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**THE STABLE MANIFOLD THEOREM
FOR SDE'S**

UC Irvine, March 3, 1998.

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Outline

- Formulate a *Local Stable Manifold Theorem* for stochastic differential equations (SDE's) (Stratonovich or Itô SDE's-driven by Brownian motion or spatial Kunita-type semimartingales with stationary ergodic increments.)
- Start with the existence of a stochastic flow for SDE.
- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward /backward anticipating SDE for all time (Stratonovich case).
- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.
- The stable and unstable manifolds are dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.
- Proof based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and anticipating stochastic calculus.

Formulation of The Theorem

Stratonovich SDE

$$dx(t) = h(x(t)) dt + \sum_{i=1}^m g_i(x(t)) \circ dW_i(t), \quad (I)$$

on \mathbf{R}^d driven by m -dimensional Brownian motion $W := (W_1, \dots, W_m)$.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) :=$ canonical filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \rightarrow \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space \mathbf{R}^m , with compact open topology;

$\mathcal{F} :=$ Borel σ -field of Ω ;

$\mathcal{F}_t :=$ sub- σ -field of \mathcal{F} generated by the evaluations $\omega \rightarrow \omega(u)$, $u \leq t$, $t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω .

$h, g_i : \mathbf{R}^d \rightarrow \mathbf{R}^d$, $1 \leq i \leq m$, vector fields on \mathbf{R}^d . For some $k \geq 1, \delta \in (0, 1)$, h is $C_b^{k, \delta}$, viz. h has all derivatives $D^j h$, $1 \leq j \leq k$, continuous and globally bounded, $D^k h$ Hölder continuous with exponent δ .

g_i , $1 \leq i \leq m$, globally bounded and in $C_b^{k+1, \delta}$.

$\theta : \mathbf{R} \times \Omega \rightarrow \Omega$ is the (ergodic) Brownian shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

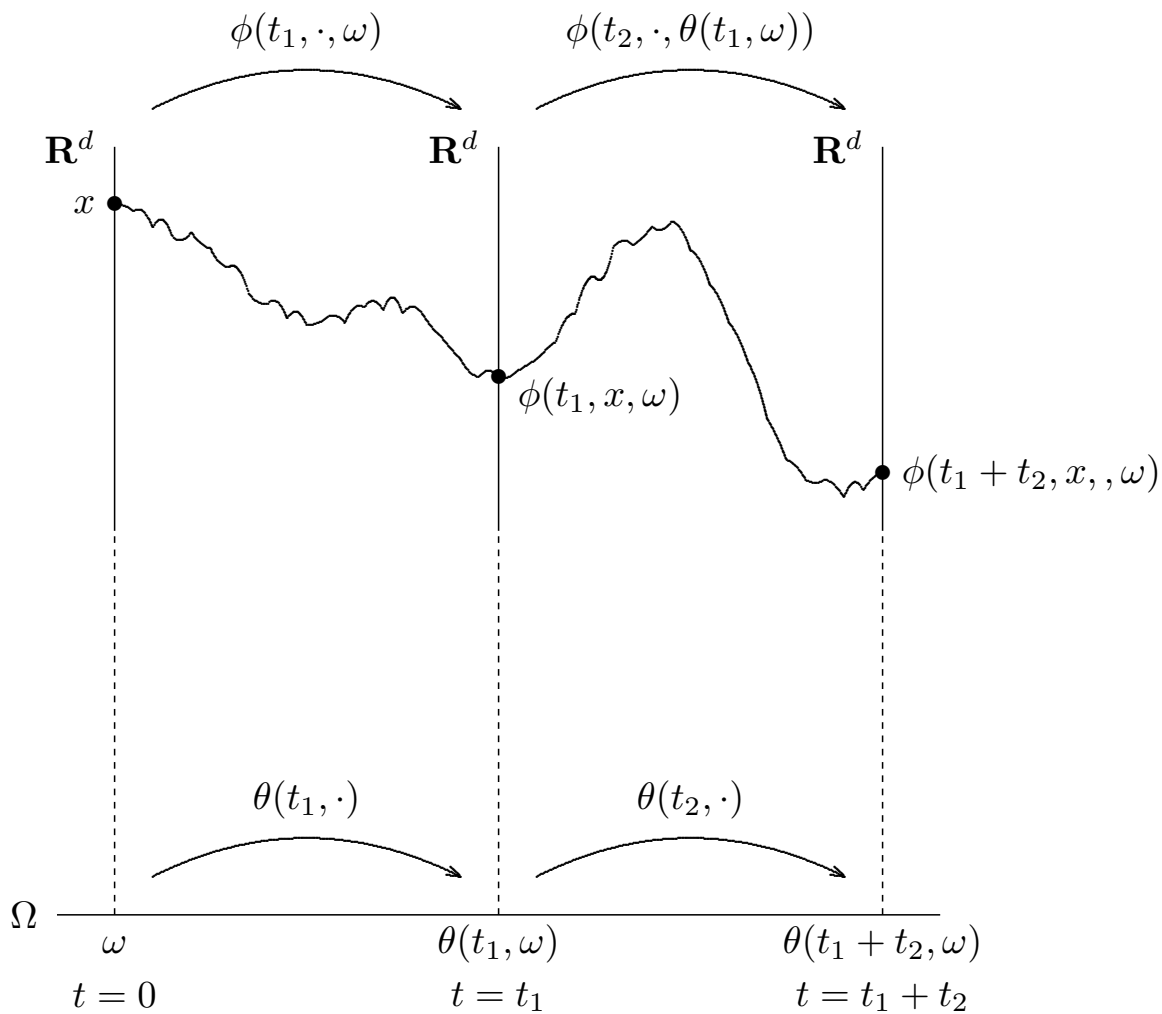
Let $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$ be the stochastic flow generated by (I) ($\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t, \omega))]^{-1}, t < 0$). Then ϕ is a perfect cocycle:

$$\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),$$

for all $s, t \in \mathbf{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of \mathbf{R}^d . (ϕ, θ) is a “random vector-bundle morphism” over the “base” probability space Ω .

The Cocycle



Definition

The SDE (I) has a *stationary trajectory* if there exists an \mathcal{F} -measurable random variable $Y : \Omega \rightarrow \mathbf{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \quad (1)$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y(\theta(t))$.

If (1) holds on a sure event Ω_t that may depend on t , then there are “perfect” versions of the stationary random variable Y and of the flow ϕ such that (1) and the cocycle property hold *for all* $\omega \in \Omega$ ([Sc]).

Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of ϕ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$ -matrix-valued cocycle. Using Kolmogorov’s theorem, the random variables

$$\sup_{x \in \mathbf{R}^d} \frac{|D_2\phi(t, x)|}{(1 + |x|^\gamma)}, \quad \gamma > 0,$$

have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ |D_2\phi(1, Y)| < \infty$. Apply Oseledec's Theorem to get a *non-random* finite Lyapunov spectrum:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |D_2\phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d).$$

Spectrum takes finitely many values $\{\lambda_i\}_{i=1}^p$ with non-random multiplicities q_i , $1 \leq i \leq p$, and $\sum_{i=1}^p q_i = d$ ([Ru.1], Theorem I.6).

Definition

Stationary trajectory $\phi(t, Y)$ of (I) is *hyperbolic* if $E \log^+ |Y(\cdot)| < \infty$, and if the linearized cocycle $(D_2\phi(n, Y(\omega), \omega), \theta(n, \omega))$ has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e. $\lambda_i \neq 0$ for all $1 \leq i \leq p$.

Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Let $\rho \in \mathbf{R}^+$, $x \in \mathbf{R}^d$.

$B(x, \rho) :=$ open ball in \mathbf{R}^d , center x and radius ρ ;

$\bar{B}(x, \rho) :=$ corresponding closed ball;

$\mathcal{C}(\mathbf{R}^d) :=$ the class of all non-empty compact subsets of \mathbf{R}^d with Hausdorff metric d^* :

$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$ where $A_1, A_2 \in \mathcal{C}(\mathbf{R}^d)$;

$d(x, A_i) := \inf\{|x - y| : y \in A_i\}$, $x \in \mathbf{R}^d$, $i = 1, 2$;

$\mathcal{B}(\mathcal{C}(\mathbf{R}^d)) :=$ Borel σ -algebra on $\mathcal{C}(\mathbf{R}^d)$ with respect to the metric d^* .

Theorem 1 (The Stable Manifold Theorem) (M.+ Scheutzow, 1997)

Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

- (i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,
- (ii) \mathcal{F} -measurable random variables $\rho_i, \beta_i : \Omega^* \rightarrow [0, \infty)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

- (a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \quad (2)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$ is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega)$ and is non-random.

$$(b) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ such that

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \quad (3)$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \quad (4)$$

(d) $\tilde{\mathcal{U}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0-1} + \epsilon_2)n} \quad (5)$$

for all integers $n \geq 0$. Also

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \quad (6)$$

for all $x \in \tilde{\mathcal{U}}(\omega)$. Furthermore, the unstable subspace $\mathcal{U}(\omega)$ of $D_2\phi$ is the tangent space to $\tilde{\mathcal{U}}(\omega)$ at $Y(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. In particular, $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$ and is non-random.

$$(e) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{U}}(\omega)}} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\phi(-t, \cdot, \omega)(\tilde{\mathcal{U}}(\omega)) \subseteq \tilde{\mathcal{U}}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \quad (7)$$

Also

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0. \quad (8)$$

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.

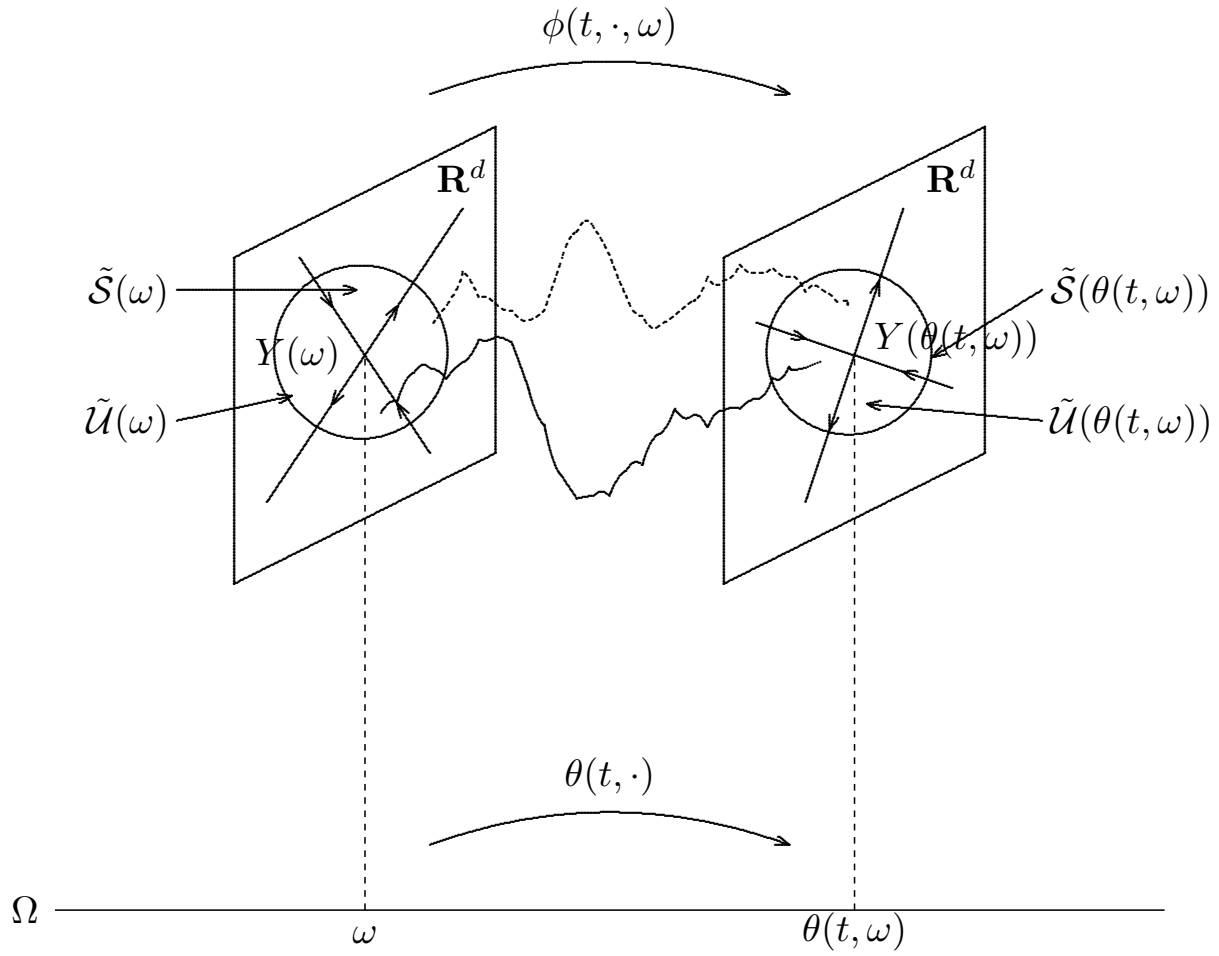
$$\mathbf{R}^d = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega). \quad (9)$$

(h) The mappings

$$\begin{aligned} \Omega &\rightarrow \mathcal{C}(\mathbf{R}^d), & \Omega &\rightarrow \mathcal{C}(\mathbf{R}^d), \\ \omega &\mapsto \tilde{\mathcal{S}}(\omega) & \omega &\mapsto \tilde{\mathcal{U}}(\omega) \end{aligned}$$

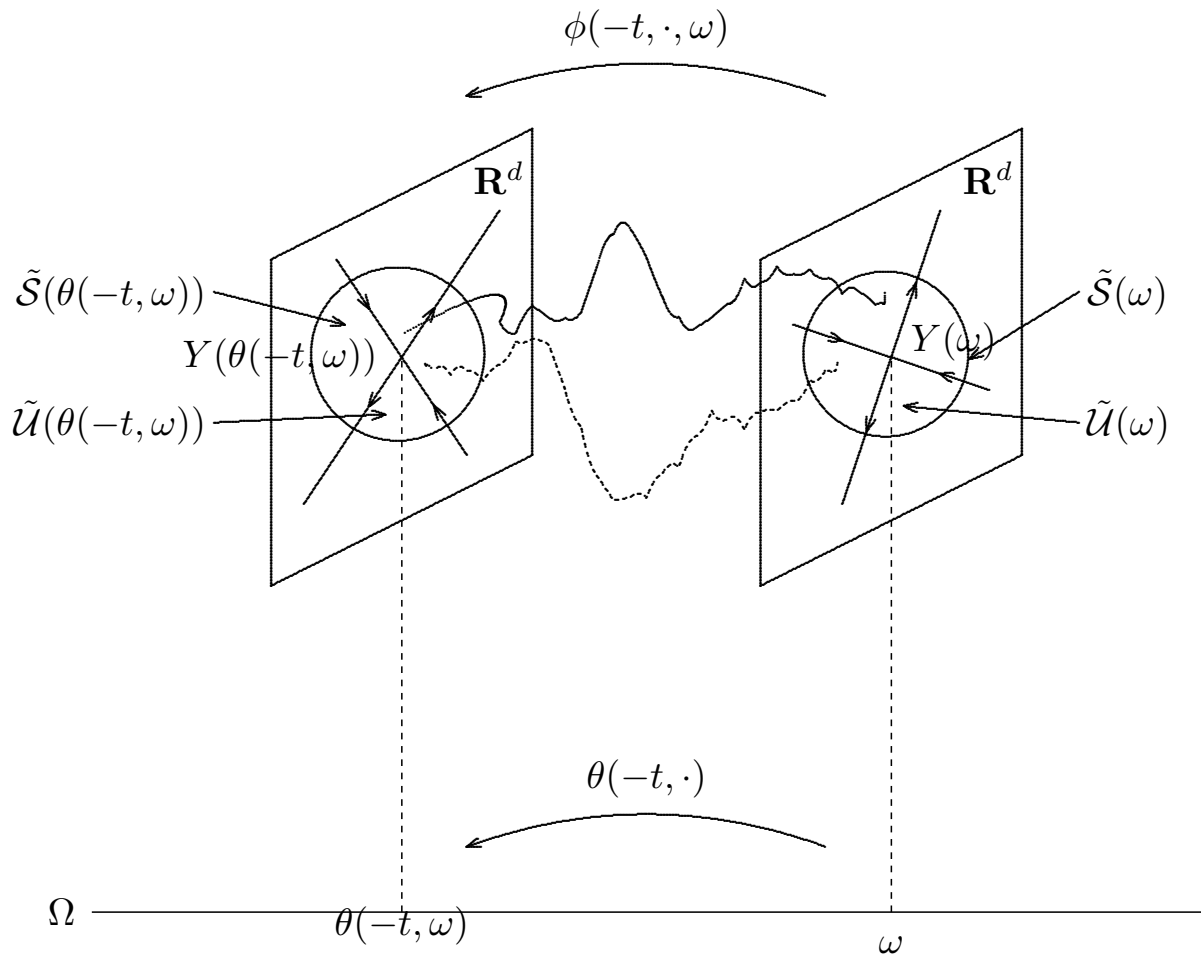
are $(\mathcal{F}, \mathcal{B}(\mathcal{C}(\mathbf{R}^d)))$ -measurable.

Assume, further, that $h, g_i, 1 \leq i \leq m$, are C_b^∞ . Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ .



$$t > \tau_1(\omega)$$

A picture is worth a 1000 words!



$$t > \tau_2(\omega)$$

Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients h, g_i . By the *Substitution Rule*, $\phi(t, Y(\omega), \omega)$ is a stationary *solution* of the *anticipating* Stratonovich SDE

$$\left. \begin{aligned} d\phi(t, Y) &= h(\phi(t, Y)) dt + \sum_{i=1}^m g_i(\phi(t, Y)) \circ dW_i(t), \quad t > 0 \\ \phi(0, Y) &= Y. \end{aligned} \right\} \quad (II)$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. Hence $D_2\phi(t, Y(\omega), \omega)$, $t \geq 0$, solves the SDE:

$$\left. \begin{aligned} dD_2\phi(t, Y) &= Dh(\phi(t, Y))D_2\phi(t, Y) dt \\ &\quad + \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t > 0 \\ D_2\phi(0, Y) &= I. \end{aligned} \right\} \quad (III)$$

D_2, D denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$\phi(t, Y), D_2\phi(t, Y), \quad t < 0,$$

solve the corresponding backward Stratonovich SDE's:

$$\left. \begin{aligned}
 d\phi(t, Y) &= -h(\phi(t, Y)) dt - \sum_{i=1}^m g_i(\phi(t, Y)) \circ \hat{d}W_i(t), & t < 0 \\
 \phi(0, Y) &= Y.
 \end{aligned} \right\} \text{(II}^-)$$

$$\left. \begin{aligned}
 dD_2\phi(t, Y) &= -Dh(\phi(t, Y))D_2\phi(t, Y) dt \\
 &\quad - \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ \hat{d}W_i(t), & t < 0 \\
 D_2\phi(0, Y) &= I.
 \end{aligned} \right\} \text{(III}^-)$$

Above SDE's (II)-(III)⁻ give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem.

Lemma 1

(i) Let $h : \Omega \rightarrow \mathbf{R}^+$ be \mathcal{F} -measurable and such that

$$\int_{\Omega} \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) dP(\omega) < \infty.$$

Then there is a sure event $\Omega_1 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1) = \Omega_1$ for all $t \in \mathbf{R}$, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$$

for all $\omega \in \Omega_1$.

(ii) Suppose $f : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R} \cup \{-\infty\}$ is a measurable process on (Ω, \mathcal{F}, P) satisfying the following conditions

(a) $E \sup_{0 \leq u \leq 1} f^+(u) < \infty, \quad E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u)) < \infty$

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and **all** $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbf{R}$, and a fixed number $f^* \in \mathbf{R} \cup \{-\infty\}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} f(t, \omega) = f^*$$

for all $\omega \in \Omega_2$.

Proof

[Mo.1], Lemma 7. \square

Theorem 2 ([O], 1968)

Let (Ω, \mathcal{F}, P) be a probability space and $\theta : \mathbf{R}^+ \times \Omega \rightarrow \Omega$ a measurable family of ergodic P -preserving transformations. Let $T : \mathbf{R}^+ \times \Omega \rightarrow L(\mathbf{R}^d)$ be measurable, such that (T, θ) is an $L(\mathbf{R}^d)$ -valued cocycle. Suppose that

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.$$

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P -measure such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n \rightarrow \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. Each $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \dots > e^{\lambda_p}$$

where the λ_i 's are distinct. Each e^{λ_i} has an eigen-space $F_i(\omega)$ and a fixed non-random multiplicity $m_i := \dim F_i(\omega)$. Define

$$E_1(\omega) := \mathbf{R}^d, \quad E_i(\omega) := \left[\bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad 1 < i \leq p.$$

Then

$$E_p(\omega) \subset \dots \subset E_{i+1}(\omega) \subset E_i(\omega) \subset \dots \subset E_2(\omega) \subset E_1(\omega) = \mathbf{R}^d$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i(\omega), \quad \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega),$$

and

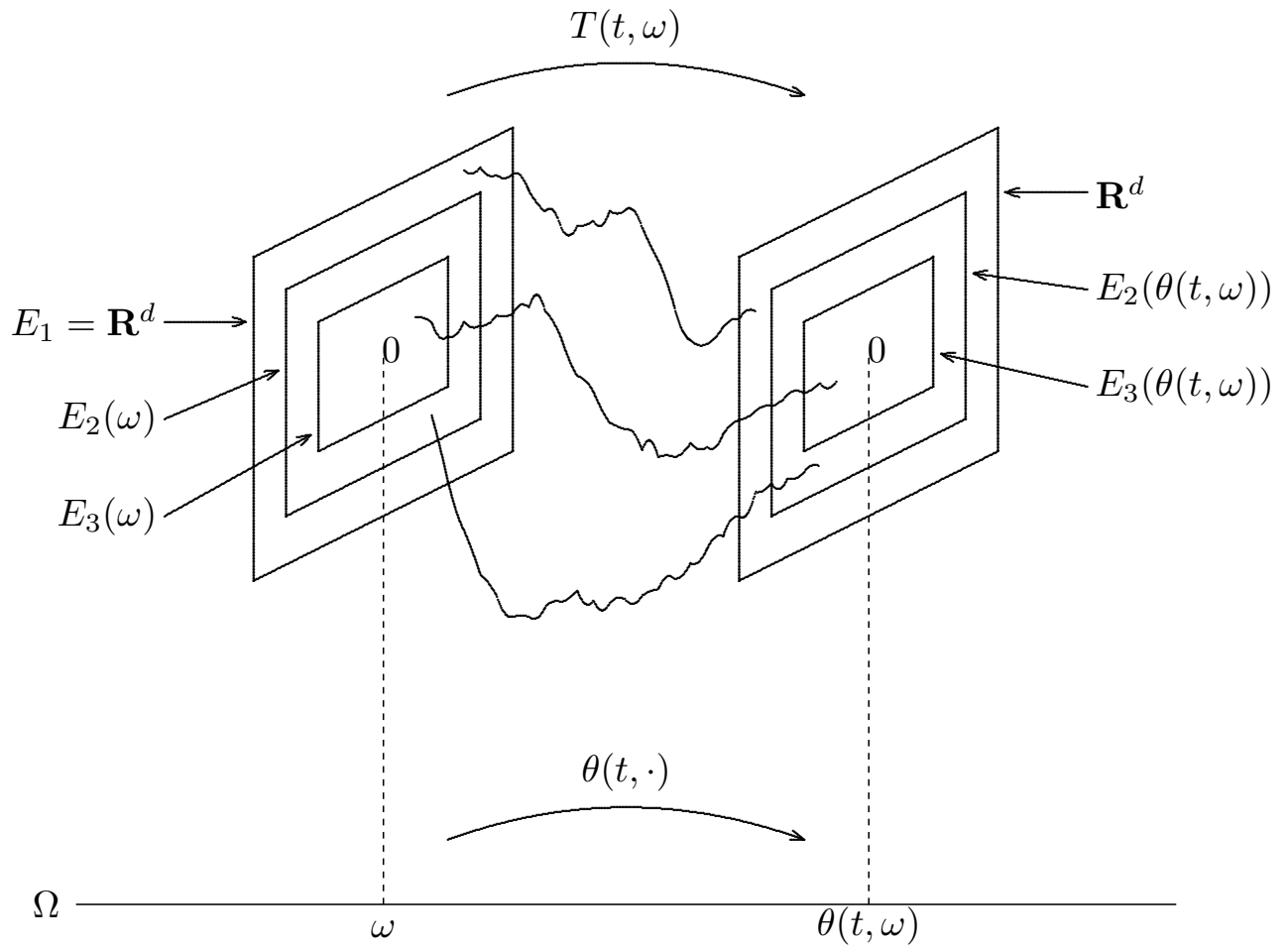
$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0$, $1 \leq i \leq p$.

Proof.

Based on the discrete version of Oseledec's multiplicative ergodic theorem and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), "perfect" infinite-dimensional version and application to SFDE's. \square

Spectral Theorem

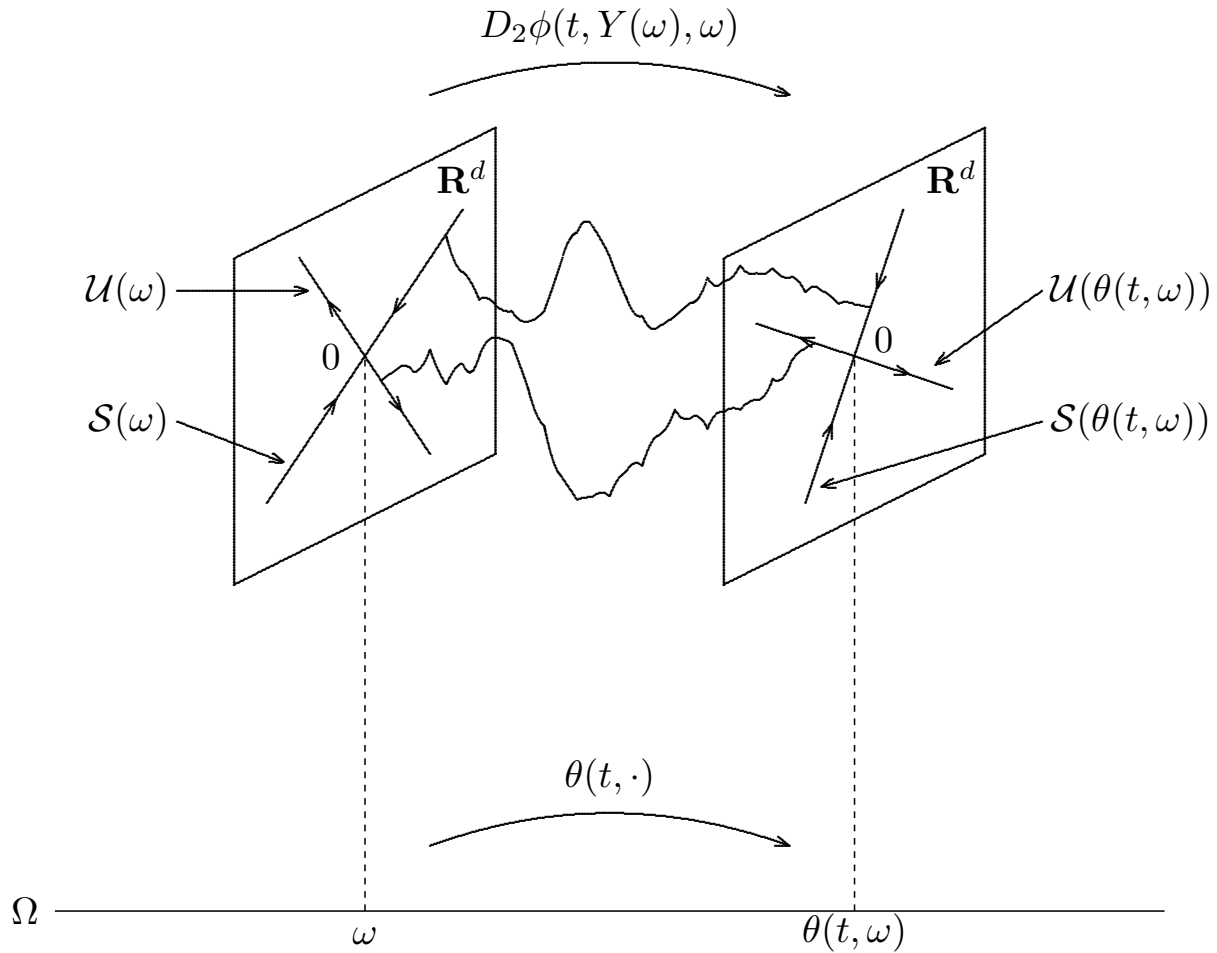


Apply Theorem 2 with $T(t, \omega) := D_2\phi(t, Y(\omega), \omega)$. Then linearized cocycle has random invariant stable and unstable subspaces $\{S(\omega), U(\omega) : \omega \in \Omega\}$:

$$D_2\phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)),$$

$$D_2\phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0.$$

[Mo.1].



Estimates on the non-linear cocycle

Theorem 3 (M. + Scheutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I), denoted by $\{\phi_{s,t}(x) : -\infty < s, t < \infty, x \in \mathbf{R}^d\}$, with the following properties:

Define $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$ by

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega, t \in \mathbf{R}.$$

Then for all $\omega \in \Omega$, $\epsilon \in (0, \delta)$, $\gamma, \rho, T > 0$, $1 \leq |\alpha| \leq k$, $\phi(t, \cdot, \omega)$ is $C^{k, \epsilon}$, $0 < \epsilon < \delta$, and the quantities

$$\begin{aligned} & \sup_{\substack{0 \leq s, t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x|(\log^+ |x|)^\gamma]}, & \sup_{\substack{0 \leq s, t \leq T, \\ x \in \mathbf{R}^d}} \frac{|D_x^\alpha \phi_{s,t}(x, \omega)|}{(1 + |x|^\gamma)}, \\ & \sup_{x \in \mathbf{R}^d} \sup_{\substack{0 \leq s, t \leq T, \\ 0 < |x' - x| \leq \rho}} \frac{|D_x^\alpha \phi_{s,t}(x, \omega) - D_x^\alpha \phi_{s,t}(x', \omega)|}{|x - x'|^\epsilon (1 + |x|)^\gamma}, \end{aligned}$$

are finite. The random variables defined by the above expressions have p -th moments for all $p \geq 1$.

$\|\cdot\|_{k,\epsilon} := C^{k,\epsilon}$ -norm on $C^{k,\epsilon}$ mappings $\bar{B}(0,\rho) \rightarrow \mathbf{R}^d$.

Lemma 2

Assume that $\log^+ |Y(\cdot)|$ is integrable. Then the cocycle ϕ satisfies

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \|\phi(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega))\|_{k,\epsilon} dP(\omega) < \infty \quad (10)$$

for any fixed $0 < T, \rho < \infty$ and any $\epsilon \in (0, \delta)$. Furthermore, the linearized flow $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$, $t \geq 0$, is an $L(\mathbf{R}^d)$ -valued perfect cocycle and

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \|D_2\phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))\|_{L(\mathbf{R}^d)} dP(\omega) < \infty \quad (11)$$

for any fixed $0 < T < \infty$. The forward cocycle

$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$ has a non-random finite Lyapunov spectrum $\{\lambda_m < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$. Each Lyapunov

exponent λ_i has a non-random multiplicity q_i , $1 \leq i \leq m$, and $\sum_{i=1}^m q_i =$

d . The backward linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$, admits a “backward” non-random finite Lyapunov spectrum:

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d),$$

taking values in $\{-\lambda_i\}_{i=1}^m$ with non-random multiplicities q_i , $1 \leq i \leq m$, and $\sum_{i=1}^m q_i = d$.

The Auxiliary Cocycle

To apply Ruelle's discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$. This a "centering" of the flow ϕ about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$.

Lemma 3

(Z, θ) is a perfect cocycle on \mathbf{R}^d and $Z(t, 0, \omega) = 0$ for all $t \in \mathbf{R}$, and all $\omega \in \Omega$.

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

Lemma 4

Suppose that $\log^+ |Y(\cdot)|$ is integrable. Then there is a sure event $\Omega_3 \in \mathcal{F}$ with the following properties:

- (i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbf{R}$,
- (ii) For every $\omega \in \Omega_3$ and any $x \in \mathbf{R}^d$, the statement

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \quad (17)$$

implies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)|. \quad (18)$$

Ruelle's Non-linear Ergodic Theorem

Theorem 4 ([Ru.1], 1979)

Let $\Omega \ni \omega \mapsto F_\omega \in C^{k,\epsilon}(\mathbf{R}^d, 0; \mathbf{R}^d, 0)$ be measurable such that $E \log^+ \|F_\cdot|_{\bar{B}(0,1)}\|_{k,\epsilon} < \infty$. Set $F^n(\omega) := F_{\theta(n-1,\omega)} \circ \cdots \circ F_{\theta(1,\omega)} \circ F_\omega$. Suppose $\lambda < 0$ is not in the spectrum of the cocycle $(DF_\omega^n(0), \theta(n, \omega))$. Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(1, \cdot)(\Omega_0) \subseteq \Omega_0$, and measurable functions $0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1$ with the following properties:

(a) If $\omega \in \Omega_0$, the set

$$V_\omega^\lambda := \{x \in \bar{B}(0, \alpha(\omega)) : |F_\omega^n(x)| \leq \beta(\omega)e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{k,\epsilon}$ submanifold of $\bar{B}(0, \alpha(\omega))$.

(b) If $x_1, x_2 \in V_\omega^\lambda$, then

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma(\omega)|x_1 - x_2|e^{n\lambda}$$

for all integers $n \geq 0$. If $\lambda' < \lambda$ and $[\lambda', \lambda]$ is disjoint from the spectrum of $(DF_\omega^n(0), \theta(n, \omega))$, then there exists a measurable $\gamma'(\omega) > 1$ such that

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}$$

for all $x_1, x_2 \in V_\omega^\lambda$ and all integers $n \geq 0$.

Proof

[Ru.1], Theorem 5.1, p. 292.

Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle (Z, θ) . Set $\tau := \theta(1, \cdot) : \Omega \rightarrow \Omega$. Define maps $F_\omega, F_\omega^n : \mathbf{R}^d \rightarrow \mathbf{R}^d$:

$$F_\omega(x) := Z(1, x, \omega) \quad x \in \mathbf{R}^d$$

$$F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$$

for all $\omega \in \Omega$. Then cocycle property for Z gives $F_\omega^n = Z(n, \cdot, \omega)$ for each $n \geq 1$. F_ω is $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) and $(DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)$.

- Integrability of the map $\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbf{R}^d)}$ (Lemma 2) implies discrete cocycle $((DF_\omega^n)(0), \theta(n, \omega), n \geq 0)$ has same non-random Lyapunov spectrum as that of linearized continuous cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$, viz. $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, where each λ_i has fixed multiplicity q_i , $1 \leq i \leq m$ (Lemma 2).
- If $\lambda_i > 0$ for all $1 \leq i \leq m$, then take $\tilde{\mathcal{S}}(\omega) := \{Y(\omega)\}$ for all $\omega \in \Omega$. Theorem is trivial in this case. Hence assume there is at least one $\lambda_i < 0$.
- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event $\Omega_1^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, \mathcal{F} -measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \rightarrow (0, \infty)$, $\rho_1 < \beta_1$, and a random family of $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds of $\bar{B}(0, \rho_1(\omega))$ denoted by $\tilde{\mathcal{S}}_d(\omega)$, $\omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$: $\tilde{\mathcal{S}}_d(\omega)$ is the set of all $x \in \bar{B}(0, \rho_1(\omega))$ such that

$$|Z(n, x, \omega)| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}, \quad n \in \mathbf{Z}^+ \quad (21)$$

$\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Therefore $\dim \tilde{\mathcal{S}}_d(\omega)$ is non-random by ergodicity of θ . Also

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \left[\sup_{\substack{x_1 \neq x_2, \\ x_1, x_2 \in \tilde{\mathcal{S}}_d(\omega)}} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}. \quad (22)$$

The $\theta(t, \cdot)$ -invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

- For each $\omega \in \Omega_1^*$, let $\tilde{\mathcal{S}}(\omega)$ be as defined in part (a) of the theorem. Then by definition of $\tilde{\mathcal{S}}_d(\omega)$ and Z :

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega). \quad (23)$$

Since $\tilde{\mathcal{S}}_d(\omega)$ is a $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifold of $\bar{B}(0, \rho_1(\omega))$, then $\tilde{\mathcal{S}}(\omega)$ is a $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifold of $\bar{B}(Y(\omega), \rho_1(\omega))$. Furthermore, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Hence $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega) = \sum_{i=i_0}^m q_i$, and is non-random.

- (22) implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \quad (24)$$

for all ω in Ω_1^* and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Lemma 4 implies there is a sure event $\Omega_2^* \subseteq \Omega_1^*$ such that $\theta(t, \cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbf{R}$, and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \quad (25)$$

for all $\omega \in \Omega_2^*$ and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore (2) holds.

- To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{\mathcal{S}}_d(\omega)$) such that $Z(n, x, \omega) \in \bar{B}(0, 1)$ for all $n \geq N_0$. Let $\Omega_4^* := \Omega_2^* \cap \Omega_3$, where Ω_3 is the shift-invariant sure event defined in the proof of Lemma 4. Then Ω_4^* is a sure event and $\theta(t, \cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbf{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).
- To prove the invariance property (4), apply the Osledec theorem to $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$. Get a sure $\theta(t, \cdot)$ -invariant event, also denoted by Ω_1^* , such that $D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$. Equality holds because $D_2\phi(t, Y(\omega), \omega)$ is injective and $\dim \mathcal{S}(\omega) = \dim \mathcal{S}(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$.
- To prove the asymptotic invariance property (3), use ideas from Ruelle's Theorems 5.1 and 4.1 in [Ru.1], to pick random variables ρ_1, β_1 and a sure event (also denoted by) Ω_1^* such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, and for any $\epsilon \in (0, \epsilon_1)$ and every $\omega \in \Omega_1^*$, and the inequalities

$$\begin{aligned} \rho_1(\theta(t, \omega)) &\geq \rho_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)t}, \\ \beta_1(\theta(t, \omega)) &\geq \beta_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)t} \end{aligned} \tag{26}$$

hold for all $t \geq 0$ and every $\omega \in \Omega_1^*$. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t, \cdot)(\Omega_5^*) = \Omega_5^*$ for all

$t \in \mathbf{R}$, and for any $0 < \epsilon < \epsilon_1$ and $\omega \in \Omega_4^*$, there exists $\beta^\epsilon(\omega) > 0$ (independent of x) with

$$|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} \quad (27)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$, $t \geq 0$. Fix $t \geq 0$, $\omega \in \Omega_5^*$ and $x \in \tilde{\mathcal{S}}(\omega)$. Let n be a non-negative integer. Then the cocycle property and (27) imply that

$$\begin{aligned} & |\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| \\ &= |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \\ &\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)(n+t)} \\ &\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}. \end{aligned} \quad (28)$$

If $\omega \in \Omega_5^*$, then it follows from (26), (27), (28) and the definition of $\tilde{\mathcal{S}}(\theta(t, \omega))$ that there exists $\tau_1(\omega) > 0$ such that $\phi(t, x, \omega) \in \tilde{\mathcal{S}}(\theta(t, \omega))$ for all $t \geq \tau_1(\omega)$. This proves asymptotic invariance.

- Prove (d), the existence of the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$, by running both the flow ϕ and the shift θ backward in time getting the cocycle $(\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)$:

$$\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \quad \tilde{Z}(t, x, \omega) := Z(-t, x, \omega), \quad \tilde{\theta}(t, \omega) := \theta(-t, \omega)$$

for all $t \geq 0, \omega \in \Omega$. The linearized flow $(D_2 \tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)$ is an $L(\mathbf{R}^d)$ -valued perfect cocycle with a non-random finite Lyapunov spectrum $\{-\lambda_1 < -\lambda_2 < \dots < -\lambda_i < -\lambda_{i+1} < \dots < -\lambda_m\}$ where $\{\lambda_m < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$ is the Lyapunov spectrum of the forward linearized flow

$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$. Apply first part of the proof to get *stable manifolds* for the backward flow $\tilde{\phi}$ satisfying assertions (a), (b), (c). This gives *unstable manifolds* for the original flow ϕ , and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

$$\tilde{\mathcal{S}}(\omega) = Y(\omega) + \tilde{\mathcal{S}}_d(\omega) \quad (29)$$

$$\tilde{\mathcal{S}}_d(\omega) = \lim_{n \rightarrow \infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^n f_i(\cdot, \omega)^{-1}(\bar{B}(0, 1)) \quad (30)$$

$$f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0} + \epsilon_1)i} Z(i, x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega_1^*,$$

for all integers $i \geq 0$. (Above limit is taken in the metric d^* on $\mathcal{C}(\mathbf{R}^d)$.) Use joint continuity of translation and measurability of Y , f_i , ρ_1 , finite intersections and the continuity of the maps

$$\mathbf{R}^+ \ni r \mapsto \bar{B}(0, r) \in \mathcal{C}(\mathbf{R}^d).$$

$$\text{Hom}(\mathbf{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in \mathcal{C}(\mathbf{R}^d).$$

- For h, g_i in C_b^∞ , can adapt above argument to give a sure event in \mathcal{F} , also denoted by Ω^* such that $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. \square

Examples of Stationary Solutions

1. Fixed points:

$$d\phi(t) = h(\phi(t)) dt + \sum_{i=1}^m g_i(\phi(t)) \circ dW_i(t)$$

$$h(x_0) = g_i(x_0) = 0, \quad 1 \leq i \leq m$$

Take $Y(\omega) = x_0$ for all $\omega \in \Omega$.

2. Linear affine case $d = 1$:

$$d\phi(t) = \lambda\phi(t) dt + dW(t)$$

$\lambda > 0$ fixed, $W(t) \in \mathbf{R}$. Take

$$Y(\omega) := - \int_0^\infty e^{-\lambda u} dW(u),$$

$$\theta(t, \omega)(s) = \omega(t + s) - \omega(t).$$

Check that $\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega))$, using integration by parts and variation of parameters.

3. Affine linear SDE in $d = 2$:

$$d\phi(t) = A\phi(t) dt + GdW(t)$$

with A a fixed hyperbolic 2×2 -diagonal matrix; G a constant matrix.

4. Non-linear transforms of (3) under a global diffeomorphism.

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