the case $v = k$, here we have two infinite families with k variable:

(iv) $\mathfrak{F}= \{\mathcal{X}_k : \mathcal{X}_k$ is the tight scccd$(2k-2, k, 2k-3)$ of Theorem 5.4, $k \geq 3\}$;

(v) $\mathfrak{G}= \{\mathcal{Y}_k : \mathcal{Y}_k$ is the tight scccd$(2k-1, k, 2k-1)$ of Theorem 5.5, $k \geq 2\}$.

(Note that $\mathcal{Y}_3$ is shown in Fig. 1(b).)

If a scccd has the same parameters as a member of $\mathfrak{F}$ or $\mathfrak{G}$ then it is isomorphic to that member:

Theorem 6.1. For a fixed $k \geq 3$ let $\mathcal{X}$ be a scccd$(2k-2, k, 2k-3)$.
Then $\mathcal{X} \cong \mathcal{X}_k \in \mathfrak{F}$.

Proof. The parameters of $\mathcal{X}$ indicate that it is tight. Let $\mathcal{X}$ be based on $[v]$ where $v = 2k-2$.

Now, $\mathcal{X}$ has $v-1$ blocks, so, from the comments preceding Theorem 5.3, it contains some element, say v, in every block. Any other element $1, \ldots, v-1$ is introduced exactly once and remains in $k-1$ successive blocks because it must appear in $v-1 = 2k-3$ covered pairs; it appears in $k-1$ covered pairs in its first block and in one covered pair in each of its $k-2$ successive blocks.

Hence, up to a permutation of $[v]$, we may construct $\mathcal{X}$ as follows: first, put v in every block; then, for $i = 1, \ldots, v-1$, introduce i in B_i, and leave it there for $k-1$ successive blocks. Then $B_{k-1} = (1, 2, \ldots, k-1, v)$, and a cyclic shift of the blocks making this the first block will produce $\mathcal{X}_k$. Hence $\mathcal{X} \cong \mathcal{X}_k$.

■

Similarly for the family $\mathfrak{G}$.

Theorem 6.2. For a fixed $k \geq 2$ let $\mathcal{Y}$ be a scccd$(2k-1, k, 2k-1)$.
Then $\mathcal{Y} \cong \mathcal{Y}_k \in \mathfrak{G}$.

■

7. The numbers $\mathbf{t_j}$ and $\mathbf{f_j}$ for a tight design

Now we consider constructions of tight scccds for $v > 2k$. First we need some preparatory material, much of which is similar to that of §4 in [2].

In an arbitrary tight scccd(v, k) let $T_j \subseteq [v]$ denote the set of elements which are introduced j times, $j \geq 0$, and let $t_j = |T_j|$.

Now consider t_0, the number of elements not introduced. From the proof of Lemma 5.1 we must have $t_0 = 0$ or 1. Let $\mathcal{Z}$ be a tight scccd(v, k) in which $t_0 = 1$, and let v be the element not introduced, so v is in every block. Any other element $z = 1, \ldots, v-1$ is introduced exactly once, for, if some z is introduced twice or more, then the pair $\{z, v\}$ is covered twice or more; a contradiction because $\mathcal{Z}$ is tight. Hence, each of $1, \ldots, v-1$ is introduced exactly once, and $b = v - 1 = \dfrac{v(v-1)}{2(k-1)}$, i.e., $v = 2k-2$. Thus $\mathcal{Z}$ is a tight scccd$(2k-2, k, 2k-3)$ and, by Theorem 6.1, lies in $\mathfrak{F}$.

Thus, all tight scccds with $t_0 = 1$ are known; they are members of $\mathfrak{F}$ with $v = 2k-2$. As we are interested in tight designs with $v > 2k$, we assume that $t_0 = 0$ and consider only t_j for which $j \geq 1$.

Let $\mathcal{C}$ be an arbitrary tight scccd(v, k, b), and, for any $x \in [v]$, let $f_{\{x\}}$ denote the number of blocks that contain x.

Let $x \in T_j$. Each time x is introduced $k-1$ pairs containing x are covered. There are $v-1$ pairs containing x to be covered, so we may let $j \leq \left\lfloor \dfrac{v-1}{k-1} \right\rfloor = A$ because $t_j = 0$ for $j > A$.

Now $x \in T_j$, so there are j blocks in which x is introduced; $(k-1)$ pairs containing x are covered in each of these blocks. There are $f_{\{x\}} - j$ blocks that contain x but in which it is not introduced; only 1 pair containing x is covered in each of these blocks. This gives $v - 1 = j(k-1) + (f_{\{x\}} - j)1$, i.e., $f_{\{x\}} = (v-1) - j(k-2)$. Hence $f_{\{x\}}$ is constant on T_j, so we let $f_j = (v-1) - j(k-2)$ be the number of blocks that contain any fixed element from T_j. In particular, $f_1 = v - k + 1$.

We have, for $1 \leq j \leq A$,

$$A = \left\lfloor \frac{v-1}{k-1} \right\rfloor, \qquad \sum_{j=1}^{A} t_j = v, \qquad \sum_{j=1}^{A} j t_j = b, \qquad f_j = (v-1) - j(k-2). \tag{2}$$

Some further properties of the numbers t_j and f_j are given below.

Lemma 7.1. For $k > 2$ and any j for which $1 \leq j \leq A$, we have

(i) $f_A < f_{A-1} < \cdots < f_2 < f_1$;
(ii) $f_j \geq j$, and, if $f_j < j$, then $t_\ell = 0$ for all $\ell \geq j$;
(iii) if $f_j = j$, then $t_j = 0$ or 1;
(iv) if $f_j = j+1$, then $t_j = 0$, 1, or 2.

Proof.

(i) Clear because $f_j = (v-1) - j(k-2)$ and $k > 2$.

(ii) For each of the j times when $x \in T_j$ is introduced it appears in at least 1 block, so $f_j \geq j$. So, clearly, if $f_j < j$ then $t_j = 0$. Also, for any $\ell > j$, we have $f_\ell < f_j < j < \ell$ by (i), hence $t_\ell = 0$.

(iii) For a fixed j, suppose $f_j = j$ but $t_j = |T_j| \geq 2$, and let x and $y \in T_j$. Now, because x is introduced j times and appears in j blocks, each time it is introduced it must be immediately removed; similarly for y. But pair $\{x, y\}$ must appear in some block, hence both x and y must be introduced in this block, a contradiction.

(iv) Now suppose $f_j = j + 1$ but $t_j \geq 3$, and let x, y, and $z \in T_j$. By the pigeonhole principle for one of the j times when x is introduced it must stay for 2 successive blocks; similarly for y and z. So the configurations $\begin{smallmatrix} x \\ x \end{smallmatrix}$, $\begin{smallmatrix} y \\ y \end{smallmatrix}$, and $\begin{smallmatrix} z \\ z \end{smallmatrix}$ occur once each. In order to cover the pairs $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ we must have the arrangement $\begin{smallmatrix} xz\cdot \\ xy\cdot \\ zy\cdot \end{smallmatrix}$ in the design, i.e., $b = 3$. But there are only 2 tight scccds with $b = 3$: one is $\mathcal{X}_3 \in \mathfrak{F}$, which we have excluded; the other is $\mathcal{Y}_2 \in \mathfrak{G}$, which is also excluded because this design has $k = 2$ and we are restricted to $k > 2$.

■

8. Start-Finish arrays for a tight design; Criteria for a tight design with v > 2k

This section is mainly concerned with the subset T_1 of elements introduced exactly once in $\mathcal{C}$, an arbitrary tight scccd(v, k).

Suppose $T_1 \neq \emptyset$ and let $x \in T_1$, and consider the $f_{\{x\}} = f_1 = v - k + 1$ successive blocks in $\mathcal{C}$ which contain x; call these blocks $\mathcal{B}_x = \{B_{x,1}, \ldots, B_{x,f_1}\}$, see Fig. 6(a). We may write x as the leftmost element in each of these blocks. We say that x *starts*, S, in $B_{x,1}$ (i.e., is introduced there), and *finishes*, F, in B_{x,f_1}, see Fig. 6(a). Call this occurrence of x in f_1 successive blocks the *run* containing x. Now let y be some other element in T_1. The pair $\{x, y\}$ must be covered in $\mathcal{B}_x$ and so, because $f_{\{y\}} = f_1$ also, either the S or the F of y must appear in $\mathcal{B}_x$. Similarly for all the other elements in T_1.

Thus, as we run through the elements in T_1, each adds its S or its F to the array in the final column of Fig. 6(a). We call this the *Start-Finish array*, or SF-array, for element x, and denote it by SF_x. Call the Ss and Fs *symbols*.

In Fig. 6(a), block $B_{x,i}$ gives rise to the i-th *row*, $R_{x,i}$, of SF_x, which contains x and, perhaps, some symbols; to illustrate this we write $B_{x,i} \to R_{x,i}$. If a row contains no symbols it is *empty* $(-)$.

$B_{x,1}$	$\rightarrow R_{x,1}$	$x\,S$
$B_{x,2}$	$\rightarrow R_{x,2}$	x
$\cdot$	$\cdot \quad \cdot$	$\cdot$
$\cdot$	$\cdot \quad \cdot$	$\cdot$
$\cdot$	$\cdot \quad \cdot$	$\cdot$
B_{x,f_1-1}	$\rightarrow R_{x,f_1-1}$	x
B_{x,f_1}	$\rightarrow R_{x,f_1}$	$x\,F$
		SF_x

(a)

1 2				
3 2	$R_{3,1}$	$3\,S$	$R_{4,1}$	$4\,S\,F$
3 1	$R_{3,2}$	$3\,-$	$R_{4,2}$	$4\,-$
3 4	$R_{3,3}$	$3\,F\,S$	$R_{4,3}$	$4\,F$
2 4				
1 4				
		SF_3		SF_4

a tight scccd$(4,2,6)$
$T_1 = \{3,4\}$
(b)

1 2 3		
2 3 4	$R_{x,1}$	$x\,S\,F$
3 4 5	$R_{x,2}$	$x\,S\,F$
4 5 1	$R_{x,3}$	$x\,F\,S$
5 1 2		
		SF_x

$\mathcal{Y}_3$
$x \in T_1 = \{1,2,3,4,5\}$
(c)

Fig. 6: SF-arrays and examples.

Fig. 6(b) shows a tight scccd$(4,2,6)$ with elements 3 and $4 \in T_1$. The SF-array for element 3, SF_3, is shown first; here the S in $R_{3,3}$ appears because element 4 starts in $B_{3,3}$. The array SF_4 is shown next; here $F \in R_{4,1}$ because 3 finishes in $B_{4,1}$. Fig. 6(c) shows the tight scccd$(5,3,5)$, $\mathcal{Y}_3$; it has $T_1 = \{1,2,3,4,5\}$ and, for all $x \in T_1$, the arrays SF_x are identical.

The main idea of this section is to place restrictions on the structure of an SF-array of a tight scccd(v,k) when $v > 2k$. In the following section we 'extend' these SF-arrays to tight scccds for $(v,k) = (9,4)$ and $(10,4)$.

So, let us assume that a tight scccd(v,k,b), $\mathcal{C}$, exists whose set of elements introduced exactly once is T_1, and let $x \in T_1$. Then, using the following Observations (1)-(10), we will establish Criteria (1)-(10) that SF_x must satisfy. In the Observations, R denotes an arbitrary row of SF_x, with corresponding block in $\mathcal{B}_x$ denoted by B_R.

Observations

(1) Each row R of SF_x contains at most one S and at most one F. For suppose R contains two or more Ss say, then two or more elements are introduced in B_R, a contradiction. Similarly for the Fs because the reverse of a scccd is again a scccd. Clearly the order of the symbols in a row does not matter.

(2) The number of empty rows between any row containing F and the next (different) row containing S as we go down SF_x is $\geq b - 2f_1 + 1$. To see this let $y \in T_1$ finish in any row of SF_x and let $z \in T_1$ start in a later one, with α empty rows between them. Now, the pair $\{y,z\}$ is not covered in $\mathcal{B}_x$, so the runs containing y and z must meet outside $\mathcal{B}_x$.

That is, $\alpha + f_{\{y\}} + f_{\{z\}} \geq b+1$, so $\alpha \geq b - 2f_1 + 1$.

Furthermore, let $FS(v,k) = b - 2f_1 + 1$. Now $FS(v,k) \geq 0$ with equality if and only if $v = 2k-2$ or $2k-1$. Now, because we are interested only in $v > 2k$, we may assume that $FS(v,k) \geq 1$, i.e., that there is always at least 1 empty row between a F and the next S.

(3) Suppose the three elements x, y, and $z \in T_1$ (where, without loss of generality, the first is the x of our SF_x) are introduced in three successive blocks in $\mathcal{C}$. See Fig. 7(a) where y starts in $B_{x,2}$ changing p, and z starts in $B_{x,3}$ changing q; see (b) for the SF-array so formed. To cover the pairs $\{p,y\}$ and $\{q,z\}$ the elements p and q must occur in the two blocks immediately succeeding B_{x,f_1}, as shown in (c). Hence, pair $\{p,q\}$ must be in all remaining blocks outside $\mathcal{B}_x$, for, if not, it will be covered more than once. So both p and $q \in T_1$, which forces $b = 2f_1 - 1$, i.e., $v = 2k-2$ or $2k-1$. So, with our restriction of $v > 2k$, we may assume that the configurations $\begin{smallmatrix}S\\S\\S\end{smallmatrix}$ and $\begin{smallmatrix}F\\F\\F\end{smallmatrix}$ do not appear (in $\mathcal{C}$ and so) in SF_x. (Such a triple $\{x,y,z\}$ is called a *persistent triple*, it persists through $v-k-1$ blocks, see Phillips and Wallis [1]. Here we have shown that if a tight scccd contains a persistent triple then it must belong to one of the families $\mathfrak{F}$ or $\mathfrak{G}$; see the constructions in §5 and in the proof of Theorem 6.1; all designs in both these families contain persistent triples.)

(a)		(b)		(c)	(d)	
	$\cdot\ \cdot\ \cdot\ \cdot$			$\cdot\ \cdot\ \cdot\ \cdot$		
	$\cdot\ \cdot\ \cdot\ \cdot$			$\cdot\ \cdot\ \cdot\ \cdot$		
$B_{x,1}$	$x\,p\,q\ \cdot$	$R_{x,1}$	$x\,S$	$x\,p\,q\ \cdot$	$R_{x,1}$	$x\,S$
$B_{x,2}$	$x\,y\,q\ \cdot$	$R_{x,2}$	$x\,S$	$x\,y\,q\ \cdot$	$R_{x,2}$	$x\,S$
$B_{x,3}$	$x\,y\,z\ \cdot$	$R_{x,3}$	$x\,S$	$x\,y\,z\ \cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot$	$\cdot$	$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot$	$\cdot$	$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot$	$\cdot$	$\cdot$	$\cdot\ \cdot\ \cdot\ \cdot$	R_{x,f_1-1}	$x\,F$
B_{x,f_1}	$x\,y\,z\ \cdot$	R_{x,f_1}	$x\,F$	$x\,y\,z\ \cdot$	R_{x,f_1}	$x\,F$
	$\cdot\ y\,z\ \cdot$			$p\,y\,z\ \cdot$		
	$\cdot\ \cdot\ z\ \cdot$			$p\,q\,z\ \cdot$		
	$\cdot\ \cdot\ \cdot\ \cdot$			$\cdot\ \cdot\ \cdot\ \cdot$		
	$\cdot\ \cdot\ \cdot\ \cdot$			$\cdot\ \cdot\ \cdot\ \cdot$		

Fig. 7: Persistent triples and their corresponding forbidden configurations.

Similarly, the configuration shown in Fig. 7(d) cannot occur in SF_x; for, if it did, then the element which starts in $B_{x,2}$ must finish in the block succeeding B_{x,f_1}, thus producing the forbidden configuration $\begin{smallmatrix}F\\F\\F\end{smallmatrix}$ (in $\mathcal{C}$).

(4) The S of x lies in $R_{x,1}$ and the F in R_{x,f_1}; and then each of the remaining $t_1 - 1$ elements in T_1 have either their S or their F present in SF_x, (but not both, for, if element y, say, has both its S and its F present, then its S must appear after its F, and so $\{x, y\}$ is covered twice). This gives a total of $t_1 + 1$ symbols.

(5) Consider R, an arbitrary row of SF_x; for each S in or above R there will be an element from T_1 in B_R; similarly for each F in or below R, except that the S and F of x contribute only one element (x itself) to B_R. Now $\mathcal{C}$ has block size k, so we must have:

$$\left\{\begin{matrix}\text{the number of } S\text{s}\\ \text{in or above } R\end{matrix}\right\} + \left\{\begin{matrix}\text{the number of } F\text{s}\\ \text{in or below } R\end{matrix}\right\} - 1 \leq k.$$

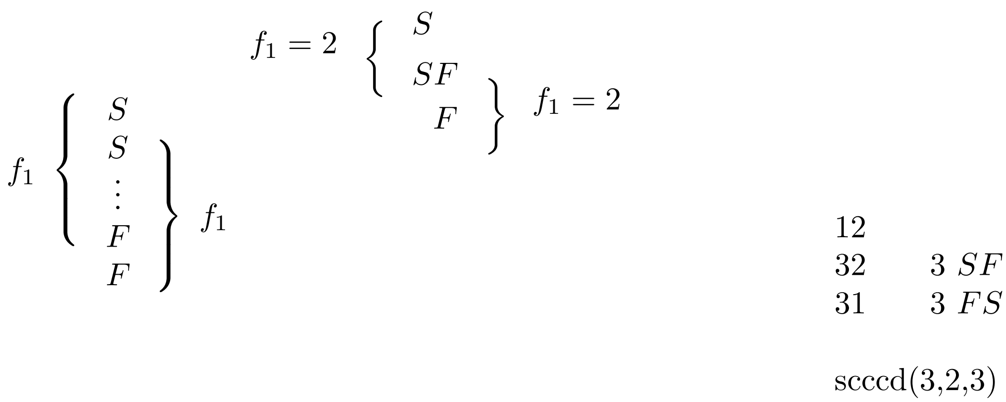
Call the left-hand side of the above equation the *weight* of R, $wt(R)$; it is the number of elements from T_1 in B_R.

(6) Let R be the last row of SF_x; then $wt(R)$ is the number of Ss in SF_x, which is $\leq k$ by Observation (5). Similarly, the number of Fs in SF_x is $\leq k$. Also, using Observation (4), the total number of symbols, $t_1 + 1$, is $\leq 2k$. Thus, $t_1 \leq 2k - 1$. There is a tight scccd(4, 2, 6) with $(t_1, t_2, t_3) = (3, 0, 1)$ for which this inequality is sharp; it is also sharp for any design in the family $\mathfrak{G}$. (Cf. §4 of [2], where $t_1 \leq k$.)

(7) Suppose that two adjacent rows in SF_x each have weight k, then, the corresponding adjacent blocks in $\mathcal{C}$ each contain only elements from T_1. Let the single-change between these two blocks be caused by y finishing in the first block and z starting in the next. Then, in SF_x, there are no empty rows between the row containing the F of y and the row containing the S of z, a contradiction to Observation (2). Hence, two rows of weight k cannot be adjacent.

(8) The configuration $\begin{smallmatrix}S\\S\\S\end{smallmatrix}$ does not occur in SF_x so the number of configurations $\begin{smallmatrix}S\\S\end{smallmatrix}$ in SF_x is $\leq \lfloor k/2 \rfloor$, otherwise the first inequality of Observation (6) is violated. Similarly for the configuration $\begin{smallmatrix}F\\F\end{smallmatrix}$.

(9) A *persistent pair* in $\mathcal{C}$, see [1] and §4 of [2], is a pair of elements from T_1 which start in successive blocks; thus they persist together through $v - k$ blocks. Each persistent pair has a configuration $\begin{smallmatrix}S\\S\end{smallmatrix}$ and $\begin{smallmatrix}F\\F\end{smallmatrix}$. We claim that our SF_x contains exactly one of the configurations $\begin{smallmatrix}S\\S\end{smallmatrix}$ or $\begin{smallmatrix}F\\F\end{smallmatrix}$ for each persistent pair of $\mathcal{C}$ (except if $\mathcal{C} \in \mathfrak{F}$ or $\mathfrak{G}$).



(a) (b)

Fig. 8: Configurations corresponding to a persistent pair.

For any persistent pair $\{y, z\}$ of $\mathcal{C}$ its configurations $\genfrac{}{}{0pt}{}{S}{S}$ and $\genfrac{}{}{0pt}{}{F}{F}$ can be arranged in one of the two ways shown in Fig. 8(a), where the upper S belongs to y. Our claim is clearly true if $x = y$ or z, so assume $x \neq y$, z. Now, the f_1 rows of SF_x must include either the S or the F of y, and either the S or the F of z. That is, they must include either the upper S or the upper F in (a), and either the lower S or the lower F, not both in each case. If we choose our f_1 rows of SF_x such that this is true and that neither the whole of the $\genfrac{}{}{0pt}{}{S}{S}$ nor the whole of the $\genfrac{}{}{0pt}{}{F}{F}$ is included, then, without loss of generality, the first row must be the row containing the lower F and, as it proceeds downwards and cycles around, its last row must be the row containing the upper S. Hence, $b = 2f_1 - 1$, i.e., $v = 2k - 2$ or $2k - 1$, so $\mathcal{C} \in \mathfrak{F}$ or $\mathfrak{G}$, as in Observation (3). Thus, because $v > 2k$, we may assume that, for any persistent pair of $\mathcal{C}$, our SF_x contains either the whole of the pair's $\genfrac{}{}{0pt}{}{S}{S}$ or the whole of the pair's $\genfrac{}{}{0pt}{}{F}{F}$, but not both.

Now, because each persistent pair in $\mathcal{C}$ contains 2 elements from T_1, and different persistent pairs contain distinct elements, and $t_1 \leq 2k-1$ from Observation (6), we see that the total number of appearances of $\genfrac{}{}{0pt}{}{S}{S}$ and $\genfrac{}{}{0pt}{}{F}{F}$ in SF_x is $\leq \lfloor t_1/2 \rfloor \leq \lfloor (2k-1)/2 \rfloor = k - 1$. (For even k this upper bound is 1 smaller than the upper bound of $2\lfloor k/2 \rfloor$ obtained by adding the upper bounds for the number of appearances of $\genfrac{}{}{0pt}{}{S}{S}$ and $\genfrac{}{}{0pt}{}{F}{F}$ in Observation (8); for odd k they are the same.)

(10) From Observation (4) SF_x contains a total of $t_1 + 1$ symbols, of which at least 1 is F. Hence, the number of appearances of S is $\leq t_1$. So, via Observation (6), the number of appearances of S is $\leq min\{t_1, k\}$. Hence, the number of appearances of F is

$\geq t_1 + 1 - min\{t_1, k\}$; similarly for the number of appearances of S. So, finally, we have: $t_1 + 1 - min\{t_1, k\} \leq |S| \leq min\{t_1, k\}$, and similarly for $|F|$.

So, to summarize our 10 Observations, let $\mathcal{C}$ be an arbitrary tight scccd(v, k) with $v > 2k$, and let $x \in T_1$ and let R be an arbitrary row in SF_x. Then, corresponding to the 10 Observations above, SF_x must satisfy Criteria (1)–(10) below, where $|C|$ denotes the number of appearances of configuration C.

Criteria

(1) R contains at most one S and at most one F.

(2) Between any F and the next S there are $\geq FS(v, k) = b - 2f_1 + 1 \geq 1$ empty rows.

(3) The configurations $\begin{smallmatrix} S \\ S \\ S \end{smallmatrix}$, $\begin{smallmatrix} F \\ F \\ F \end{smallmatrix}$, and the configuration of Fig. 7(d) do not appear.

(4) $S \in R_{x,1}$, $F \in R_{x,f_1}$, $|S| + |F| = t_1 + 1$.

(5) $wt(R) \leq k$.

(6) $1 \leq t_1 \leq 2k - 1$.

(7) Two rows of $wt(k)$ cannot be adjacent.

(8) $\left|\begin{smallmatrix} S \\ S \end{smallmatrix}\right| \leq \lfloor k/2 \rfloor$, $\left|\begin{smallmatrix} F \\ F \end{smallmatrix}\right| \leq \lfloor k/2 \rfloor$.

(9) $\left|\begin{smallmatrix} S \\ S \end{smallmatrix}\right| + \left|\begin{smallmatrix} F \\ F \end{smallmatrix}\right| \leq k - 1$.

(10) $t_1 + 1 - min\{t_1, k\} \leq |S|, |F| \leq min\{t_1, k\}$.

Finally, some comments relevant to Observation (9).

To see that our claim fails in a design from $\mathfrak{F}$ or $\mathfrak{G}$, consider the tight scccd$(3, 2, 3)$ $\mathcal{Y}_2 \in \mathfrak{G}$ shown in Fig. 8(b). (This corresponds to the second arrangement in (a) where $f_1 = 2$, so $b = 3$.) All 3 pairs $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are persistent pairs; however $k - 1 = 1$. The array SF_3 contains the $\begin{smallmatrix} S \\ S \end{smallmatrix}$ of persistent pair $\{1, 3\}$ and the $\begin{smallmatrix} F \\ F \end{smallmatrix}$ of persistent pair $\{2, 3\}$, but neither the $\begin{smallmatrix} S \\ S \end{smallmatrix}$ nor the $\begin{smallmatrix} F \\ F \end{smallmatrix}$ of persistent pair $\{1, 2\}$. Similarly for all other designs in $\mathfrak{F}$ or $\mathfrak{G}$, where the number of persistent pairs is equal to the number of blocks.

When $v > 2k$, because SF_x must contain either the $\begin{smallmatrix} S \\ S \end{smallmatrix}$ or the $\begin{smallmatrix} F \\ F \end{smallmatrix}$ of every persistent pair in $\mathcal{C}$, the total number of persistent pairs in $\mathcal{C}$ equals the total number of appearances of $\begin{smallmatrix} S \\ S \end{smallmatrix}$s and $\begin{smallmatrix} F \\ F \end{smallmatrix}$s in SF_x, which is $\leq k - 1$ (by Criterion (9)); this upper bound is sharp for the tight scccd$(9, 4, 12)$, $\mathcal{U}_2$, which contains 3 persistent pairs, see §9. (Cf. §4 of [2], where the number of persistent pairs is $\leq k/2$.)

9. Constructions of tight scccd(9,4,12)s and scccd(10,4,15)s using SF-arrays

We now illustrate the method of constructing tight designs with $v > 2k$ using SF-arrays.

(v,k)=(9,4) First we construct all non-isomorphic tight scccd$(9,4,12)$s. To start, we must find all SF-arrays for $(v,k) = (9,4)$ that satisfy Criteria (1)–(10) of §8.

Equation (2) of §7 gives $A = 2$, $t_1 + t_2 = 9$, and $t_1 + 2t_2 = 12$; hence $t_1 = 6$ and $t_2 = 3$. We also have $f_1 = 6$ and $f_2 = 4$. From Criterion (2), $FS(9,4) = 1$, i.e., there must be at least 1 empty row between any F and the next S in our SF-arrays. Let $T_1 = \{1,2,3,4,5,6\}$ and $T_2 = \{7,8,9\}$, and let $x \in T_1$.

There are exactly 8 SF-arrays that satisfy Criteria (1)–(10). Of these, 6 are shown in Fig. 9(a)–(f), and the remaining 2 in (a) and (b) of Fig. 12.

	(a)	(b)	(c)	(d)	(e)	(f)
$R_{x,1}$	$x\,S\,F$	$x\,S\,F$	$x\,S\,F$	$x\,S$	$x\,S\,F$	$x\,S\,F$
$R_{x,2}$	$x\,F$	$x\,-$	$x\,-$	$x\,S\,F$	$x\,-$	$x\,-$
$R_{x,3}$	$x\,-$	$x\,S\,F$	$x\,S$	$x\,-$	$x\,S\,F$	$x\,S\,F$
$R_{x,4}$	$x\,S\,F$	$x\,-$	$x\,S\,F$	$x\,S\,F$	$x\,-$	$x\,F$
$R_{x,5}$	$x\,-$	$x\,S$	$x\,-$	$x\,-$	$x\,S\,F$	$x\,-$
$R_{x,6}$	$x\,F\,S$	$x\,F\,S$	$x\,F\,S$	$x\,F\,S$	$x\,F$	$x\,F\,S$

Fig. 9: 6 of the 8 SF-arrays which satisfy Criteria (1)–(10) when $(v,k) = (9,4)$. These 6 form an equivalence class of SF-arrays.

In an arbitrary scccd $\mathcal{C}$ each of the t_1 elements of T_1 has 2 symbols, a S and a F; this gives a total of $2t_1$ symbols, of which $t_1 + 1$ appear in SF_x. We now consider the $t_1 - 1$ 'missing' symbols.

See Fig. 10(a) which contains the SF_x of Fig. 9(a). Let us enlarge this SF_x from $f_1 = 6$ rows to $b = 12$ rows by including the $t_1 - 1 = 5$ missing symbols and dropping the xs, see Fig. 10(b). For example, the element which starts in row $R_{x,4}$ of (a), i.e., in R_4 of (b), must finish $f_1 = 6$ rows later in R_9, hence, the F belonging to this element lies in R_9, as shown. Call this new array with b rows a SF-*skeleton*; note that the SF-skeleton (b) is uniquely determined from the SF_x in (a).

Now, in Fig. 10(b), let the $S \in R_1$ correspond to element 1, the $S \in R_4$ correspond to 2, the $S \in R_6$ to 3, the $S \in R_8$ to 4, the $S \in R_9$ to 5, and, finally, the $S \in R_{11}$ to element 6.

In the SF-skeleton of Fig. 10(b) the SF-array SF_1 (i.e., Fig. 9(a) with $x = 1$) appears as rows R_1–R_6; also, SF_2 (Fig. 9(b) with $x = 2$) appears as rows R_4–R_9; SF_3 as rows R_6–R_{11}; SF_4 as rows R_8–R_1; SF_5 as rows R_9–R_2; and, finally, SF_6 (Fig. 9(f) with $x = 6$) appears as rows R_{11}–R_4. Thus all 6 of the SF-arrays in Fig. 9 occur in the SF-skeleton of Fig. 10(b). We say that these 6 SF-arrays are *equivalent* ($\sim$) to one another.

In order to begin extending Fig. 10(b) to a tight scccd$(9, 4)$, consider Fig. 10(c), which is a potential tight scccd$(9, 4)$ with all elements from $T_1 = \{1, 2, 3, 4, 5, 6\}$ present.

$R_{x,1}$	$x\,S\,F$		R_1	$S\,F$	B_1	1 4 6 5	R_1	$-$
$R_{x,2}$	$x\,F$		R_2	F	B_2	1 $*$ 6 5	R_2	$S\,F$
$R_{x,3}$	$x\,-$	$\longrightarrow$	R_3	$-$	B_3	1 $*$ 6 $*$	R_3	$-$
$R_{x,4}$	$x\,S\,F$		R_4	$S\,F$	B_4	1 2 6 $*$	R_4	$S\,F$
$R_{x,5}$	$x\,-$		R_5	$-$	B_5	1 2 $*$ $*$	R_5	F
$R_{x,6}$	$x\,F\,S$		R_6	$S\,F$	B_6	1 2 3 $*$	R_6	$-$
			R_7	$-$	B_7	$*$ 2 3 $*$	R_7	$S\,F$
			R_8	S	B_8	4 2 3 $*$	R_8	$-$
			R_9	$S\,F$	B_9	4 2 3 5	R_9	$S\,F$
			R_{10}	$-$	B_{10}	4 $*$ 3 5	R_{10}	$-$
			R_{11}	$S\,F$	B_{11}	4 6 3 5	R_{11}	S
			R_{12}	$-$	B_{12}	4 6 $*$ 5	R_{12}	$S\,F$
SF_x (a)			SF-skeleton of SF_x (b)		(c)		reverse of (b) (d)	

Fig. 10: An SF-array, its SF-skeleton, and the corresponding potential tight scccd$(9, 4)$; the reverse of the SF-skeleton of SF_x.

We must now add on the elements in $T_2 = \{7, 8, 9\}$; each is introduced twice, and appears in $f_2 = 4$ blocks.

See Fig. 11(a). Without loss of generality start element 7 in B_2, then $7 \in B_3$; for, if not, then both 7 and 5 finish in B_2, a contradiction. Without loss of generality start 8 in B_3; this produces (a).

See Fig. 11(b). Now either $7 \in B_4$ or $8 \in B_4$. If $7 \in B_4$ then (because $7 \in T_2$ and $f_2 = 4$) 7 must start once more in a block containing elements 3, 4, and 9 because pairs $\{7, 3\}$, $\{7, 4\}$, and $\{7, 9\}$ will not have been covered, but this is impossible; hence $8 \in B_4$. Further, we have (b) by similar reasoning to the above.

See Fig. 11(c). Now if $8 \in B_7$, then pair $\{8,9\}$ will be covered twice; so $7 \in B_7$ and 7 must occur for 2 successive blocks because $f_2 = 4$ and it has already occurred in 2 blocks. To finish we must have $8 \in B_{10}$ and $9 \in B_{12}$, producing Fig. 11(c).

Let $\mathcal{U}_1$ denote this tight scccd$(9,4,12)$; clearly, by its construction, it is unique up to labelling.

	(a)		(b)		(c)
B_1	1 4 6 5	B_1	1 4 6 5	B_1	1 4 6 5
B_2	1 **7** 6 5	B_2	1 7 6 5	B_2	1 7 6 5
B_3	1 **7** 6 **8**	B_3	1 7 6 8	B_3	1 7 6 8
B_4	1 2 6 $*$	B_4	1 2 6 **8**	B_4	1 2 6 8
B_5	1 2 $*$ $*$	B_5	1 2 **9** **8**	B_5	1 2 9 8
B_6	1 2 3 $*$	B_6	1 2 3 **9**	B_6	1 2 3 9
B_7	$*$ 2 3 $*$	B_7	$*$ 2 3 **9**	B_7	**7** 2 3 9
B_8	4 2 3 $*$	B_8	4 2 3 $*$	B_8	4 2 3 **7**
B_9	4 2 3 5	B_9	4 2 3 5	B_9	4 2 3 5
B_{10}	4 $*$ 3 5	B_{10}	4 $*$ 3 5	B_{10}	4 **8** 3 5
B_{11}	4 6 3 5	B_{11}	4 6 3 5	B_{11}	4 6 3 5
B_{12}	4 6 $*$ 5	B_{12}	4 6 $*$ 5	B_{12}	4 6 **9** 5

$\mathcal{U}_1$, scccd$(9,4,12)$ (c)

Fig. 11: Extending a potential tight scccd$(9,4)$ to a tight scccd$(9,4)$.

In general, for a given (v,k) with $v > 2k$, let $\mathcal{S}$ denote the set of SF-arrays that satisfy Criteria (1)–(10) and let SF_x and SF'_x be two arbitrary SF-arrays in $\mathcal{S}$. We define an equivalence relation $\sim$ on $\mathcal{S}$ as follows:

$$SF_x \sim SF'_x \quad \text{if and only if} \quad \begin{array}{c} SF_x \text{ appears as consecutive rows} \\ \text{in the } SF\text{-skeleton determined by } SF'_x. \end{array}$$

But, because the skeleton determined by an arbitrary SF-array in $\mathcal{S}$ is unique, we can redefine $\sim$ as:

$$SF_x \sim SF'_x \quad \text{if and only if} \quad \begin{array}{c} \text{the } SF\text{-skeleton determined by } SF_x \\ \text{is a cyclic shift of the } SF\text{-skeleton determined by } SF'_x. \end{array}$$

It is straightforward to prove that $\sim$ is an equivalence relation.

Each equivalence class of SF-arrays gives rise to one SF-skeleton; we say that this SF-skeleton *represents* the class. Instead of attempting to extend all SF-arrays (or, rather, their SF-skeletons) from a particular equivalence class to tight designs, we need only attempt to extend the SF-skeleton that represents this class. (Also, if two SF-skeletons represent different classes, and both can be extended to tight designs, then these designs are non-isomorphic.) However, a single SF-skeleton can be extended to non-isomorphic designs, see Fig. 14.

The remaining 2 SF-arrays for $(v, k) = (9, 4)$ are shown in (a) and (b) of Fig. 12; they form another equivalence class. The SF-skeleton of (b) is (c) which extends uniquely (up to labelling) to the tight scccd$(9, 4, 12)$ shown in (d); call it $\mathcal{U}_2$. The SF-arrays (a) and (b) each occur 3 times each amongst the $t_1 = 6$ SF-arrays of $\mathcal{U}_2$. This tight design contains $k - 1 = 3$ persistent pairs, which, by the comments at the end of §8, is the maximum number allowed in a tight scccd(v, k) with $v > 2k$.

	(a)	(b)			(c) SF-skeleton of (b)		(d) $\mathcal{U}_2$, scccd$(9,4,12)$
$R_{x,1}$	$x\,S\,F$	$x\,S$		R_1	S	B_1	1 9 5 6
$R_{x,2}$	$x\,F$	$x\,S\,F$		R_2	$S\,F$	B_2	1 2 5 6
$R_{x,3}$	$x\,-$	$x\,F$	$\rightarrow$	R_3	F	B_3	1 2 7 6
$R_{x,4}$	$x\,S$	$x\,-$		R_4	$-$	B_4	1 2 7 8
$R_{x,5}$	$x\,S\,F$	$x\,S$		R_5	S	B_5	1 2 3 8
$R_{x,6}$	$x\,F$	$x\,F\,S$		R_6	$S\,F$	B_6	1 2 3 4
				R_7	F	B_7	9 2 3 4
				R_8	$-$	B_8	9 7 3 4
				R_9	S	B_9	5 7 3 4
				R_{10}	$S\,F$	B_{10}	5 6 3 4
				R_{11}	F	B_{11}	5 6 8 4
				R_{12}	$-$	B_{12}	5 6 8 9

Fig. 12: The remaining 2 SF-arrays for $(v, k) = (9, 4)$, which form another equivalence class; the SF-skeleton which represents this class and the corresponding tight scccd$(9, 4)$.

Now, because the SF-skeletons from which $\mathcal{U}_1$ and $\mathcal{U}_2$ were formed represent different equivalence classes, we have $\mathcal{U}_1 \ncong \mathcal{U}_2$. This gives us:

Theorem 9.1. There are 2 non-isomorphic tight scccd$(9, 4, 12)$s, namely $\mathcal{U}_1$ and $\mathcal{U}_2$ shown above.

■

As mentioned at the end of §4, the reverse of a scccd(v,k,b), $\mathcal{B}$, is another scccd(v,k,b), rev$(\mathcal{B})$; and, if a scccd $\mathcal{C}$ is tight then rev$(\mathcal{C})$ is also tight. Hence, from Theorem 9.1, rev$(\mathcal{U}_1) \cong \mathcal{U}_1$ or $\mathcal{U}_2$.

For an arbitrary scccd $\mathcal{C}$, to obtain the SF-skeleton of rev$(\mathcal{C})$ from the SF-skeleton of $\mathcal{C}$ we reverse the order of its rows and switch $S \leftrightarrow F$.

The SF-skeleton of $\mathcal{U}_1$ is shown in Fig. 10(b) and the SF-skeleton of rev$(\mathcal{U}_1)$ in Fig. 10(d); it is a cyclic shift of the SF-skeleton of $\mathcal{U}_1$. Hence, because the extension of the SF-skeleton of $\mathcal{U}_1$ to a tight design is unique up to labelling, we have rev$(\mathcal{U}_1) \cong \mathcal{U}_1$. Thus $\mathcal{U}_1$ and $\mathcal{U}_2$ are *self-reverse*.

(v,k)=(10,4) We now construct all non-isomorphic tight scccd$(10,4,15)$s.

Equation (2) of §7 yields the three solutions: $(t_1,t_2,t_3) = (7,1,2)$, $(6,3,1)$, and $(5,5,0)$. Here $f_3 = 3$, so Lemma 7.1(iii) with $j = 3$ disposes of the first solution. For the remaining two let $\mathcal{V}$ be a tight scccd$(10,4,15)$.

(i) $(t_1,t_2,t_3) = (6,3,1)$. We could use SF-arrays here but, for variety, we prefer the following approach which is justified by the result: there are 2 non-isomorphic tight scccd$(10,4,15)$s, both of which can be constructed by 'expanding' $\mathcal{U}_2$ of Fig. 12(d).

First some definitions, see §7 of [2]. For any scccd(v,k,b), $\mathcal{B} = \{B_1,\ldots,B_b\}$, and for any $i = 1,\ldots,b-1$ let U_i be the subset of $k-1$ elements which survives from B_i to B_{i+1}; we call U_i the *unchanged subset at location* i. Also, let U_b be the subset of $k-1$ elements which survives from B_b to B_1, the unchanged subset at location b.

As before, let $T_1 = \{1,2,3,4,5,6\}$ and $T_2 = \{7,8,9\}$, and so $T_3 = \{10\}$. Now, because $f_3 = 3$, each time element 10 is introduced into a block it is immediately changed. So the arrangement of blocks shown below must occur 3 times, at the pairs of consecutive locations: ℓ_1, ℓ_1+1, and ℓ_2, ℓ_2+1, and ℓ_3, ℓ_3+1. We have shown the arrangement at the pair of locations ℓ, $\ell+1$ for any $\ell \in \{\ell_1,\ell_2,\ell_3\}$.

$$
\begin{array}{llllll}
 & & \ell & & & \\
B_\ell & x_1\ x_2\ x_3\ x_4 & \swarrow & & U_\ell & = \{x_2,x_3,x_4\} \\
B_{\ell+1} & 10\ x_2\ x_3\ x_4 & & & & \\
B_{\ell+2} & x_5\ x_2\ x_3\ x_4 & \nwarrow & & U_{\ell+1} & = \{x_2,x_3,x_4\} \\
 & & \ell+1 & & &
\end{array}
$$

Here $U_\ell = U_{\ell+1}$, i.e., the two unchanged subsets for this arrangement of blocks are equal. Hence, each of the 3 pairs of locations produces an unchanged subset which survives through the pair of locations; and, because $\mathcal{V}$ is tight, these 3 unchanged subsets partition the set $\{1,2,\ldots,9\}$.

Now if we remove the 3 blocks B_{ℓ_1+1}, B_{ℓ_2+1}, and B_{ℓ_3+1} that contain element 10 from $\mathcal{V}$ we obtain a tight scccd$(9,4,12)$ with 3 unchanged subsets which partition $\{1,2,\ldots,9\}$, i.e., with an *expansion set of locations*, see §7 of [2]. Of $\mathcal{U}_1$ and $\mathcal{U}_2$, only $\mathcal{U}_2$ has an expansion set of locations, in fact it has two:

$\{1,2,7\}$ at location 3,		$\{1,2,8\}$ at location 4,
$\{9,3,4\}$ at location 7,	and	$\{7,3,4\}$ at location 8,
$\{5,6,8\}$ at location 11,		$\{5,6,9\}$ at location 12.

Expanding $\mathcal{U}_2$ at the first expansion set above with element 10 gives us the tight scccd$(10,4,15)$, $\mathcal{V}_1$ shown in Fig. 13(a); and, $\mathcal{V}_2$ in (b) comes from using the second expansion set.

B_1	1 9 5 6	S	S_1F_2
B_2	1 2 5 6	$S\,F$	S_1F_1
B_3	1 2 7 6	F	S_2F_1
B_4	1 2 7 **10**	–	S_3F_3
B_5	1 2 7 8	–	S_2F_2
B_6	1 2 3 8	S	S_1F_2
B_7	1 2 3 4	$S\,F$	S_1F_1
B_8	9 2 3 4	F	S_2F_1
B_9	9 **10** 3 4	–	S_3F_3
B_{10}	9 7 3 4	–	S_2F_2
B_{11}	5 7 3 4	S	S_1F_2
B_{12}	5 6 3 4	$S\,F$	S_1F_1
B_{13}	5 6 8 4	F	S_2F_1
B_{14}	5 6 8 **10**	–	S_3F_3
B_{15}	5 6 8 9	–	S_2F_2

$\mathcal{V}_1$, scccd$(10,4,15)$
(a)

B_1	1 9 5 6	S	S_1F_2
B_2	1 2 5 6	$S\,F$	S_1F_1
B_3	1 2 7 6	F	S_2F_1
B_4	1 2 7 8	–	S_2F_2
B_5	1 2 **10** 8	–	S_3F_3
B_6	1 2 3 8	S	S_1F_2
B_7	1 2 3 4	$S\,F$	S_1F_1
B_8	9 2 3 4	F	S_2F_1
B_9	9 7 3 4	–	S_2F_2
B_{10}	**10** 7 3 4	–	S_3F_3
B_{11}	5 7 3 4	S	S_1F_2
B_{12}	5 6 3 4	$S\,F$	S_1F_1
B_{13}	5 6 8 4	F	S_2F_1
B_{14}	5 6 8 9	–	S_2F_2
B_{15}	5 6 **10** 9	–	S_3F_3

$\mathcal{V}_2$, scccd$(10,4,15)$
(b)

Fig. 13: The 2 non-isomorphic tight scccd$(10,4)$s with $(t_1,t_2,t_3)=(6,3,1)$, their SF-skeletons and complete SF-skeletons. Both of these designs come from expanding $\mathcal{U}_2$ of Fig. 12(d).

Now we show that $\mathcal{V}_1$ and $\mathcal{V}_2$ are non-isomorphic even though their SF-skeletons are cyclic shifts of each other.

See Fig. 13. Consider the *complete SF-skeleton* shown to the right of the SF-skeletons. In any block of a scccd one element starts and one finishes. Suppose that an element from T_j starts and that one from $T_{j'}$ finishes, then the corresponding row of the complete SF-skeleton is $S_j\, F_{j'}$. Thus, the complete SF-skeleton includes start-finish information about all elements in $[v]$, not just those in T_1. Clearly, if two designs are isomorphic, then their complete SF-skeletons must be cyclic shifts of one another; this is not so for $\mathcal{V}_1$ and $\mathcal{V}_2$, hence $\mathcal{V}_1 \ncong \mathcal{V}_2$. So there are exactly 2 non-isomorphic tight scccd$(10, 4, 15)$s with $(t_1, t_2, t_3) = (6, 3, 1)$, namely $\mathcal{V}_1$ and $\mathcal{V}_2$.

The reverse of a tight scccd(v, k), $\mathcal{C}$, is another tight scccd(v, k), rev$(\mathcal{C})$. Moreover, for $j = 1, \ldots, A$, we have equality amongst the sets T_j for $\mathcal{C}$ and rev$(\mathcal{C})$, and so equality amongst the numbers t_j for $\mathcal{C}$ and rev$(\mathcal{C})$.

So rev$(\mathcal{V}_1)$ also has $(t_1, t_2, t_3) = (6, 3, 1)$, and thus rev$(\mathcal{V}_1) \cong \mathcal{V}_1$ or $\mathcal{V}_2$. Now if rev$(\mathcal{V}_1) \cong \mathcal{V}_1$, then the complete SF-skeleton of rev$(\mathcal{V}_1)$ must be a cyclic shift of the complete SF-skeleton of $\mathcal{V}_1$, but this is not the case. Hence, rev$(\mathcal{V}_1) \cong \mathcal{V}_2$. So $\mathcal{V}_1$ is not isomorphic to its reverse, similarly for $\mathcal{V}_2$.

For two scccds, $\mathcal{B}$ and $\mathcal{B}'$, we write $\mathcal{B}\, \mathrm{r}\, \mathcal{B}'$ if $\mathcal{B} \ncong \mathcal{B}'$, but rev$(\mathcal{B}) \cong \mathcal{B}'$ (or, equivalently, rev$(\mathcal{B}') \cong \mathcal{B}$). Thus $\mathcal{V}_1\, \mathrm{r}\, \mathcal{V}_2$.

(ii) $(t_1, t_2, t_3) = (5, 5, 0)$. There are 32 SF-arrays that satisfy Criteria (1)–(10), and 8 equivalence classes of SF-arrays, 6 of size 5 and 2 of size 1.

The SF-skeletons of the 8 classes are shown in Fig. 14; underneath each is the number of its extensions to non-isomorphic designs, and the names of the designs.

So there are 10 non-isomorphic tight scccd$(10, 4)$s with $(t_1, t_2, t_3) = (5, 5, 0)$, namely $\mathcal{V}_m$ for $3 \leq m \leq 12$. We also have: $\mathcal{V}_3\, \mathrm{r}\, \mathcal{V}_5$, $\mathcal{V}_4\, \mathrm{r}\, \mathcal{V}_6$, $\mathcal{V}_7\, \mathrm{r}\, \mathcal{V}_8$, $\mathcal{V}_9\, \mathrm{r}\, \mathcal{V}_{10}$, and $\mathcal{V}_{11}\, \mathrm{r}\, \mathcal{V}_{12}$.

The 2 designs from the previous case give us:

Theorem 9.2. There are 12 non-isomorphic tight scccd$(10, 4)$s, namely $\mathcal{V}_m$ for $m = 1, \ldots, 12$.

■

$S\,F$	$-$	$S\,F$	$-$	$S\,F$	F	$S\,F$	$S\,F$
F	$-$	F	F	F	$-$	F	$-$
$-$	$S\,F$	$-$	$-$	$-$	$-$	$-$	$-$
F	$-$	$-$	$-$	$-$	$-$	$-$	$S\,F$
$-$	F	$S\,F$	$S\,F$	S	$S\,F$	$-$	$-$
$-$	F	$-$	F	F	F	S	$-$
$S\,F$	$-$	F	$-$	F	$-$	$S\,F$	$S\,F$
$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$-$	$S\,F$	$-$	S	$-$	S	$-$	$-$
S	$-$	S	$-$	S	S	S	$S\,F$
S	$-$	$S\,F$	$S\,F$	$S\,F$	F	S	$-$
$-$	S	$-$	$-$	$-$	$-$	F	$-$
$S\,F$	$-$	$-$	$-$	$-$	$-$	F	$S\,F$
$-$	S	S	S	$-$	S	$-$	$-$
$-$	$S\,F$	$-$	$S\,F$	S	$S\,F$	$-$	$-$
2	2	1	1	1	1	0	2
$\mathcal{V}_3$, $\mathcal{V}_4$	$\mathcal{V}_5$, $\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$	$\mathcal{V}_9$	$\mathcal{V}_{10}$		$\mathcal{V}_{11}$, $\mathcal{V}_{12}$

Fig. 14: The 8 SF-skeletons which represent the 8 equivalence classes of SF-arrays for $(v,k) = (10,4)$ and $(t_1,t_2,t_3) = (5,5,0)$. Underneath each SF-skeleton is the number of its extensions to non-isomorphic designs, and the names of the designs.

10. Non-existence of some tight designs

In this section we consider three parameter sets for (v,k):

(i) $\{(3k-3,k)\colon\ k$ even and $\geq 2\}$;

(ii) $\{(3k-2,k)\colon\ k$ even and $\geq 2\}$;

(iii) $\{((i+1)^2/4, (i^2+7)/8)\colon\ i$ odd and $\geq 3\}$.

Every (v,k) in (i), (ii), and (iii) above satisfies the division requirement $2(k-1)|v(v-1)$ for a tight design to exist; however, for (i) and (ii), tight designs only exist when $k=2$ or 4, and, for (iii), only when $i=3$ or 5.

Using the notation of [5], we denote by $SCD(v,k,b)$ a single-change (non-circular) covering design on $[v]$ with b blocks of size k. We let $f(v,k)$ be the smallest b for which there exists a $SCD(v,k,b)$. The function $f(v,k)$ is studied in [5], [4], and [3].

Now a scccd(v,k,b) is also a $SCD(v,k,b)$. In particular, a minimal scccd$(v,k,b_*(v,k))$ is a $SCD(v,k,b_*(v,k))$, so we have $f(v,k) \leq b_*(v,k)$. (There are many (v,k)s for which equality holds.)

In §§5 and 6 we considered $\mathfrak{F}$ and $\mathfrak{G}$, two families of tight scccd(v, k)s for $v = 2k - 2$ and $2k - 1$ respectively. In the following two theorems we consider $v = 3k - 3$ (case (i) above) and $3k - 2$ ((ii) above) respectively.

Theorem 10.1. A tight scccd$(3k - 3, k, (9k - 12)/2)$ exists only when $k = 2$ or 4.

Proof. Here k is even. Consider the pair $(3k - 3, k)$ for $k \geq 6$, from Theorem 3.3 of [5] we have $f(3k - 3, k) = 5k - 8$. If a tight scccd$(3k - 3, k, (9k - 12)/2)$ exists then we must have $5k - 8 \leq (9k - 12)/2$, a contradiction. Thus a tight scccd$(3k - 3, k)$ does not exist for $k \geq 6$.

For $k = 2$ we have a tight scccd$(3, 2, 3)$, $\mathcal{Y}_2$, and for $k = 4$ a tight scccd$(9, 4, 12)$, e.g., $\mathcal{U}_1$.

■

So we have infinitely many pairs $(v, k) = (3k - 3, k)$ where k is even and ≥ 6, for which $2(k - 1)|v(v - 1)$ but a tight scccd(v, k) does not exist, e.g., a tight scccd$(15, 6, 21)$ does not exist.

Similarly for $v = 3k - 2$:

Theorem 10.2. A tight scccd$(3k - 2, k, (9k - 6)/2)$ exists only when $k = 2$ or 4.

■

When $k = 2$ we have a tight scccd$(4, 2, 6)$ and $k = 4$ a tight scccd$(10, 4, 15)$, e.g., $\mathcal{V}_1$.

Theorems 10.1 and 10.2 can also be proved using SF-arrays.

The final result in this section, Theorem 10.4, will, for variety and interest, be proved using the following lemma, although it can also be proved in a similar manner to the above.

Lemma 10.3. Let $\mathcal{C}$ be a tight scccd(v, k) with $v > 2k$, $x \in T_1$, $t_1 = |T_1|$, and $f_1 = v - k + 1$. Then

(i) the total number of symbols in any r successive rows of SF_x is $\leq r + 1$;

(ii) $t_1 \leq f_1$.

Proof.

(i) A straightforward proof by induction on r.

(ii) For any $x \in T_1$, the SF-array SF_x has f_1 rows, hence $\leq f_1 + 1$ symbols. But, by Criterion (4), it has exactly $t_1 + 1$ symbols. Hence $t_1 \leq f_1$.

■

The inequality $t_1 \leq f_1$ is sharp for both of the tight scccd$(9, 4, 12)$s $\mathcal{U}_1$ and $\mathcal{U}_2$ of §9.

Let $\mathcal{C}$ be a tight scccd(v, k). From Observation (2) we have $2f_1 \leq b + 1$, with equality if and only if $\mathcal{C} \in \mathfrak{F}$ or $\mathfrak{G}$. So, for tight designs other than those in $\mathfrak{F}$ or $\mathfrak{G}$, we have $2f_1 \leq b$.

We now classify tight designs with $2f_1 = b$, so $v \geq 2k$.

Theorem 10.4. A tight scccd(v, k, b) with $2f_1 = b$ is a tight scccd$(4, 2, 6)$ or a tight scccd$(9, 4, 12)$.

Proof. Let $\mathcal{D}$ be a tight scccd(v, k) with $2f_1 = b$ and $v \geq 2k$.

We have $2(v-k+1) = \dfrac{v(v-1)}{2(k-1)}$, i.e., $v = (4k-3+\sqrt{(8k-7)})/2$. So let $k = (i^2+7)/8$ where i is odd and ≥ 3. Hence, $(v, k) = ((i+1)^2/4, (i^2+7)/8)$ ((iii) above), and $f_1 = (i^2+4i+3)/8$.

Equation (2) from §7 then gives $f_3 = (v-1) - 3(k-2) = (21+4i-i^2)/8$. So, for $i \geq 7$, we have $f_3 \leq 0 < 3$. Thus, from Lemma 7.1(ii) with $j = 3$, we have $t_\ell = 0$ for $\ell \geq 3$, i.e., every element in $\mathcal{D}$ is introduced once or twice. So, equation (2) yields:

$$t_1 + t_2 = v = \frac{(i+1)^2}{4}; \qquad t_1 + 2t_2 = b = \frac{(i^2+4i+3)}{4}.$$

This gives $t_1 = (i^2-1)/4$. Now, for $i \geq 7$ we have $v > 2k$, so, via Lemma 10.3(ii), we must have $t_1 \leq f_1$; but this is false when $i \geq 7$. Thus, a tight scccd$((i+1)^2/4, (i^2+7)/8)$ does not exist for $i \geq 7$.

For $i = 3$ a tight scccd$(4, 2, 6)$ exists and for $i = 5$ a tight scccd$(9, 4, 12)$ exists.

■

11. Perfect designs; column-regular designs; element-regular designs

Again, let $\mathcal{B} = \{B_1, \ldots, B_b\}$ be an arbitrary scccd(v, k, b), and, for each $i = 1, \ldots, b$, let sf$(\mathcal{B}_i)$ be the standardized form of its i-th cyclic shift $\mathcal{B}_i$, see §4.

Now consider sf$(\mathcal{B}_i)$ for any fixed $i = 1, \ldots, b$; its first block is $(1, 2, \ldots, k)$. For $r = 1, \ldots, k$, let its r-th column be the column beginning with r, and let $\eta_{i,r}$ be the number of elements introduced into this column. Now let A_i be the ordered k-tuple $[\eta_{i,1}, \ldots, \eta_{i,k}]$; call this the *column-array* of sf$(\mathcal{B}_i)$.

Consider again $\mathcal{E} = \{B_1, \ldots, B_8\}$, the scccd$(6, 3, 8)$ from Fig. 3(a) and its standardized form sf$(\mathcal{E}_1) = \{L_1, \ldots, L_8\}$ from Fig. 3(b) shown again in Fig. 15(a); we have $A_1 = [3, 2, 3]$. Now consider Fig. 15(b), which shows sf$(\mathcal{E}_5)$, the standardized form of $\mathcal{E}_5 = \{B_5, B_6, B_7, B_8, B_1, B_2, B_3, B_4\}$, this design has $A_5 = [1, 4, 3]$. So, for a fixed $\mathcal{B}$, we may have different A_i for different i.

			1 2 3		
			1 2 4		
			1 5 4		
			1 6 4		
			7 6 4		1 2
			7 8 4		1 3
L_1	1 2 3	1 2 3	7 3 4		4 3
L_2	1 2 4	1 2 4	7 3 5		5 3
L_3	1 5 4	1 5 4	8 3 5		5 2
L_4	1 6 4	1 5 2	6 3 5	1 2 3	5 1
L_5	3 6 4	6 5 2	6 2 5	1 2 4	4 1
L_6	3 6 5	6 5 3	6 2 8	1 5 4	4 5
L_7	2 6 5	6 4 3	1 2 8	3 5 4	4 2
L_8	2 6 3	6 1 3	1 2 7	3 5 2	3 2
	sf($\mathcal{E}$)	sf($\mathcal{E}_5$)	scccd$(8, 3, 14)$	rsf($\mathcal{Y}_3$)	scccd$(5, 2, 10)$
	$A_1 = [3, 2, 3]$	$A_2 = [1, 4, 3]$			
	(a)	(b)	(c)	(d)	(e)

(c) the representative standardized form of a perfect scccd$(8, 3, 14)$: each A_i is a permutation of $A_1 = [4, 5, 5]$,

(d) rsf($\mathcal{Y}_3$): each A_i is a permutation of $A_1 = [2, 1, 2]$, not perfect, element-regular with $\mu = 1$,

(e) scccd$(5, 2, 10)$: column-regular with $\eta = 5$, perfect, element-regular with $\mu = 2$.

Fig. 15: Standardized designs and their column-arrays, and other properties.

A standardized scccd(v, k, b) $\mathcal{B} = \{B_1, \ldots, B_b\}$ is *perfect* if each of the unchanged elements between B_b and B_1 is in the same column in B_b as in B_1. So, the two ends of a perfect standardized scccd can be 'joined-up' to give a circular version of requirement (4) in the definition of standardization (§4). For any $\mathcal{B}$, all of its b standardized forms are perfect or none are. Hence, a standardized $\mathcal{B}$ is perfect if and only if rsf($\mathcal{B}$) is perfect. The standardized form of $\mathcal{Y}_3$ (from Fig. 1(b)) is shown in Fig. 15(d); this is also rsf($\mathcal{Y}_3$), it is not perfect. See Fig. 15(c) for the representative standardized form of a perfect scccd$(8, 3, 14)$.

An interesting property of perfect standardized scccds is:

Theorem 11.1.
Let $A_1, \dots, A_b$ be the column-arrays of a perfect standardized scccd(v, k, b). Then each A_i is a permutation of A_1, for $i = 1, \dots, b$.

Proof. For any fixed $r = 1, \dots, k$, consider the r-th column in a perfect standardized scccd(v, k, b) $\mathcal{B} = \mathrm{sf}(\mathcal{B}_1) = \{B_1, \dots, B_b\}$. The elements in this column in B_b and B_1 are either the same, or different if the single-change between B_b and B_1 occurs in this column. In either case, we may write the elements of this column in a circle. Then $\eta_{1,r}$, the number of introductions in this column, is counted starting at B_1; this number is fixed no matter where on the circle we start. Now let $\mathrm{sf}(\mathcal{B}_2)$ be formed from $\mathcal{B}_2 = \{B_2, B_3, \dots, B_b, B_1\}$ by a permutation of $[v]$ and a permutation ϕ of $[k]$, i.e., of the columns. Then $\eta_{2,r}$, the number of introductions in column r of $\mathrm{sf}(\mathcal{B}_2)$, is equal to the number of introductions in column $\phi^{-1}(r)$ of $\mathrm{sf}(\mathcal{B}_1)$ when starting counting at B_2, which is the same as starting at B_1; this number is $\eta_{1,\phi^{-1}(r)}$. Thus $\eta_{2,r} = \eta_{1,\phi^{-1}(r)}$ for $r = 1, \dots, k$. That is, A_2 is a permutation of A_1, and so on for A_i, $i = 3, \dots, b$.

■

For example, in the perfect standardized scccd$(8, 3, 14)$ in Fig. 15(c), each A_i is a permutation of $A_1 = [4, 5, 5]$. The scccd$(5, 3, 5)$ shown in Fig. 15(d) is $\mathrm{rsf}(\mathcal{Y}_3)$. Each column-array of this design is a permutation of $A_1 = [2, 1, 2]$, even though it is not perfect, so the converse of Theorem 11.1 is not true.

Consider $\mathrm{sf}(\mathcal{B}_i)$ for a fixed i, if the number of introductions into each column is the same, then we say that $\mathrm{sf}(\mathcal{B}_i)$ is *column-regular*, see §4 of [2]. So $\eta_{i,r} = \eta = b/k$ for each $r = 1, \dots, k$, and $A_i = [\eta, \dots, \eta]$. Also, $\mathcal{B}$ itself is *column-regular* if $\mathrm{sf}(\mathcal{B}_i)$ is column-regular for each $i = 1, \dots, b$. So a scccd(v, k, b) is column-regular if each of its b standardized forms is itself column-regular.

Although we have defined a column-array only for a standardized scccd we can also define it for a column-strict scccd. So, the column-array of a column-strict scccd(v, k, b), $\mathcal{B}$, is the ordered k-tuple $[\eta_1, \dots, \eta_k]$ where η_r is the number of elements introduced into the r-th column of $\mathcal{B}$, for each $r = 1, \dots, k$.

The definitions of 'perfect' and 'column-regular' can also be carried over to column-strict scccds; and a scccd(v, k, b), $\mathcal{B} = \{B_1, \dots, B_b\}$, is column-regular if the column-strict representation of each $\mathcal{B}_i$ is itself column-regular.

We can now prove:

Theorem 11.2. A standardized column-regular scccd(v, k, b) is perfect.

Proof. Let $\mathcal{B} = \mathrm{sf}(\mathcal{B}_1) = \{B_1, \dots, B_b\}$ be a standardized column-regular scccd(v, k, b), and let $\eta = b/k$. Now $B_1 = (1, \dots, k)$, without loss of generality let element 1 be introduced in B_1 and let $1'$ be changed from B_b, and suppose that 1 and $1'$ are in different columns; let $1'$ be in the s-th column where $s \neq 1$. Now $\mathcal{B}$ is column-regular and so the column-array of the column-strict representation of each $\mathcal{B}_i$ is $A_1 = [\eta, \ldots, \eta]$.

Now consider the column-strict $\mathcal{B}_2 = \{B_2, B_3, \ldots, B_b, B_1\}$, where the elements in B_2 are in the same order as they were in $\mathcal{B}$, element 1 is now in the same column as $1'$, the s-th column; hence the s-th element in the column-array of the column-strict $\mathcal{B}_2$ is $\eta + 1$, a contradiction. So 1 and $1'$ are in the same column in $\mathcal{B}$.

Now consider element r for any fixed $r \in \{2, \ldots, k\} = B_1 \cap B_b$; let it be changed first from B_{i_r} (i.e, $r \in B_1, \ldots, B_{i_r}$), and replaced by r' in B_{i_r+1}. See the column-strict arrangement $\mathcal{B}_{i_r+1} = \{B_{i_r+1}, \ldots, B_b, B_1, \ldots, B_{i_r}\}$ in Fig. 16(a); elements r' and r lie in the same column by the previous argument.

B_{i_r+1}	r'	r'
.	.	.
.	.	.
.	.	.
B_b	.	r
B_1	.	r
.	.	"
.	.	"
.	.	"
B_{i_r}	r	r
	(a)	(b)

Fig. 16: Figure for Theorem 11.2.

Now, $r \in B_b$ and $r \in B_1, \ldots, B_{i_r}$, so we have Fig. 16(b). We can retrieve $\mathcal{B} = \{B_1, \ldots, B_b\}$ from (b) without changing the columns in which r and r' appear. Hence, in $\mathcal{B}$, the $r \in B_1$ lies in the same column as the $r \in B_b$, and, because r was arbitrarily chosen from $B_1 \cap B_b$, so $\mathcal{B}$ is perfect.

■

Combining Theorems 11.1 and 11.2 we have the following theorem in which all designs are assumed to be column-strict.

Theorem 11.3. Let $\mathcal{B} = \{B_1, \ldots, B_b\}$ be an arbitrary scccd(v, k, b). Then

(i) if $\mathcal{B}$ is perfect the column-array of $\mathcal{B}_i$ is a permutation of the column-array of $\mathcal{B}$ for each $i = 1, \ldots, b$;

(ii) if $\mathcal{B}$ is column-regular then $\mathcal{B}$ is perfect.

■

An arbitrary scccd(v, k, b), $\mathcal{B}$, is *element-regular* if each of the v elements from $[v]$ is introduced the same number of $\mu = b/v$ times. In the notation of §7 we have $t_\mu = v$.

If $\mathcal{C}$ is tight and element-regular with $\mu = 1$ then $v = b = 2k - 1$, and so $\mathcal{C}$ is a scccd$(2k-1, k, 2k-1)$ and, by Theorem 6.2, is isomorphic to $\mathcal{Y}_k \in \mathfrak{G}$.

Our final example is shown in Fig. 15(e). It is the representative standardized form of a tight scccd$(5, 2, 10)$ which is column-regular with $\eta = 5$, and so perfect, and element-regular with $\mu = 2$.

Fig. 17 gives some of the numbers of non-isomorphic tight scccd(v, k)s for $v \leq 10$.

Fig. 17 here. Please contact the author for the latest version

Fig. 17: Table showing some of the numbers of non-isomorphic tight scccd(v, k)s for $v \geq 2k - 2$ and $v \leq 10$, $(k = 2, v \geq 3)$. The number of perfect designs is shown in parenthesis (). For $k \geq 3$ the 1 in column $v = 2k - 2$ corresponds to $\mathcal{X}_k \in \mathcal{F}$, and for $k \geq 2$ the 1 in column $v = 2k - 1$ to $\mathcal{Y}_k \in \mathcal{G}$. The symbol $-$ means that $2(k-1) \not| \, v(v-1)$ and so a tight design with parameters (v, k) cannot exist. The missing numbers are currently being computed.

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References

[1] N.C.K. Phillips and W.D. Wallis, Persistent pairs in single-change covering designs, Proceedings of the 24-th Southeastern International Conference on Combinatorics, Graph Theory, and Computing, (Boca Raton FL, 1993). Congressus Numerantium, 96 (1993) 75–82.

[2] D.A. Preece, R.L. Constable, G. Zhang, J.L. Yucas, W.D. Wallis, J.P. McSorley, and N.C.K. Phillips, Tight single-change covering designs, Utilitas Mathematica, 47 (1995) 55–84.

[3] G.H.J. van Rees, Single-change covering designs II, Congressus Numerantium, 92 (1993) 29–32.

[4] W.D. Wallis, J.L. Yucas, and G.-H. Zhang, Single-change covering designs, Designs, Codes and Cryptography, 3 (1993) 9–19.

[5] G.-H. Zhang, Some new bounds on single-change covering designs, SIAM J. Discrete Math, 7 (1994) 166–171.