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Linearizable Feedforward Systems: A Special Class

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Linearizable Feedforward Systems: A Special Class

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***Abstract*— We address the problem of linearizability of systems in feedforward form. In a recent paper [22] we completely solved the linearizability for strict feedforward systems. We extend here those results to a special class of feedforward systems. We provide an algorithm, along with explicit transformations, that linearizes the system by change of coordinates when some easily checkable conditions are met. We also re-analyze type II class of linearizable strict feedforward systems provided by Krstic in [9] and we show that this class is the unique linearizable among the class of *quasi-linear strict feedforward systems* (see Definition III.1). Our results allow an easy computation of the linearizing coordinates and thus provide a stabilizing feedback controller for the original system. They can also be implemented via software like mathematica/matlab/maple using simple integrations, derivations of functions.**

I. Introduction

LINEAR systems constitute, without doubt, the most well-known class of control systems. Their importance resides in the fact that several physical systems can be modeled using linear dynamics making thus their analysis and design very simple. The controllability, observability, reachability, and realization of linear systems have been expressed in very simple algebraic terms. Another crucial property of linear controllable systems is that they can be stabilized by linear feedback controllers. Although not all systems can be modeled using linear dynamics, the approximation of nonlinear phenomena by linear models has proved to be a satisfactory tool for their study. It is not then surprising that the question of transforming nonlinear control systems into linear ones has attracted much attention. To give a brief account of that, consider a control system

$$\Sigma \ : \ \dot{x} = f(x) + g(x)u, \quad (x, u) \in X \times U \subseteq \Re^n \times \Re^m,$$

defined in an open neighborhood $X \times U$ of $(0, 0) \in \Re^n \times \Re^m$. The two problems below were investigated in the early 80's:
Problem 1. Does there exist a diffeomorphism $z = \Phi(x)$ giving rise to new coordinates system $z = (z_1, \ldots, z_n)^\top$ in which the system Σ takes the linear form

$$\dot{z} = Fz + Gu, \qquad z \in \Re^n, \ u \in \Re^m \, ?$$

Problem 2. Did there exist a change of coordinates $z = \Phi(x)$ coupled with an invertible feedback $u = \alpha(x) + \beta(x)v$ that transform Σ into a linear system $\dot{z} = Fz + Gv$?

Both problems were solved independently by Jakubczyk and Respondek [6], and Hunt and Su [4], who gave necessary and sufficient geometric conditions in terms of Lie brackets of vector fields defining the system (see Theorem II.2).

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Although those conditions did provide a way of testing the (feedback) linearizability of a system, they offer little on how to find the linearizing change of coordinates (and feedback) except solving a system of partial differential equations (PDEs).

In [9] (see also [10]), Krstic considered two classes of nonlinear systems in *strict feedforward form*, (type I and type II), and showed that they are linearizable by providing explicit coordinates changes. A single-input control system

$$\Sigma \ : \ \dot{x} = f(x) + g(x)u, \quad (x, u) \in X \times U \subseteq \Re^n \times \Re,$$

defined in an open neighborhood $X \times U$ of $(0, 0) \in \Re^n \times \Re$ is in *strict feedforward form* if

$$(\text{SFF}) \begin{cases} f_j(x) = f_j(x_{j+1}, \ldots, x_n), & 1 \le j \le n-1, \\ f_n(x) = 0, \\ g_j(x) = g_j(x_{j+1}, \ldots, x_n), & 1 \le j \le n-1, \\ g_n \in \Re^* = \Re \setminus \{0\} \end{cases}$$

and the system is in *feedforward form* (FF) if

$$(\text{FF}) \begin{cases} f_j(x) = f_j(x_j, \ldots, x_n), & 1 \le j \le n, \\ g_j(x) = g_j(x_j, \ldots, x_n), & 1 \le j \le n. \end{cases}$$

In providing those classes, Krstic mentioned the difficulty of finding the linearizing diffeomorphism saying *there is no systematic way of finding those changes of coordinates*. Inspired by his work, we extended the two classes to all linearizable (resp. feedback linearizable) strict feedforward systems and proved that there is indeed a systematic way of finding the linearizing coordinates [22] (resp. feedback linearzing coordinates [23]). Let us mention that (strict) feedforward systems have been introduced as early as in the papers of Teel [24], [25] which have been followed since by a growing literature [13], [7], [18], [8], [19], [3], [12], [16], [1], [2], [14], [21], [9], [10], [11], [17], [20], [22]. What made strict feedforward systems appealing is that a stabilizing feedback controller can always be constructed when their linear approximation around the equilibrium is controllable [24], [25].

The objective of this paper is to tackle the feedforward case. If there is a component f_j or g_j that is nonlinear with respect to the variable x_j, then finding an explicit linearizing coordinates becomes almost impossible and would necessitate solving PDEs. For that reason we will restrict our study to a special class of feedforward systems, called *feedforward-nice*, for which the components f_j and g_j are affine with respect to the variable x_j for all $1 \le j \le n$. In Section II we will give our first result as an algorithm allowing to construct explicitly the linearizing coordinates in a finite number of steps ($\leq \frac{(n-1)(n+2)}{2}$). In Section III we will consider a subclass of type II and give necessary and sufficient conditions for their linearizability.

II. Main Results

For reasons mentioned previously, we consider here a subclass of feedforward systems (FF) for which the components $f_j(x_j,\ldots,x_n)$ and $g_j(x_j,\ldots,x_n)$ are affine with respect to the variable x_j for all $1 \leq j \leq n$. This subclass, call it *feedforward-nice* (FFnice), is described by

$$\text{(FFnice)} \begin{cases} \dot{x} \;=\; f(x) + g(x)u, \\ f_j(x) \;=\; x_j \tilde{f}_j(x_{j+1},\ldots,x_n) + \hat{f}_j(x_{j+1},\ldots,x_n), \\ g_j(x) \;=\; x_j \tilde{g}_j(x_{j+1},\ldots,x_n) + \hat{g}_j(x_{j+1},\ldots,x_n) \end{cases}$$

with $\hat{f}_j(0) = 0, \hat{g}_j(0) = 0, f_n = 0,$ and $\tilde{f}_j(0), g_n \in \Re^*$.

If, in addition, the control vector field g is rectified in the coordinates x, that is, $g(x) = (0,\ldots,0,1)^\top \in \Re^n$, we say that the (FFnice)-system is *control-normalized*.

The first result of this paper provides an algorithm for linearizing (FFnice)-systems and is stated below.

Theorem II.1 (i) *Consider a system* Σ *in* (FFnice). *There exists an explicit change of coordinates* $z = \Phi(x)$ *that transforms* Σ *into a control-normalized* (FFnice)

$$\bar{\Sigma} \;:\; \dot{z} = \bar{f}(z) + \bar{g}(z)u, \quad z \in \Re^n,$$

that is, such that

$$\begin{cases} \bar{f}_j(z) = z_j \tilde{\bar{f}}_j(z_{j+1},\ldots,z_n) + \hat{\bar{f}}_j(z_{j+1},\ldots,z_n), \\ 1 \leq j \leq n-1 \\ \bar{f}_n(z) = 0, \\ \bar{g}(z) = (0,\ldots,0,1)^\top. \end{cases}$$

(ii) *Any* (FFnice)-*system that is linearizable can be transformed into a linear controllable system via a diffeomorphism* $\tilde{x} = \Psi(x)$ *whose components are obtained by composing, differentiating, inverting, and integrating those of the* (FFnice)-*system using a maximum of* $\frac{(n-1)(n+2)}{2}$ *steps.*

Let us first point out that (FFnice)-systems form a well-defined class of feedforward systems that remain invariant under any change of coordinates $\tilde{x} = \Psi(x)$ of the form

$$\begin{cases} \Psi_j(x) = x_j \phi_j(x_{j+1},\ldots,x_n) + \psi_j(x_{j+1},\ldots,x_n), \\ \Psi_n(x) = x_n, \\ \psi_j(0) = 0, \qquad \phi_j(0) \neq 0, \end{cases} \tag{II.1}$$

for all $1 \leq j \leq n-1$.

A consequence of Theorem II.1 is that, for linearizable (FFnice)-systems, we can construct a stabilizing feedback controller $u = -\sum_{j=1}^{n} k_j \Psi_j(x)$, where $\Psi = (\Psi_1,\ldots,\Psi_n)^\top$ is the linearizing diffeomorphism given by (ii) and the polynomial $p(\lambda) = \lambda^n + \sum_{j=1}^{n} k_j \lambda^{j-1}$ is Hurwitz.

Proof of Theorem II.1

A. We will first prove (i) by constructing an explicit change of coordinates that normalizes the control vector field g.

Step 1. Consider the system $\Sigma \;:\; \dot{x} = f(x) + g(x)u$ in (FFnice) form.

We can assume without loss of generality that $g_n = 1$; otherwise replace x_n by x_n/g_n. We look for a change of coordinates $z = \Phi(x)$ of the form (II.1) whose components are given by

$$\begin{aligned} z_j &= \Phi_j(x) = x_j, \; j \neq n-1, \\ z_{n-1} &= \Phi_{n-1}(x) = x_{n-1}\phi_{n-1}(x_n) + \psi_{n-1}(x_n) \end{aligned}$$

in order to annihilate the component g_{n-1} of the transformed vector field. This is possible only if we have

$$\begin{cases} \phi'_{n-1}(x_n) + \tilde{g}_{n-1}(x_n)\phi_{n-1}(x_n) = 0, \\ \psi'_{n-1}(x_n) + \hat{g}_{n-1}(x_n)\phi(x_n) = 0. \end{cases}$$

A solution is given by

$$\begin{aligned} \phi_{n-1}(x_n) &= \exp\Big(-\int_0^{x_n} \tilde{g}_{n-1}(s)\,\mathrm{ds}\Big), \\ \psi_{n-1}(x_n) &= -\int_0^{x_n} \hat{g}_{n-1}(s)\phi_{n-1}(s)\,\mathrm{ds}. \end{aligned}$$

Hence we can transform the system so as to annihilate the component g_{n-1} of the system. Because the change of coordinates is of the form (II.1), the transformed system is also in a (FFnice) form.

For simplicity of the exposition, we will always reset the variable of the transformed system to x after having applied a change of coordinates $z = \Phi(x)$. Before we proceed to the general step, let us notice that the inverse $x = \Phi^{-1}(z)$ of the diffeomorphism $z = \Phi(x)$ is easily computable as:

$$\begin{aligned} x_j &= z_j, \; j \neq n-1, \\ x_{n-1} &= \Big[z_{n-1} - \psi_{n-1}(z_n)\Big] \exp \int_0^{z_n} \tilde{g}_{n-1}(s)\,\mathrm{ds}. \end{aligned}$$

General Step. We assume that the system has been transformed, via a sequence of coordinates changes of the form (II.1), into a (FFnice) form $\Sigma \;:\; \dot{x} = f(x) + g(x)u$ for which the components $g_{n-1}, g_{n-2}, \ldots, g_{k+1}$ are all zero. Consider the kth component of the system which decomposes as

$$g_k(x) = x_k \tilde{g}_k(x_{k+1},\ldots,x_n) + \hat{g}_k(x_{k+1},\ldots,x_n).$$

We then look for a change of coordinates $z = \Phi(x)$ whose components are given by

$$\begin{aligned} z_j &= \Phi_j(x) = x_j, \; j \neq k, \\ z_k &= \Phi_k(x) = x_k \phi_k(x_{k+1},\ldots,x_n) + \psi_k(x_{k+1},\ldots,x_n) \end{aligned}$$

in order to annihilate the component g_k of the transformed vector field. Because

$$\begin{aligned} \dot{z}_k &= \phi_k(\cdot)\Big(f_k(x_k,\ldots,x_n) + g_k(x_k,\ldots,x_n)u\Big) \\ &+ \sum_{j=k+1}^{n-1} \Big(x_k \frac{\partial \phi_k}{\partial x_j} + \frac{\partial \psi_k}{\partial x_j}\Big) f_j(\cdot) + \Big(x_k \frac{\partial \phi_k}{\partial x_n} + \frac{\partial \psi_k}{\partial x_n}\Big)u, \end{aligned}$$

we obtain the following PDEs:

$$\begin{cases} \dfrac{\partial \phi_k}{\partial x_n} + \tilde{g}_k(x_{k+1},\ldots,x_n)\phi_k(x_{k+1},\ldots,x_n) = 0, \\ \dfrac{\partial \psi_k}{\partial x_n} + \hat{g}_k(x_{k+1},\ldots,x_n)\phi_k(x_{k+1},\ldots,x_n) = 0. \end{cases}$$

A simple solution can be chosen to be

$$\phi_k(\cdot) \,=\, \exp\Big(-\int_0^{x_n} \tilde{g}_k(x_{k+1},...,x_{n-1},s)\,\mathrm{ds}\Big),$$

$$\psi_k(\cdot) \,=\, -\!\int_0^{x_n} \hat{g}_k(x_{k+1},...,x_{n-1},s)\phi_k(x_{k+1},...,x_{n-1},s)\,\mathrm{ds}.$$

We can thus annihilate the kth component of the control vector field by a change of coordinates (II.1) and still transform the system into a (FFnice). The inverse $x = \Phi^{-1}(z)$ of the diffeomorphism $z = \Phi(x)$ is here given by:

$$x_j = z_j,\ \ j \neq k,$$

$$x_k = \Big[z_k - \psi_k(z_{k+1},...,z_n)\Big]\exp\int_0^{z_n} \tilde{g}_k(z_{k+1},...,z_{n-1},s)\,\mathrm{ds}.$$

This completes the proof of Theorem II.1 (i) and provides an algorithm allowing to transform a (FFnice)-system into a control-normalized (FFnice) form. □

B. Consider a (FFnice)-system. By Theorem II.1 (i), bring it to a control-normalized (FFnice)-form (keep same notation)

$$(\text{FFnice})\begin{cases} \quad\dot{z} \,=\, \bar{f}(z) + \bar{g}(z)u, \\ \bar{f}_j(z) \,=\, z_j\bar{\tilde{f}}_j(z_{j+1},\ldots,z_n) + \bar{\hat{f}}_j(z_{j+1},\ldots,z_n), \\ \bar{f}_n(z) \,=\, 0, \\ \bar{g}(z) \,=\, (0,\ldots,0,1)^\top \in \Re^n. \end{cases}$$

Before we proceed to the linearization algorithm, recall the following from [6] and [4] (see also [5], [15]).

Theorem II.2 *A control-affine system $\Sigma : \dot{z} = f(z) + ug(z)$ is locally equivalent, via a change of coordinates $\tilde{x} = \Phi(z)$, to a linear controllable system $\dot{\tilde{x}} = A\tilde{x} + bu$ if and only if*
(S1) $\dim \operatorname{span}\{ad_f^q g(z),\ \ 0 \le q \le n-1\} = n$;
(S2) $[ad_f^q g, ad_f^r g] = 0,\ \ 0 \le q < r \le n.$

Above, $ad_f^k g$ stands for the kth iterative Lie bracket:

$$ad_f^0 g = g, ad_f g = [f,g], \ldots, ad_f^k g = [f, ad_f^{k-1} g]$$

and (A,b) for the Brunovský canonical pair.

Step 1. Condition (S2) of Theorem II.2 for $q = 0$, $r = 1$ (denoted by $(\pounds_n)$) implies

$$(\pounds_n) \Longrightarrow \frac{\partial^2 \bar{f}_j}{\partial z_n^2} \equiv 0, \quad \text{for all } 1 \le j \le n-1.$$

The condition $(\pounds_n)$ stands for a very strong necessary condition that is equivalent of saying that all components of the system should be affine in the variable z_n. Thus, if it fails to be satisfied, the algorithm stops: a change of coordinates linearizing the system can't be found. So let us assume that the condition is satisfied. Since the system is (FFnice) we have $\bar{f}_{n-1}(z) = z_{n-1}\bar{\tilde{f}}_{n-1}(z_n) + \bar{\hat{f}}_{n-1}(z_n)$ and using $(\pounds_n)$ we have $\bar{\tilde{f}}_{n-1}(z_n) = \alpha z_n$ and $\bar{\hat{f}}_n(z_n) = \beta z_n$ which implies $\bar{f}_{n-1}(z) = (\alpha z_{n-1} + \beta)z_n$ with $\beta \in \Re^*$. Replacing z_{n-1} by $\int_0^{z_{n-1}} \frac{\mathrm{ds}}{\alpha s + \beta}$ we obtain $\bar{f}_{n-1}(z) = z_n$.

Now, let us assume that we have found a change of coordinates that brings the (FFnice)-system into a new control-normalized (FFnice)-system (f, g) for which $\bar{f}_{n-1}(z_n) = z_n$ and $\bar{f}_j(z) = \bar{f}_j(z_j,\ldots,z_{n-1})$ for all $j = n-2,\ldots,k+1$ for some $1 \le k \le n-2$.

Consider the kth component and decompose it in the form

$$\begin{aligned} \bar{f}_k(z) \,&=\, z_k\bar{\tilde{f}}_k(z_{k+1},\ldots,z_n) + \bar{\hat{f}}_k(z_{k+1},\ldots,z_n) \\ &=\, F_k(z_k,\ldots,z_{n-1}) \\ &\quad+\, z_n\Big(z_k\theta_k(z_{k+1},\ldots,z_{n-1}) + \mu_k(z_{k+1},\ldots,z_{n-1})\Big), \end{aligned}$$

where $F_k(\cdot) = z_k\tilde{F}_k(z_{k+1},\ldots,z_{n-1}) + \hat{F}_k(z_{k+1},\ldots,z_{n-1})$ is affine in z_k and independent of the variable z_n.

We apply a change of coordinates of the form

$$\tilde{x}_j = \Psi_j(z) = z_j,\ \ j \neq k,$$

$$\tilde{x}_k = \Psi_k(z) = z_k\phi_k(z_{k+1},\ldots,z_{n-1}) + \psi_k(z_{k+1},\ldots,z_{n-1})$$

that transforms the kth component. We have

$$\begin{aligned} \dot{\tilde{x}}_k =\, &\phi_k\Big[F_k + z_n\Big(z_k\theta_k(z_{k+1},...,z_{n-1}) + \mu_k(z_{k+1},...,z_{n-1})\Big)\Big] \\ &+ \sum_{j=k+1}^{n-2}\Big(z_k\frac{\partial \phi_k}{\partial z_j} + \frac{\partial \psi_k}{\partial z_j}\Big)\bar{f}_j(\cdot) + \Big(z_k\frac{\partial \phi_k}{\partial z_{n-1}} + \frac{\partial \psi_k}{\partial z_{n-1}}\Big)z_n. \end{aligned}$$

Collecting the affine terms in z_k that are coefficients of z_n, we obtain the following PDEs:

$$\begin{cases} \dfrac{\partial \phi_k}{\partial z_{n-1}} + \theta_k(z_{k+1},\ldots,z_{n-1})\phi_k(z_{k+1},\ldots,z_{n-1}) = 0, \\ \dfrac{\partial \psi_k}{\partial z_{n-1}} + \mu_k(z_{k+1},\ldots,z_{n-1})\phi_k(z_{k+1},\ldots,z_{n-1}) = 0. \end{cases}$$

A simple solution can be chosen to be

$$\phi_k(\cdot) = \exp\Big(-\int_0^{z_{n-1}} \theta_k(z_{k+1},...,z_{n-2},s)\,\mathrm{ds}\Big),$$

$$\psi_k(\cdot) = -\!\int_0^{z_{n-1}} \mu_k(z_{k+1},...,z_{n-2},s)\phi_k(z_{k+1},...,z_{n-2},s)\,\mathrm{ds}.$$

Thus, the proposed change of coordinates allows to cancel the terms containing the variable z_n in the kth component of the system. Moreover, it preserves the (FFnice)-form and the fact that the system is control-normalized because

$$\frac{\partial \Psi_j}{\partial z_n} = 0, 1 \le j \le n-1 \quad \text{ and } \ \Psi_n(z) = z_n.$$

The inverse $z = \Psi^{-1}(\tilde{x})$ is easily obtained as following:

$$z_j = \tilde{x}_j,\ \ j \neq k,$$

$$z_k = \Big[\tilde{x}_k - \psi_k(\tilde{x}_{k+1},..,\tilde{x}_{n-1})\Big]\exp\int_0^{\tilde{x}_{n-1}} \theta_k(\tilde{x}_{k+1},..,\tilde{x}_{n-2},s)\,\mathrm{ds}.$$

It follows that a change of coordinates can be found that transforms the system into a new control-normalized (FFnice)-form (we keep same notation)

$$\bar{\Sigma}\ :\ \begin{cases} \dot{z} = \bar{f}(z) + \bar{g}(z)u, \\ \bar{f}_j(z) = z_j \tilde{\bar{f}}_j(z_{j+1}, \ldots, z_{n-1}) + \hat{\bar{f}}_j(z_{j+1}, \ldots, z_{n-1}), \\ 1 \le j \le n-2, \\ \bar{f}_{n-1}(z) = z_n, \quad \bar{f}_n(z) = 0, \\ \bar{g}(z) = (0, \ldots, 0, 1)^\top \in \Re^n. \end{cases}$$

General Step. Consider the projection $\pi : \Re^n \longrightarrow \Re^{n-1}$ defined as $\pi(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1})^\top$. Let $\pi(\Sigma)$ be the projection of Σ via π : obviously $\pi(\Sigma)$ is a (FFnice)-system in $\Re^{n-1}$ control-normalized with control input $v = z_n$. Thus the necessary condition for linearizability becomes

$$(\mathcal{L}_{n-1}) \Longrightarrow \frac{\partial^2 \bar{f}_j}{\partial z_{n-1}^2} \equiv 0, \quad \text{for all } 1 \le j \le n-2.$$

We can repeat the same procedure as in **Step 1** to construct the change of coordinates that annihilate the terms containing the variable z_{n-1}. The same procedure will be repeated $(n-2)$ times as long as the corresponding $(\mathcal{L}_k)$ conditions are satisfied for the corresponding reduced system. The diffeomorphism Φ transforming a system into a control-normalized (FFnice)-form as well as the diffeomorphism taking the control-normalized (FFnice) into the Brunovský canonical form are obtained by integrations, derivations, compositions of the components of the (FFnice)-system, and control-normalized (FFnice), respectively. It can be easily verified that there is a maximum of $\frac{(n-1)(n+2)}{2}$ steps. □

Example II.3 Consider the (FFnice)-system

$$\begin{cases} \dot{x}_1 = x_2 + a x_1 x_2 + (1 + a x_1)\left(\frac{1}{2}x_2 - \frac{1}{12}x_3 x_4\right) u, \\ \dot{x}_2 = x_3 + \frac{1}{2}x_3 u, \\ \dot{x}_3 = x_4 + x_4 u, \\ \dot{x}_4 = u \end{cases}$$

with $a \in \Re$ a real parameter. We can apply the change of coordinates

$$\bar{x}_1 = \int_0^{x_1} \frac{1}{1+as}\,\mathrm{ds}, \qquad \bar{x}_3 = x_3 - \tfrac{1}{2}x_4^2,$$
$$\bar{x}_2 = x_2, \qquad \bar{x}_4 = x_4$$

to transform the system into

$$\begin{cases} \dot{\bar{x}}_1 = \bar{x}_2 + \frac{1}{24}\left(12\bar{x}_2 - 2\bar{x}_3\bar{x}_4 - \bar{x}_4^3\right) u, & \dot{\bar{x}}_3 = \bar{x}_4, \\ \dot{\bar{x}}_2 = \bar{x}_3 + \frac{1}{2}\bar{x}_4^2 + \frac{1}{2}\left(\bar{x}_3 + \frac{1}{2}\bar{x}_4^2\right) u, & \dot{\bar{x}}_4 = u. \end{cases}$$

Then, the change of coordinates

$$\hat{x}_1 = \bar{x}_1, \qquad \hat{x}_2 = \bar{x}_2 - \tfrac{1}{2}\left(\bar{x}_3\bar{x}_4 + \tfrac{1}{6}\bar{x}_4^3,\right)$$
$$\hat{x}_3 = \bar{x}_3, \qquad \hat{x}_4 = \bar{x}_4$$

transforms the system into

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \frac{1}{2}\hat{x}_3\hat{x}_4 + \frac{1}{12}\hat{x}_4^3 + \left(\frac{1}{2}\hat{x}_2 + \frac{1}{6}\hat{x}_3\hat{x}_4\right) u, & \dot{\hat{x}}_3 = \hat{x}_4, \\ \dot{\hat{x}}_2 = \hat{x}_3, & \dot{\hat{x}}_4 = u. \end{cases}$$

Finally, we apply the change of coordinates

$$\tilde{x}_1 = \hat{x}_1 - \tfrac{1}{2}\hat{x}_2\hat{x}_4 - \tfrac{1}{12}\hat{x}_3\hat{x}_4^2, \quad \tilde{x}_3 = \hat{x}_3,$$
$$\tilde{x}_2 = \hat{x}_2, \qquad \tilde{x}_4 = \hat{x}_4$$

to transform the system into a linear one

$$\dot{\tilde{x}}_1 = \tilde{x}_2, \quad \dot{\tilde{x}}_2 = \tilde{x}_3, \quad \dot{\tilde{x}}_3 = \tilde{x}_4, \quad \dot{\tilde{x}}_4 = u.$$

The linearizing diffeomorphism is then computed as

$$\tilde{x}_1 = \int_0^{x_1} \frac{\mathrm{ds}}{1+as} - \frac{1}{2}x_2 x_4 + \frac{1}{6}x_3 x_4^2 - \frac{1}{24}x_4^4, \; \tilde{x}_3 = x_3 - \tfrac{1}{2}x_4^2$$
$$\tilde{x}_2 = x_2 - \tfrac{1}{2}x_3 x_4 + \tfrac{1}{6}x_4^3, \qquad \tilde{x}_4 = x_4. \quad \triangle$$

III. Linearizable Systems of Type II

Consider a subclass of (SFF)-forms (type II) given in [9]:

$$\begin{cases} \dot{x}_1 = x_2 + g_1(x_2, \ldots, x_n)u, \\ \dot{x}_2 = x_3 + g_2(x_3, \ldots, x_n)u, \\ \quad \cdots \\ \dot{x}_{n-1} = x_n + g_{n-1}(x_n)u, \\ \dot{x}_n = u. \end{cases} \tag{III.1}$$

Krstic [9] defined recursively $\mu_n(x_n), \ldots, \mu_1(x_n)$ as:

$$\begin{cases} \mu_n = \dfrac{1}{x_n}\displaystyle\int_0^{x_n} g_{n-1}(s)\,\mathrm{ds}, \\ \mu_i = \dfrac{1}{x_n}\displaystyle\int_0^{x_n}\Big(g_{n-1}(s) - \sum_{j=i+1}^{n} \mu_j(s) g_{i+n-j}(0, ..., 0, s)\Big)\,\mathrm{ds} \end{cases} \tag{III.2}$$

for $i = n-1, \ldots, 1$. Next, he defined the functions $\gamma_k(x_n)$ for $k = 1, \ldots, n$ in terms of $\mu_1, \ldots, \mu_n$ recursively as:

$$\begin{cases} \gamma_1 = \mu_n'(x_n), \\ \gamma_k = \displaystyle\sum_{l=1}^{k-1} \gamma_l(x_n)\mu_{l+n+1-k}(x_n) + \mu_{n+1-k}'(x_n). \end{cases} \tag{III.3}$$

He then showed that (III.1) is linearizable if

$$g_i(x_{i+1}, .., x_n) = \sum_{j=i+1}^{n-1} \gamma_{j-i}(x_n) x_j + g_i(0, .., 0, x_n). \tag{III.4}$$

Notice that (III.4) is in the form

$$g_i(x_{i+1}, \ldots, x_n) = \sum_{j=i+1}^{n} \beta_{i,j}(x_n) x_j, \; 1 \le i \le n-1 \tag{III.5}$$

with $\beta_{i,j}(x_n) = \gamma_{j-i}(x_n)$ and $\beta_{i,j}(x_n) = g_i(0, .., 0, x_n)$.

Definition III.1 We say that (III.1) is a *quasi-linear* (SFF)-system if (III.5) holds for some functions $\beta_{i,j}(x_n)$.

It is clear that (III.1) given by (III.4) is a quasi-linear (SFF)-system that is linearizable provided the coefficients γ_k satisfy (III.3) with μ_k given by (III.2). Now, does any quasi-linear (SFF)-form that is linearizable satisfy (III.2)-(III.3)-(III.4)?

Let us first notice that quasi-linear (SFF)-systems can be represented in the more compact form

$$\dot{x} = Ax + g(x)u = Ax + bu + B(x_n)xu, \tag{III.6}$$

where (A, b) is the Brunovský canonical pair, and

$$B(x_n) = \begin{pmatrix} 0 & \beta_{1,2}(x_n) & \beta_{1,3}(x_n) & \cdots & \beta_{1,n}(x_n) \\ 0 & 0 & \beta_{2,3}(x_n) & \cdots & \beta_{2,n}(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{n-1,n}(x_n) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is an upper triangular matrix (u.t.m) whose entries are smooth (resp. analytic) functions of x_n. We have the following characterization about the linearizability of (III.6).

Theorem III.2 *The following conditions are equivalent.*

(i) *System (III.6) is linearizable by change of coordinates.*

(ii) *There exists an invertible u.t.m $M = M(x_n)$ such that*

$$\begin{cases} AM = MA, \\ Mb + (MB + M')x = b \quad \text{for all } x \in \Re^n. \end{cases} \tag{III.7}$$

(iii) *System (III.6) is in the form (III.4) with γ_k given by (III.3) and μ_k given by (III.2).*

A consequence of Theorem III.2 is the following algorithm: Linearization Procedure. Consider the (SFF)-system (III.6).

Step 1: Take the matrix $\hat{B}$ obtained from B by deleting the last row and last column. If $\hat{B}$ is not an u.t.m in Toeplitz form, then STOP the system is not linearizable.

Step 2: If $\hat{B}$ is an u.t.m in Toeplitz form, then solve the first-order ordinary differential equation $\hat{M}\hat{B} + \hat{M}' = 0$. Then solve the 1st equation of (III.9) for $i = 2$ to find $\alpha_{1,n}$. Because M has to be Toeplitz the obtention of $\hat{M}$ and $\alpha_{1,n}$ yields that of M by extending the last column and last row.

Step 3: If B_n and M_n denote the last columns of B and M, respectively, verify if $M_n + x_n(MB_n + M'_n) = b$. If yes the system is linearizable by $z = M(x_n)x$; otherwise it is not.

Proof of Theorem III.2 (ii) $\Rightarrow$ (i) Let $M = M(x_n)$ be an invertible u.t.m satisfying (III.7). Then $z = M(x_n)x$ implies

$$\begin{aligned} \dot{z} &= M(x_n)\dot{x} + M'(x_n)x\dot{x}_n \\ &= M(x_n)Ax + \Big(Mb + (MB + M')x\Big)u \\ &= AM(x_n)x + bu = Az + bu. \end{aligned}$$

• (i) $\Rightarrow$ (ii) We already proved (see [22]) that a system (III.1) in $\Re^3$, with $g_1(0) = g_2(0) = 0$, is linearizable if and only if

$$\gamma_1(x_2, x_3) = x_2\gamma_1(x_3) + \theta_1(x_3)$$

where

$$\gamma_1(x_3) = \frac{d}{dx_3}\left(\frac{1}{x_3}\int_0^{x_3} g_2(s)\mathrm{d}s\right).$$

If the linearizability condition holds, then the coordinates

$$z = M(x_3)x \triangleq \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1(x_3) & \alpha_2(x_3) \\ 0 & 1 & \alpha_1(x_3) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with $\alpha_1(x_3) = -\frac{1}{x_3}\int_0^{x_3} g_2(s)\mathrm{d}s$ and

$$\alpha_2(x_3) = -\frac{1}{x_3}\Big(\int_0^{x_3}\gamma_1(s)\mathrm{d}s - \int_0^{x_3}\alpha_1(s)g_2(s)\mathrm{d}s\Big)$$

linearize the system. Thus, the matrix M satisfies (III.7).

Assume the implication true for $(n-1)$-dimensional systems. Let (III.1) be an n-dimensional system linearizable by

$$z = \Phi(x) = (\Phi_1(x_1, \ldots, x_n), \Phi_2(x_2, \ldots, x_n), \ldots, \Phi_n(x_n))^\top.$$

Define the projection

$$\rho \ : \ \Re^n \longrightarrow \Re^{n-1} \quad \text{by } \rho(x_1, \ldots, x_n) = (x_2, \ldots, x_n)^\top.$$

The projection $\rho(\Sigma)$ is a $(n-1)$-dimensional system linearizable by $\rho(\Phi(x)) = (\Phi_2(x_2, \ldots, x_n), \ldots, \Phi_n(x_n))^\top$. By the induction argument we have $\Phi_i(x) = x_i + \sum_{j=i+1}^{n}\alpha_{i,j}(x_n)x_j$ for $2 \le i \le n$ with $\alpha_{i,j} = \alpha_{i-1,j-1}$. Because $z = \Phi(x)$ linearizes the system, we obtain

$$\dot{z} = \frac{\partial\Phi}{\partial x}Ax + \frac{\partial\Phi}{\partial x}bu + \frac{\partial\Phi}{\partial x}B(x_n)xu = A\Phi(x) + bu$$

which implies, in particular, that $A\Phi(x) = L_{Ax}\Phi(x)$. In one hand side $\Phi_2(x) = L_{Ax}\Phi_1(x)$, and in the other hand side

$$\Phi_2(x) = x_2 + \sum_{j=3}^{n}\alpha_{2,j}(x_n)x_j = L_{Ax}\Big(x_1 + \sum_{j=2}^{n-1}\alpha_{1,j}(x_n)x_j\Big).$$

We then deduce that $\Phi_1(x) = x_1 + \sum_{j=2}^{n-1}\alpha_{1,j}(x_n)x_j + \theta(x_n)$. Thus the change of coordinates $z = \Phi(x)$ is in the form $z = M(x_n)x$, where $M(x_n)$ satisfies (III.7) necessarily.

• (iii) $\Rightarrow$ (ii) Consider (III.1) and define μ_k as in (III.2) and γ_k as in (III.3). Let $\beta_{i,n}(x_n) = g_i(0, \ldots, 0, x_n)/x_n$ and $\beta_{i,j}(x_n) = \gamma_{j-i}(x_n)$ for $1 \le i < j \le n-1$. We denote by $\alpha_{i,j}(x_n) = -\mu_{i+1+n-j}(x_n)$ for $1 \le i < j \le n$ and take $\alpha_{i,i} = 1$ for $1 \le i \le n$. Notice that

$$\begin{aligned} \alpha_{i,j}(x_n) &= \alpha_{i+1,j+1}(x_n) = \cdots = \alpha_{i+n-j,n}(x_n), \\ \beta_{i,j}(x_n) &= \beta_{i+1,j+1}(x_n) = \cdots = \beta_{i+n-j-1,n-1}(x_n). \end{aligned} \tag{III.8}$$

Define the invertible u.t.m $M(x_n) = \Big(\alpha_{i,j}(x_n)\Big)_{1 \le i \le j \le n}$ and the nilpotent u.t.m B by $B(x_n) = \Big(\beta_{i,j}(x_n)\Big)_{1 \le i < j \le n}$. We want to show that (M, B) satisfies (III.7).

Because M is Toeplitz, $AM = MA$ is trivial. Remark that $Mb + (MB + M')x = b$ for all $x \in \Re^n$ is equivalent to:

$$\alpha_{i-1,n} + \sum_{k=i-1}^{n}\sum_{j=k+1}^{n}\alpha_{i-1,k}\beta_{k,j}x_j + \sum_{j=i-1}^{n}\alpha'_{i-1,j}x_j = 0$$

for any $2 \le i \le n$ and for all $x \in \Re^n$. This latter condition is equivalent to the following two conditions:

$$\begin{cases} \alpha_{i-1,n} + \sum_{s=i-1}^{n-1}\alpha_{i-1,s}\beta_{s,n}x_n + \alpha'_{i-1,n}x_n = 0, \\ \beta_{i-1,j} + \sum_{s=i}^{j-1}\alpha_{i-1,s}\beta_{s,j} + \alpha'_{i-1,j} = 0 \end{cases} \tag{III.9}$$

for all $2 \le i \le j \le n-1$. The 1st equation of (III.9) rewrites

$$\mu_i(x_n) - \beta_{i-1,n}(x_n)x_n + \sum_{s=i}^{n-1}\mu_{i+n-s}(x_n)g_s(x_n) + \mu'_i(x_n)x_n = 0$$

which is equivalent to (we recognize (III.2) after integration)

$$(x_n\mu_i)' = g_{i-1}(0,\ldots,0,x_n) - \sum_{j=i+1}^{n} \mu_j g_{i+n-j}(0,\ldots,0,x_n).$$

The 2^{nd} equation of (III.9) is equivalent to

$$\gamma_{j-i+1}(x_n) - \sum_{s=i}^{j-1} \mu_{i+n-s}(x_n)\gamma_{j-s}(x_n) - \mu'_{i+n-j}(x_n) = 0.$$

Taking $k = j-i+1$ and $l = j-s$, the above is equivalent to (compare with (III.3))

$$\gamma_k(x_n) = \sum_{l=1}^{k-1} \mu_{l+n+1-k}(x_n)\gamma_l(x_n) + \mu'_{n+1-k}(x_n).$$

• (iii) $\Rightarrow$ (ii) Consider the system (III.6) and assume that $M(x_n) = \Big(\alpha_{i,j}(x_n)\Big)_{1\le i\le j\le n}$ exists and satisfies (III.7). Define μ_k and γ_k such that for $1 \le i < j \le n-1$ we have $\beta_{i,j}(x_n) = \gamma_{j-i}(x_n)$ and for $1 \le i < j \le n$ we have $\alpha_{i,j}(x_n) = -\mu_{i+1+n-j}(x_n)$. This is possible provided that (III.8) is satisfied. That's the only point we need to clarify because $Mb + (MB + M')x = b$ for all $x \in \Re^n$ is already equivalent to (III.2)-(III.3)-(III.4). The 1^{st} condition of (III.8) is satisfied since $AM = MA$. Let $\hat{M}$ and $\hat{B}$ denote respectively the matrices M and B with the last row and last column deleted. $Mb + (MB + M')x = b$ for all $x \in \Re^n$ implies that $\hat{M}\hat{B} + \hat{M}' = 0$. The matrix $\hat{M}$ being in Toeplitz form (hence is $\hat{M}'$) and invertible, it thus follows that $\hat{B} = \hat{M}^{-1}\hat{M}'$ is also in Toeplitz form. □

Example III.3 Reconsider Example II.3 and assume $a = 0$:

$$\begin{cases} \dot{x}_1 = x_2 + \left(\frac{1}{2}x_2 - \frac{1}{12}x_3x_4\right)u, & \dot{x}_3 = x_4 + x_4u, \\ \dot{x}_2 = x_3 + \frac{1}{2}x_3u, & \dot{x}_4 = u. \end{cases}$$

This is a quasi-linear (SFF)-system in $\Re^n$ ($n = 4$) with

$$B = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{12}x_4 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \hat{B} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{12}x_4 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

We search for an invertible u.t.m

$$\hat{M} = \begin{pmatrix} 1 & \alpha_{1,2}(x_4) & \alpha_{1,3}(x_4) \\ 0 & 1 & \alpha_{1,2}(x_4) \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\hat{M}\hat{B} + \hat{M}' = 0$. This yields the system of ODEs:

$$\alpha'_{1,3}(x_4) - \frac{1}{12}x_4 + \frac{1}{2}\alpha_{1,2}(x_4) = 0, \quad \alpha'_{1,2}(x_4) + \frac{1}{2} = 0.$$

A solution is $\alpha_{1,2}(x_4) = -\frac{1}{2}x_4$ and $\alpha_{1,3}(x_4) = \frac{1}{6}x_4^2$. The 1^{st} equation of (III.9) for $i = 2, \beta_{1,4} = \beta_{2,4} = 0, \beta_{3,4} = 1$ gives

$$\alpha_{1,4}(x_4) + \alpha_{1,3}(x_4)x_4 + \alpha'_{1,4}(x_4)x_4 = 0.$$

Its solution is $\alpha_{1,4}(x_4) = -\frac{1}{24}x_4^3$. Thus $z = M(x_4)x$ is as

$$z_1 = x_1 - \tfrac{1}{2}x_2x_4 + \tfrac{1}{6}x_3x_4^2 - \tfrac{1}{24}x_4^4, \quad z_3 = x_3 - \tfrac{1}{2}x_4^2,$$
$$z_2 = x_2 - \tfrac{1}{2}x_3x_4 + \tfrac{1}{6}x_4^3, \quad z_4 = x_4.$$

We rediscover the coordinates in Example II.3 for $a = 0$. △

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