PERIODIC PRIME KNOTS AND TOPOLOGICALLY TRANSITIVE FLOWS ON 3-MANIFOLDS

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ABSTRACT. Suppose that φ is a nonsingular (fixed point free) C^1 flow on a smooth closed 3-dimensional manifold M with $H_2(M) = 0$. Suppose that φ has a dense orbit. We show that there exists an open dense set $N \subseteq M$ such that any knotted periodic orbit which intersects N is a nontrivial prime knot.

1. INTRODUCTION

We need some standard terminology from knot theory. For presentation of knots in dynamical systems see the book [5] by Ghrist, Holmes, and Sullivan. Let $\Gamma \subset M$ denote a knot. By this we mean that Γ is the image of a continuous injective function from the circle to a 3-dimensional manifold M. We shall say that Γ is a trivial knot if it bounds a disk. We say that Γ is a composite knot if there exists a 2-sphere S in M such that $S \cap \Gamma$ is two points, z and w, and the intersection of each component of $\Gamma - \{z, w\}$ together with a segment in S from z to w is a nontrivial knot. We shall say that Γ is a prime knot if it is neither composite or trivial. When the knot is of class C^1 and

$$\Theta : \Gamma \times \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} \to M$$

is a C^1 embedding such that, for all $x \in \Gamma$, $\Theta((x, 0, 0)) = x$, the concepts of trivial, composite, and prime extend to the solid torus which is the image of Θ.

Our main theorem is Theorem 1. As a consequence of this theorem, for any topologically transitive C^1 nonsingular flow on S^3, there is an open dense set $N \subseteq S^3$ such that any periodic orbit intersecting N is a nontrivial prime knot.

THEOREM 1. *Let M be a smooth closed (compact, no boundary) 3-dimensional manifold with $H_2(M) = 0$. Suppose φ is a C^1 nonsingular (fixed point free) topologically transitive (φ has a dense orbit) flow on M. There exists an open dense set $N \subseteq M$ such that if is γ a periodic orbit with $\gamma \cap N \neq \varnothing$ then γ is a nontrivial prime knot.*

REMARK: It is possible that some periodic orbits are trivial. As an example, Harrison and Pugh in [7] define a nonsingular flow on S^3 with a a dense orbit by Birkhoff suspending Katok diffeomorphisms of a disk. The flow has a dense orbit but the diffeomorphism of the disk has a fixed point which corresponds to a trivial knot in the flow.

For the rest of this paper, let M be a smooth closed 3-dimensional manifold with $H_2(M) = 0$, and let φ be a C^1 nonsingular topologically transitive flow on M.

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Our motivation for this result is a Theorem 2 below, which appears as Theorem 1 from [3]. Let p be any point in the dense orbit of φ. Let D be a compact disk containing p which is transverse to the flow. That is, D is a compact disk and there is an open disk E containing D that is transverse to the flow. We call such a disk a transverse disk, and if D is in addition a global cross section we will call it a global transverse disk. Let $q \in D$ be a point in the forward orbit of p and let $\overrightarrow{pq}$ denote the orbit segment beginning at p and ending q. Let $[pq]$ denote a compact segment in $D - \overrightarrow{pq} \cap D$ connecting p to q. Let $\Gamma = \overrightarrow{pq} \cup [pq]$.

THEOREM 2. *If q is close enough to p then Γ is a nontrivial prime knot. The result holds in the case $H_2(M) \neq 0$ if the flow has no periodic orbits.*

For a point $x \in M$ we use γ_x to denote the orbit through x. Theorem 3 below is proven as Theorem 2.1 in [6]. We use it to prove a periodic orbit forms a prime knot under our specified conditions.

THEOREM 3. *A solid torus T contained in M is a (nontrivial) prime knot if there exists a transversely orientable bidimensional C^2 foliation $\mathcal{F}$ on $\mathcal{V} = \overline{M - T}$ such that:*

(1) *$\mathcal{F}$ is transversal to $\partial\mathcal{V}$. Moreover, every leaf of $\mathcal{F}$ has nonempty intersection with $\partial\mathcal{V}$.*
(2) *The one-dimensional foliation $\mathcal{F}|_{\partial\mathcal{V}}$ on $\partial\mathcal{V}$ contains a meridian σ as a leaf. Moreover, $\mathcal{F}|_{\partial\mathcal{V}}$ contains no Reeb components.*
(3) *If $\mathcal{F}$ has a compact leaf K, there are finitely many discs $D_1, D_2, ..., D_s$ contained in T such that the union of K with $\cup_{i=1}^{s} D_i$ is a torus L satisfying $L \cap \partial T = K \cap \partial T = \cup_{i=1}^{s} \partial D_i$*
(4) *Let $B = \{(x,y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 \leq 9 \text{ and } x \leq 2\}$ and decompose its boundary ∂B as the union of $B_1 = \{(x,y) \in B | x^2+y^2 = 1\}$, $B_2 = \{(x,y) \in B | x = 2\}$ and $B_3 = \{(x,y) \in B | x^2+y^2 = 9\}$. There exists an embedding $\lambda : B \times [-1,1] \to \mathcal{V}$ such that*
    (a) *$\lambda : (B_1 \cup B_2) \times [-1,1]$ is precisely the intersection of $\partial\mathcal{V}$ with the image $Im(\lambda)$ of λ.*
    (b) *The complement of $\lambda(B_1 \times (-1/2, 1/2))$ in $\partial\mathcal{V}$ is a union of meridians of $\partial\mathcal{V}$ which are leaves of $\mathcal{F}|_{\partial\mathcal{V}}$.*
    (c) *For all $p \in B$, the segments $\lambda(\{p\} \times [-1,1])$ are transversal to $\mathcal{F}$.*
    (d) *Let H be a half straight line of $\mathbb{R}^2$ starting at the origin. Then, for all $z \in [-1,1]$, $\lambda((H \cap B) \times \{z\})$ is contained in a leaf of $\mathcal{F}$. Also, for all $z \in [-1,-1/2) \cup (1/2,1]$, $\lambda(B \times \{z\})$ is a plaque of $\mathcal{F}$.*

Proof. (of Theorem 1)

Let p be any point in the dense orbit. We will prove that there is a neighborhood N_p of p such that if $a \in N_p$ and γ_a is periodic then γ_a is a nontrivial prime knot. Once this is proven for every p in the dense orbit, the set $N = \cup_p N_p$ is the open (it is the union of open sets) dense (it contains the dense orbit) set required in the theorem.

The idea of the proof is simple. In [3], Theorem 2 is proven by showing that there exists a solid torus neighborhood of $\Gamma = [pq] \cup \overrightarrow{pq}$ and a foliation satisfying the criteria of Theorem 3 proving that this solid torus is a prime knot, and hence Γ is a prime knot. We show that for any periodic point a in a small neighborhood of p, this foliation can be moved by a small amount so that a torus neighborhood of γ_a is a prime knot, and hence that γ_a itself is a prime knot.

Let D be a global transverse disk containing p. In [2] it is proven that any non-singular C^1 flow on a manifold of dimension greater than 2 has a global transverse disk. We can assume that the disk contains p, for if D is any global transverse disk and t_p is any time such that $\varphi(t_p, p) \in D$ then, $\varphi(-t_p, D)$ is a global transverse disk containing p.

It is proven in [3] that there is a disk $D_1 \subset D$ containing p, a foliation $\mathcal{F}$ on M, a solid torus neighborhood T of $\overrightarrow{pq} \cup [pq]$, and an imbedding λ satisfying the conditions of Theorem 3, proving that T is a prime solid torus. (See Figure 3 of [3] and Figure 1.) This can be chosen so that the embedding $\lambda : B \to M$ has its image in a flowbox W whose base is D_1, whose top is a disk $U \subset D$, and such that $W \cap D = D_1 \cup U$ and $D_1 \cap U = \varnothing$. Moreover, we can assume that $T \cap W$ is a pair of cylindrical flow boxes T_1 and T_2.

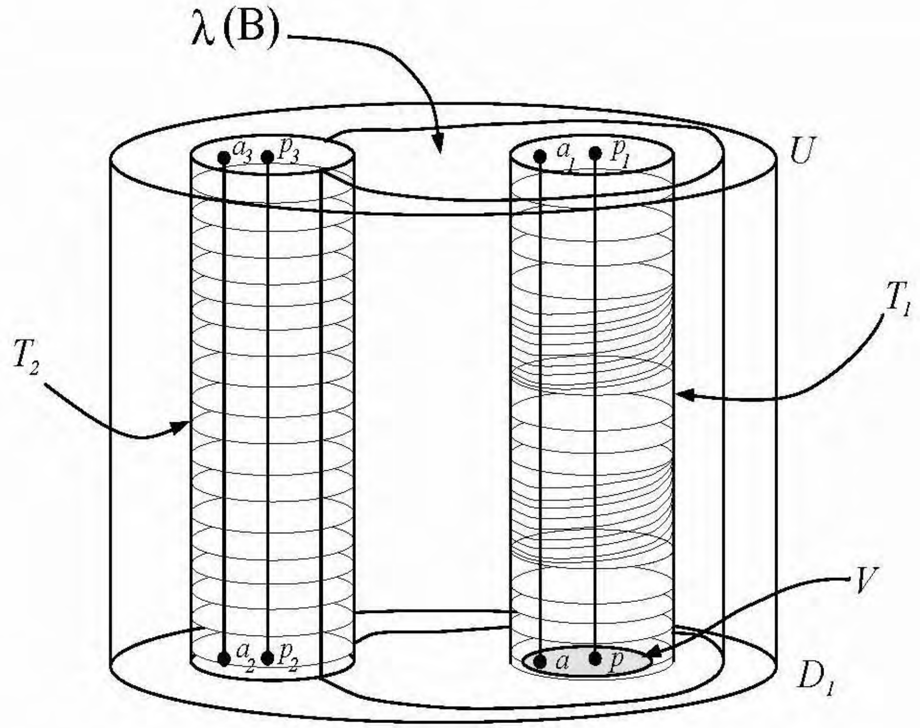

FIGURE 1. The imbedding $\lambda(B)$ inside the flowbox W.

Let V denote the interior of the base of T_1. Note that V is an open disk. Let a be any periodic point in V. Then the orbit beginning at a follows the orbit beginning at p through the cylinders T_1 and T_2. Define p_1, p_2, and p_3 by

$$p_1 = \varphi(t_1, p), \text{ where } t_1 = \min\{t > 0 : \varphi(t, p) \in U\}$$
$$p_2 = \varphi(t_2, p), \text{ where } t_2 = \min\{t > t_1 : \varphi(t, p) \in D_1\}$$
$$p_3 = \varphi(t_3, p), \text{ where } t_3 = \min\{t > t_2 : \varphi(t, p) \in U\}$$

Define a_1, a_2, and a_3 in the same manner. (See Figure 1.) Perturb the foliation $\mathcal{F}$ from [3] so that it is defined on $M - \overrightarrow{aa_1}$ instead of $M - \overrightarrow{pp_1}$. Specifically, there is a homeomorphism ϕ of T_1 that fixes the vertical boundary, is constant on the vertical coordinate, and takes $\overrightarrow{aa_1}$ to $\overrightarrow{pp_1}$. Define the new foliation $\mathcal{F}'$ to be equal to $\mathcal{F}$ on $M - T_1$ and to be the pullback by ϕ of $\mathcal{F}$ on T_1. Then define T' to be a small tubular neighborhood of γ_a.

By reducing the size of D_1 so that $\gamma_a \cap D_1$ is two points a and a_2 if necessary, if T' is chosen small enough (with T' a torus neighborhood of γ_a) then $T' \cap W$ has two components. Let T_1' be the component containing $\overrightarrow{aa_1}$ and T_2' be the other component. As in [3], we can then define $\lambda : B \to B$ satisfying the criteria of Theorem 3 and the solid torus T' is a prime knot. Hence the periodic orbit through a is a prime knot.

Let $\epsilon > 0$ and define $N_p = \varphi((-\epsilon, \epsilon), V)$. If ϵ is small enough then N_p is an open neighborhood of p and any periodic orbit which intersects N_p intersects V and hence is a nontrivial prime knot. □

We conclude with two questions:

- Under the assumptions of Theorem 1, is it true that every orbit is either prime or trivial?
- Can the assumption that $H_2(M) = 0$ be removed?

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