1995

Crosscorrelation Functions of DS Codes and Probability of Error Evaluations for a Multiple Access Channel

Osmar Coronel
Southern Illinois University Carbondale

R. Viswanathan
Southern Illinois University Carbondale, viswa@engr.siu.edu

Crosscorrelation Functions of DS Codes and Probability of Error Evaluations for a Multiple Access Channel

Osmar Coronel and R. Viswanathan
Department of Electrical Engineering
Southern Illinois University at Carbondale
Carbondale, IL 62901-6603

Abstract

We derive a computationally feasible closed form error expression for probability of error for a small multiple access DS-BPSK system operating in a Rayleigh fading environment. Two properties of the crosscorrelation functions of the DS codes help reduce the number of error probability evaluations required when one considers all possible phase shifts of the users' codes. Examination of the error rates for different phases of three maximal-length codes reveal that the probability of error is only weakly dependent on the phases.

I Introduction

The evaluation of the system capacity of a multi-user DS communication system is in general complicated. An approach is to find the maximum multiple-access interference for a given model. When the peak interference does not appear very often, this scheme yields pessimistic results. A more realistic approach that leads to the signal to noise ratio as a measure of performance is the use of mean-squared value of the multiple-access interference [1]. However, what really measures the performance of a system and hence determines the system capacity is the probability of bit error, P_e. In a direct sequence multiple access system, P_e is a function of crosscorrelation of the direct sequence codes. In a simple but not accurate model, the effect of the multi-user interference on the output of the receiver correlator has been assumed to be Gaussian[2,4]. The numerical evaluation of P_e that does not invoke Gaussian assumption for multiple access interference is in general computationally intensive[5-7]. For system optimization purposes, it is of interest to know whether the relative phases of the codes have some influence on the probability of error. For instance in [8], the authors optimize the phases of a set of codes with respect to the signal to noise ratio.

This study provides a computable closed form solution for the probability of bit error of an asynchronous direct sequence-binary phase shift keying (DS-BPSK) code division multiple access(CDMA) system operating in a Rayleigh fading channel with a few number of users. It examines whether the relative phases of the codes play a significant role in the performance of a multi-user system, and also provides some insights on the properties of crosscorrelation functions of the codes. In section II we specify our transmitter, channel, and receiver model. A closed form expression for the P_e is derived. In Appendix A some properties of the cross correlation functions of DS codes are stated. In section III, P_e is evaluated for different phase shifts of the codes. Section IV concludes this study.

II Multiple Access System

Assume a star configuration with K users connected to a base station using a DS-BPSK system. In our analysis we consider user number one the as the reference user.

2.1 Transmission Model

The k^{th} user information signal is represented by

$$b_k(t) \equiv \sum_{i=-\infty}^{\infty} b_i^k P_T(t-iT), \tag{1}$$

where b_i^k represents the k^{th} user data taking on values from the set $\{\pm 1\}$ with equal probability during the i^{th} timing interval, $P_T(t)$ is a rectangular waveform of unit height over $(0,T)$ and zero elsewhere. The pseudo-random spreading signature sequence with period N is given by

$$a_k(t) \equiv \sum_{j=-\infty}^{\infty} a_j^k P_{T_c}(t-jT_c), \tag{2}$$

where a_j^k represents the k^{th} user chip at the j^{th} timing interval and $P_{T_c}(.)$ is a rectangular waveform of unit height and T_c seconds duration. We assume that each bit b_i^k of $b_k(t)$ spans one period of $a_k(t)$ which takes $T = NT_c$ seconds, where T and T_c are the bit and the chip duration respectively. Each user has a different signature sequence. Then the transmitted signal by the k^{th} user is

$$s_k(t) = Ab_k(t)a_k(t)\cos(\omega t + \phi_k), \tag{3}$$

where ϕ_k is a random variable (r.v.) representing an arbitrary carrier phase.

2.2 Channel Model

The channel is assumed to be Rayleigh fading with low pass equivalent impulse response

$$h_k(t) \equiv \beta_k \delta(t-\tau_k)e^{j\varphi_k}, \tag{4}$$

for the k^{th} user, where $\delta(.)$ is the Dirac delta function, τ_k is a r.v. uniformly distributed over $[0,T)$ representing an arbitrary time delay, φ_k is a uniform r.v. representing an arbitrary phase shift due to the channel, β_k is a Rayleigh r.

0-7803-2489-7/95 $4.00 © 1995 IEEE

v. with second moment equal to $2\sigma_k^2$. All the channel variables are time invariant, mutually independent and independent from user to user. This could be an approximate model of a cellular communication system. The received signal is given by

$$r(t) = \sum_{k=1}^{K} A\beta_k b_k(t-\tau_k) a_k(t-\tau_k)\cos(\omega t+\theta_k)+\eta(t), \quad (5)$$

$$\theta_k \equiv \phi_k + \varphi_k + \omega\tau_k, \quad (6)$$

ϕ_k and φ_k are such that θ_k is a r.v. uniformly distributed over $[0,2\pi)$, $\eta(t)$ is AWGN with double sided power spectral density (psd) of $N_0/2$.

2.3 Receiver Model

The receiver of the desired user is assumed to coherently recover the carrier phase and delay by locking onto the arriving desired signal[3,5]. In the receiver side, we have user number one's despreading signature, a correlator and a threshold detector. θ_1 and τ_1, of the user 1 received signal can be assumed to be zero with no loss of generality. The signal sample at the reference user's receiver integrator output can be called x' and is given by

$$x' = \int_0^T r(t)a_1(t)\cos(\omega t)dt = \frac{AT}{2}\beta_1 b_0^1 + \frac{AT}{2}z + v' \quad (7)$$

where $\frac{AT}{2}\beta_1 b_0^1$ is the desired component, z is the multi-user interference term defined as

$$z \equiv \sum_{k=2}^{K} \beta_k \cos(\theta_k)\frac{1}{T}\int_0^T b_k(t-\tau_k)a_1(t)a_k(t-\tau_k)dt \quad (8)$$

$v' \sim N(0,\frac{N_0 T}{4})$, and $N(m,\sigma^2)$ denotes a normal density with mean m, and variance σ^2.

We can rewrite the integrator's output as

$$x' = \sigma_1\frac{AT}{2}x, \quad (9)$$

where the decision variable x is given by

$$x = \frac{\beta_1}{\sigma_1}b_0^1 + \frac{z}{\sigma_1} + \frac{v'}{(\sigma_1 AT/2)}. \quad (10)$$

In a Rayleigh fading environment, the actual received signal to noise ratio is $\beta_1^2\frac{E_b'}{N_0}$, where $E_b' = \frac{A^2 T}{2}$ and the average signal to noise ratio is

$$\frac{E_{b1}}{N_0} = E(\beta_1^2)\frac{E_b'}{N_0}. \quad (11)$$

It is clear that the third term on the right hand side of (10) is distributed as normal with zero mean and variance equal to $\frac{1}{E_{b1}/N_0}$. The receiver decision is based on the following:

$$x \underset{H_0}{\overset{H_1}{\gtrless}} 0 \quad (12)$$

where the hypotheses are

H_0: $b_0^1 = -1$

H_1: $b_0^1 = +1$

A set of crosscorrelation functions that contribute to the probability of error through the multi-user interference term are the continuous periodic crosscorrelation function[1,2,9]

$$\mathcal{R}_{ji}(\tau) \equiv \int_0^T a_i(t)a_j(t-\tau)dt, \quad (13)$$

and the continuous odd crosscorrelation function

$$\hat{\mathcal{R}}_{ji}(\tau) \equiv \int_\tau^T a_i(t)a_j(t-\tau)dt - \int_0^\tau a_i(t)a_j(t-\tau)dt. \quad (14)$$

Let

$$\mathcal{R}_{k1b}(\tau_k) \equiv \begin{cases} \mathcal{R}_{k1}(\tau_k) \text{ if } b_{-1}^k b_0^k = +1 \\ \hat{\mathcal{R}}_{k1}(\tau_k) \text{ if } b_{-1}^k b_0^k = -1 \end{cases} \quad (15)$$

Rewriting the multi-user term z in terms of the crosscorrelation functions yields,

$$z = \sum_{k=2}^{K}\frac{\mathcal{R}_{k1b}(\tau_k)}{T}\beta_k\cos(\theta_k). \quad (16)$$

We note that $\beta_k\cos(\theta_k)$ is a Gaussian r.v. with zero mean and variance σ_k^2. Therefore the conditional distribution of z given $\{\mathcal{R}_{k1b}(\tau_k), 2\le k\le K\}$ is $N(0, \sum_{k=2}^{K}\frac{\mathcal{R}_{k1b}^2(\tau_k)}{T^2}\sigma_k^2)$.

The detector makes a wrong decision if x is negative while $b_0^1 = +1$, or if x is positive and $b_0^1 = -1$. During the detection interval of b_0^1, the other two bits of the k^{th} interferer namely b_{-1}^k and b_0^k for $k\neq 1$ can independently take on $\{\pm 1\}$. It is clear from equations (10), (12), (15), and (16) that,

$$f_X(x|\beta_1, b_0^1, \mathcal{R}_{k1b}(\tau_k), 2\le k\le K) = N(\frac{\beta_1}{\sigma_1}b_0^1, \sum_{k=2}^{K}\frac{\mathcal{R}_{k1b}^2(\tau_k)}{T^2}\frac{\sigma_k^2}{\sigma_1^2} + \frac{1}{E_{b1}/N_0}) \quad (17)$$

Let

$$S_b \equiv \{b_{-1}^k, b_0^k \mid 2\le k\le K, b_{-1}^k\in\{\pm 1\}, b_0^k\in\{\pm 1\}\}, \quad (18)$$

$$S_\tau \equiv \{\tau_k \mid 2\le k\le K, 0\le\tau_k<T\}. \quad (19)$$

The conditional probability of error is expressed by,

$$P_{e|\beta_1,S_b,S_\tau}=\frac{1}{2}\{P(x'<0|H_1)+P(x'>0|H_0)\}$$

$$=Q\left(\frac{\beta_1/\sigma_1}{\sqrt{\sum_{k=2}^{K}\frac{\mathcal{R}_{k1b}^2(\tau_k)}{T^2}\frac{\sigma_k^2}{\sigma_1^2}+\frac{1}{E_{b1}/N_0}}}\right). \tag{20}$$

Above Q(.) is one minus the standard normal cdf. $\left(\frac{\beta_1}{\sigma_1}\right)^2$ has an exponential distribution with mean equal to 2, because β_1 is a Rayleigh r.v. with second moment equal to $2\sigma_1{}^2$. Let

$$\gamma_o\equiv\frac{1}{\sum_{k=2}^{K}\frac{\mathcal{R}_{k1b}^2(\tau_k)}{T^2}\frac{\sigma_k^2}{\sigma_1^2}+\frac{1}{E_{b1}/N_0}} \tag{21}$$

Averaging (20) with respect to β_1 yields,

$$P_{e|S_b,S_\tau}=\frac{1}{2}\left[1-\frac{1}{\sqrt{1+1/\gamma_o}}\right] \tag{22}$$

Up on substituting for γ_o,

$$P_{e|S_b}=\frac{1}{2}-\frac{1}{2}\int_{S_\tau}\!\!\cdot\!\int\!\cdot\!\cdot\!\int\frac{d\tau_K\ldots d\tau_2}{\sqrt{1+\sum_{k=2}^{K}\frac{\mathcal{R}_{k1b}^2(\tau_k)}{T^2}\frac{\sigma_k^2}{\sigma_1^2}+\frac{1}{E_{b1}/No}}}. \tag{23}$$

As we see, due to $\mathcal{R}_{k1b}^2(\tau_k)$, the probability of bit error is controlled by the sign of the product $b_{-1}^k b_0^k$ instead of b_{-1}^k and b_0^k individually. The product $b_{-1}^k b_0^k$ gives the relative sign between the two bits of the k^{th} user, which are overlapping the bit of the desired user.

For a three users system, let us define the following.

$$S_0\equiv\{b_{-1}^2,b_0^2,b_{-1}^3,b_0^3\,|\,b_{-1}^2=b_0^2,b_{-1}^3=b_0^3\}, \tag{24}$$
$$S_1\equiv\{b_{-1}^2,b_0^2,b_{-1}^3,b_0^3\,|\,b_{-1}^2=b_0^2,b_{-1}^3=-b_0^3\}, \tag{25}$$
$$S_2\equiv\{b_{-1}^2,b_0^2,b_{-1}^3,b_0^3\,|\,b_{-1}^2=-b_0^2,b_{-1}^3=b_0^3\}, \tag{26}$$
$$S_3\equiv\{b_{-1}^2,b_0^2,b_{-1}^3,b_0^3\,|\,b_{-1}^2=-b_0^2,b_{-1}^3=-b_0^3\}. \tag{27}$$

Hence,

$$P_{e|S_0}=\frac{1}{2}-\frac{1}{2}\int_0^T\!\!\int_0^T\frac{d\tau_2\,d\tau_3}{\sqrt{1+\frac{\sigma_2^2}{\sigma_1^2}\frac{\mathcal{R}_{21}^2(\tau_2)}{T^2}+\frac{\sigma_3^2}{\sigma_1^2}\frac{\mathcal{R}_{31}^2(\tau_3)}{T^2}+\frac{1}{E_{b1}/N_0}}} \tag{28}$$

In (28), replacing $\mathcal{R}_{31}(\tau_3)$ ($\mathcal{R}_{21}(\tau_2)$) with $\hat{\mathcal{R}}_{31}(\tau_3)$ ($\hat{\mathcal{R}}_{21}(\tau_2)$) gives $P_{e|S_1}$ ($P_{e|S_2}$). $P_{e|S_3}$ is obtained from (28) by replacing $\mathcal{R}_{31}(\tau_3)$ by $\hat{\mathcal{R}}_{31}(\tau_3)$ and $\mathcal{R}_{21}(\tau_2)$ by $\hat{\mathcal{R}}_{21}(\tau_2)$. Appendix A presents several properties of the crosssorrelation functions discussed in this section. As discussed in the next section, properties IV and V help to reduce the number of evaluations of the conditional error probabilities $P_{e|S_0}$, $P_{e|S_1}$, and $P_{e|S_2}$, when one considers all the phase shifts of the codes of the three users. Finally,

$$P_e=\frac{1}{4}[P_{e|S_0}+P_{e|S_1}+P_{e|S_2}+P_{e|S_3}]. \tag{29}$$

III Probability of Error for Three Users

In a system with K users, assume each user has an m-sequence signature. For N=31 or 63 there are only three non reciprocal m-sequences available. Hence assume K=3 and N=31 from now on. Let σ_i^2 be equal for all i, which means each user experiences a similar Rayleigh fading. The integrals involved in the probability of error expressions such as (28) can be evaluated in closed form as described in [10]. Let $Pe_i(b,c,d)$ be the probability of bit error of the i^{th} user given that the user number one is using its signature with phase b, user two with phase c, and user three with phase d. Because the crosscorrelation functions are a function of the sequences' phases, an optimization of the probability of error can be performed by adjusting the phases of the codes such that $Pe_{peak}(b,c,d)\equiv\max\{Pe_i(b,c,d):1\le i\le 3\}$ is minimized. To generate the signatures we are using a shift register configuration of figure 8-5 of [11]. From [11] we obtain the coefficients for the feedback tap g_5, g_4, g_3, g_2, g_1, 1. For user 1, the feedback tap is $45_8=100101_2$, for user 2, the feedback tap is $75_8=111101_2$, and for user 3, the feedback tap is $67_8=110111_2$. We need to find the values of $(b\;c\;d)$ that achieve

$Pe_{\max}\equiv\max\{Pe_{peak}(b,c,d):1\le b,c,d\le N\}$,

and $Pe_{\min}\equiv\min\{Pe_{peak}(b,c,d):1\le b,c,d\le N\}$. Let the corresponding values of $(b\;c\;d)$ be $(b_M\;c_M\;d_M)$ and $(b_m\;c_m\;d_m)$, respectively. To do this we need to calculate Pe, N^3 times corresponding to all shifts of the three users. However, according to the property IV, $P_{e|S_0}$ Eq. (28) is invariant with respect to any phase shift of any of the codes. Therefore, we have to calculate it just once, instead of N^3 times. Because of property V, we need to calculate $P_{e|S_1}$ and $P_{e|S_2}$ N^2 times each one, instead of N^3

times each one. Finally, $P_{e|S_3}$ needs to be calculated N^3 times. Garber and Pursley [8] performed a phase optimization for m-sequences of length 31 generated by shift registers with the same feedback taps that we have used, but they maximized the signal to noise ratio defined in [2]. For K=3, N=31 and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2$ we have found

b_M=28, c_M=16, d_M=16,
$Pe_{\max} = 1.3476\text{x}10^{-2}$ for E_b / N_o =30dB,
$Pe_{\max} = 3.4818\text{x}10^{-2}$ for E_b / N_o =10dB;
b_m=6, c_m=27, d_m=19,
$Pe_{\min} = 0.8685\text{x}10^{-2}$ for E_b / N_o =30dB,
$Pe_{\min} = 3.0623\text{x}10^{-2}$ for E_b / N_o =10dB.

Table I shows the phases of the codes that yield $Pe_{\min}$ and $Pe_{\max}$ and Table II from [8] shows the phases that yield minimum and maximum multi- user interference variance. It should be noted that Table I corresponds to the shift register configuration of Fig. 8.5 of [11], with the most significant bit of initial loading occupying the left most cell of the shift register, whereas Table II corresponds to the configuration in [12]. It can be verified that the group of phases that produce minimum probability of bit error and minimum multi- user interference variance are not the same and a similar statement holds true for the maximum probability of error and the maximum interference variance. On the other hand, from these results we observe that $Pe_{\max}$ is less than two times $Pe_{\min}$. Hence, for Rayleigh fading channel, the probability of error does not change significantly with the phase shifts of the users' codes. Our results seem to validate the results in [8], where the spread of the interference variance over different phases of the codes was also found to be of the same order, namely the ratio of maximum to minimum interference variances is less than two. Because of fading, change of *SNR* by a factor 2 would roughly translate to a factor of only 2 in error probability also. This is in contrast to an AWGN channel where the changes in codes' phases can change E_b / N_0 by as much as 2dB at an error rate of 10^{-5}[6]. Also, only a slight decrease in error probability, when E_b / N_0 is changed from 10dB to 30dB, indicates that the multiple access system considered is essentially interference limited. With a processing gain of 31, the system is able to support only three users at 10^{-2} error rate. In [4] the author considers error rates when interfererers' signal range from one-fifth to one-hundredth of the reference user's signal. But the effect of code phases was not considered.

IV Conclusions

In this paper we derived a computationally feasible closed form error expression for a three user DS-BPSK system operating in a Rayleigh fading environment. The results obtained indicate that the probability of error is only weakly dependent on the phases of the DS codes.

Appendix A
Properties of Crosscorrelation Functions

Below we state some properties of different crosscorrelation functions defined in section II. The properties can be proved by the application of periodicity of $a_i(t)$ and change of variable of integration.

All the integrals that appear below are assumed to exist. Also, $g(\ )$ and $f(\ )$ are arbitrary functions of appropriate variables.

Let n, m, and q be integers, l=n-m,

$$a_i(t) \equiv a_1(t + nT_c), \tag{A1}$$

$$a_j(t) \equiv a_k(t + mT_c), \tag{A2}$$

$$a_p(t) \equiv a_r(t + qT_c). \tag{A3}$$

I

$$\mathcal{R}_{k1}(\tau) = \int_0^T a_1(t) a_k(t-\tau) dt = \int_c^{T+c} a_1(t) a_k(t-\tau) dt. \tag{A4}$$

II

$$\mathcal{R}_{ji}(\tau) = \mathcal{R}_{k1}(\tau + lT_c) \tag{A5}$$

III

$$\int_0^T g(\mathcal{R}_{ji}(\tau)) d\tau = \int_0^T g(\mathcal{R}_{k1}(\tau)) d\tau \tag{A6}$$

This property states that $\int_0^T g(\mathcal{R}_{ji}(\tau)) d\tau$ is invariant to any shifts of $a_k(t)$ and $a_1(t)$, since m and n are arbitrary. This means (A6) also equals $\int_0^T g(\mathcal{R}_{ki}(\tau)) d\tau = \int_0^T g(\mathcal{R}_{j1}(\tau)) d\tau$.

IV

$$\int_0^T\int_0^T f(\mathcal{R}_{ji}(\tau_j), \mathcal{R}_{pi}(\tau_p)) d\tau_j d\tau_p = \int_0^T\int_0^T f(\mathcal{R}_{k1}(\tau_k), \mathcal{R}_{r1}(\tau_r)) d\tau_k d\tau_r \tag{A7}$$

V

$$\int_0^T\int_0^T f\left(\mathcal{R}_{ji}(\tau_j), \hat{\mathcal{R}}_{pi}(\tau_p)\right) d\tau_j d\tau_p = \int_0^T\int_0^T f\left(\mathcal{R}_{ki}(\tau_k), \hat{\mathcal{R}}_{pi}(\tau_p)\right) d\tau_k d\tau_p \tag{A8}$$

References

[1] M.B. Pursley, "Spread Spectrum Multiple-access Communications," in *Multi-User Communication*

Systems, Ed., G. Longo, New York: Springer-Verlag, pp. 139-199, 1981.

[2] M. B. Pursley, "Performance Evaluation for Phase-Coded Spread-Spectrum Multiple-Access Communication-Part I: System Analysis", IEEE Trans. on Comm.,pp. 795-799, 1977.

[3] M. Kavehrad and P. J. McLane, "Performance of Low-Complexity Channel Coding and Diversity for Spread Spectrum in Indoor, Wireless Communication", AT&T Tech. Journal, vol. 64, pp. 1927-1965, Oct. 1985.

[4] E. Geraniotis, "Direct-Sequence Spread-Spectrum Multiple-Access Communications Over Nonselective and Frequency-Selective Rician Fading Channels," IEEE Trans. On Comm., pp. 756-764, Aug. 1986.

[5] M. Kavehrad, "Performance of Nondiversity Receivers for Spread Spectrum in Indoor, Wireless Communication", AT&T Tech. Journal, vol. 64, No 6, pp. 1181-1210, July-August 1985.

[6] M.B. Pursley, D.V. Sarwate, and W.E. Stark, "Error Probability for Direct Sequence Spread Spectrum Multiple-access Communications- Part I: Upper and Lower Bounds, IEEE Trans. on Comm., pp. 975-984, May 1982.

[7] E.A. Geraniotis and M.B. Pursley,"Error Probability for Direct Sequence Spread Spectrum Multiple-access Communications- Part II: Approximations, IEEE Trans. on Comm., pp. 985-995, May 1982.

[8] F. D. Garber and M. B. Pursley, "Optimal Phases of Maximal Length Sequences for Asynchronous Spread-Spectrum Multiplexing", Electron. Lett., vol. 16, pp. 756-757, Sept. 1980.

[9] D. V. Sarwate and M. B.Pursley, "Crosscorrelation properties of pseudorandom and related sequences", Proc. IEEE, vol. 68, pp. 593-619, May 1980.

[10] O.A. Coronel, "Crosscorrelation Functions of DS Codes and Probability of Error Evaluation for Multiple Access Channels," M.S. thesis, Department of EE, SIUC, Carbondale, Illinois, Sept. 1994.

[11] Rodger E. Ziemer and Rodger L. Peterson, *Digital Communications and Spread Spectrum Systems*, New York: Macmillan Publishing Co.,1985.

[12] M.B. Pursley and H.F.A. Roefs, "Numerical Evaluation of Correlation Parameters for Optimal Phases of Binary Shift-Register Sequences," IEEE Trans. On Comm., pp. 1597-1604, Oct. 1979.

Feedback Connection	Initial Loading min	max
100101	00110	11100
111101	11011	10000
110111	10011	10000

TABLE I
Initial Conditions of a 5-Stage Shift Register for Maximum and Minimum Probability of Error

Feedback Connection	Initial Loading min	max
100101	00110	11111
111101	10111	11001
110111	11110	11100

TABLE II
Initial Conditions of a 5-Stage Shift Register for Maximum and Minimum Interference Variance[8]