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Analysis of the Stochastic Channel Model by Saleh & Valenzuela via the Theory of Point Processes

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Abstract—In this paper we revisit the classical channel model by Saleh & Valenzuela via the theory of spatial point processes. By reformulating this model as a particular point process and by repeated application of Campbell's Theorem we provide concise and elegant access to its overall structure and underlying features, like the intensity function of the component delays and the delay-power intensity. The flexibility and clarity of the mathematical instruments utilized to obtain these results lead us to conjecture that the theory of spatial point processes provides a unifying mathematical framework to define, analyze, and compare most channel models already suggested in literature and that the powerful tools of this framework have not been fully exploited in this context yet.

I. INTRODUCTION

Literature regarding channel models for (indoor) radio propagation dates back earlier than 1960, and most commonly the wireless multipath channel is characterized via its (time and space varying) impulse response [1]. Two classic and seminal contributions within channel modeling are those by Turin et al. [2] and Saleh & Valenzuela [3]. To some extent the (indoor) model by Saleh & Valenzuela can be seen as a generalization of the (urban) model by Turin. Specifically, the generalization aimed at mimicking cluster alike behavior since this effect was reported to have been observed experimentally.

Ever since the model by Saleh & Valenzuela (for short the S-V model) was proposed in 1987, many refined or marginally extended variants have appeared, see e.g. [4] and [5]. Unfortunately, these channel models have not been developed within any unifying mathematical framework. Instead their treatment is of rather ad-hoc nature and, as a result, their inherent features remain essentially veiled and any two different models are not easily comparable.

Recently the authors of [6] and [7] reformulated and outlined the S-V model in terms of marked point processes. The S-V model has also been revisited in [8] by use of shot-noise tools and point process theory. Among other things the analysis in [7] and [8] show that the overall intensity of the relative delays of multipath components grows linearly with the propagation delay. Unfortunately, the mathematical tools used in [7] to extract the features of the model are not directly associated with the general theory of point processes. On the other hand, the tools used in [8] are rather advanced and the derivations less transparent. Accordingly, the potential theoretical benefits arising through these point process reformulations are not immediately evident.

In this paper we showcase how the general theory of spatial point processes provides an insightful view upon the inherent structure and features of the classical S-V model. Like [7] and [8] we revisit the model and reformulate it as a particular point process. Aligned with [7] we show that the component delays consist of the union of a Poisson point process and a Cox point process and we derive the associated intensity function as an immediate consequence of Campbell's Theorem. The derivation in [7] is similar but with no reference to Campbell's Theorem. Furthermore, and in contrast to the involved proofs relying on shot-noise tools in [8], we obtain the delay-power intensity in a simple and direct way by invoking once more Campbell's Theorem. These results demonstrate the potential of this well-known theorem from the theory of spatial point processes in the context of stochastic channel modeling. In view of this, our conclusion is that the theory of spatial point processes and its powerful tools have not been fully exploited yet to analyze the properties of most proposed stochastic channel models. This theory appears to provide the necessary unifying framework for which these models can be contrasted within.

II. POINT PROCESS FRAMEWORK

We assume familiarity with the basics of the theory of spatial point processes (see [9, Sec. 1.3, Chap. 2] and [10, Sec. 1.5, 6.2] for highly recommendable introductions). Concepts from abstract measure theory will be kept at a minimum.

A. Locally finiteness and simplicity

Denote by Y a *locally finite* and *simple* point process defined on a d-dimensional space $S \subseteq \mathbb{R}^d$. For intuitive, practical and mathematical reasons, these two properties are convenient to impose since several technical aspects can then be disregarded. A point process is locally finite if the number of points falling in every *bounded Borel set* $B \subseteq S$ is *almost surely* finite. A point process is simple if, almost surely, no two points of the process coincide. Accordingly, any realization of the point process Y can be identified as a countable set of points $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \dots\}$, $\boldsymbol{y}_i \in S$, where the index i of $\boldsymbol{y}_i$ serves solely as a dummy label. Thus, the index is used only to distinguish points and to indicate countability. It does not indicate any ordering of the points.

B. The intensity function and Campbell's Theorem

Consider the *counting function* defined, using a generic indicator function $\mathbb{1}[\cdot] \in \{0, 1\}$, as

$$N_Y(B) := \sum_{\boldsymbol{y} \in Y} \mathbb{1}[\boldsymbol{y} \in B],$$

which equals the random number of points from Y falling in the set B. For any fixed and *bounded* B, the count $N_Y(B)$ is

a non-negative integer-valued random variable. The expected value of the counting function $\mu_Y(B) := \mathbb{E}\big[N_Y(B)\big]$ defines a *measure* on S, the so-called *intensity measure* of Y. If the intensity measure can be expressed as

$$\mu_Y(B) = \int_B \varrho_Y(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad B \subseteq S,$$

for a *locally integrable* function $\varrho_Y : S \to [0, \infty)$, then ϱ_Y is called the *intensity function* of Y. The case when the intensity function exists is by far the most important for applications [11]. The importance of the intensity function is evident from the following result, often referred to as *Campbell's Theorem*.

Campbell's Theorem. *Let Y be a point process on $S \subseteq \mathbb{R}^d$ with intensity function ϱ_Y. Then for a real or complex-valued measurable function $h : S \to \mathbb{R}$ (or $\mathbb{C}$), the random variable $\sum_{\boldsymbol{y} \in Y} h(\boldsymbol{y})$ has expected valued*

$$\mathbb{E}\left[\sum_{\boldsymbol{y} \in Y} h(\boldsymbol{y}) \right] = \int_S h(\boldsymbol{y})\varrho_Y(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{1}$$

provided that the integral on the right exists.

Proofs with varying degrees of detail can be found in [9, Sec. 3.2], [11, Prop. 4.1] and [12, Thm. 2.2]. Often, the theorem is stated only for non-negative functions h, since the equality in (1) is then unconditionally true, i.e. the integral is always well-defined but possibly divergent. When h is real-valued some care must be taken since the integral at the right hand side of (1) has no meaning if the positive and the negative part of h are not integrable. Similar care must be taken for complex h.

C. Poisson and Cox point processes

We now define two classes of point processes which are particularly important for our treatment in the forthcoming section, namely Poisson point processes and Cox point processes. These definitions can be found in many text books covering the theory of spatial point processes. Our treatment is directly inspired by [11] and the interested reader may consult [10]–[12] for further details.

Definition. *A point process Y on $S \subseteq \mathbb{R}^d$ is called a Poisson point process with intensity function ϱ_Y if:*
  (i) *For any $B \subseteq S$ with $\mu_Y(B) = \int_B \varrho_Y(\boldsymbol{s})\mathrm{d}\boldsymbol{s} < \infty$ the count $N_Y(B)$ is Poisson distributed with mean $\mu_Y(B)$.*
  (ii) *Given that $N_Y(B) = n \in \mathbb{N}$ where $0 < \mu_Y(B) < \infty$, the distribution of $Y \cap B$ is the same as that of n points drawn i.i.d. according to f_B, where*

$$f_B(\boldsymbol{s}) := \frac{\varrho_Y(\boldsymbol{s})\mathbb{1}[\boldsymbol{s} \in B]}{\mu_Y(B)}.$$

We write $Y \sim \texttt{PoissonPP}\big(S, \varrho_Y\big)$.

Definition. *Let $Z(\boldsymbol{s})$, $\boldsymbol{s} \in S$, be a non-negative random field such that, almost surely, every realization of Z is a locally integrable function on S. If a point process Y, conditioned on Z, is a Poisson point process with intensity function Z, then Y is called a Cox point process driven by Z.*

Cox point processes (also often referred to as *doubly stochastic Poisson point processes* [10]) are flexible models for clustered point patterns. Specifically, the two-level construction most commonly entails the Cox class to exhibit so-called *over-dispersion* compared to the Poisson class [11, Sec. 5.2].

III. THE MODEL BY SALEH & VALENZUELA

In this section we analyze the impulse response of the classical S-V model within the framework of spatial point processes. The main purpose of this effort is to straightforwardly derive the features of this model through a flexible and powerful theory. Several relevant aspects of the model are revealed through this reformulation, e.g. its overall delay intensity, a concise and clear derivation of the average power gain and, a simple derivation of the delay-power intensity as well.

A. Classical formulation

Saleh & Valenzuela define the channel impulse response with *cluster* and *within-cluster* delays as [3, Eq. (25)]

$$h(t) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \beta_{k,\ell} \exp(j\theta_{k,\ell})\delta\big(t - (T_\ell + \tau_{k,\ell})\big), \tag{2}$$

where δ is the Dirac delta and j is the imaginary unit. The index ℓ indicates a certain cluster and k is the within-cluster index. By definition in [3], $T_0 = 0$ and $\tau_{0,\ell} = 0$ for each $\ell \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Beside these fixed delay components, a sequence of Poisson point processes are suggested such that

- $\{T_\ell\}_{\ell \in \mathbb{N}} \sim \texttt{PoissonPP}\big(\mathbb{R}_+, \Lambda\big)$
- $\{\tau_{k,\ell}\}_{k \in \mathbb{N}} \sim \texttt{PoissonPP}\big(\mathbb{R}_+, \lambda\big)$ for each $\ell \in \mathbb{N}_0$,

with $\Lambda, \lambda > 0$ being two parameters. Moreover, conditional second-order moments are modeled such that [3, Eq. (26)]

$$\mathbb{E}\big[\beta_{k,\ell}^2 \,\big|\, T_\ell, \tau_{k,\ell}\big] = Q \exp\big(-T_\ell/\Gamma\big) \exp\big(-\tau_{k,\ell}/\gamma\big), \tag{3}$$

with $\Gamma, \gamma > 0$ and Q being the average power gain of the first component within the first cluster (i.e. corresponding to the fixed delay T_0). Conditioned on all T_ℓ's and all $\tau_{k,\ell}$'s, the $\beta_{k,\ell}$'s are assumed to be mutually independent random variables. Specifically, each power gain $\beta_{k,\ell}^2$, conditioned on T_ℓ and $\tau_{k,\ell}$, should follow an exponential distribution with mean parameter decaying as described by (3). Fig. 1 illustrates the Poisson point processes involved in the S-V model.

Finally, it was mentioned in [3] that practically the doubly-infinite sum in (2) should "stop" whenever each of the exponentially decaying terms in (3) had become small enough. Through the insight gained via the forthcoming reformulation of this classical channel model we are able to motivate a less heuristic "stopping criterion".

B. Point process formulation

Naturally, we select the space $S = \mathbb{R}_+$ and let $T_0 = 0$ as above. In addition, we introduce the point processes:

$$C := \{T_\ell\}_{\ell \in \mathbb{N}} \qquad \text{(all \textit{cluster} delays except } T_0)$$

$$W_\ell := \{T_\ell + \tau_{k,\ell}\}_{k \in \mathbb{N}} \qquad \text{(delays \textit{within} the } \ell\text{'th cluster)}$$

$$W := \bigcup_{\ell=0}^{\infty} W_\ell \qquad \text{(all \textit{within-cluster} delays)}$$

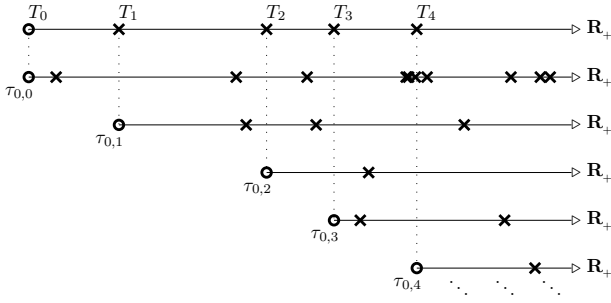$$Y := C \cup W \qquad \text{(all propagation delays except } T_0).$$

Figure 1. Realization of Poisson point processes corresponding to the S-V model. Circle points indicate fixed delay components. The top process occurs with rate Λ while each of the lower processes occurs with rate λ. A new point process is initialized whenever a new point emerges from the top process.

Figure 2. Intensity functions associated with the S-V model.

Notice that C is the Poisson point process specified at first in the previous paragraph. Its intensity function has a simple form, namely $\varrho_C(t) = \Lambda$ for all $t \in S$. By conditioning, we immediately identify a sequence of Poisson point processes

$$W_\ell | T_\ell \sim \texttt{PoissonPP}\Big(\mathbb{R}_+, \; \lambda \mathbb{1}\big[t > T_\ell\big]\Big), \quad \ell \in \mathbb{N}_0,$$

and since the Poisson class is stable with respect to countable superpositions [11, Prop. 3.6], we see that

$$W|C \sim \texttt{PoissonPP}\Big(\mathbb{R}_+, \; \widetilde{\varrho}_W\Big),$$

with the staircase-alike intensity function

$$\widetilde{\varrho}_W(t) = \lambda + \lambda \sum_{c \in C} \mathbb{1}\big[t > c\big], \quad t \in S. \tag{4}$$

Accordingly, we identify that the point process W, without conditioning on C, is a Cox point process driven by a stochastic process Z having the same functional form as $\widetilde{\varrho}_W$ in (4) but with C being random. The intensity function of the Cox point process W is $\varrho_W(t) = \mathbb{E}\big[Z(t)\big]$ [11, Sec. 5.2], and by direct application of Campbell's Theorem we get

$$\varrho_W(t) = \lambda + \lambda \mathbb{E}\left[\sum_{c \in C} \mathbb{1}\big[t > c\big]\right] = \lambda + \lambda \Lambda t, \quad t \in S.$$

Since $Y = C \cup W$ is a union of almost surely disjoint point processes, its associated intensity function reads [10, Sec. 6.2.3]

$$\varrho_Y(t) = \varrho_C(t) + \varrho_W(t) = \Lambda + \lambda + \lambda \Lambda t, \quad t \in S.$$

It is interesting to notice that the entire set of propagation delays (excluding the first component T_0) is the *union of a Poisson point process and a Cox point process*. Of course, the realization of W depends upon the realization of C, i.e. these two point processes are not independent. In [7] this interpretation was inherently adopted, without being explicitly mentioned. Another interesting yet expected observation is that the intensity function ϱ_Y rises linearly with propagation delay, see Fig. 2. The jump of height $\Lambda + \lambda$ at $T_0 = 0$ in the graph of ϱ_Y appears due to the cluster delays and the delays within the very first cluster. The term $\lambda \Lambda t$ result from the fact that, on average, a total of Λt additional clusters emerge during the interval $[0, t]$, with each and every one of them spawning further delay components at rate λ.
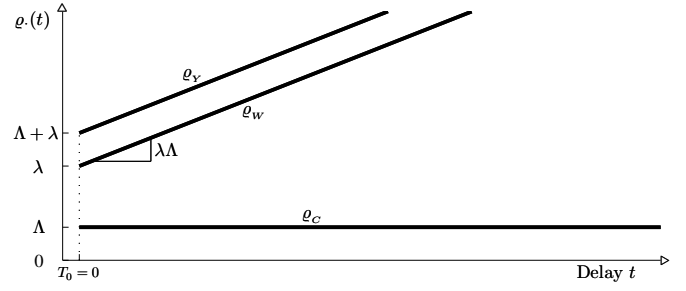
C. Multipath power gain

Analogous to the approach in [3], we consider the following non-negative random variable

$$G := \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \beta_{k,\ell}^2, \tag{5}$$

referred to as the total *multipath power gain* [3]. By splitting G into three terms corresponding to T_0 and arrivals in C and W, its expectation can be calculated as

$$\mathbb{E}\big[G\big] = \mathbb{E}\big[\beta_{0,0}^2\big] + \mathbb{E}\left[\underbrace{\sum_{\ell=1}^{\infty} \beta_{0,\ell}^2}_{(\star)}\right] + \mathbb{E}\left[\underbrace{\sum_{\ell=0}^{\infty} \sum_{k=1}^{\infty} \beta_{k,\ell}^2}_{(\diamond)}\right].$$

As in [3] we write $\beta\big(T_\ell, \tau_{k,\ell}\big)$ in substitute for $\beta_{k,\ell}$ to facilitate a comprehensible notation in the following. For additional clarity we introduce the function

$$f\big(t, \tilde{t}\big) := Q \exp\big(-t/\Gamma - \tilde{t}/\gamma\big), \quad t, \tilde{t} \in S.$$

Notice that $f\big(T_\ell, \tau_{k,\ell}\big) = f\big(T_\ell, (T_\ell + \tau_{k,\ell}) - T_\ell\big)$ coincides with the expression in (3). Then, by intermediate conditioning on C, we calculate the expectation of the term $(\star)$ as

$$\mathbb{E}\big[(\star)\big] = \mathbb{E}\left[\sum_{c \in C} \beta^2(c,0)\right] = \mathbb{E}\left[\sum_{c \in C} \underbrace{\mathbb{E}\big[\beta^2(c,0)\,|\,C\big]}_{f(c,0)}\right] = Q\Lambda\Gamma,$$

where the final step follows by application of Campbell's Theorem. Next, by defining $C_0 := \{T_0\} \cup C$ and with a similar sequence of manipulations involving intermediate conditioning and Campbell's Theorem, we find the expected value of $(\diamond)$ to be[1]

$$\mathbb{E}\big[(\diamond)\big] = \mathbb{E}\left[\sum_{c \in C_0} \sum_{w \in W_c} \beta^2(c, w-c)\right] \tag{6}$$

$$= \mathbb{E}\left[\sum_{c \in C_0} \mathbb{E}\left[\sum_{w \in W_c} \underbrace{\mathbb{E}\big[\beta^2(c, w-c)\,|\,c, w\big]}_{f(c, w-c)}\right]\right] = Q\big(1 + \Lambda\Gamma\big)\lambda\gamma.$$

Accordingly, the average total power gain is given by

$$\mathbb{E}\big[G\big] = \mathbb{E}\big[\beta_{0,0}^2\big] + \mathbb{E}\big[(\star)\big] + \mathbb{E}\big[(\diamond)\big]$$
$$= \quad Q \quad + \quad Q\Lambda\Gamma \quad + \quad Q\big(1 + \Lambda\Gamma\big)\lambda\gamma$$
$$= Q\big(1 + \lambda\gamma\big)\big(1 + \Lambda\Gamma\big), \tag{7}$$

[1]Note that in (6) we abuse notation since the collections W_c are not explicitly defined. We only defined these as W_ℓ via the counting index ℓ.

as was also reported in a footnote in [3]. Yet, the original sequence of arguments used to obtain this result may appear less instructive, see [3, Eq. (27), (31)] for comparison. Notice that, depending on how we choose to write out the product in (7), we end up with different interpretations of individual average power contributions.

D. Delay-power intensity

Motivated by the definition of G in (5) together with the relationship in (7), we consider

$$p(t) := \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \beta_{k,\ell}^2 \delta\big(t - (T_\ell + \tau_{k,\ell})\big).$$

We wish to calculate how the average power gains are distributed across delay. From (7) we already know the mean total power gain, yet we seek to obtain further insight. The above definition of $p(t)$ is motivated by the fact that $\int_0^\infty p(t)\mathrm{d}t = G$, and since $\mathbb{E}[G]$ is finite, the non-negative random variable G is itself finite almost surely. Accordingly, we define

$$P(t) := \mathbb{E}\big[p(t)\big], \quad t \in S,$$

and we refer to this function as the *delay-power intensity*. By similar manipulations as in the previous paragraph (conditioning, Campbell's Theorem, etc.) we find

$$\frac{P(t)}{Q} = \delta(t) + \begin{cases} k_1 \exp\big(-\frac{1}{\Gamma}t\big) + k_2 \exp\big(-\frac{1}{\gamma}t\big), & \Gamma \neq \gamma \\ \varrho_Y(t) \exp\big(-\frac{1}{\gamma}t\big), & \Gamma = \gamma \end{cases}$$

where we have conveniently introduced the two constants

$$k_1 := \Lambda\Big(1 + \lambda\frac{\Gamma\gamma}{\Gamma-\gamma}\Big) \quad \text{and} \quad k_2 := \lambda\Big(1 - \Lambda\frac{\Gamma\gamma}{\Gamma-\gamma}\Big).$$

The same expression for $P(t)$ is obtained in [8, Chap. 2,3] using rather involved shot-noise tools with weighty notational overhead. Notice the particular relationship

$$\mathbb{E}[G] = \mathbb{E}\bigg[\int_0^\infty p(t)\mathrm{d}t\bigg] = \int_0^\infty \mathbb{E}\big[p(t)\big]\mathrm{d}t = \int_0^\infty P(t)\mathrm{d}t.$$

The delay-power intensity of the S-V model is depicted in Fig. 3. Notice that $P(t)$ is *not exponentially decaying*, not even when $\Gamma = \gamma$ since ϱ_Y rises linearly (compare with the dotted line in Fig. 3).

Finally, as mentioned in the beginning of this section, we are now able to motivate a simple "stopping criterion" suitable, e.g. for simulation purposes. Specifically, select a delay threshold $t_{\max}(\alpha)$ such that

$$\int_0^{t_{\max}(\alpha)} P(t)\mathrm{d}t = \alpha\mathbb{E}[G],$$

for a relevant choice of $\alpha \in (0,1)$, e.g. $\alpha = 0.99$.

IV. CONCLUSION

In this contribution we have revisited the radio channel model by Saleh & Valenzuela (the S-V model) within the framework of spatial point processes. We have shown that the component delays in the S-V model emerge from the union of a Poisson point process and a Cox point process. Furthermore, we have demonstrated that the intensity function of the component delays and the delay-power intensity can
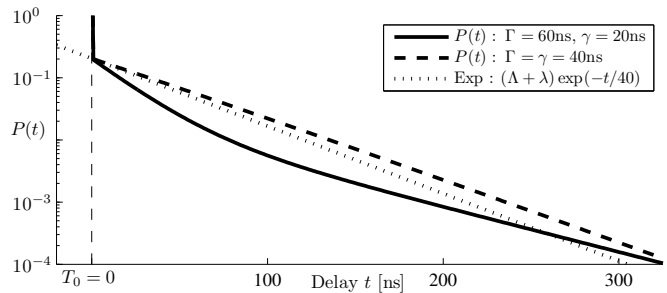

Figure 3. Delay-power intensity of the S-V model (solid line). The parameter values of $\Lambda, \lambda, \Gamma, \gamma$ correspond to the estimates reported in [3]. The dashed curve correspond to the selection $\Gamma = \gamma = 40$ns. For comparison, the dotted curve provides a purely exponential decay.

be derived in a straightforward and rigorous manner as an immediate consequence of Campbell's Theorem.

The above results indicate that the theory of spatial point processes yields a natural, unifying theoretical framework for dealing with stochastic channel models. This applies in particular to most channel models already suggested in literature, including the models by Turin et al. [2], Spencer et al. [4], and Chong et al. [5]. Our results also reveal that the powerful tools available in this framework, like Campbell's Theorem, have not been exploited to their full extent in this context yet. Overall the considered application to the S-V model and to some extent the work in [6]–[8] show that the resulting mathematical treatments inherit clarity and conciseness, in addition to rigorousness, in contrast to the traditionally used ad-hoc and heuristic arguments.

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