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Analytic Normal Forms and Symmetries of Strict Feedforward Control Systems

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SUMMARY

This paper deals with the problem of convergence of normal forms. We identify a n-dimensional subclass of control-affine systems, called *special strict feedforward form*, shortly (SSFF), possessing a normal form which is a smooth (resp. analytic) counterpart of the formal normal form of Kang. We provide a constructive algorithm and illustrate by several examples. The second part of the paper is concerned about symmetries of single-input control systems. We show that any symmetry of a smooth system in special strict feedforward form is conjugated to a *scaling translation* and any 1-parameter family of symmetries is conjugated to a family of scaling translations along the first variable. We compute explicitly those symmetries by finding the conjugating diffeomorphism. We illustrate our results by computing the symmetries of the Cart-Pole system.

KEY WORDS: Normal Forms, Feedback Transformation, Convergence, Strict Feedforward, Symmetries

1. Introduction

In the past twenty five years the problem of feedback equivalence of control systems under change of coordinates and input has been studied extensively. Several methods have been proposed to deal with the problem of transforming the nonlinear control system

$$\Pi: \ \dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

into a simpler form

$$\tilde{\Pi}: \ \dot{z} = \tilde{f}(z,v), \quad z \in \mathbb{R}^n, \ v \in \mathbb{R}^m$$

by an invertible feedback transformation of the form

$$\Gamma: \quad \begin{array}{rcl} z & = & \phi(x) \\ u & = & \gamma(x,v), \end{array}$$

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where the dynamics of the equivalent system are given by

$$\tilde{f}(z,v) = \mathrm{d}\phi(\phi^{-1}(z)) \cdot f(\phi^{-1}(z), \gamma(\phi^{-1}(z), v)).$$

When the system $\tilde{\Pi}$ takes its simplest form $\dot{z} = Az + Bv$, that is $\tilde{\Pi}$ is linear, then the system Π is said to be linearizable via feedback. Necessary and sufficient geometric conditions for this to be the case have been given in [16] and [19]. Except for the planar case, those conditions turn out to be restrictive and a natural problem of finding normal forms for non linearizable systems arose and has been extensively studied during the last two decades (see, e.g., [6], [7], [20], [22], [23], [24], [27], [37], [61], [62] and the recent survey [42]).

A very fruitful approach leading to normal forms has been proposed by Kang and Krener [24] and then followed by Kang [22], [23]. Their idea, which is closely related with classical Poincaré's technique for linearization of dynamical systems (see e.g. [1]), is to analyze, step by step, the action of the Taylor series expansion of the feedback transformation Γ on the Taylor series expansion of the system Π. Using that approach, results on normal forms of single-input control systems with controllable linearization have been obtained by Kang and Krener [24] for the quadratic terms, and then generalized by Kang [22] for higher order terms. The results of Kang and Krener [24],[22] have been completed by Tall and Respondek who obtained canonical forms and dual canonical forms for single-input nonlinear control systems with controllable linearization [47], [48] and then with uncontrollable linearization [49] (see also [29]). Recently those results have been generalized by Tall [45], [46] to multi-input nonlinear control systems.

The theory of normal forms, although formal, has been very useful in analyzing control systems. Using this method, bifurcations of nonlinear systems were treated in [25], [26] and the references therein, a complete description of symmetries around equilibria were presented in [38], [39], a characterization of systems equivalent to feedforward forms in [52], [53]. Their counterparts, in the discrete case, have also been obtained using a similar approach [9, 10, 11, 12, 13, 14, 15].

The convergence of these normal forms and their normalizing transformations in the C^∞ and analytic categories is still an open problem (see [4]).

A starting point is a result of Kang [22] derived from [27], and [28] (see also [17]) stating that, if an analytic control system is linearizable by a formal transformation, then it is linearizable by an analytic transformation. Kang [22] also gave a class of non linearizable 3-dimensional analytic control systems which are equivalent to their normal forms by analytic transformations. In [57], we gave the largest class ever of n-dimensional systems, namely the subclass of special strict feedforward forms, that can be brought to their normal form via smooth and analytic feedback transformations. Notice however, that C^∞-smooth and/or analytic normal forms were obtained in [6], [18], [20], [36], [43], [63] via singularity theory methods.

We will first address the problem of convergence of normal forms in section 3 by providing the largest class ever of smooth (resp. analytic) control systems that can be brought to their normal forms via smooth (resp. analytic) feedback transformations. The class of *special strict feedforward forms* we consider here (see definition later) is a generalization of that, of the same name, studied in [56] with the difference that the linearization is uncontrollable.

Although it is not clear if any smooth (resp. analytic) strict feedforward system can be brought to its smooth (resp. analytic) normal form, we will define in section 4 a *strict feedforward normal form*, that is close as much as possible to the normal form, to which any smooth (resp. analytic) strict feedforward form can be transformed via smooth (resp. analytic) feedback transformation.

For simplicity of notations, we will deal with single-input nonlinear control system of the form

$$\Pi \ : \ \dot{x} = f(x, u),$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. This system is in *strict feedforward form* if we have

$$(SFF) \qquad \begin{cases} \dot{x}_1 &= f_1(x_2, \ldots, x_n, u) \\ & \ldots \\ \dot{x}_{n-1} &= f_{n-1}(x_n, u) \\ \dot{x}_n &= f_n(u). \end{cases}$$

A basic structural property of systems in strict feedforward form is that their solutions can be found by quadratures. Indeed, knowing $u(t)$ we integrate $f_n(u(t))$ to get $x_n(t)$, then we integrate $f_{n-1}(x_n(t), u(t))$ to get $x_{n-1}(t)$, we keep doing that, and finally we integrate $f_1(x_2(t), \ldots, x_n(t), u(t))$ to get $x_1(t)$.

Another property, crucial in applications, of systems in (strict) feedforward form is that we can construct for them a stabilizing feedback. This important result goes back to Teel [59] and has been followed by a growing literature on stabilization and tracking for systems in (strict) feedforward form (see e.g. [21], [32], [44], [60], [3], [33]).

The natural question of which systems are equivalent to (strict) feedforward forms arose and has been investigated by several authors. In [31], the problem of transforming a system, affine with respect to controls, into (strict) feedforward form via a diffeomorphism, i.e., via a nonlinear change of coordinates, was studied. A geometric description of systems in feedforward form has been given in [2]. Using the formal approach, we proposed a step-by-step constructive method to bring a system into a feedforward form in [52] and strict feedforward form in [53]. Recently, (see [40]), we have shown that feedback equivalence (resp. state-space equivalence) to the strict feedforward form can be characterized by the existence of a sequence of infinitesimal symmetries (resp. strong infinitesimal symmetries) of the system.

Another topic of interest that we have been investigating is about symmetries of nonlinear systems. We showed, even in the formal case, that there is a strong connection between the existence of symmetries and the feedback equivalence to (strict) feedforward systems.

We will further that topic here by providing explicit symmetries of systems in strict feedforward form via the smooth (resp. analytic) normalizing feedback transformation constructed in section 3.

The paper is organized as follows. In section 2 we will recall the Kang normal form and our canonical form for single-input control systems. Analytic normal forms for strict feedforward and special strict feedforward systems are given in section 3, followed by their proofs. In section 4, we discuss symmetries of control systems. Illustrative examples (cart-pole, Kapitsa pendulum, etc) are given throughout the sections.

2. Normal and Canonical Forms

We start by briefly reviewing the results on normal and canonical forms obtained using the formal approach.

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2.1. Formal Normal Forms

All objects, i.e., functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth (or real analytic, if explicitly stated). Let h be a smooth function. By

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

we denote its Taylor expansion around zero, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m.

Similarly, for a map ϕ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp. for a vector field f on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the term of degree m of its Taylor expansion at zero, i.e., each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m in x.

Consider the Taylor series expansion of the single-input system Π, given by

$$\Pi^\infty : \dot{x} \quad = \quad f(x,u) = Fx + Gu + \sum_{m=2}^{\infty} f^{[m]}(x,u), \tag{2.1}$$

where $F = \frac{\partial f}{\partial x}(0,0)$ and $G = \frac{\partial f}{\partial u}(0,0)$. Except otherwise stated, we will assume the linear approximation around the origin to be controllable.

Consider also the Taylor series expansion Γ^∞ of the feedback transformation Γ given by

$$\Gamma^\infty : \begin{array}{rcl} z & = & \phi(x) = Tx + \sum_{m=2}^{\infty} \phi^{[m]}(x) \\ u & = & \gamma(x,v) = Kx + Lv + \sum_{m=2}^{\infty} \gamma^{[m]}(x,v), \end{array} \tag{2.2}$$

where the matrix T is invertible and $L \neq 0$. The action of Γ^∞ on the system Π^∞ step by step leads to the following normal form obtained by Kang [22] (see also [24] and [47]).

Theorem 2.1. The control system Π^∞, defined by (2.1), is feedback equivalent, by a formal transformation Γ^∞ of the form (2.2), to the formal normal form

$$\Pi^\infty_{NF} : \dot{z} = Az + Bv + \sum_{m=2}^{\infty} \bar{f}^{[m]}(z,v),$$

where (A,B) is the Brunovský canonical form and for any $m \geq 2$, we have

$$\bar{f}_j^{[m]}(z,v) = \begin{cases} \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}^{[m-2]}(z_1,\ldots,z_i), & 1 \leq j \leq n-1, \\ 0, & j = n, \end{cases} \tag{2.3}$$

with $P_{j,i}^{[m-2]}$ being homogeneous polynomials of degree $m-2$ of the indicated variables, and $z_{n+1} = v$.

The Kang normal form has been re-normalized [47] to obtain a canonical form

$$\Pi^{\infty}_{CF} \ : \dot{z} = Az + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(z,v),$$

with the components $\bar{f}^{[m]}_j(z,v)$ satisfying (2.3) and, in addition

$$\frac{\partial^{m_0} \bar{f}^{[m_0]}_{j^*}}{\partial z_1^{i_1} \cdots \partial z_{n-s}^{i_{n-s}}} = \pm 1. \tag{2.4}$$

Moreover, for any $m \geq m_0 + 1$,

$$\frac{\partial^{m_0} \bar{f}^{[m]}_{j^*}}{\partial z_1^{i_1} \cdots \partial z_{n-s}^{i_{n-s}}}(z_1, 0, \ldots, 0) = 0.$$

For the definitions of the integers m_0, j^*, and s, we refer the reader to [47].

The importance of the canonical form resides in the fact that two systems Σ_1^{∞} and Σ_2^{∞} are formally feedback equivalent if and only if their canonical forms $\Sigma^{\infty}_{1,CF}$ and $\Sigma^{\infty}_{2,CF}$ coincide. The canonical form played a key role in computing the symmetries of control systems.

If the linearization of the system around the origin is uncontrollable, we introduced weights corresponding to the uncontrollable variables [49]. Assuming the linearly uncontrollable part to be of dimension s, we split the coordinates as $(x_1, \ldots, x_s)$, denoting the uncontrollable variables, and $(x_{s+1}, \ldots, x_n)$, denoting the controllable variables. We then proved [49] (see [55]) that any single-input system, with uncontrollable linearization, is feedback equivalent to a weighted normal form

$$\Pi^{\infty}_{NF} : \begin{cases} \dot{z}_j &= R_j(z_1, \ldots, z_s) + z_{s+1} S_j(z_1, \ldots, z_s) + \sum\limits_{i=s+1}^{n+1} z_i^2 Q^{\infty}_{j,i}(z_1, \ldots, z_i), \ 1 \leq j \leq s \\ \dot{z}_j &= z_{j+1} + \sum\limits_{i=j+2}^{n+1} z_i^2 P^{\infty}_{j,i}(z_1, \ldots, z_i), \quad s+1 \leq j \leq n-1 \\ \dot{z}_n &= z_{n+1} = v\,, \end{cases}$$

where the functions $Q^{\infty}_{j,i}$ and $P^{\infty}_{j,i}$ are formal power series in the controllable variables $z_{s+1}, \ldots, z_n, v$ whose coefficients are smooth (resp. analytic) functions of the uncontrollable variables $(z_1, \ldots, z_s)$. Those results stand for the single-input case, and have been generalized in the multi-input case [45, 46].

The problem whether an analogous result holds in the smooth (resp. analytic) category is actually a challenging question, which can be formulated as whether for a smooth (resp. analytic) system Π the normalizing feedback transformation Γ^{∞} gives rise to a smooth (resp. convergent) Γ and thus leads to a smooth (resp. analytic) normal form Π_{NF} or canonical form Π_{CF}. One of the difficulties resides in the fact that it is not clear at all how to express, in terms of the original system, homogeneous invariants transformed via an infinite composition of homogeneous feedback transformations. We will study in this paper a special class of smooth (resp. analytic) control systems, namely special strict feedforward systems, that can be brought to their thus normal form by smooth (resp. analytic) transformations.

3. Smooth and Analytic Normal Forms

Let start by recalling the results obtained by Kang for 3-dimensional systems. He pointed out that any system of the form

$$\begin{cases} \dot{x}_1 &= x_2 + f_{1,2}(x_2) + x_3 f_{1,3}(x_2) + x_3^2 P(x_1, x_2, x_3) \\ \dot{x}_2 &= x_3 + x_3 f_{2,3}(x_2, x_3) \\ \dot{x}_3 &= u, \end{cases}$$

where $f_{1,2}, f_{1,3}$, and $f_{2,3}$ are analytic functions, is feedback equivalent to its normal form

$$\begin{cases} \dot{z}_1 &= z_2 + z_3^2 \tilde{P}(z_1, z_2, z_3) \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= v. \end{cases}$$

Indeed, the change of coordinates and feedback

$$\begin{aligned} z_1 &= x_1 - \int_0^{x_2} f_{1,3}(\epsilon)\mathrm{d}\epsilon \\ z_2 &= x_2 + f_{1,2}(x_2) \\ z_3 &= x_3 + x_3 f_{2,3}(x_2, x_3) \\ v &= \dot{z}_3 = (\partial z_3/\partial x_1)\dot{x}_1 + (\partial z_3/\partial x_2)\dot{x}_2 + (\partial z_3/\partial x_3)\dot{x}_3 \end{aligned}$$

do the job.

He also gave a class of 3-dimensional systems with one uncontrollable mode, namely, systems of the form

$$\begin{cases} \dot{x}_0 &= \lambda x_0 + f_0(x_0, x_1, x_2) \\ \dot{x}_1 &= x_2 + f_1(x_0, x_1, x_2) \\ \dot{x}_2 &= u, \end{cases}$$

that can be brought to a normal form

$$\begin{cases} \dot{z}_0 &= \lambda z_0 + z_1 Q_0(z_0) + z_1^2 Q_1(z_0, z_1) + z_2^2 P(z_0, z_1, z_2) \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= v \end{cases}$$

with $Q_0(z_0) \equiv 0$ if $\lambda \neq 0$.

Notice that while the first class is linearly controllable, the second class has a controllability index $p = 2$ and is the analytic counterpart of our weighted normal form when $s = 1$ and $n = 3$.

In the following we will give an n-dimensional class of smooth (resp. analytic) control systems, with uncontrollable linearization, that can be brought to their normal form (weighted normal form) via smooth (resp. analytic) feedback transformation.

Consider the class of *smooth (resp. analytic)* single-input control systems

$$\Pi \ : \ \dot{x} = f(x, u),$$

either locally in a neighborhood $X \times U$ of $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ or globally on $\mathbb{R}^n \times \mathbb{R}$, in *strict feedforward form* (SFF), that is, such that

$$(SFF) \qquad f_j(x, u) = f_j(x_{j+1}, \ldots, x_n, u), \quad 1 \leq j \leq n.$$

Notice that for any $1 \le i \le n$, the subsystem Π^i, defined as the projection of Π onto $\mathbb{R}^{n-i+1}$ via $\pi(x_1, \ldots, x_n) = (x_i, \ldots, x_n)$, is a well defined system whose dynamics are given by

$$\dot{x}_j = f_j(x_{j+1}, \ldots, x_n, u)$$

for $i \le j \le n$. Define the linearizability index of the (SFF)-system to be the largest integer p such that the subsystem Π^r, where $p+r=n$, is feedback linearizable. Clearly, the linearizability index is feedback invariant and hence the linearizability indices of two feedback equivalent systems coincide.

Notice that each component of a strict feedforward system (SFF) decomposes uniquely, locally or globally, as:

$$\begin{cases} f_j(x,u) = h_j(x_{j+1}) + F_j(x_{j+1}, \ldots, x_n, u) \\ F_j(x_{j+1}, 0, \ldots, 0) = 0, \quad 1 \le j \le n, \\ F_n = 0. \end{cases} \tag{3.1}$$

Let s be the smallest integer such that

$$\frac{\partial h_j}{\partial x_{j+1}}(0) \ne 0 \quad \text{for } s+1 \le j \le n, \tag{3.2}$$

where $x_{n+1} = u$. This means, in particular, that the linearization of the system around the origin is controllable when $s = 0$ and is uncontrollable when $s > 0$.

A strict feedforward form for which

$$h_j(x_{j+1}) = k_j x_{j+1}, \qquad s+1 \le j \le r-1, \tag{3.3}$$

for some non zero real numbers $k_{s+1}, \ldots, k_{r-1}$, will be called a *special strict feedforward form*, shortly, (SSFF).

The first result of this paper is stated as following.

Theorem 3.1. *Consider a smooth (resp. analytic) special strict feedforward form (SSFF) given by (3.1)-(3.2)-(3.3) in a neighborhood of* $(0,0) \in \mathbb{R}^n \times \mathbb{R}$*. There exists a smooth (resp. analytic) local feedback transformation that brings the system (3.1)-(3.2)-(3.3) into the normal form*

$$\Pi_{SSFNF}: \begin{cases} \dot{z}_j = R_j(z_{j+1}, \ldots, z_s) + z_{s+1} S_j(z_{j+1}, \ldots, z_s) + \sum_{i=s+1}^{n+1} z_i^2 Q_{j,i}(z_{j+1}, \ldots, z_i), & \text{if } 1 \le j \le s \\ \dot{z}_j = z_{j+1} + \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}(z_{j+1}, \ldots, z_i), & \text{if } s+1 \le j \le r \\ \dot{z}_{r+1} = z_{r+2}, \\ \quad \cdots \\ \dot{z}_{n-1} = z_n, \\ \dot{z}_n = z_{n+1} = v\,, \end{cases} \tag{3.4}$$

where $R_j(z_{j+1}, \ldots, z_s)$ *and* $S_j(z_{j+1}, \ldots, z_s)$ *are smooth (resp. analytic) functions depending on the uncontrollable variables only,* $Q_{j,i}(z_{j+1}, \ldots, z_i)$ *and* $P_{j,i}(z_{j+1}, \ldots, z_i)$ *are smooth (resp. analytic) functions of the indicated variables and* $z_{n+1} = v$*. Moreover, if the system is defined globally on* $\mathbb{R}^n \times \mathbb{R}$*, then so are the feedback transformation and the normal form.*

This result, although stated for strict feedforward systems, remains true even if the uncontrollable part corresponding to the variables $(x_1, \ldots, x_s)^\top$ is not in strict feedforward form. In other words, if the projection Π^s is in strict feedforward form (with s defined as above), then the system is smoothly (resp. analytically) feedback equivalent to its normal form. This provides the largest class ever of nonlinear control systems that can be brought to their normal form via a smooth (resp. analytic) feedback transformation.

When $s = 0$, that is, the linearization about the origin is controllable, the normal form reduces to (see [41], [56])

$$\Pi_{SSFNF} : \begin{cases} \dot z_1 &= z_2 + \sum_{i=3}^{n+1} z_i^2 P_{1,i}(z_2, \ldots, z_i), \\ & \cdots \\ \dot z_j &= z_{j+1} + \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}(z_{j+1}, \ldots, z_i), \\ & \cdots \\ \dot z_r &= z_{r+1} + \sum_{i=r+2}^{n+1} z_i^2 P_{r,i}(z_{r+1}, \ldots, z_i), \\ \dot z_{r+1} &= z_{r+2}, \\ & \cdots \\ \dot z_{n-1} &= z_n, \\ \dot z_n &= v. \end{cases}$$

A main observation is that the above normal form Π_{SSFNF} given by (3.4) is itself a (SSFF)-system and, on the other hand, it constitutes a smooth (resp. analytic) counterpart Π_{NF} of the formal normal form Π_{NF}^∞ (actually, the weighted normal form) given by Theorem 2.1. However, the convergence to the canonical form is only guaranteed in the analytic case.

A question of importance is whether we can always transform a strict feedforward form, say (3.1)-(3.2), into a special strict feedforward form (3.1)-(3.2)-(3.3). To answer that question, consider another smooth (resp. analytic) system

$$\tilde\Pi \; : \; \dot z = \tilde f(z, v),$$

in strict feedforward form (SFF), that is, such that

$$\begin{cases} \tilde f_j(z, u) = \tilde h_j(z_{j+1}) + \tilde F_j(z_{j+1}, \ldots, z_n, v), \\ \tilde F_j(z_{j+1}, 0, \ldots, 0) = 0, \quad 1 \le j \le n, \\ \tilde F_n = 0. \end{cases} \tag{3.5}$$

Let $\tilde q$ denote the smallest integer such that

$$\frac{\partial \tilde h_j}{\partial z_{j+1}}(0) \neq 0, \quad \text{for } \tilde s + 1 \le j \le n. \tag{3.6}$$

It is in the special strict feedforward form (SSFF) if

$$\tilde h_j(z_{j+1}) = \tilde k_j z_{j+1}, \quad \tilde s + 1 \le j \le \tilde r - 1 \tag{3.7}$$

for some non zero real numbers $\tilde k_{\tilde s+1}, \ldots, \tilde k_{\tilde r-1}$.

Theorem 3.2. *If two smooth (resp. analytic) (SFF)-systems given, respectively, by (3.1)-(3.2) and (3.5)-(3.6) are feedback equivalent, then* $s = \tilde{s}$, $r = \tilde{r}$ *and*

$$\tilde{h}_j(l_{j+1}z_{j+1}) = l_j h_j(z_{j+1}), \qquad s+1 \le j \le r-1,$$

for some non zero real numbers $l_{s+1}, \dots, l_{r-1}$.

The following corollary provides the answer to the question stated above.

Corollary 3.3. *A strict feedforward system (SFF), given by (3.1)-(3.2), is feedback equivalent to the special strict feedforward form (SSFF), given by (3.5)-(3.6)-(3.7), if and only if*

$$h_j(x_{j+1}) = k_j x_{j+1},$$

for $s+1 \le j \le r-1$, *that is, the* $(x_{s+1}, \dots, x_{r-1})$*-part of the system is already in (SSFF) in its original coordinates.*

Notice that the terms $h_j(x_{j+1})$ for $s+1 \le j \le n$ are feedback linearizable (actually h_j form exactly the feedback linearizable terms of the form Π_{SFNF} given by Theorem 3.1). Basically, Theorem 3.2 and Corollary 3.3 imply that the linearizable terms $h_j(x_{j+1})$, for $s+1 \le j \le r-1$, of a strict feedforward form (SFF) cannot be linearized (unless they are already linear) via any feedback transformation that preserves the strict feedforward structure of the system. This means that special strict feedforward forms (SSFF) define the only subclass of strict feedforward systems that can be brought to the Kang normal form Π_{NF} still being in the strict feedforward form. Whether it is possible to bring a (SFF)-system into its normal form Π_{NF} by a smooth (resp. analytic) transformation is unclear but if true, then the normal form Π_{NF} will loose the structure of (SFF) (unless the system is (SSFF)). On the other hand, any strict feedforward form (SFF) can be brought to a form Π_{SFNF}, called *strict feedforward normal form* (introduced by the authors in [52] in the formal category), which is close as much as possible to the normal form Π_{NF} that is generalized here as

$$\Pi_{SSFNF}: \begin{cases} \dot{z}_j = R_j(z_{j+1},..,z_s) + z_{s+1}S_j(z_{j+1},..,z_s) + \sum_{i=s+1}^{n+1} z_i^2 Q_{j,i}(z_{j+1},..,z_i), & \text{if } 1 \le j \le s \\ \dot{z}_j = h_j(z_{j+1}) + \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}(z_{j+1},..,z_i), & \text{if } s+1 \le j \le r \\ \dot{z}_{r+1} = z_{r+2}, \\ \quad\dots \\ \dot{z}_{n-1} = z_n, \\ \dot{z}_n = v. \end{cases}$$

Moreover, if the system is defined globally on $\mathbb{R}^n \times \mathbb{R}$, then so are the feedback transformation and the normal form.

In order to explain relations between various results proved in this section, let us recall that according to Theorem 2.1, we can bring the infinite Taylor expansion Π^∞ of a control system into its formal normal form Π^∞_{NF}:

$$\Pi^\infty \xrightarrow{formal} \Pi^\infty_{NF}.$$

Theorem 3.1 says that any smooth (resp. analytic) strict feedforward system can be transformed into its strict feedforward normal forms via a smooth (resp. analytic) feedback transformation, locally or globally:

$$\Pi_{SFF} \xrightarrow[C^\omega]{C^\infty} \Pi_{SFNF}.$$

Provided that the linear approximation is controllable, the linearizability index of a general (SFF)-system on $\mathbb{R}^2$ is at least one while the linearizability index of a general control-affine system on $\mathbb{R}^3$ is at least two. It follows that in those two cases the functions h_j are not invariant (compare Theorem 3.2), which implies the following:

Corollary 3.4. *Any smooth (resp. analytic) strict feedforward form (SFF) on $\mathbb{R}^2$, given by (3.1)-(3.2), is feedback equivalent to the* normal form

$$\Pi_{SSFNF}: \begin{cases} \dot z_1 &= z_2 + v^2 P_{1,3}(z_2, v) \\ \dot z_2 &= v, \end{cases}$$

where $P_{1,3}$ is a smooth (resp. analytic) function of the indicated variables.

Any smooth (resp. analytic) control-affine strict feedforward (SFF) on $\mathbb{R}^3$ is feedback equivalent to the normal form

$$\Pi_{SSFNF}: \begin{cases} \dot z_1 &= z_2 + z_3^2 P_{1,3}(z_2, z_3) \\ \dot z_2 &= z_3 \\ \dot z_3 &= v. \end{cases}$$

where $P_{1,3}$ is a smooth (resp. analytic) function of the indicated variables.

Normal forms for strict feedforward systems on $\mathbb{R}^2$ with noncontrollable linearization are given in [40].

Examples

Example 3.5. Cart-Pole System. In this example we consider a cart-pole system that is represented by a cart with an inverted pendulum on it [34], [58]. The Lagrangian equations of motion for the cart-pole system are

$$\begin{aligned} (m_1+m_2)\ddot q_1 + m_2 l\cos(q_2)\ddot q_2 &= m_2 l \sin(q_2)\dot q_2^2 + F \\ \cos(q_2)\ddot q_1 + l\ddot q_2 &= g\sin(q_2), \end{aligned}$$

where m_1 and q_1 are the mass and position of the cart, m_2, l, $q_2 \in (-\pi/2, \pi/2)$ are the mass, length of the link, and angle of the pole, respectively.

Taking $\ddot q_2 = u$ and applying the feedback law (see [34])

$$F = -ul(m_1 + m_2\sin^2(q_2))/\cos(q_2 + (m_1+m_2)g\tan(q_2) - m_2 l\sin(q_2)\dot q_2^2,$$

the dynamics of the cart-pole system are transformed into

$$\begin{cases} \dot x_1 &= x_2, \\ \dot x_2 &= g\tan(x_3) - lu/cos(x_3), \\ \dot x_3 &= x_4, \\ \dot x_4 &= u, \end{cases} \tag{3.8}$$

where we take $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$.

This system is in strict feedforward form (SFF) with the linearizability index $p = 2$. We showed in [38] that the diffeomorphism

$$z = \sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x))^\top$$

defined by

$$\begin{aligned} z_1 &= \sigma_1(x) = \mu x_1 + \mu l \int_0^{x_3} \frac{\mathrm{d}s}{\cos s}, \\ z_2 &= \sigma_2(x) = \mu x_2 + \mu l \frac{x_4}{\cos x_3} \\ z_3 &= \sigma_3(x) = \mu g \tan x_3, \\ z_4 &= \sigma_4(x) = \mu g \frac{x_4}{\cos^2 x_3} \end{aligned}$$

takes the system into its canonical form Σ_{SFCF}:

$$\begin{cases} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3 + \frac{z_3}{(1+(g/l)z_3^2)^{3/2}} z_4^2, \\ \dot{z}_3 &= z_4, \\ \dot{z}_4 &= v. \end{cases}$$

In the next example, we consider a case where the linearization about the origin is not controllable.

Example 3.6. (**Kapitsa Pendulum**) We consider in this example the Kapitsa Pendulum whose equations (see [5] and [8]) are given by

$$\begin{cases} \dot{\alpha} &= p + \frac{w}{l} \sin\alpha \\ \dot{p} &= (gl - \frac{w^2}{l^2}\cos\alpha)\sin\alpha - \frac{w}{l} p \cos\alpha \\ \dot{z} &= w, \end{cases} \tag{3.9}$$

where α denotes the angle of the pendulum with the vertical axis z, w the velocity of the suspension point z, p is proportional to the generalized impulsion, g is the gravity constant and l the length of the pendulum.

Assume we control the acceleration $a = \dot{w}$. Introducing the coordinates system $(x_1, x_2, x_3, x_4) = (\alpha, p, z/l, w/l)$, we take $u = a/l$ as a control.

The system (3.9) considered around an equilibrium point $(\alpha_0, p_0, z_0, u_0) = (k\pi, 0, 0, 0)$, rewrites

$$\begin{cases} \dot{x}_1 &= x_2 + x_1 x_4 + x_4 P_1(x_1) \\ \dot{x}_2 &= \epsilon g_0 x_1 - x_2 x_4 + x_2 x_4 P_2(x_1) + x_4^2 Q_2(x_1) + R_2(x_1) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u, \end{cases} \tag{3.10}$$

where $g_0 = g\,/l$, $\epsilon = \pm 1$, P_1, P_2, R_2 are analytic functions whose 1-jets at $(k\pi, 0, 0, 0)$ vanish and Q_2 an analytic function vanishing at $(k\pi, 0, 0, 0)$. Above, $\epsilon = 1$ corresponds to $\alpha_0 = 2n\pi$ and $\epsilon = -1$ to $\alpha_0 = (2n + 1)\pi$.

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We can notice that the linearization about any equilibrium point is uncontrollable with a 2-dimensional controllable part. Since the projection of the system on the controlled variables is in linear canonical form, hence in strict feedforward form, we expect the system to be brought to its normal form via analytic feedback transformation (according to Theorem 3.1).

Indeed, one can easily check that the quadratic feedback transformation

$$\Gamma^2 \ : \ \begin{array}{rcl} y_1 &=& x_1 - x_1x_3 \\ y_2 &=& x_2 + x_2x_3 \\ y_3 &=& x_3 \\ y_4 &=& x_4 \end{array}$$

brings the system (3.10) into the system

$$\begin{cases} \dot{y}_1 &=& y_2 + y_2y_3\tilde{P}_1(y_3) + y_4\tilde{Q}_1(y_1, y_3) \\ \dot{y}_2 &=& \epsilon g_0 y_1 + y_1y_3\tilde{P}_2(y_1, y_3) + y_4\tilde{Q}_2(y_1, y_2, y_3) + y_4^2\tilde{R}_2(y_1, y_3) + \tilde{S}_2(y_1) \\ \dot{y}_3 &=& y_4 \\ \dot{y}_4 &=& u \end{cases} \tag{3.11}$$

where $\tilde{P}_1, \tilde{Q}_1, \tilde{P}_2, \tilde{Q}_2, \tilde{R}_2$ and $\tilde{S}_2$ are analytic functions.

Since the vector field defined in $\mathbb{R}^3$ by

$$f = \tilde{Q}_1(y_1, y_3)\partial/\partial y_1 + \tilde{Q}_2(y_1, y_2, y_3)\partial/\partial y_2 + \partial/\partial y_3$$

does not vanish neither at $(0, 0, 0)^\top \in \mathbb{R}^3$ nor at $(\pi, 0, 0)^\top \in \mathbb{R}^3$, there exists an analytic transformation $z = \Phi(y)$ of the form

$$\begin{array}{rcl} z_1 &=& \phi_1(y_1, y_2, y_3) \\ z_2 &=& \phi_2(y_1, y_2, y_3) \\ z_3 &=& y_3 \end{array}$$

such that

$$(\Phi_* f)(z) = \partial/\partial z_3.$$

This latter transformation, completed by $z_4 = y_4$ and $u = v$, brings the system (3.11) into its normal form

$$\begin{cases} \dot{z}_1 &=& z_2 + R_1(z_1, z_2) + z_3\bar{P}_1(\bar{z}_3) + z_4^2 Q_{1,4}(\bar{z}_3) \\ \dot{z}_2 &=& \epsilon g_0 z_1 + R_2(z_1, z_2) + z_3\bar{P}_2(\bar{z}_3) + z_4^2 Q_{2,4}(\bar{z}_3) \\ \dot{z}_3 &=& z_4 \\ \dot{z}_4 &=& v, \end{cases}$$

where $\bar{z}_3 = (z_1, z_2, z_3)$. Notice that

$$\bar{P}_j(\bar{z}_3) = S_j(z_1, z_2) + z_3 P_{j,3}(z_1, z_2, z_3), \quad j = 1, 2.$$

3.1. Proof of Theorem 3.1

Consider a system Π_{SFF} in strict feedforward form with linearizability index $p = n - r$ and uncontrollable linearization of dimension s. The system Π_{SFF} is given by (3.1)-(3.2)-(3.3). Since the projection Π^{s+1} on $\mathbb{R}^{n-q}$ depends exclusively on the controllable variables $x_{s+1}, \ldots, x_n$ we will first show that Π^{s+1} can be brought to its normal form. For simplicity in the notation, we will assume $s = 0$. Notice that a short constructive proof was given in [41] and an alternative proof in [56] in the case of controllable linearization. For the sake of completeness we will provide a more detailed proof here that generalizes to the uncontrollable linearization. Without loss of generality we assume the system in the form

$$\begin{cases} \dot{x}_1 &= h_1(x_2) + F_1(x_2, \ldots, x_n, u) \\ \dot{x}_2 &= h_2(x_3) + F_2(x_3, \ldots, x_n, u) \\ & \cdots \\ \dot{x}_r &= h_r(x_{r+1}) + F_r(x_{r+1}, \ldots, x_n, u) \\ \dot{x}_{r+1} &= x_{r+2} \\ & \cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u, \end{cases} \tag{3.12}$$

where h_j, and F_j are smooth functions such that

$$\begin{aligned} h_j(x_{j+1}) &= k_j x_{j+1} \\ F_j(x_{j+1}, 0, \ldots, 0) &= 0 \end{aligned} \tag{3.13}$$

for any $1 \leq j \leq r - 1$.

We will provide a constructive algorithmic proof by defining explicit changes of coordinates whose composition takes the system into its normal form. The algorithm will be divided into n major steps. The first step consists of normalizing linear terms in u in the first $n-1$ components. Then, in the second step we will normalize linear terms in x_n in the first $n-2$ components, and so on. The algorithm consists of at most $(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n-1)}{2}$ changes of coordinates. Actually, there are fewer changes of coordinates if the linearizability index $p > 2$.

Applying the change of coordinates and feedback

$$\begin{aligned} z_j &= k_1 \cdots k_{j-1} x_j, & 1 \leq j \leq r \\ z_{r+1} &= k_1 \cdots k_{r-1} h_r(x_{r+1}), & \\ z_{j+1} &= \dot{z}_j, & r+1 \leq j \leq n-1 \\ v &= \dot{z}_n, & \end{aligned}$$

we can assume $h_j(x_{j+1}) = x_{j+1}$ for $1 \leq j \leq r$.

Step 1. Denote the system (3.12)-(3.13) by Π_{n+1}. We first decompose the component $F_r(x_{r+1}, \ldots, x_n, u)$ uniquely as

$$F_r(x_{r+1}, \ldots, x_n, u) = \bar{F}_r(x_{r+1}, \ldots, x_n) + u\Theta_{r,n+1}(x_{r+1}, \ldots, x_n) + u^2 P_{r,n+1}(x_{r+1}, \ldots, x_n, u)$$

with $\bar{F}_r(x_{r+1}, 0, \ldots, 0) = 0$.

The diffeomorphism $z = \sigma_{\boldsymbol{r}}^{n+1}(x)$ whose components are

$$\begin{aligned} z_j &= \sigma_{\boldsymbol{r}j}^{n+1}(x) = x_j, \ \text{if } j \neq r \\ z_r &= \sigma_{\boldsymbol{rr}}^{n+1}(x) = x_r - \int_0^{x_n} \Theta_{r,n+1}(x_{r+1}, \ldots, x_{n-1}, \epsilon)\mathrm{d}\epsilon, \end{aligned}$$

allows to normalize the linear terms in u in the rth component, and transforms it as

$$\dot{z}_r = z_{r+1} + \hat{F}_r(z_{r+1}, \ldots, z_n) + u^2 P_{r,n+1}(z_{r+1}, \ldots, z_n, u),$$

where

$$\hat{F}_r(z_{r+1}, \ldots, z_n) = \bar{F}_r(z_{r+1}, \ldots, z_n) - \sum_{k=r+1}^{n-1} z_{k+1} \int_0^{z_n} \frac{\partial \Theta_{r,n+1}}{\partial z_k}(z_{r+1}, \ldots, z_{n-1}, \epsilon)\mathrm{d}\epsilon.$$

Notice that the inverse of $z = \sigma_{\boldsymbol{r}}^{n+1}(x)$, say $x = \eta_{\boldsymbol{r}}^{n+1}(z)$, is given by

$$\begin{aligned} x_j &= \eta_{\boldsymbol{r}j}^{n+1}(z) = z_j, \ \text{if } j \neq r \\ x_r &= \eta_{\boldsymbol{rr}}^{n+1}(z) = z_r + \int_0^{z_n} \Theta_{r,n+1}(z_{r+1}, \ldots, z_{n-1}, \epsilon)\mathrm{d}\epsilon, \end{aligned}$$

Assume that the linear terms in u are normalized in the rth component through the $(i+1)$st component, that is,

$$\begin{cases} \dot{x}_1 &= x_2 + F_1(x_2, \ldots, x_n, u) \\ \dot{x}_2 &= x_3 + F_2(x_3, \ldots, x_n, u) \\ & \cdots \\ \dot{x}_r &= x_{r+1} + F_r(x_{r+1}, \ldots, x_n, u) \\ \dot{x}_{r+1} &= x_{r+2} \\ & \cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u, \end{cases}$$

where F_j are smooth (resp. analytic) functions such that

$$F_j(x_{j+1}, 0, \ldots, 0) = 0$$

and moreover,

$$F_j(x_{j+1}, \ldots, x_n, u) = \hat{F}_j(x_{j+1}, \ldots, x_n) + u^2 P_{j,n+1}(x_{j+1}, \cdots, x_n, u)$$

for any $i + 1 \leq j \leq r$.

Decompose the ith component $F_i(x_{i+1}, \ldots, x_n, u)$ uniquely as

$$F_i(x_{i+1}, \ldots, x_n, u) = \bar{F}_i(x_{i+1}, \ldots, x_n) + u\Theta_{i,n+1}(x_{i+1}, \ldots, x_n) + u^2 P_{i,n+1}(x_{i+1}, \ldots, x_n, u)$$

with $\bar{F}_i(x_{i+1}, 0, \ldots, 0) = 0$.

The diffeomorphism $z = \sigma_{\boldsymbol{i}}^{n+1}(x)$ whose components are

$$\begin{aligned} z_j &= \sigma_{\boldsymbol{i}j}^{n+1}(x) = x_j, \ \text{if } j \neq i \\ z_i &= \sigma_{\boldsymbol{ii}}^{n+1}(x) = x_i - \int_0^{x_n} \Theta_{i,n+1}(x_{i+1}, \ldots, x_{n-1}, \epsilon)\mathrm{d}\epsilon, \end{aligned}$$

allows to normalize the linear terms in u in the ith component, and transforms that component as

$$\dot{z}_i = z_{i+1} + \hat{F}_i(z_{i+1}, \ldots, z_n) + u^2 P_{i,n+1}(z_{i+1}, \ldots, z_n, u),$$

where

$$\hat{F}_i(z_{i+1}, \ldots, z_n) = \bar{F}_i(z_{i+1}, \ldots, z_n) - \sum_{k=i+1}^{n-1} z_{k+1} \int_0^{z_n} \frac{\partial \Theta_{i,n+1}}{\partial z_k}(z_{i+1}, \ldots, z_{n-1}, \epsilon) \mathrm{d}\epsilon.$$

Notice that the inverse of $z = \sigma_{\boldsymbol{i}}^{n+1}(x)$, say $x = \eta_{\boldsymbol{i}}^{n+1}(z)$, is given by

$$\begin{aligned} x_j &= \eta_{\boldsymbol{i}j}^{n+1}(z) = z_j, \text{ if } j \neq i \\ x_i &= \eta_{\boldsymbol{i}i}^{n+1}(z) = z_i + \int_0^{z_n} \Theta_{i,n+1}(z_{i+1}, \ldots, z_{n-1}, \epsilon) \mathrm{d}\epsilon. \end{aligned}$$

The recursive method can be then applied to define diffeomorphisms $\sigma_{\boldsymbol{r}}^{n+1}, \ldots, \sigma_{\boldsymbol{1}}^{n+1}$, allowing to normalize the linear terms in u of the corresponding component.

The composition $\sigma^{n+1} = \sigma_{\boldsymbol{1}}^{n+1} \circ \cdots \circ \sigma_{\boldsymbol{r}}^{n+1}$ of the successive coordinates changes transforms the system into (we reset the variable to x)

$$\begin{cases} \dot{x}_1 = x_2 + F_1(x_2, \ldots, x_n, u) \\ \dot{x}_2 = x_3 + F_2(x_3, \ldots, x_n, u) \\ \quad \cdots \\ \dot{x}_r = x_{r+1} + F_r(x_{r+1}, \ldots, x_n, u) \\ \dot{x}_{r+1} = x_{r+2} \\ \quad \cdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = u, \end{cases}$$

where F_j are smooth (resp. analytic) functions such that

$$F_j(x_{j+1}, 0, \ldots, 0) = 0, \quad 1 \leq j \leq r,$$

and moreover,

$$F_j(x_{j+1}, \ldots, x_n, u) = \hat{F}_j(x_{j+1}, \ldots, x_n) + u^2 P_{j,n+1}(x_{j+1}, \ldots, x_n, u)$$

for any $1 \leq j \leq r$.

Step 2. Let $\Pi_n = \sigma^{n+1}(\Pi_{n+1})$ be the system Π_{n+1} transformed via the diffeomorphism σ^{n+1}.

We deduce from above that $\hat{F}_j(x_{j+1}, 0, \ldots, 0) = 0$ for $1 \leq j \leq r$. Let decompose the component $\hat{F}_r(x_{r+1}, \ldots, x_n)$ uniquely as following

$$\hat{F}_r(x_{r+1}, \ldots, x_n) = \bar{F}_r(x_{r+1}, \ldots, x_{n-1}) + x_n \Theta_{r,n}(x_{r+1}, \ldots, x_{n-1}) + x_n^2 P_{r,n}(x_{r+1}, \ldots, x_n)$$

with $\bar{F}_r(x_{r+1}, 0, \ldots, 0) = 0$.

The diffeomorphism $z = \sigma_{\boldsymbol{r}}^n(x)$ whose components are

$$\begin{aligned} z_j &= \sigma_{\boldsymbol{r}j}^n(x) = x_j, \text{ if } j \neq r \\ z_r &= \sigma_{\boldsymbol{r}r}^n(x) = x_r - \int_0^{x_{n-1}} \Theta_{r,n}(x_{r+1}, \ldots, x_{n-2}, \epsilon) \mathrm{d}\epsilon, \end{aligned}$$

allows to normalize the linear terms in x_n in the rth component, and transforms the component as

$$\dot{z}_r = z_{r+1} + \hat{F}_r(z_{r+1}, \ldots, z_{n-1}) + z_n^2 P_{r,n}(z_{r+1}, \ldots, z_n) + u^2 P_{r,n+1}(z_{r+1}, \ldots, z_n, u),$$

where

$$\hat{F}_r(z_{r+1}, \ldots, z_{n-1}) = \bar{F}_r(z_{r+1}, \ldots, z_{n-1}) - \sum_{k=r+1}^{n-2} z_{k+1} \int_0^{z_{n-1}} \frac{\partial \Theta_{r,n}}{\partial z_k}(z_{r+1}, \ldots, z_{n-2}, \epsilon) d\epsilon.$$

Following the same line as in *step 1*, we would suppose that coordinates changes $\sigma_{\boldsymbol{r}}^n, \ldots, \sigma_{\boldsymbol{i+1}}^n$ has been defined such that their composition transforms the original system into (we keep the variable as x)

$$\begin{cases} \dot{x}_1 = x_2 + F_1(x_2, \ldots, x_n, u) \\ \dot{x}_2 = x_3 + F_2(x_3, \ldots, x_n, u) \\ \quad \cdots \\ \dot{x}_r = x_{r+1} + F_r(x_{r+1}, \ldots, x_n, u) \\ \dot{x}_{r+1} = x_{r+2} \\ \quad \cdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = u, \end{cases}$$

where F_j are smooth (resp. analytic) functions such that

$$F_j(x_{j+1}, 0, \ldots, 0) = 0, \quad 1 \le j \le r,$$

and moreover,

$$F_j(z_{j+1}, \ldots, z_n) = \hat{F}_j(x_{j+1}, \ldots, x_{n-1}) + x_n^2 P_{j,n}(x_{j+1}, \ldots, x_n) + u^2 P_{j,n+1}(x_{j+1}, \ldots, x_n, u)$$

for any $i+1 \le j \le r$.

Decompose the ith component $\hat{F}_i(x_{i+1}, \ldots, x_n, u)$ as

$$\begin{aligned} \hat{F}_i(x_{i+1}, \ldots, x_n) = {} & \bar{F}_i(x_{i+1}, \ldots, x_{n-1}) + x_n \Theta_{i,n}(x_{i+1}, \ldots, x_{n-1}) \\ & + x_n^2 P_{i,n}(x_{i+1}, \ldots, x_n) + u^2 P_{i,n+1}(x_{i+1}, \ldots, x_n, u) \end{aligned}$$

with $\bar{F}_i(x_{i+1}, 0, \ldots, 0) = 0$.

The diffeomorphism $z = \sigma_{\boldsymbol{i}}^n(x)$ whose components are

$$\begin{aligned} z_j &= \sigma_{\boldsymbol{ij}}^n(x) = x_j, \text{ if } j \ne i \\ z_i &= \sigma_{\boldsymbol{ii}}^n(x) = x_i - \int_0^{x_{n-1}} \Theta_{i,n+1}(x_{i+1}, \ldots, x_{n-2}, \epsilon) d\epsilon, \end{aligned}$$

allows to normalize the linear terms in x_n in the ith component, and transforms the ith component as

$$\dot{z}_i = z_{i+1} + \hat{F}_i(z_{i+1}, \ldots, z_{n-1}) + z_n^2 P_{i,n}(z_{i+1}, \ldots, z_n) + u^2 P_{i,n+1}(z_{i+1}, \ldots, z_n, u),$$

where

$$\hat{F}_i(z_{i+1},\dots,z_{n-1}) = \bar{F}_i(z_{i+1},\dots,z_{n-1}) - \sum_{k=i+1}^{n-2} z_{k+1}\int_0^{z_{n-1}} \frac{\partial\Theta_{i,n}}{\partial z_k}(z_{i+1},\dots,z_{n-2},\epsilon)\mathrm{d}\epsilon.$$

Keeping up with the algorithm we define, successively, $\sigma^n_{\boldsymbol{r}},\dots,\sigma^n_{\mathbf{1}}$ whose composition is the diffeomorphism $\sigma^n = \sigma^n_{\mathbf{1}}\circ\cdots\circ\sigma^n_{\boldsymbol{r}}$ that takes the system into

$$\begin{cases} \dot{x}_1 &= x_2 + F_1(x_2,\dots,x_n,u)\\ \dot{x}_2 &= x_3 + F_2(x_3,\dots,x_n,u)\\ &\dots\\ \dot{x}_r &= x_{r+1} + F_r(x_{r+1},\dots,x_n,u)\\ \dot{x}_{r+1} &= x_{r+2}\\ &\dots\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= u, \end{cases}$$

where F_j are smooth (resp. analytic) functions such that

$$F_j(z_{j+1},\dots,z_n) = \hat{F}_j(x_{j+1},\dots,x_{n-1}) + x_n^2P_{j,n}(x_{j+1},\dots,x_n) + u^2P_{j,n+1}(x_{j+1},\dots,x_n,u)$$

for any $1\le j\le r$ with

$$\hat{F}_j(x_{j+1},0,\dots,0) = 0.$$

We notice that all coordinate changes defined in *step 2* depend only on the variables $x_1,\dots,x_{n-1}$ but not on the variables (x_n,u), which is the reason why the linear terms in u are not created after the completion of *step 1*. Now, *step 2*, like the remaining steps of the algorithm, could have been viewed as *step 1* carried over on lower dimensional systems. Indeed, taking $\hat{\Pi}_n$ as the restriction of system Π_n on $\mathbb{R}^{n-1}$ with coordinates $(x_1,\dots,x_{n-1})$ and control x_n, *step 1* would be applied to normalize the linear terms in the new control x_n.

Starting from the original system Π_{n+1}, we then define a successive sequence of diffeomorphisms σ^{k+1} given in each step as $\sigma^{k+1} = \sigma^{k+1}_{\mathbf{1}}\circ\cdots\circ\sigma^{k+1}_{\boldsymbol{r}}$ for $k = n, n-1,\dots,2$ yielding a successive sequence of strict feedforward systems $\Pi_n,\Pi_{n-1},\dots,\Pi_2$, where for any $2\le k\le n$, the system Π_k is the transform of Π_{k+1} via σ^{k+1}. Moreover, each system Π_k is in the form

$$\begin{cases} \dot{x}_1 &= x_2 + F_1(x_2,\dots,x_n,u)\\ \dot{x}_2 &= x_3 + F_2(x_3,\dots,x_n,u)\\ &\dots\\ \dot{x}_r &= x_{r+1} + F_r(x_{r+1},\dots,x_n,u)\\ \dot{x}_{r+1} &= x_{r+2}\\ &\dots\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= u, \end{cases}$$

where for any $1\le j\le r$

$$F_j(x_{j+1},\dots,x_n,u) = \hat{F}_j(x_{j+1},\dots,x_k) + \sum_{i=k+1}^{n+1} x_i^2P_{j,i}(x_{j+1},\dots,x_i)$$

with $\hat{F}_j(x_{j+1}, 0, \ldots, 0) = 0$. The functions $P_{j,i}(x_{j+1}, \ldots, x_i)$ are smooth (resp. analytic) in their arguments, and are zero if $i \leq j+1$.

The composition $\sigma(x) = \sigma^3 \circ \cdots \circ \sigma^{n+1}(x)$ of these diffeomorphisms transforms the original system Π_{n+1} into its strict feedforward normal form, which indeed coincides with Π_2.

To complete the proof we need to show that the uncontrollable part can be brought to a normal form without changing the components of the controllable part, already in normal form.

Reconsider Π_{SFF} given by (3.1)-(3.2)-(3.3), and assume its linear uncontrollable part to be of dimension s. We will denote, for the sake of clarity, the controllable variables $x_{s+1}, \ldots, x_n$ by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$, where $m = n - s$. Without loss of generality, we can assume Π_{SFF} in the form

$$\Pi_{SFF} : \begin{cases} \dot{x}_1 &= F_1(x_2, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u), \\ & \cdots \\ \dot{x}_j &= F_j(x_{j+1}, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u), \\ & \cdots \\ \dot{x}_s &= F_s(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u), \\ \dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2 + \sum_{i=3}^{n+1} \boldsymbol{x}_i^2 P_{1,i}(\boldsymbol{x}_2, \ldots, \boldsymbol{x}_i) \\ & \cdots \\ \dot{\boldsymbol{x}}_j &= \boldsymbol{x}_{j+1} + \sum_{i=j+2}^{n+1} \boldsymbol{x}_i^2 P_{j,i}(\boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_i) \\ & \cdots \\ \dot{\boldsymbol{x}}_{m-1} &= \boldsymbol{x}_m + u^2 P_{m-1,m+1}(\boldsymbol{x}_m, u) \\ \dot{\boldsymbol{x}}_m &= u. \end{cases}$$

The projection Π^{s+1} is already in normal form following the normalization algorithm underlined above.

This part of the proof follows a similar line as previously. Decompose the component $F_s(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u)$, uniquely as:

$$F_s(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u) = \hat{F}_s(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) + uQ_s(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) + u^2 P_s(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u).$$

The change of coordinates $(z, \boldsymbol{x}) = (\tau_{\boldsymbol{s}}^{m+1}(x), \boldsymbol{x})$ whose components are defined by

$$\begin{aligned} z_j &= \tau_{\boldsymbol{s}j}^{m+1}(x) = x_j, \quad 1 \leq j \leq s-1, \\ z_s &= \tau_{\boldsymbol{s}s}^{m+1}(x) = x_s - \int_0^{\boldsymbol{x}_m} Q_s(\boldsymbol{x}_1, .., \boldsymbol{x}_{m-1}, \epsilon) \mathrm{d}\epsilon \end{aligned}$$

allows to cancel the linear terms in u in the sth component. Assume that linear terms in u in the sth-component through the $(i+1)$st component have been canceled. Then decompose the ith component $F_i(x_{i+1}, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u)$ uniquely as follows:

$$\begin{aligned} F_i(x_{i+1}, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u) &= \hat{F}_i(x_{i+1}, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) + uQ_i(x_{i+1}, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \\ &\quad + u^2 P_i(x_{i+1}, \ldots, x_s, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, u). \end{aligned}$$

The change of coordinates $(z, \boldsymbol{x}) = (\tau_{\boldsymbol{i}}^{m+1}(x), \boldsymbol{x})$ whose first s components are defined by

$$\begin{aligned} z_j &= \tau_{\boldsymbol{ij}}^{m+1}(x) = x_j, \quad 1 \le j \le i-1, \\ z_i &= \tau_{\boldsymbol{ii}}^{m+1}(x) = x_i - \int_0^{\boldsymbol{x}_m} Q_i(x_{i+1}, .., x_s, \boldsymbol{x}_1, .., \boldsymbol{x}_{m-1}, \epsilon)\mathrm{d}\epsilon \\ z_j &= \tau_{\boldsymbol{ij}}^{m+1}(x) = x_j, \quad i+1 \le j \le s \end{aligned}$$

allows to cancel the linear terms in u in the ith component.

The composition $\tau^{m+1} = \tau_{\boldsymbol{1}}^{m+1} \circ \cdots \circ \tau_{\boldsymbol{s}}^{m+1}$ allows to cancel all linear terms in u in all s components.

Similarly, we define transformations $\tau^m, \ldots, \tau^2$, where for any $2 \le k \le m+1$, $\tau^k = \tau_{\boldsymbol{1}}^k \circ \cdots \circ \tau_{\boldsymbol{s}}^k$ is the transformation that linearizes the terms in $\boldsymbol{x}_k$ in all the s components. Notice that the linear terms in $\boldsymbol{x}_1$ cannot be canceled in any of the first s components. The composition $\tau = \tau^2 \circ \cdots \circ \tau^{m+1}$ takes the system into its normal form. Each transformation is smooth (resp. analytic) and the algorithm involves a finite number of such transformations whose inverses are also smooth (resp. analytic). Moreover, due to the structure of the strict feedforward form, the components of the controllable part remain unchanged during the normalization of the uncontrollable part. This completes the proof of the theorem.

4. Symmetries of Nonlinear Systems

We will first recall our results on symmetries obtained in the single-input case using the canonical form. In the second subsection, we will give explicit symmetries of strict feedforward systems, and finally we will discuss, in term of symmetries, the feedback equivalence to a strict feedforward system

4.1. Symmetries via Canonical Form

Consider the single-input control-affine system

$$\Sigma : \ \dot{x} = f(x) + g(x)u,$$

where $x \in X$, is an open subset of $\mathbb{R}^n$, and $u \in U = \mathbb{R}$, f and g are smooth vector fields on X. The *field of admissible velocities* is the field of affine lines

$$\mathcal{A}(x) = \{f(x) + ug(x) \ : \ u \in \mathbb{R}\} \subset T_xX.$$

A diffeomorphism $\psi : X \longrightarrow X$ is a symmetry of Σ if it preserves the field of affine lines $\mathcal{A}$ (in other words, the affine distribution $\mathcal{A}$ of rank 1), that is, if $\psi_*\mathcal{A} = \mathcal{A}$.

A *local symmetry* at $p \in X$ is a local diffeomorphism ψ of X_0 onto X_1, where X_0 and X_1 are, respectively, neighborhoods of p and $\psi(p)$, such that

$$(\psi_*\mathcal{A})(q) = \mathcal{A}(q) \ \text{ for any } q \in X_1.$$

A local symmetry ψ at p is called a *stationary symmetry* if $\psi(p) = p$ and a *nonstationary symmetry* if $\psi(p) \neq p$.

Symmetries take a very simple form if we bring the system into its canonical form. Indeed, we have the following result (see [38] and [39] for proofs and details):

Proposition 4.1. *Assume that the system Σ is analytic, the linear approximation (F, G) of Σ at an equilibrium point p is controllable and Σ is not locally feedback linearizable at p. Assume, moreover, that the local feedback transformation, bringing Σ into its canonical form Σ_{CF}, is analytic at p.*

(i) *Σ admits a nontrivial local stationary symmetry if and only if the drift $\bar{f}(x) = Ax + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(x)$ of the canonical form Σ_{CF}^{∞} satisfies*

$$\bar{f}(x) = -\bar{f}(-x),$$

that is, the system is odd.

(ii) *Σ admits a nontrivial local nonstationary symmetry if and only if the drift $\bar{f}(x)$ of the canonical form Σ_{CF}^{∞} satisfies*

$$\bar{f}(x) = \bar{f}(x_1 + c_1, x_2, \ldots, x_n), \quad \text{for some } c_1 \in \mathbb{R}$$

that is $\bar{f}$ is periodic with respect to x_1.

(iii) *Σ admits a nontrivial local 1-parameter family of symmetries if and only if the drift $\bar{f}(x)$ of the canonical form Σ_{CF}^{∞} satisfies*

$$\bar{f}(x) = \bar{f}(x_2, \ldots, x_n).$$

In the case the diffeomorphism transforming the system Σ into its canonical form Σ_{CF} is not analytic, we have obtained formal symmetries (see [39, 38]).

4.2. Explicit Symmetries of Strict Feedforward Systems

In this subsection we consider the case when Π is affine in control, that is, the class of *smooth* (resp. *analytic*) single-input control systems in *strict feedforward form* (SFF)

$$\Sigma_{SFF} \; : \begin{cases} \dot{x} = f(x) + g(x)u, \\ f_j(x) = f_j(x_{j+1}, \ldots, x_n), \quad 1 \le j \le n-1, \\ g_j(x) = g_j(x_{j+1}, \ldots, x_n), \quad 1 \le j \le n-1 \\ f_n(x) = f_n \in \mathbb{R} \;\text{ and }\; g_n(x) = g_n \in \mathbb{R}^*. \end{cases}$$

Following section 3, for any $1 \le i \le n$, the subsystem Σ_{SFF}^i denotes the projection of Σ_{SFF} onto $\mathbb{R}^{n-i+1}$ via $\pi_i(x_1, \ldots, x_n) = (x_i, \ldots, x_n)$ with dynamics given by

$$\dot{x}_j = f_j(x_{j+1}, \ldots, x_n) + g_j(x_{j+1}, \ldots, x_n)u, \quad i \le j \le n.$$

The linearizability index of Σ_{SFF} is thus the largest integer p such that the subsystem Σ_{SFF}^{r+1}, where $p + r = n$, is feedback linearizable. We will assume here that the linear approximation around the origin is controllable which implies that $p \ge 2$. The set $\mathcal{E} = \{x^e \in \mathbb{R}^n \,|\, f(x^e) = 0\}$ of equilibrium points consists of lines parallel to the x_1-axis; in other words, any equilibrium point is of the form $x^e = (c_1, \ldots, c_{r+1}, 0, \ldots, 0)^\top$. For any nonzero real numbers $\lambda_1, \ldots, \lambda_r, \lambda \in \mathbb{R}^*$ and any $c_1, \ldots, c_{r+1} \in \mathbb{R}$, put $\Lambda = (\lambda_1, \ldots, \lambda_r, \lambda, \ldots, \lambda)$ and $C = (c_1, \ldots, c_{r+1}, 0, \ldots, 0)$ and define a *scaling translation* by

$$\mathbb{T}_{\Lambda,C}(x) = (\lambda_1 x_1 + c_1, \ldots, \lambda_n x_n + c_n)^\top,$$

with $c_{r+2} = \cdots = c_n = 0$ and $\lambda_{r+1} = \cdots = \lambda_n = \lambda$.

Theorem 4.2. *Consider a smooth system Σ_{SFF} in strict feedforward form with linearizability index $p = n - r$. Any symmetry ψ of Σ_{SFF} is of the form*

$$\psi = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma,$$

for a fixed (Λ, C), where $z = \sigma(x)$ is the diffeomorphism of the transformation taking Σ_{SFF} into its strict feedforward normal form Σ_{SFNF} given by Definition 4.3 below. Any local 1-parameter family of symmetries ψ_{c_1} of Σ_{SFF} is of the same form with $c_1 \in (-\epsilon_1, \epsilon_1)$.

Theorem 4.2 says basically that strict feedforward systems have 1-parameter families of symmetries conjugated to *scaling translations*. Recall that in [38] we showed that any symmetry is conjugated to at most two 1-parameter families of translations along the first variable; those translations being the only symmetries of the canonical form (Proposition 4.1).

The constant parameters $\lambda_1, \ldots, \lambda_r, \lambda$ are likely to be either $+1$ or -1 and will be uniquely determined by $c_2, \ldots, c_r$ (given by other equilibrium point) because, together, they should satisfy some strong conditions (SC), see below. The only free parameter is c_1. In Example 4.9 we provide a case where some of the parameters $\lambda_1, \ldots, \lambda_r, \lambda$ are not equal to $+1$ or -1 as well as some constants $c_2, \ldots, c_{r+1}$ that are non zero. We then compare the results obtained here with those of [38], and show no ambiguity between them.

The importance of this result is that we can always put a (SFF)-system into a strict feedforward normal form (SFNF) via smooth feedback transformation while the canonical form is only guaranteed in the formal category. Moreover, the feedback transformation taking the system into its strict feedforward normal form (SFNF) can be constructed explicitly, for smooth systems, see section 3.1.

The notion of strict feedforward normal form plays a crucial role in proving Theorem 4.2 and, in the affine case, takes the following form.

Definition 4.3. A smooth *strict feedforward normal form*, denoted Σ_{SFNF}, is a strict feedforward form

$$\begin{cases} \dot{x}_1 &= \hat{F}_1(x_2, \ldots, x_n) \\ & \ldots \\ \dot{x}_r &= \hat{F}_r(x_{r+1}, \ldots, x_n) \\ \dot{x}_{r+1} &= x_{r+2} \\ & \ldots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u \end{cases}$$

for which $p = n - r$ is the linearizability index and

$$(SFNF) \qquad \hat{F}_j(x) = h_j(x_{j+1}) + \sum_{i=j+2}^{n} x_i^2 P_{j,i}(x_{j+1}, \ldots, x_i)$$

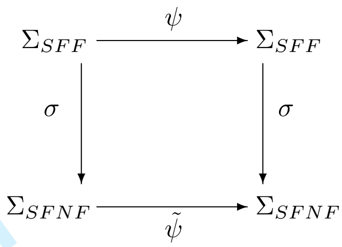
for any $1 \le j \le r$, where h_j and $P_{j,i}$ are smooth functions of the indicated variables.

The above strict feedforward normal form Σ_{SFNF} was introduced in [41], where we proved the following:

Theorem 4.4. *Any smooth strict feedforward form can be transformed into a strict feedforward normal form via smooth feedback transformation.*

Remark 4.5. (i) The explicit construction of the feedback transformation (in particular, the diffeomorphism $z = \sigma(x)$) taking a (SFF)-system into its (SFNF), was given in the proof of Theorem 3.1. We have assume there, without loss of generality, that $h_j(x_{j+1}) = x_{j+1}$ but the algorithm remains the same.

Then using the commutative diagram



where $\tilde{\psi}$ is a symmetry of the strict feedforward normal form Σ_{SFNF}, all we will have to prove is that all $\tilde{\psi}$'s are exhausted by *scaling translations* $\mathbb{T}_{\Lambda,C}$ defined above.

(ii) We will use this item to deduce, as a corollary, necessary and sufficient condition for a system to be brought to a strict feedforward form (see Theorem II.4 of [40]).

B1. Proof of Theorem 4.2

We will prove Theorem 4.2 by showing that symmetries of systems in strict feedforward normal form Σ_{SFNF} are exhausted by *scaling translations* $\mathbb{T}_{\Lambda,C}$ defined above. Let us consider a system in the strict feedforward normal form Σ_{SFNF}, given by definition 4.3.

Notice that if $\tilde{x} = \tilde{\psi}(x)$ is a symmetry of Σ_{SFF} (in particular, of Σ_{SFNF}), then it preserves the structure of the strict feedforward form. Hence (see [51]), we have $\tilde{x}_j = \tilde{\psi}_j(x) = \tilde{\psi}_j(x_j, \ldots, x_{n-1})$ for $1 \leq j \leq n-1$. This implies that $\pi_r(\tilde{\psi}) = (\tilde{\psi}_r(x), \ldots, \tilde{\psi}_n(x))$ is a symmetry of the projection Σ^{r+1}_{SFNF} of Σ_{SFF} whose dynamics are

$$\begin{cases} \dot{x}_r &= h_r(x_{r+1}) + \sum\limits_{i=r+2}^{n} x_i^2 P_{r,i}(x_{r+1}, \ldots, x_i) \\ \dot{x}_{r+1} &= x_{r+2} \\ & \cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u. \end{cases}$$

We claim that $\tilde{\psi}_j(x) = \tilde{\psi}_j(x_j)$ for any $r \leq j \leq n-1$. Indeed, we have $\tilde{\psi}_{n-1}(x) = \tilde{\psi}_{n-1}(x_{n-1})$. Let k be the largest integer, $r \leq k \leq n-2$, such that $\dfrac{\partial \tilde{\psi}_k}{\partial x_s} \neq 0$ for some $s \geq k+1$ (we can take s to be the largest integer that yields this property). Thus

$$\dot{\tilde{x}}_k = \frac{\partial \tilde{\psi}_k}{\partial x_k} \dot{x}_k + \cdots + \frac{\partial \tilde{\psi}_k}{\partial x_s} x_{s+1} = \tilde{x}_{k+1} = \tilde{\psi}_{k+1}(x)$$

gives a contradiction because $\tilde{\psi}_{k+1}(x) = \tilde{\psi}_{k+1}(x_{k+1})$. We conclude that $\tilde{\psi}_j(x) = \tilde{\psi}_j(x_j)$ for $r \leq j \leq n-1$. Since

$$\dot{\tilde{x}}_j = \tilde{\psi}'_j(x_j) x_{j+1} = \tilde{x}_{j+1} = \tilde{\psi}_{j+1}(x_{j+1}),$$

we deduce that $\tilde{\psi}_j(x_j) = \lambda_j x_j + c_j$ for all $r+1 \leq j \leq n-1$. Similarly we get $\tilde{\psi}_r(x_r) = \lambda_r x_r + c_r$ and hence

$$\pi_r(\tilde{\psi}(x)) = (\lambda_r x_r + c_r, \lambda_{r+1}x_{r+1} + c_{r+1}, \ldots, \lambda_n x_n + c_n)^\top.$$

In fact, it is easy to see that $\lambda_{r+1} = \cdots = \lambda_n = \lambda$ and $c_{r+2} = \cdots = c_n = 0$ but for homogeneity of notation, we will carry those constants as such.

Notice that λ_r, and the pairs (λ_k, c_k), $r+1 \leq k \leq n$ should satisfy the strong condition:

$$(\mathrm{SC})_r \qquad \hat{F}_r(\lambda_{r+1}x_{r+1} + c_{r+1}, \ldots, \lambda_n x_n + c_n)) = \lambda_r \hat{F}_r(x_{r+1}, \ldots, x_n),$$

and

$$\hat{F}_r(x_{r+1}, \ldots, x_n) = h_r(x_{r+1}) + \sum_{i=r+2}^{n} x_i^2 P_{r,i}(x_{r+1}, \ldots, x_i).$$

We can remark that $(\mathrm{SC})_r$ is equivalent to the conditions

$$(\mathrm{SC})_a \qquad h_r(\lambda_{r+1}x_{r+1} + c_{r+1}) = \lambda_r h_r(x_{r+1})$$

$$(\mathrm{SC})_b \qquad P_{r,i}(\lambda_{r+1}x_{r+1} + c_{r+1}, \ldots, \lambda_i x_i + c_i) = \frac{\lambda_r}{\lambda_i^2} P_{r,i}(x_{r+1}, \ldots, x_i), \; r+2 \leq i \leq n.$$

A similar argument will imply that $\tilde{\psi}_j(x) = \tilde{\psi}(x_j)$ for all $1 \leq j \leq r-1$. Taking $j = r-1$, we should have

$$\dot{\tilde{x}}_{r-1} = \tilde{\psi}'_{r-1}(x_{r-1})\hat{F}_{r-1}(x_r, \ldots, x_n) = \hat{F}_{r-1}(\tilde{x}_r, \ldots, \tilde{x}_n)$$

which implies that $\tilde{\psi}'_{r-1}(x_{r-1}) = \lambda_{r-1}$, and consequently, $\tilde{\psi}_{r-1}(x_{r-1}) = \lambda_{r-1}x_{r-1} + c_{r-1}$.

A straightforward recurrence shows that for any $1 \leq j \leq r$, we have $\tilde{\psi}_j(x_j) = \lambda_j x_j + c_j$. At each step, the constant λ_j is related to the pairs (λ_k, c_k), for $j+1 \leq k \leq n$, by the strong conditions

$$(\mathrm{SC})_j \qquad \hat{F}_j(\lambda_{j+1}x_{j+1} + c_{j+1}, \ldots, , \lambda_i x_n + c_n) = \lambda_j \hat{F}_j(x_{j+1}, \ldots, x_n),$$

and

$$\hat{F}_j(x_{j+1}, \ldots, x_n) = h_j(x_{j+1}) + \sum_{i=j+2}^{n} x_i^2 P_{j,i}(x_{r+1}, \ldots, x_i).$$

Notice that the constant c_1 can be chosen arbitrarily. The proof of Theorem 4.2 is then completed by the commutative diagram giving the explicit diffeomorphism $z = \sigma(x)$ of the feedback transformation bringing Σ_{SFF} into its strict feedforward normal form (see Proof of Theorem 3.1).

B2. Feedback Equivalence to Strict Feedforward Systems

The problem of transforming a system, affine with respect to controls, into (strict) feedforward form via a nonlinear change of coordinates was studied in [31], and a geometric description of systems in feedforward form has been given in [2]. We proposed a step-by-step constructive method to bring a system into a feedforward form in [54] and into a strict feedforward form in [50].

Recently (see [40]), we have proved that feedback equivalence (resp. state-space equivalence) to the strict feedforward form can be characterized by the existence of a sequence of infinitesimal symmetries (resp. strong infinitesimal symmetries) of the system. We give here as a corollary, a restatement of the equivalence conditions obtained in [40] in terms of the symmetries of strict feedforward systems.

Corollary 4.6. *Consider a smooth affine system* Σ *with linearizability index* $p = n - r$. *The following conditions are equivalent.*

(i) Σ *is, locally at* $q \in X$, *feedback equivalent to the affine strict feedforward form (SFF);*

(ii) Each system Σ_1, $\Sigma_2, \ldots, \Sigma_r$ *possesses an infinitesimal symmetry* v_i, *whose local flow* $\gamma_{c_i}^{v_i}$ *is conjugated to a* scaling translation

$$\gamma_{c_i}^{v_i} = \sigma_i^{-1} \circ \mathbb{T}_{\Lambda,C}^i \circ \sigma_i, \quad c_i \in (-\epsilon_i, \epsilon_i),$$

where Σ_1 *is the restriction of* Σ *to a neighborhood* X_q *and*

$$\Sigma_{i+1} = \Sigma_i/_{\sim_{v_i}}, \quad 1 \le i \le r-1.$$

Above, the equivalence relation $\sim_{v_i}$ is induced by the local action of the 1-parameter local group $\gamma_{c_i}^{v_i}$ defined by v_i, that is, such that $q_1 \sim_{v_i} q_2$ if and only if they belong to the same integral curve of v_i, and for any $1 \le i \le r-1$ the scaling translation $\mathbb{T}_{\Lambda,C}^i$ is the composition of $\mathbb{T}_{\Lambda,C}$ with the projection π_i:

$$\mathbb{T}_{\Lambda,C}^i(x) = (\lambda_i x_i + c_i, \ldots, \lambda_r x_r + c_r, \lambda x_{r+1}, \ldots, \lambda x_n)^\top.$$

Examples

Example 4.7. (**Cart-Pole Cont'd**) Reconsider the cart-pole system given in Example 3.5.

It has been shown that the cart-pole system can be put into its canonical form

$$\begin{cases} \dot z_1 &= z_2, \\ \dot z_2 &= z_3 + \dfrac{z_3}{(1+(g/l)z_3^2)^{3/2}} z_4^2, \\ \dot z_3 &= z_4, \\ \dot z_4 &= v \end{cases}$$

via analytic feedback transformation. It is straightforward to verify that

$$\mathbb{T}_{\mathrm{Id},C}(z) = (z_1 + c_1, z_2, z_3, z_4)^\top \quad \text{and} \quad \mathbb{T}_{-\mathrm{Id},C}(z) = (-z_1 + c_1, -z_2, -z_3, -z_4)^\top$$

constitute two 1-parameter families of symmetries for the canonical form. By Theorem 4 (see [38]), they exhaust all possible symmetries of the canonical form.

The symmetries of (3.8) are obtained by computing

$$\psi(x) = \sigma^{-1} \circ \mathbb{T}_{\pm\mathrm{Id},C}(x) \circ \sigma(x)$$

where the inverse $x = \eta(z) = \sigma^{-1}(z)$ is given by

$$\begin{aligned} x_1 &= \eta_1(z) = \tilde{\mu} g z_1 + \theta(z_3), \\ x_2 &= \eta_2(z) = \tilde{\mu} g z_2 - \tilde{\mu} l \frac{z_4}{\sqrt{1+(\tilde{\mu} z_3)^2}} \\ x_3 &= \eta_3(z) = \arctan(\tilde{\mu} z_3), \\ x_4 &= \eta_4(z) = \frac{\tilde{\mu} z_4}{1+(\tilde{\mu} z_3)^2} \end{aligned}$$

for a suitable function $\theta(z_3)$. It follows easily that

$$\sigma^{-1} \circ \mathbb{T}_{\mathrm{Id},C} \circ \sigma(x) = \mathbb{T}_{\mathrm{Id},\bar{C}}(x) \quad \text{and} \quad \sigma^{-1} \circ \mathbb{T}_{-\mathrm{Id},C} \circ \sigma(x) = \mathbb{T}_{-\mathrm{Id},\bar{C}}(x)$$

are both 1-parameter families of translations along the first component x_1 of $(x_1, x_2, x_3, x_4)^\top$. The meaning of the symmetries here, when expressed back in the original physical coordinates $(\tilde{q}_1, \dot{\tilde{q}}_1, \tilde{q}_2, \dot{\tilde{q}}_2) = (q_1 + c_1, \dot{q}_1, q_2, \dot{q}_2)$, is that the experiment conducted on two similar carts traveling at the same speed (either forward or backward) yields the same conclusions.

In the following, we give examples showing that symmetries as nonstationary translations and scaling translations are possible.

Example 4.8. Consider the system in $\mathbb{R}^4$ described by

$$\begin{cases} \dot{x}_1 &= \sin x_2 + x_4^2 \sin x_3, \\ \dot{x}_2 &= \sin x_3 + x_4^3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= u. \end{cases}$$

This system is clearly in (SFNF) with linearizability index $p = 2$. It is easy to check that the forward and backward translations

$$\mathbb{T}_{\mathrm{Id},C}(x) = (x_1+c_1, x_2+c_2, x_3+c_3, x_4)^\top \quad \text{and} \quad \mathbb{T}_{-\mathrm{Id},C}(x) = (-x_1+c_1, -x_2+c_2, -x_3+c_3, -x_4)^\top$$

are symmetries, where c_2 and c_3 are any multiples of 2π.

Example 4.9. Consider the system

$$\Sigma_{SFF} : \begin{cases} \dot{x}_1 &= x_2 + 2x_2 e^{x_3} \sin x_3 + 2x_2 e^{x_3} x_4^2, \\ \dot{x}_2 &= e^{x_3} \sin x_3 + e^{x_3} x_4^2, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= u, \end{cases}$$

in strict feedforward form with linearizability index $p = 2$. Due to the terms $2x_2 e^{x_3} \sin x_3$, this system is not in strict feedforward normal form. However, it is straightforward to check that the diffeomorphism $z = \sigma(x)$ defined by

$$z_1 = x_1 - x_2^2, \; z_2 = x_2, \; z_3 = x_3, \; z_4 = x_4$$

takes Σ_{SFF} into the strict feedforward normal form

$$\Sigma_{SFNF} : \begin{cases} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= e^{z_3} \sin z_3 + e^{z_3} z_4^2, \\ \dot{z}_3 &= z_4, \\ \dot{z}_4 &= u. \end{cases}$$

We can notice that the scaling translations

$$\tilde{z} = \mathbb{T}_{\Lambda,C}(z) = (\lambda z_1 + c_1, \lambda z_2, z_3 + c_3, z_4)^\top$$

with $c_3 = 2k\pi,\ k \in \mathbb{Z}$, and $\lambda = e^{c_3}$ form a family of symmetries of Σ_{SFNF} parameterized by c_1. Indeed, it is easy to see that they map Σ_{SFNF} into Σ_{SFNF} given, around the equilibrium $q = (0, 0, c_3, 0)^\top$, by

$$\Sigma^{(q)}_{SFNF} : \begin{cases} \dot{\tilde{z}}_1 &= \tilde{z}_2, \\ \dot{\tilde{z}}_2 &= e^{\tilde{z}_3} \sin \tilde{z}_3 + e^{\tilde{z}_3} \tilde{z}_4^2, \\ \dot{\tilde{z}}_3 &= \tilde{z}_4, \\ \dot{\tilde{z}}_4 &= u. \end{cases}$$

The composition $\tilde{x} = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma(x)$ expresses the coordinates $\tilde{x}$ in terms of the coordinates x as follows

$$\begin{aligned} \tilde{x}_1 &= \lambda x_1 + (\lambda^2 - \lambda) x_2^2 + c_1, \\ \tilde{x}_2 &= \lambda x_2, \\ \tilde{x}_3 &= x_3 + c_3 \\ \tilde{x}_4 &= x_4, \end{aligned}$$

where $c_3 = 2\pi$ and $\lambda = e^{c_3}$.

A straightforward calculation shows that

$$\begin{aligned} \dot{\tilde{x}}_1 &= \lambda(x_2 + 2x_2 e^{x_3} \sin x_3 + 2x_2 e^{x_3} x_4^2) + 2(\lambda^2 - \lambda) x_2 (e^{x_3} \sin x_3 + e^{x_3} x_4^2) \\ &= \lambda x_2 + 2\lambda^2 x_2 (e^{x_3} \sin x_3 + e^{x_3} x_4^2) \\ &= \tilde{x} + 2\tilde{x}_2 e^{\tilde{x}_3} \sin \tilde{x}_3 + 2\tilde{x}_2 e^{\tilde{x}_3} \tilde{x}_4^2 \end{aligned}$$

because $\lambda x_2 = \tilde{x}_2$ and

$$\lambda e^{x_3} \sin x_3 = e^{x_3 + c_3} \sin(x_3 + c_3) = e^{\tilde{x}_3} \sin \tilde{x}_3.$$

Similarly, we can show that

$$\dot{\tilde{x}}_2 = \lambda(e^{x_3} \sin x_3 + e^{x_3} x_4^2) = e^{\tilde{x}_3} \sin \tilde{x}_3 + e^{\tilde{x}_3} \tilde{x}_4^2.$$

Since $\dot{\tilde{x}}_3 = \tilde{x}_4$ and $\dot{\tilde{x}}_4 = u$, it follows that the composition $\tilde{x} = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma(x)$ maps Σ_{SFF}, defined around the equilibrium $(0, 0, 0, 0)^\top$, into Σ_{SFF} described, around the equilibrium $q = (0, 0, 2\pi, 0)^\top$, by the same dynamics

$$\Sigma_{SFF} : \begin{cases} \dot{\tilde{x}}_1 &= \tilde{x}_2 + 2\tilde{x}_2 e^{\tilde{x}_3} \sin \tilde{x}_3 + 2\tilde{x}_2 e^{\tilde{x}_3} \tilde{x}_4^2, \\ \dot{\tilde{x}}_2 &= e^{\tilde{x}_3} \sin \tilde{x}_3 + e^{\tilde{x}_3} \tilde{x}_4^2, \\ \dot{\tilde{x}}_3 &= \tilde{x}_4, \\ \dot{\tilde{x}}_4 &= u. \end{cases}$$

We have the commutative diagram

$$\begin{array}{ccc} \Sigma_{SFF}^{(o)} & \xrightarrow{\;\psi\;} & \Sigma_{SFF}^{(q)} \\ \sigma \downarrow & & \downarrow \sigma \\ \Sigma_{SFNF}^{(o)} & \xrightarrow[\mathbb{T}_{\Lambda,C}]{} & \Sigma_{SFNF}^{(q)} \end{array}$$

Hence $\tilde{x} = \psi(x) = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma(x)$ is a 1-parameter family of symmetries of Σ_{SFF}.

For convenience of notation, we will denote Σ_{SFF}, defined around $(0,0,0,0)^\top$, by $\Sigma_{SFF}^{(o)}$ and the system Σ_{SFF}, defined around $q = (0,0,2\pi,0)^\top$, by $\Sigma_{SFF}^{(q)}$. The same notations apply to the systems $\Sigma_{SFNF}^{(o)}$ and $\Sigma_{SFNF}^{(q)}$.

Now, in view of the results obtained in [41], we will compute the canonical form of $\Sigma_{SFF}^{(o)}$ and the transformations taking $\Sigma_{SFF}^{(o)}$ and $\Sigma_{SFF}^{(q)}$ to this canonical form.

It is easy to verify that $y = \Phi(x)$, given by

$$\begin{aligned} y_1 &= x_1 - x_2^2, \\ y_2 &= x_2, \\ y_3 &= e^{x_3} \sin x_3 \\ y_4 &= e^{x_3}(\sin x_3 + \cos x_3) x_4, \end{aligned}$$

followed by an appropriate feedback, takes the system $\Sigma_{SFF}^{(o)}$ into its canonical form

$$\Sigma_{SFCF} : \begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3 + \Theta(y_3) y_4^2, \\ \dot{y}_3 = y_4, \\ \dot{y}_4 = v, \end{cases}$$

where $\Theta(y_3) = \dfrac{1}{e^{x_3}(\sin x_3 + \cos x_3)^2}\Big|_{x_3 = \theta^{-1}(y_3)}$ with $\theta(x_3) = e^{x_3} \sin x_3$.

On the other hand, applying the translation

$$\hat{x} = T(\tilde{x}) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 - c_3, \tilde{x}_4)$$

to the system $\Sigma_{SFF}^{(q)}$, we can shift back the equilibrium point to $(0,0,0,0)$. In the new coordinates, $\Sigma_{SFF}^{(q)}$ becomes

$$\tilde{\Sigma}_{SFF}^{(o)} : \begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + 2\lambda \hat{x}_2 \left(e^{\hat{x}_3} \sin \hat{x}_3 + e^{\hat{x}_3} \hat{x}_4^2 \right), \\ \dot{\hat{x}}_2 = \lambda \left(e^{\hat{x}_3} \sin \hat{x}_3 + e^{\hat{x}_3} \hat{x}_4^2 \right), \\ \dot{\hat{x}}_3 = \hat{x}_4, \\ \dot{\hat{x}}_4 = u, \end{cases}$$

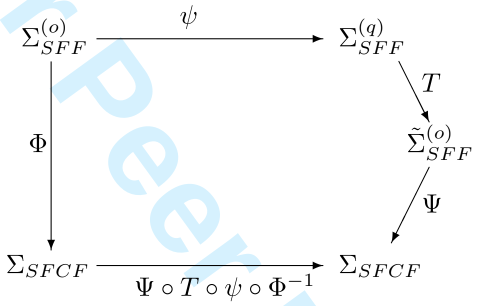
where $\lambda = e^{c_3}$. The diffeomorphism $\tilde{y} = \Psi(\hat{x})$ given by

$$\begin{aligned}
\tilde{y}_1 &= \lambda^{-1}(\hat{x}_1 - \hat{x}_2^2),\\
\tilde{y}_2 &= \lambda^{-1}\hat{x}_2,\\
\tilde{y}_3 &= e^{\hat{x}_3}\sin\hat{x}_3\\
\tilde{y}_4 &= e^{\hat{x}_3}(\sin\hat{x}_3 + \cos\hat{x}_3)\hat{x}_4,
\end{aligned}$$

followed by an appropriate feedback, takes the system $\tilde{\Sigma}^{(o)}_{SFF}$ into its canonical form

$$\Sigma_{SFCF} : \begin{cases}
\dot{\tilde{y}}_1 = \tilde{y}_2,\\
\dot{\tilde{y}}_2 = \tilde{y}_3 + \Theta(\tilde{y}_3)\tilde{y}_4^2,\\
\dot{\tilde{y}}_3 = \tilde{y}_4,\\
\dot{\tilde{y}}_4 = v.
\end{cases}$$

It follows that the composition $\tilde{y} = \Psi \circ T \circ \psi \circ \Phi^{-1}(y)$ is a 1-parameter family of symmetries of the canonical form according to the diagram.



We explicitly find this family of symmetries by expressing the coordinates $\tilde{y}$ as functions of the coordinates y:

$$\begin{aligned}
\tilde{y}_1 &= \lambda^{-1}(\hat{x}_1 - \hat{x}_2^2) = \lambda^{-1}(\tilde{x}_1 - \tilde{x}_2^2)\\
&= \lambda^{-1}\left(\lambda x_1 + (\lambda^2 - \lambda)x_2^2 + c_1 - \lambda^2 x_2^2\right)\\
&= x_1 - x_2^2 + \tilde{c}_1 = y_1 + \tilde{c}_1.
\end{aligned}$$

Similarly, we get

$$\tilde{y}_2 = \lambda^{-1}\hat{x}_2 = \lambda^{-1}\tilde{x}_2 = \lambda^{-1}(\lambda x_2) = x_2 = y_2;$$

$$\tilde{y}_3 = e^{\hat{x}_3}\sin\hat{x}_3 = e^{\tilde{x}_3 + 2\pi}\sin(\tilde{x}_3 + 2\pi) = e^{x_3}\sin x_3 = y_3$$

and

$$\begin{aligned}
\tilde{y}_4 &= e^{\hat{x}_3}(\sin\hat{x}_3 + \cos\hat{x}_3)\hat{x}_4\\
&= e^{\tilde{x}_3 + 2\pi}(\sin(\tilde{x}_3 + 2\pi) + \cos(\tilde{x}_3 + 2\pi))\tilde{x}_4\\
&= e^{x_3}\sin x_3 + e^{x_3}x_4^2 = y_4.
\end{aligned}$$

We conclude that the symmetries of the canonical form are exhausted here by a 1-parameter family of translations along the first variable. This is in concordance with the results in [38]. Notice that the composition $\Phi \circ \psi \circ \Phi^{-1}$ does not yield a symmetry for the canonical form. The reason is that, the system $\Sigma^{(q)}_{SFF}$, being defined around the equilibrium q, is not transformed into the canonical form Σ_{SFCF} by the same diffeomorphism Φ as $\Sigma^{(o)}_{SFF}$ is.

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