12-2009

Explicit Feedback Linearization of Control Systems

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Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

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Abstract— **This paper addresses the problem of feedback linearization of nonlinear control systems via state and feedback transformations. Necessary and sufficient geometric conditions were provided in the early eighties but finding the feedback linearizing coordinates is subject to solving a system of partial differential equations and had remained open since then. We will provide in this paper a complete solution to the problem (see the companion paper where the state linearization has been addressed) by defining an algorithm that allows to compute explicitly the linearizing state coordinates and feedback for any nonlinear control system that is truly feedback linearizable. Each algorithm is performed using a maximum of $n-1$ steps (n being the dimension of the system) and they are made possible by explicitly solving the Flow-box or straightening theorem. A possible implementation via software like mathematica/matlab/maple using simple integrations, derivations of functions might be considered.**

I. Introduction and Preliminaries

In the late seventies and early eighties the problem of transforming a nonlinear control system, via change of coordinates and feedback, into a linear one, has been introduced and is known today as feedback linearization. The feedback classification was applied first to linear systems for which a complete picture has been made possible. The controllability, observability, reachability, and realization of linear systems have been expressed in very simple algebraic terms. A crucial property of linear controllable systems is that they can be stabilized by linear feedback controllers. Because of the simplicity of their analysis and design; because several physical systems can be modeled using linear dynamics, and due to the observation that some nonlinear phenomena are just hidden linear systems, it is thus not surprising that the linearization problems were (and still are) of paramount importance and have attracted much attention. Uncovering the hidden linear properties of nonlinear control systems turns out to be useful in analyzing the latter systems though some global properties might be lost during the operation. To give a brief description of the linearization problems we will start first by recalling some basic facts about linear systems.

A. Linear Systems

We consider linear systems of the form

$$\Lambda : \begin{cases} \dot{x} = Fx + Gu = Fx + \sum\limits_{i=1}^{m} G_i u_i, \\ y = Hx \end{cases}$$

where $x \in \mathbb{R}^n$, Fx and $G_1, \ldots, G_m$ are, respectively, linear and constant vector fields on $\mathbb{R}^n$, Hx a linear vector field on $\mathbb{R}^p$, and $u = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$. To any linear system Λ we attach two geometric objects: **(a)** the controllability space

$$\mathcal{C}_n = \operatorname{span}[G\, FG \cdots F^{n-1}G]$$

as a $n \times (nm)$ matrix whose columns are those of the matrices $F^{k-1}G, k = 1, \ldots, n$, and **(b)** the observability space

$$\mathcal{O}_n = \operatorname{span}[H^\top\, (HF)^\top \cdots (HF^{n-1})^\top]^\top,$$

as a $(np) \times n$ matrix whose rows are those of the matrices $HF^{k-1}, k = 1, \ldots, n$. The system Λ is *controllable* (resp. *observable*) if and only if $\dim \mathcal{C}_n = n$ (resp. $\operatorname{rank} \mathcal{O}_n = n$).

By a linear change of coordinates $\tilde{x} = Tx$ and a linear feedback $u = Kx + Lv$, where T, K, and L are matrices of appropriate sizes, T and L being invertible, the system Λ is transformed into a linear equivalent one

$$\tilde{\Lambda} : \begin{cases} \dot{\tilde{x}} = \tilde{F}\tilde{x} + \tilde{G}v, \\ \tilde{y} = \tilde{H}\tilde{x} \end{cases}$$

with $\tilde{F}\tilde{x} = T(F + GK)T^{-1}$, $\tilde{G} = TGL$ and $\tilde{H} = HT^{-1}$.

It is shown in the literature [1], [6] that the dimension of $\mathcal{C}_n$ and the rank of $\mathcal{O}_n$, (hence the controllability and observability), are two invariants of the feedback classification of linear systems. The problem of feedback classification for linear systems Λ is to find linear state coordinates $w = Tx$ and linear feedback $u = Kx + Lv$ that map Λ into a simpler linear system $\tilde{\Lambda}$. It is a classical result of the linear control theory (see, e.g., [1], [6]) that any linear controllable system is feedback equivalent to the following Brunovský canonical form (single-input case):

$$\Lambda_{Br} : \dot{w} = Aw + bv, w \in \mathbb{R}^n, v \in \mathbb{R},$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

In the case of multi-input linear control systems we can find positive integers $\rho_1 \geq \cdots \geq \rho_m$, $\sum\limits_{i=1}^{m} \rho_i = n$ (called controllability, Brunovský or Kronecker indices) such that Λ_{Br} is a cascade of single-input linear systems $\Lambda_{Br}^1, \ldots, \Lambda_{Br}^m$:

$$\Lambda_{Br}^i : \dot{w}_i = A_i w_i + b_i v_i, w_i \in \mathbb{R}^{\rho_i}, v_i \in \mathbb{R},$$

with $A = \operatorname{diag}\{A_1, \ldots, A_m\}$ and $b = \operatorname{diag}\{b_1, \ldots, b_m\}$.

For a complete description and geometric interpretation of the Brunovský controllability indices we refer to the literature [1], [3], [4] , [5], [6], [10] and references therein.

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B. Nonlinear Systems and Feedback Linearization Problem.

Consider a smooth (resp. analytic) control-affine system

$$\Sigma: \ \dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, x \in \mathbb{R}^n$$

around an equilibrium (x_e, u_e), that is, $f(x_e)+g(x_e)u_e = 0$. We assume that $f, g_1, \ldots, g_m$ are smooth (resp. analytic) and $(x_e, u_e) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ or simply $f(0) = 0$. Let

$$\tilde{\Sigma}: \ \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})v = \tilde{f}(\tilde{x}) + \sum_{i=1}^{m} \tilde{g}_i(\tilde{x})v_i, \tilde{x} \in \mathbb{R}^n$$

be another smooth (resp. analytic) control-affine system. The systems Σ and $\tilde{\Sigma}$ are called *feedback equivalent* if there exist

$$\Gamma : \begin{cases} \tilde{x} &= \phi(x) \\ u &= \alpha(x) + \beta(x)v \end{cases}$$

a transformation that maps Σ into $\tilde{\Sigma}$, that is, such that

$$(PDEs) \quad \begin{cases} \mathrm{d}\phi(x) \cdot (f(x) + g(x)\alpha(x)) &= \tilde{f}(\phi(x)) \\ \mathrm{d}\phi(x) \cdot (g(x)\beta(x)) &= \tilde{g}(\phi(x)). \end{cases}$$

We will briefly write $\Gamma = (\phi, \alpha, \beta)$ and put $\Gamma_* \Sigma = \tilde{\Sigma}$. When Σ and $\tilde{\Sigma}$ are *state equivalent* we simply write $\phi_* \Sigma = \tilde{\Sigma}$.

The following two problems were considered in the late seventies and early eighties by Krener [7], and Brockett [2].

Problem 1. When does there exist a local diffeomorphism $w = \phi(x)$ defining new coordinates $w = (w_1, \ldots, w_n)^\top$ in which the transformed system $\phi_* \Sigma$ takes the linear form

$$\Lambda : \dot{w} = Fw + Gu = Fw + \sum_{i=1}^{m} G_i u_i, \ w \in \mathbb{R}^n, \ u \in \mathbb{R}^m \,?$$

Problem 2. When did there exist a (local)feedback transformation $\Gamma = (\phi, \alpha, \beta)$ that takes Σ into a linear system

$$\Lambda : \dot{w} = Aw + Bv = Aw + \sum_{i=1}^{m} b_i v_i, \ w \in \mathbb{R}^n, \ v \in \mathbb{R}^m \,?$$

When **Problem 1** (resp. **Problem 2**) is solvable, then the system Σ is called *state linearizable*, shortly $\mathcal{S}$-linearizable (resp. *feedback linearizable*, shortly, $\mathcal{F}$-linearizable). **Problem 1** was completely solved by Krener [7] and **Problem 2** partially by Brockett [2] for $m = 1$ and β constant. A generalization was obtained independently by Hunt and Su [3], Jakubczyk and Respondek [5], who gave necessary and sufficient geometric conditions in terms of Lie brackets of vector fields defining the system. Indeed, attach to Σ the sequence of nested distributions $\mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^n$, where

$$\mathcal{D}^k = \left\{ ad_f^q g_i, \quad 0 \le q \le k-1, \ 1 \le i \le m \right\}, k = 1, \ldots, n$$

with $ad_f^0 g_i = g_i$ and $ad_f^l g_i = [f, ad_f^{l-1} g_i]$ for all $l \ge 1$.

Theorem I.1 *A control system $\Sigma : \dot{x} = f(x) + g(x)u$ is locally equivalent, via change of coordinates $w = \phi(x)$ and feedback $v = \alpha(x) + \beta(x)u$, to a linear controllable system $\Lambda : \dot{w} = Aw + bv$ if and only if*

(F1) $\dim \mathcal{D}^n(x) = n$

(F2) *$\mathcal{D}^{n-1}$ is involutive, that is, $[\mathcal{D}^{n-1}, \mathcal{D}^{n-1}] \subseteq \mathcal{D}^{n-1}$.*

If the transformation $\Gamma = (\phi, \alpha, \beta)$ linearizes Σ, then $(PDEs)$ should hold with $\tilde{f}(\phi(x)) = A\phi(x)$, $\tilde{g}(\phi(x)) = B$.

Although the conditions $(F1)$ and $(F2)$ provide a way of testing the feedback linearizability of a system, they offer little on how to find the feedback linearizing group Γ except by solving $(PDEs)$ which is, in general, not straightforward. Indeed, for the single-input case, the solvability of $(PDEs)$ is equivalent of finding a function h with $h(0) = 0$ such that

$$L_g(h){=}0, L_g L_f(h){=}0, \ldots, L_g L_f^{n-2}(h){=}0, L_g L_f^{n-1}(h){\neq}0,$$

where for any vector field ν and any function h, $L_\nu(h) = \frac{\partial h}{\partial x} v(x)$ is the Lie derivative of h along ν. We propose here to give a complete solution to problem 2 without solving the partial differential equations. We will provide an algorithm giving explicit solutions in that case. Recall that we have previously obtained explicit solutions for few subclasses of control-affine systems, namely *strict feedforward forms*, *strict-feedforward nice* and *feedforward forms*, for which linearizing coordinates were found without solving the corresponding PDEs (see [11], [12], [14]). Indeed, for those subclasses we exhibited algorithms that can be performed using a maximum of $\frac{n(n+1)}{2}$ steps each involving composition and integration of functions only (but not solving PDEs) followed by a sequence of $n+1$ derivations. What played a main role in finding those algorithms were the strict feedforward form *structure*, that is, the fact that each component of the system depended only on higher variables. In this paper we consider general control-affine systems for which we provide a feedback linearizing algorithm that can be implemented using a maximum of n steps. This algorithm is, in part, based on the explicit solving of the flow-box theorem [15] and differs completely from those outlined in [11], [14] (see also [8], [9]). In what follows we will address only the single input case. We first recall the following well-known result.

Theorem I.2 *A control system $\Sigma : \dot{x} = f(x) + g(x)u$ is locally $\mathcal{F}$-equivalent to a linear controllable system if and only if it is $\mathcal{S}$-equivalent to the feedback form*

$$(FB) \quad \begin{cases} \dot{z}_1 &= \hat{f}_1(z_1, z_2) \\ \dot{z}_2 &= \hat{f}_2(z_1, z_2, z_3) \\ & \cdots \\ \dot{z}_{n-1} &= \hat{f}_{n-1}(z_1, \ldots, z_n) \\ \dot{z}_n &= \hat{f}_n(z_1, \ldots, z_n) + \hat{g}_n(z_1, \ldots, z_n)u. \end{cases}$$

The proof of Theorem I.2 is straightforward and can be found in the literature (e.g. [3], [4], [5], [10]). Let $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_n)$, $\hat{g} = (0, \ldots, 0, \hat{g}_n)$ and $\hat{h}(z) = z_1$. It follows that the feedback transformation $\Gamma \triangleq (\hat{\phi}, \hat{\alpha}, \hat{\beta})$ defined by $w = \hat{\phi}(z)$, $u = \hat{\alpha}(z) + \hat{\beta}(z)v$, where

$$\hat{\phi}_1(z) = \hat{h}(z), \hat{\phi}_2(z) = L_{\hat{f}}(\hat{h}), \ldots, \hat{\phi}_n(z) = L_{\hat{f}}^{n-1}(\hat{h})$$

$$\hat{\alpha}(z) = -\frac{L_{\hat{f}}^n(\hat{h})}{L_{\hat{g}} L_{\hat{f}}^{n-1}(\hat{h})} \quad \text{and} \quad \hat{\beta}(z) = -\frac{1}{L_{\hat{g}} L_{\hat{f}}^{n-1}(\hat{h})}$$

brings (FB) into the Brunovský canonical form Λ_{Br}.

II. Main Results: $\mathcal{F}$-Linearizable Systems

Below we give our main result, that is, an algorithm allowing to construct explicitly feedback linearizing coordinates.

Consider $\Sigma : \dot{x} = f(x) + g(x)u$ and let $1 \leq \mathbf{k} \leq n-1$. We say that Σ is in $(FB)_{\mathbf{k}}$-form, and we denote it $\Sigma_{\mathbf{k}}^{\mathrm{FB}}$, if in some coordinates $\mathbf{x_k} = (\mathbf{x_{k1}}, \ldots, \mathbf{x_{k}}_n)$, it takes the form

$$\Sigma_{\mathbf{k}}^{\mathrm{FB}} : \begin{cases} \dot{\mathbf{x}}_{\mathbf{k}j} &= F_{\mathbf{k}j}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}k+1}), \text{ if } 1 \leq j \leq k \\ \dot{\mathbf{x}}_{\mathbf{k}k+1} &= F_{\mathbf{k}k+1}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}k+2}) \\ & \cdots \\ \dot{\mathbf{x}}_{\mathbf{k}n-1} &= F_{\mathbf{k}n-1}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}n}) \\ \dot{\mathbf{x}}_{\mathbf{k}n} &= F_{\mathbf{k}n}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}n}) + u, \end{cases}$$

where $k = \mathbf{k}$. For simplicity we chose the coefficient of the control input u to be 1 but this is not a restriction. We have

Theorem II.1 *Consider a linearly controllable system*

$$\Sigma : \dot{x} = f(x) + g(x)u, \, x \in \mathbb{R}^n, \, u \in \mathbb{R}.$$

Assume it is $\mathcal{F}$-linearizable (let $\Sigma \triangleq \Sigma_{\mathbf{n}}^{FB}$ and $x \triangleq \mathbf{x_n}$).

There exists a sequence of explicit *coordinates changes $\phi_{\mathbf{n}}(\mathbf{x_n}), \phi_{\mathbf{n-1}}(\mathbf{x_{n-1}}), \ldots, \phi_{\mathbf{2}}(\mathbf{x_2})$ that gives rise to a sequence of $(FB)_{\mathbf{k}}$-forms $\Sigma_{\mathbf{n-1}}^{\mathrm{FB}}, \Sigma_{\mathbf{n-2}}^{\mathrm{FB}}, \ldots, \Sigma_{\mathbf{1}}^{\mathrm{FB}}$ such that for any $2 \leq \mathbf{k} \leq n$ we get $\Sigma_{\mathbf{k-1}}^{\mathrm{FB}} = (\phi_{\mathbf{k}})_* \Sigma_{\mathbf{k}}^{\mathrm{FB}}$.*

Moreover, in the coordinates $z \triangleq \phi_{\mathbf{2}}(\mathbf{x_2})$ the system Σ (actually $\Sigma_{\mathbf{1}}^{\mathrm{FB}}$) takes the feedback form (FB).

A direct consequence of this result is the following corollary.

Corollary II.2 *Consider a linearly controllable system Σ and assume it is $\mathcal{F}$-linearizable. Then Σ is linearizable by the feedback transformation $w = \hat{\phi} \circ \phi(x), u = \hat{\alpha}(\phi(x)) + \hat{\beta}(\phi(x))v$, where $z = \phi(x)$ is the diffeomorphism taking Σ into the feedback form (FB), and $\Gamma = (\hat{\phi}, \hat{\alpha}, \hat{\beta})$ the transformation taking (FB) into to the Brunovský form Λ_{Br}.*

The proof of Theorem II.1 follows from the algorithm below.

A. Feedback Linearizing Coordinates: $(\mathcal{F}\pounds)$-Algorithm.

Consider $\Sigma : \dot{x} = f(x) + g(x)u, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}$ and assume it is $\mathcal{F}$-linearizable. Applying a linear feedback $z = Tx, u = Kx + Lv$, if necessary, we assume that $\frac{\partial f}{\partial x}(0) = A$ and $g(0) = b$, where (A, b) is the Brunovský canonical pair. The algorithm below consists of a maximum of $n-1$ steps.

Step 1. Set $\Sigma \triangleq \Sigma_{\mathbf{n}}^{\mathrm{FB}}$ and $x \triangleq \mathbf{x_n} = (\mathbf{x_{n1}}, \ldots, \mathbf{x_{nn}})^\top$. Apply Theorem II.2 ([16]) with $\nu = g(x)$ to construct a change of coordinates $z = \phi(x)$ such that $\phi_*(g)(z) = \partial_{z_n}$. If we denote $\mathbf{x_{n-1}} \triangleq z$ and $\phi_{\mathbf{n}} \triangleq \phi$, it thus follows that the change of coordinates $\mathbf{x_{n-1}} = \phi_{\mathbf{n}}(\mathbf{x_n})$ takes $\Sigma_{\mathbf{n}}^{\mathrm{FB}}$ into

$$\Sigma_{\mathbf{n-1}}^{\mathrm{FB}} : \begin{cases} \dot{\mathbf{x}}_{\mathbf{n-1}1} &= F_{\mathbf{n-1}1}(\mathbf{x}_{\mathbf{n-1}1}, \ldots, \mathbf{x}_{\mathbf{n-1}n}) \\ \dot{\mathbf{x}}_{\mathbf{n-1}2} &= F_{\mathbf{n-1}2}(\mathbf{x}_{\mathbf{n-1}1}, \ldots, \mathbf{x}_{\mathbf{n-1}n}) \\ & \cdots \\ \dot{\mathbf{x}}_{\mathbf{n-1}n-1} &= F_{\mathbf{n-1}n-1}(\mathbf{x}_{\mathbf{n-1}1}, \ldots, \mathbf{x}_{\mathbf{n-1}n}) \\ \dot{\mathbf{x}}_{\mathbf{n-1}n} &= F_{\mathbf{n-1}n}(\mathbf{x}_{\mathbf{n-1}1}, \ldots, \mathbf{x}_{\mathbf{n-1}n}) + u. \end{cases}$$

Remark that this first step is independent of whether Σ is $\mathcal{F}$-linearizable or not. It depends only on the fact that the vector field g is nonsingular, and hence, can be rectified.

Step $n-k$**.** Assume that a sequence of explicit coordinates changes $\phi_{\mathbf{n}}, \ldots, \phi_{\mathbf{k+1}}$ were found whose composition $\mathbf{x_k} = \phi_{\mathbf{k+1}} \circ \cdots \circ \phi_{\mathbf{n}}(\mathbf{x_n})$ takes $\Sigma_{\mathbf{n}}^{\mathrm{FB}}$ into the $(FB)_{\mathbf{k}}$-form

$$\Sigma_{\mathbf{k}}^{\mathrm{FB}} : \dot{\mathbf{x}}_{\mathbf{k}} = F_{\mathbf{k}}(\mathbf{x_k}) + bu, \;\; \mathbf{x_k} \in \mathbb{R}^n,$$

where (recall that $k = \mathbf{k}$)

$$F_{\mathbf{k}j}(\mathbf{x_k}) = \begin{cases} F_{\mathbf{k}j}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}k+1}), & 1 \leq j \leq k \\ F_{\mathbf{k}j}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}j+1}), & k+1 \leq j \leq n-1 \\ F_{\mathbf{k}j}(\mathbf{x_{k1}}, \ldots, \mathbf{x}_{\mathbf{k}n}), & j = n. \end{cases}$$

Once again reset the variable $x \triangleq \mathbf{x_k}$ and denote $\Sigma_{\mathbf{k}}^{\mathrm{FB}}$ simply by $\Sigma : \dot{x} = f(x) + g(x)u$ with $g(x) = b$ and

$$f_j(x) = \begin{cases} f_j(x_1, \ldots, x_{k+1}), & 1 \leq j \leq k \\ f_j(x_1, \ldots, x_{j+1}), & k+1 \leq j \leq n, \end{cases}$$

where the last component f_n depends only on $x_1, \ldots, x_n$. We showed in Section IV (IV.1) that there exist smooth functions $\Theta(x) = \Theta(x_1, \ldots, x_{k+1}), F_j(x) = F_j(x_1, \ldots, x_k)$ and $\nu_j(x) = \nu_j(x_1, \ldots, x_k)$ for $1 \leq j \leq k$ such that

$$f_j(x_1, \ldots, x_{k+1}) = F_j(x) + \nu_j(x)\Theta(x) \;\; 1 \leq j \leq k$$

with $\Theta(0) = 0$ and $\frac{\partial \Theta}{\partial x_{k+1}}(0) \neq 0$. This and the fact that $\frac{\partial f_k}{\partial x_{k+1}}(0) \neq 0$ imply $\nu_k(0) \neq 0$. Define the nonsingular vector field

$$\nu(x) \;=\; \nu_1(x)\partial_{x_1} + \cdots + \nu_k(x)\partial_{x_k} \;\in\; \mathbb{R}^k.$$

Apply Theorem II.2 ([16]) to construct a change of coordinates $z = \phi(x_1, \ldots, x_k) \in \mathbb{R}^k$ such that $\phi_*(\nu)(z) = \partial_{z_k}$. Extend such change of coordinates in $\mathbb{R}^n$ (still called ϕ) by

$$z = \phi(x) = (\phi_1(x), \ldots, \phi_k(x), x_{k+1}, \ldots, x_n)^\top.$$

The inverse $x = \psi(z) = \phi^{-1}(z)$ is also obtained by Theorem II.2 ([16]). Clearly, the inverse is of the form

$$x = \psi(z) = (\psi_1(z), \ldots, \psi_k(z), z_{k+1}, \ldots, z_n)^\top.$$

The change of coordinates transforms the system Σ into

$$\tilde{\Sigma} : \dot{z} = \tilde{f}(z) + \tilde{g}(z)u = \phi_* f(z) + \phi_* g(z)u,$$

where $\phi_* g(z) = (0, \ldots, 0, 1)^\top$ and

$$\begin{aligned} \tilde{f}(z) = \phi_* f(z) \;=\; & \sum_{j=1}^{k} \phi_*\Big(f_j(x_1, \ldots, x_{k+1})\partial_{x_j}\Big) \\ & + \sum_{j=k+1}^{n} \phi_*\Big(f_j(x_1, \ldots, x_{j+1})\partial_{x_j}\Big). \end{aligned}$$

It is easy to see that the second term is equivalent to

$$\sum_{j=k+1}^{n} \phi_*\Big(f_j(x_1, \ldots, x_{j+1})\partial_{x_j}\Big) = \sum_{j=k+1}^{n} f_j(\psi(z))\partial_{z_j}. \quad \text{(II.1)}$$

The first term rewrites

$$\begin{aligned} \sum_{j=1}^{k} \phi_*\Big(f_j(x)\partial_{x_j}\Big) &= \sum_{j=1}^{k} \phi_*\Big(F_j(x_1, \ldots, x_k)\partial_{x_j}\Big) \\ &\quad + \sum_{j=1}^{k} \phi_*\Big(\Theta(x)\nu_j(x_1, \ldots, x_k)\partial_{x_j}\Big) \\ &= \sum_{j=1}^{k} \tilde{F}_j(z_1, \ldots, z_k)\partial_{z_j} + \Theta(\psi(z))\partial_{z_k} \end{aligned} \quad \text{(II.2)}$$

We deduce from (II.2) that the first $k-1$ components depend only on the variables $z_1, \ldots, z_k$ and the kth component depends on $z_1, \ldots, z_{k+1}$. In the other hand (II.1) shows that the jth component ($j = k+1, \ldots, n$) depends on the variables $z_1, \ldots, z_{j+1}$. We thus conclude that

$$\tilde{f}_j(z) = \begin{cases} \tilde{f}_j(z_1, \ldots, z_k), & 1 \le j \le k-1 \\ \tilde{f}_j(z_1, \ldots, z_{j+1}), & k \le j \le n, \end{cases}$$

where the last component $\tilde{f}_n$ depends only on $z_1, \ldots, z_n$.

Denote $\mathbf{x_{k-1}} \triangleq z$ and $\phi_{\mathbf{k}} \triangleq \phi$. Thus the change of coordinates $\mathbf{x_{k-1}} = \phi_{\mathbf{k}}(\mathbf{x_k})$ brings the system $\Sigma^{\mathrm{FB}}_{\mathbf{k}}$ into

$$\Sigma^{\mathrm{FB}}_{\mathbf{k-1}} : \begin{cases} \dot{\mathbf{x}}_{\mathbf{k-1}j} &= F_{\mathbf{k-1}j}(\mathbf{x}_{\mathbf{k-1}1}, \ldots, \mathbf{x}_{\mathbf{k-1}k}) \\ & \quad \text{if } 1 \le j \le k-1 \\ \dot{\mathbf{x}}_{\mathbf{k-1}k} &= F_{\mathbf{k-1}k}(\mathbf{x}_{\mathbf{k-1}1}, \ldots, \mathbf{x}_{\mathbf{k-1}k+1}) \\ & \cdots \\ \dot{\mathbf{x}}_{\mathbf{k-1}n-1} &= F_{\mathbf{k-1}n-1}(\mathbf{x}_{\mathbf{k-1}1}, \ldots, \mathbf{x}_{\mathbf{k-1}n}) \\ \dot{\mathbf{x}}_{\mathbf{k-1}n} &= F_{\mathbf{k-1}n}(\mathbf{x}_{\mathbf{k-1}1}, \ldots, \mathbf{x}_{\mathbf{k-1}n}) + u. \end{cases}$$

This completes the induction an the algortihm; consequently, we can construct a sequence $\phi_{\mathbf{n}}(\mathbf{x_n}), \phi_{\mathbf{n-1}}(\mathbf{x_{n-1}}), \ldots, \phi_{\mathbf{2}}(\mathbf{x_2})$ of explicit coordinates changes whose composition $z = \phi_{\mathbf{2}} \circ \cdots \circ \phi_{\mathbf{n}}(\mathbf{x_n})$ takes the original system Σ into the (FB) form.

B. *Summary of Algorithm.* Start with a system

$$\Sigma : \dot{x} = f(x) + g(x)u, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}.$$

Step 0. Normalize the vector field $g \longmapsto g = (0, \ldots, 0, 1)^\top$ and apply a linear feedback to put the linearization in Brunovský form (not necessary but very recommended).

Step $n-k$**.** If the condition

$$(\mathcal{F}\pounds_{k+1}) \Longrightarrow \frac{\partial^2 f_j}{\partial x_{k+1}^2} = \gamma_{n-k}(x) \frac{\partial f_j}{\partial x_{k+1}}, \; 1 \le j \le k$$

fails ($\gamma_{n-k}(x)$ not the same for first k components) then system is not feedback linearizable and algorithm stops.

If $(\mathcal{F}\pounds_{k+1})$ is satisfied, then decompose the first k components $f_1, \ldots, f_k$ as following (see (IV.1))

$$f_j(x_1, \ldots, x_{k+1}) = F_j(x) + \nu_j(x)\Theta(x) \; 1 \le j \le k.$$

Apply Theorem II.2 ([16]) to construct a change of coordinates $z = \phi(x) \in \mathbb{R}^n$ to rectify the nonsingular vector field

$$\nu(x) = \nu_1(x)\partial_{x_1} + \cdots + \nu_k(x)\partial_{x_k} + 0 \cdot \partial_{x_{k+1}} + \cdots + 0 \cdot \partial_{x_n},$$

that is, such that $\phi_*(\nu)(z) = \partial_{z_k}$. Compute $\phi_*\Sigma$ the transform of precedent system. Repeat **Step** $n-k$ for $k = n-1, \ldots, 2$. End if system is in (FB) form or algorithm fails.

III. Examples

Example III.1 Consider a single-input control system

$$\Sigma : \dot{x} = f(x) + g(x)u \triangleq \begin{cases} \dot{x}_1 &= x_2(1+x_3) \\ \dot{x}_2 &= x_3(1+x_1) - x_2 u \\ \dot{x}_3 &= x_1 + (1+x_3)u \end{cases}$$

with $f(x) = (x_2(1+x_3), x_3(1+x_1), x_1)^\top$ and

$$g(x) = (0, -x_2, 1+x_3)^\top.$$

We first rectify the vector field $g(x)$. Put $\nu(x) = g(x)$ and apply Theorem II.2 ([16]) with $n = 3$ and $\sigma_3(x) = \frac{1}{1+x_3}$, thus $\sigma_3\nu = -\frac{x_2}{1+x_3}\partial_{x_2} + \partial_{x_3}$. Since $\nu_1 = 0$ and $\nu_2(x) = -x_2$, we have $\phi_1(x) = x_1$ in one side, and

$$L_{\sigma_3\nu}(\sigma_3\nu_2) = \frac{2x_2}{(1+x_3)^2}, \quad L^2_{\sigma_3\nu}(\sigma_3\nu_2) = -\frac{6x_2}{(1+x_3)^3}$$

in the other, and recurrently

$$L^{s-1}_{\sigma_3\nu}(\sigma_3\nu_2) = \frac{(-1)^s s! x_2}{(1+x_3)^s}.$$

It follows that

$$z_2 = \phi_2(x) = x_3 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L^{s-1}_{\sigma_3\nu}(\sigma_3\nu_2)(x) = x_2(1+x_3).$$

To calculate $\phi_3(x)$, notice that

$$L_{\sigma_3\nu}(\sigma_3) = -\frac{1}{(1+x_3)^2} \text{ and } L^2_{\sigma_3\nu}(\sigma_3) = \frac{2}{(1+x_3)^3}.$$

Thus a simple recurrence shows that

$$L^{s-1}_{\sigma_3\nu}(\sigma_3) = \frac{(-1)^{s-1}(s-1)!}{(1+x_3)^s}, \quad \text{for } s \ge 1$$

which implies

$$\begin{aligned} z_3 = \phi_3(x) &= \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_3^s}{s!} L^{s-1}_{\sigma_3\nu}(\sigma_3)(x) \\ &= \sum_{s=1}^{\infty} \frac{1}{s} \left(\frac{x_3}{1+x_3}\right)^s \\ &= \sum_{s=1}^{\infty} \int \left(\frac{x_3}{1+x_3}\right)^{s-1} \left(\frac{x_3}{1+x_3}\right)' \mathrm{d}\, x_3 \\ &= \int \frac{1}{1+x_3} \mathrm{d}\, x_3 = \ln(1+x_3). \end{aligned}$$

We apply the change of coordinates

$$z_1 = x_1, z_2 = x_2(1+x_3), z_3 = \ln(1+x_3)$$

to transform the original system into

$$\dot{z} = \hat{f}(z) + \hat{g}(z)u \triangleq \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = (1+z_1)e^{z_3}(e^{z_3}-1) + z_1 z_2 e^{-z_3} \\ \dot{z}_3 = z_1 e^{-z_3} + u. \end{cases}$$

The system is in (FB)-form and can be put into the linear Brunovský form $\Lambda_{Br} : \dot{w}_1 = w_2, \dot{w}_2 = w_3, \dot{w}_3 = v$ via

$$\begin{aligned} w_1 &= \hat{h}(z) = z_1 \\ w_2 &= L_{\hat{f}}\hat{h}(z) = z_2 \\ w_3 &= L^2_{\hat{f}}\hat{h}(z) = (1+z_1)e^{z_3}(e^{z_3}-1) + z_1 z_2 e^{-z_3} \\ v &= L^3_{\hat{f}}\hat{h}(z) + L_{\hat{g}} L^2_{\hat{f}}\hat{h}(z)u. \end{aligned}$$

The composition of the two-step changes of coordinates and feedback gives linearizing coordinates for the original system

$$\begin{aligned} w_1 &= x_1 \\ w_2 &= x_2(1+x_3) \\ w_3 &= x_3(1+x_1)(1+x_3) + x_1 x_2 \\ v &= x_2(1+x_3)(x_2+x_3+x_3^2) + x_1(1+x_1)(1+3x_3) \\ &\quad + [(1+x_1)(1+x_3)(1+2x_3) - x_1 x_2]u \end{aligned}$$

Such linearizing coordinates and feedback could have been obtained by other methods. We want to point out that the method is applicable to all feedback linearizable systems.

Example III.2 Consider a single-input control system

$$\Sigma : \dot{x} = f(x) + g(x)u \triangleq \begin{cases} \dot{x}_1 &= x_2 - x_4^2 \\ \dot{x}_2 &= x_4 + 2x_1^2 x_4 + 2x_4 u \\ \dot{x}_3 &= x_1^2 \\ \dot{x}_4 &= x_1 + x_4^2 + u \end{cases}$$

with $f(x) = (x_2 - x_4^2, x_4 + 2x_1^2 x_4, x_1^2, x_1 + x_4^2)^\top$ and $g(x) = (0, 2x_4, 0, 1)^\top$. This system is not feedback linearizable as it can be checked that $[g, ad_f g] \notin \text{span } \{g, ad_f g\}$. We want to show that the algorithm provides such information without having to compute the involutivity of the distributions.

We first start by rectifying the control vector field g. Identify $\nu = g(x)$ with $\sigma_4 = 1$. We calculate the component

$$\begin{aligned} \phi_2(x) &= x_2 + \sum_{s=1}^{\infty} \frac{(-1)^s x_4^s}{s!} L_\nu^{s-1}(\nu_2)(x) \\ &= x_2 + \sum_{s=1}^{\infty} \frac{(-1)^s x_4^s}{s!} L_\nu^{s-1}(2x_4)(x) = x_2 - x_4^2. \end{aligned}$$

Since ν_1, ν_3, ν_4 are constants, then $\phi_1(x) = x_1, \phi_3(x) = x_3$, and $\phi_4(x) = x_4$. The change of coordinates $z_1 = x_1, z_2 = x_2 - x_4^2, z_3 = x_3, z_4 = x_4$ takes the system into

$$\tilde{\Sigma} : \dot{z} = \tilde{f}(z) + \tilde{g}(z)u \triangleq \begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_4 - 2z_1 z_4 + 2z_1^2 z_4 - 2z_4^3 \\ \dot{z}_3 &= z_1^2 \\ \dot{z}_4 &= z_1 + z_4^2 + u \end{cases}$$

where $\tilde{g} = (0, 0, 0, 1)^\top$ and

$$\tilde{f}(z) = (z_2, z_4 - 2z_1 z_4 + 2z_1^2 z_4 - 2z_4^3, z_1^2, z_1 + z_4^2)^\top.$$

Clearly,

$$\frac{\partial \tilde{f}}{\partial z_4} = (0, 1 - 2z_1 + 2z_1^2 - 6z_4^2, 0, 2z_4)^\top, \frac{\partial^2 \tilde{f}}{\partial z_4^2} = (0, -12z_4, 0, 2)^\top$$

from which we deduce that $\frac{\partial^2 \tilde{f}_j}{\partial z_4^2} = \gamma_1 \frac{\partial \tilde{f}_j}{\partial z_4}$, $1 \leq j \leq 3$ fails. The algorithm ends: the system is not $\mathcal{F}$-linearizable.

Example III.3 Consider the single-input control system [4]

$$\Sigma : \dot{x} = f(x) + g(x)u \triangleq \begin{cases} \dot{x}_1 &= e^{x_2} u \\ \dot{x}_2 &= x_1 + x_2^2 + e^{x_2} u \\ \dot{x}_3 &= x_1 - x_2 \end{cases}$$

with $f(x) = (0, x_1 + x_2^2, x_1 - x_2)^\top$ and $g(x) = (e^{x_2}, e^{x_2}, 0)^\top$. We first rectify the vector field $g(x)$. Denote $\nu(x) = g(x)$ and apply Theorem II.2 ([16]) with $n = 3$ and $\sigma_2(x) = e^{-x_2}$, hence $\sigma_2 \nu = \partial_{x_1} + \partial_{x_2}$. Since $\nu_3 = 0$, then $\phi_3(x) = x_3$. Because $L_{\sigma_2 \nu}^{s-1}(\sigma_2 \nu_1) = 0$ for all $s \geq 2$, we obtain

$$\begin{aligned} z_1 = \phi_1(x) &= x_1 + \sum_{s=1}^{\infty} \frac{(-1)^s x_2^s}{s!} L_{\sigma_2 \nu}^{s-1}(\sigma_2 \nu_1)(x) \\ &= x_1 - x_2(\sigma_2 \nu_1)(x) = x_1 - x_2. \end{aligned}$$

To compute ϕ_2 notice that $L_{\sigma_2 \nu}^{s-1}(\sigma_2) = (-1)^{s-1} e^{-x_2}$ for all $s \geq 2$. It thus follows that

$$\begin{aligned} z_2 = \phi_2(x) &= \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_2^s}{s!} L_{\sigma_2 \nu}^{s-1}(\sigma_2)(x) \\ &= \sum_{s=1}^{\infty} \frac{x_2^s}{s!} e^{-x_2} = 1 - e^{-x_2}. \end{aligned}$$

The change of coordinates

$$z = \phi(x) = (x_1 - x_2, 1 - e^{-x_2}, x_3)^\top$$

whose inverse $x = \psi(z) = (z_1 - \ln(1 - z_2), -\ln(1 - z_2), z_3)^\top$ can be obtained directly or by applying Theorem II.2 (ii) (see [16]), takes the original system into

$$\begin{cases} \dot{z}_1 &= -z_1 + \ln(1 - z_2^2) - (\ln(1 - z_2))^2 \\ \dot{z}_2 &= (1 - z_2)[z_1 - \ln(1 - z_2^2) + (\ln(1 - z_2))^2] + u \\ \dot{z}_3 &= z_1. \end{cases}$$

A permutation of the variables $\tilde{z}_1 = z_3, \tilde{z}_2 = z_1, \tilde{z}_3 = z_2$ yields a system in feedback form

$$(FB) \begin{cases} \dot{\tilde{z}}_1 &= \tilde{z}_2 \\ \dot{\tilde{z}}_2 &= -\tilde{z}_2 + \ln(1 - \tilde{z}_3^2) - (\ln(1 - \tilde{z}_3))^2 \\ \dot{\tilde{z}}_3 &= (1 - \tilde{z}_3)[\tilde{z}_2 - \ln(1 - \tilde{z}_3^2) + (\ln(1 - \tilde{z}_3))^2] + u \end{cases}$$

that can be linearized by

$$\begin{aligned} w_1 &= \tilde{z}_1 \\ w_1 &= \tilde{z}_2 \\ w_3 &= -\tilde{z}_2 + \ln(1 - \tilde{z}_3^2) - (\ln(1 - \tilde{z}_3))^2 \\ v &= \dot{w}_3. \end{aligned}$$

We thus deduce that the change of coordinates

$$\begin{aligned} w_1 &= x_3 \\ w_2 &= x_1 - x_2 \\ w_3 &= -x_1 - x_2^2 \\ v &= -2x_2(x_1 + x_2^2) - (1 + 2x_2)e^{x_2} u \end{aligned}$$

brings Σ into Brunovský $\Lambda_{Br} : \dot{w}_1 = w_2, \dot{w}_2 = w_3, \dot{w}_3 = v$. Notice that such change of coordinates was given in [4]. However, the system was coupled with the given output $y = h(x) = x_3$ which made finding them straightforward.

IV. Appendix

Below we establish an equivalence between the involutivity conditions of Theorem I.1 and a sequence of easily computable conditions $(\mathcal{F}\mathcal{L}_n), \ldots, (\mathcal{F}\mathcal{L}_1)$ each stating the fact that the second derivative of f with respect to some variable is proportional to its first derivative with respect to the same variable. This constitutes the core of the algorithm.

Simple Involutivity Conditions.

Consider the system $\Sigma : \dot{x} = f(x) + g(x)u$ and assume without loss of generality that $g(x) = (0, \ldots, 0, 1)^\top$ and

$$f_j(x) = \begin{cases} f_j(x_1, \ldots, x_{k+1}) & 1 \leq j \leq k \\ f_j(x_1, \ldots, x_{j+1}) & k+1 \leq j \leq n, \end{cases}$$

where $1 \leq k \leq n - 1$ and f_n depends only on $x_1, \ldots, x_n$.

Claim: *If the following distributions*

$$\mathcal{D}^j(x) = \text{span}\left\{g(x) ad_f g(x) \ldots, ad_f^{j-1} g(x)\right\},\ 1 \le j \le n$$

are involutive, then there is a function γ_{n-k} such that

$$(\mathcal{F}\pounds_{k+1}) \Longrightarrow \frac{\partial^2 f_j}{\partial x_{k+1}^2} = \gamma_{n-k}(x)\frac{\partial f_j}{\partial x_{k+1}},\ 1 \le j \le k.$$

Moreover, functions $\Theta(x) = \Theta(x_1, \ldots, x_{k+1})$ and $F_j(x) = F_j(x_1, \ldots, x_k)$ and $\nu_j(x) = \nu_j(x_1, \ldots, x_k)$ exist such that

$$f_j(x_1, \ldots, x_{k+1}) = F_j(x) + \nu_j(x)\Theta(x)\ 1 \le j \le k \quad \text{(IV.1)}$$

with $\Theta(x)$ depending exclusively on $\gamma_{n-k}(x)$.

Proof: Remark that the vector field f can be written as

$$f(x) = \sum_{j=1}^{k} f_j(x_1, \ldots, x_{k+1})\partial_{x_j} + \sum_{j=k+1}^{n} f_j(x_1, \ldots, x_{j+1})\partial_{x_j}$$

and that the function Θ given above is independent of j; otherwise the decomposition (IV.1) would have been trivial. For any $1 \le j \le n$ denote by $\Delta^j = \text{span}\left\{\partial_{x_{n-j+1}}, \ldots, \partial_{x_n}\right\}$ the module generated over the field of smooth functions, that is, each element of Δ^j is a linear combination of the vector fields $\partial_{x_{n-j+1}}, \ldots, \partial_{x_n}$ whose coefficients are smooth functions. We first verify easily that

$$ad_f g = -\frac{\partial f_{n-1}}{\partial x_n}\partial_{x_{n-1}} - \frac{\partial f_n}{\partial x_n}\partial_{x_n} = \mu_{n-1}(x)\partial_{x_{n-1}} + \vartheta_{n-1}(x)$$

where $\mu_{n-1}(x) = -\frac{\partial f_{n-1}}{\partial x_n}$ and $\vartheta_{n-1}(x) \in \Delta^1$. An induction argument implies that for any $1 \le j \le n-k-1$ we have

$$ad_f^j g = \mu_{n-j}(x)\partial_{x_{n-j}} + \vartheta_{n-j}(x)$$

where $\mu_{n-j}(x) = (-1)^j \prod_{i=1}^{j} \frac{\partial f_{n-i}}{\partial x_{n-i+1}}$ and $\vartheta_{n-j}(x) \in \Delta^j$. In particular for $j = n-k-1$ we have

$$ad_f^{n-k-1} g = \mu_{k+1}(x)\partial_{x_{k+1}} + \vartheta_{k+1}(x)$$

where $\vartheta_{k+1}(x) \in \Delta^{n-k-1}$. The Lie bracket with f gives

$$\begin{aligned} ad_f^{n-k} g &= \sum_{j=1}^{k}\left[f_j(x_1, \ldots, x_{k+1})\partial_{x_j}, \mu_{k+1}\partial_{x_{k+1}} + \vartheta_{k+1}\right] \\ &+ \sum_{j=k+1}^{n}\left[f_j(x_1, \ldots, x_{j+1})\partial_{x_j}, \mu_{k+1}\partial_{x_{k+1}} + \vartheta_{k+1}\right] \\ &= -\mu_{k+1}(x)\sum_{j=1}^{k}\frac{\partial f_j}{\partial x_{k+1}}\partial_{x_j} + \tilde{\vartheta}_k, \end{aligned}$$

where $\tilde{\vartheta}_k(x) \in \Delta^{n-k} = \text{span}\left\{\partial_{x_{k+1}}, \ldots, \partial_{x_n}\right\}$. This is due to the following facts:

(i) $ad_f^{n-k-1} g \in \Delta^{n-k}$;

(ii) $f_j(x_1, \ldots, x_{j+1})\partial_{x_j} \in \Delta^{n-k},\ \ k+1 \le j \le n$;

(iii) $[f_j(x_1, \ldots, x_{k+1})\partial_{x_j}, \Delta^{n-k}] = f_j(\cdot)[\partial_{x_j}, \Delta^{n-k}]$;

(iv) $[\Delta^{n-k}, \Delta^{n-k}] \subseteq \Delta^{n-k}$.

A simple calculation shows (using items (i)-(iv)) that

$$\left[ad_f^{n-k} g, ad_f^{n-k-1} g\right] = \mu_{k+1}^2(x)\sum_{j=1}^{k}\frac{\partial^2 f_j}{\partial x_{k+1}^2}\partial_{x_j} + \hat{\vartheta}_k(x),$$

where $\hat{\vartheta}_k \in \Delta^{n-k} = \text{span}\left\{\partial_{x_{k+1}}, \ldots, \partial_{x_n}\right\}$.

The involutivity of $\mathcal{D}^{n-k+1}$ implies that

$$\begin{aligned} \left[ad_f^{n-k} g, ad_f^{n-k-1} g\right] &= \sum_{j=k}^{n}\delta_{n-j} ad_f^{n-j} g \\ &= \delta_{n-k} ad_f^{n-k} g + \breve{\vartheta}_k \end{aligned}$$

for some smooth functions $\delta_0, \delta_1, \ldots, \delta_{n-k}$.

Comparing the two Lie brackets it follows that

$$(\mu_{k+1})^2 \cdot \sum_{j=1}^{k}\frac{\partial^2 f_j}{\partial x_{k+1}^2}\partial_{x_j} = -(\mu_{k+1})\delta_{n-k} \cdot \sum_{j=1}^{k}\frac{\partial f_j}{\partial x_{k+1}}\partial_{x_j},$$

that is, the condition

$$(\mathcal{F}\pounds_{k+1}) \Longrightarrow \frac{\partial^2 f_j}{\partial x_{k+1}^2} = \gamma_{n-k}(x)\frac{\partial f_j}{\partial x_{k+1}},\ 1 \le j \le k.$$

Notice that $\gamma_{n-k} = \gamma_{n-k}(x_1, \ldots, x_{k+1})$ depends exclusively on the variables $x_1, \ldots, x_{k+1}$ since the components f_j depend only on such variables. A double integration shows that there exist functions $F_j(x)$ and $\nu_j(x)$, $1 \le j \le k$ such that

$$f_j(x_1, \ldots, x_{k+1}) = F_j(x_1, \ldots, x_k) + \nu_j(x_1, \ldots, x_k)\Theta(x)$$

where

$$\Theta(x) = \int_0^{x_{k+1}} \exp\left(\int_0^t \gamma_{n-k}(x_1, \ldots, x_k, \mathrm{s})\mathrm{ds}\right)\mathrm{dt}$$

depends exclusively on γ_{n-k} but not on the components. This achieves the proof of the claim.

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