EQUIVARIANT FLOW EQUIVALENCE FOR SHIFTS OF FINITE TYPE, BY MATRIX EQUIVALENCE OVER GROUP RINGS

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ABSTRACT. Let G be a finite group. We classify G-equivariant flow equivalence of nontrivial irreducible shifts of finite type in terms of (i) elementary equivalence of matrices over $\mathbb{Z}G$ and (ii) the conjugacy class in $\mathbb{Z}G$ of the group of G-weights of cycles based at a fixed vertex. In the case $G = \mathbb{Z}/2$, we have the classification for *twistwise flow equivalence*. We include some algebraic results and examples related to the determination of $\mathrm{E}(\mathbb{Z}G)$ equivalence, which involves $\mathrm{K}_1(\mathbb{Z}G)$.

CONTENTS

1. INTRODUCTION

Let G denote a finite group. In this paper, by a G shift of finite type (G-SFT) we will mean an SFT together with a continuous G-action which commutes with the shift, where in addition the action is

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free and the SFT is irreducible and is nontrivial (contains more than one orbit). We will classify these systems up to G-flow equivalence. This equivalence relation can be described in terms of G-SFTs, skew products or suspension flows (Sec 2). For example, two G-SFTs are G-flow equivalent if and only if there exists an orientation preserving homeomorphism between their mapping tori which commutes with the induced G actions.

A G-SFT can be presented by a finite square matrix A over $\mathbb{Z}_+G$, the positive cone of the integral group ring $\mathbb{Z}G$ [30]. Let $(I-A)_\infty$ denote the $\mathbb{N}\times\mathbb{N}$ matrix whose upper left corner is $I-A$ and which otherwise equals the infinite identity matrix. Let $\mathrm{E}(\mathbb{Z}G)$ be the group of $\mathbb{N}\times\mathbb{N}$ matrices generated by basic elementary matrices (those which differ from I in at most one entry, which must be off-diagonal) over $\mathbb{Z}G$. Let $W(A)$ denote the *weight class* of A (4.1): the conjugacy class in G of the group of weights of loops based at a fixed vertex. We show the weight class is an invariant of G-flow equivalence. When $W(A)=W(B)=G$, we will show that G-SFTs presented by matrices A and B are G-flow equivalent if and only if there are matrices U,V in $\mathrm{E}(\mathbb{Z}G)$ such that $U(I-A)_\infty V=(I-B)_\infty$ (Thm. 6.1). The complete classification up to G-flow equivalence, which allows the possibility $W(A)\subsetneqq G$, has a more complicated statement (Thm. 6.4).

In the case that G is trivial, our classification reduces to the familiar classification of Franks [15] by cokernel group and determinant. When G is nontrivial, the classification up to $\mathrm{E}(\mathbb{Z}G)$ is much more difficult and interesting, and remains an open problem. We consider these algebraic issues in Sections 8 and 9. In Section 8, we give the modest requisite K-theory terminology and background, and for the case $G=\mathbb{Z}/2$ we give a constructive partial result (Theorem 8.1) and some very concrete illustrative examples (8.6, 8.7) which indicate how the $\mathbb{Z}G$ equivalence problem becomes more difficult when G is nontrivial (i.e. $\mathbb{Z}G\neq\mathbb{Z}$). In Section 9, we consider $\mathrm{E}(\mathbb{Z}G)$ equivalence of injective matrices. In this case, $\mathrm{GL}(\mathbb{Z}G)$ equivalence amounts to isomorphism of cokernel modules, and the refinement to $\mathrm{E}(\mathbb{Z}G)$ equivalence is classified by $\mathrm{K}_1(\mathbb{Z}G)/H$ for an associated subgroup H of $\mathrm{SK}_1(\mathbb{Z}G)$. As one consequence, if G is abelian and $\det(I-A)$ is not a zero divisor in $\mathbb{Z}G$, then $\det(I-A)$ determines the G-flow equivalence class up to finitely many possibilities (Theorem 6.5). Some of the algebra here works more generally and in particular has a consequence for invariants of SFTs with Markov measures (9.10,9.11).

Algebraic invariants over $\mathbb{Z}$ for isomorphism and flow equivalence of SFTs are paralleled by the algebraic invariants over $\mathbb{Z}G$ for G-equivariant isomorphism and flow equivalence of G-SFTs. The first key step, classification of G-SFTs by strong shift equivalence over $\mathbb{Z}_+G$ of defining matrices, is due to Parry (Prop. 2.7.1). We use a systematic conversion [9, Thm. 7.2] from the realm of strong shift equivalence to the realm of "positive K-theory" to establish necessary matrix conditions for G-flow equivalence. We generalize existing positive K-theory constructions [7] to establish sufficient conditions in the case $W(A) = W(B) = G$. To understand the reduction to this case, we draw on ideas of Holt, Parry and Schmidt [29, 31, 36].

Among motivations for studying G-SFTs, we mention three. First, there are two systematic frameworks for classifying systems related to SFTs: the ideas around strong shift equivalence growing out of Williams paper [42], and the ideas of positive K-theory growing out of the Kim-Roush-Wagoner papers [22]. (See [5, 6, 9].) The G-FE classification fills in another piece of both frameworks. Second, in the study of "symmetric chaos" [12], G-SFTs arise as important tools for the study of equivariant basic sets [11, 13], and can equal such sets. (We emphasize that we are not addressing the important but quite different case of nonfree actions.) Finally (and in fact our initial motivation), we are interested in *twistwise flow equivalence*, which arose [37, 38, 39] in the study of basic sets of Smale flows on 3-manifolds. Twistwise flow equivalence amounts to equivariant flow equivalence of G-SFTs with $G = \mathbb{Z}/2$, so our results include a classification up to twistwise flow equivalence, along with constructive techniques resolving some open questions (Sec. 7).

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2. G-flow equivalence and SFTs

In this section we give background for flow equivalence, G-flows and G-SFTs.

2.1. **Notational conventions.** Except in part of Section 9, G denotes a finite group. All our G actions are assumed to be continuous (each g acts by a homeomorphism), from the right ($(x, g) \mapsto xg$), and free (if g fixes any x then g is the identity in G). Here are other notations.

- Let $x = \sum_{g \in G} n_g g$ be an element of $\mathbb{Z}G$. We define $\pi_h(x) = x_h = n_h$ for each $h \in G$. If $x_g > 0$, we say g is a *summand* of x.
- For a and b in $\mathbb{Z}G$ we say $a \gg b$ if $\pi_g(a) > \pi_g(b)$ for each $g \in G$, and $a > b$ if $\pi_g(a) \geq \pi_g(b)$ for each $g \in G$ and $\pi_g(a) > \pi_g(b)$ for

at least one $g \in G$. We define $\ll$ and $<$ similarly and extend these notations to matrices if they hold entry-by-entry.
- Let A be matrix over $\mathbb{Z}G$. We say A is *very positive* if $A \gg 0$ and A is *strictly positive* if $A > 0$.
- The *augmentation map* $\alpha\colon \mathbb{Z}G \to \mathbb{Z}$ sends an element $\sum n_g g$ to $\sum n_g$. Applying α entry-wise to a matrix A with entries in $\mathbb{Z}G$ produces a matrix $\alpha(A)$ with entries in $\mathbb{Z}$.
- In this paper, a ring means a ring with 1. Let $\mathcal{R}$ be a ring. Then $E(\mathcal{R})$ has already been defined; $E(n, \mathcal{R})$ is defined likewise, for $n \times n$ rather than $\mathbb{N} \times \mathbb{N}$ matrices. See the beginning of Section 8 for more.

2.2. **Flows and sections.** Let Y be a compact metrizable space. In this paper, a *flow* on Y will be an $\mathbb{R}$-action on Y, given by a continuous map $\gamma\colon \mathbb{R} \times Y \to Y$, where γ is locally injective (the flow has no rest points). Two flows are *topologically conjugate*, or *conjugate*, if there is a homeomorphism intertwining their $\mathbb{R}$-actions. Two flows are *equivalent* if there is a homeomorphism between their domains taking $\mathbb{R}$-orbits to $\mathbb{R}$-orbits and preserving orientation (i.e. respecting the direction of the flow).

A compact subset C of Y is a *cross section* of the flow if the restriction of γ to $\mathbb{R} \times C$ is a surjective local homeomorphism. (In this case, the return map to C is a well defined homeomorphism $R : C \to C$; the return time r is a continuous function on C; and the given flow is topologically conjugate to the "flow under the function" built from R and r.) We say that R is a *section* to the flow. Two homeomorphisms are *flow equivalent* if they are topologically conjugate to sections of a common flow. (Homeomorphisms f, g are topologically conjugate if there is a homeomorphism h such that $hf = gh$.) Sections of two flows are flow equivalent if and only if the flows are equivalent.

In the case that T_1 and T_2 are homeomorphisms of zero dimensional compact metrizable spaces, Parry and Sullivan [32] showed that T_1 and T_2 are flow equivalent if and only if there is a third homeomorphism T such that there are discrete towers T_1' and T_2' over T which are topologically conjugate respectively to T_1 and T_2. (A discrete tower is a homeomorphism (X', T') built from (X, T) by partitioning X into finitely many closed open sets C_i, picking for each i a positive integer n_i, making X' the disjoint union of the sets $C_i \times \{j\}$, $1 \le j \le n_i$, and for $x \in C_i$ setting $T'(x, k) = (x, k+1)$ when $k < n_i$, and $T'(x, n_i) = (Tx, 1)$. Here (X, T) is called the *base* of the tower.)

2.3. G-**flows and** G-**sections.** By a *G-flow* we mean a flow together with a continuous free right G-action which commutes with the flow

$(t(yg) = (ty)g)$. By a *G-homeomorphism* we mean a homeomorphism together with a continuous free right G-action with which it commutes. Two G-flows are *G-conjugate* if the flows are topologically conjugate by a map which intertwines the G-actions. Two G-flows are *G-equivalent* if the flows are equivalent by a map which intertwines the G-actions $(f(xg) = (fx)g)$. A *G-cross section* to a G-flow is a cross section C which is G-invariant. Then there is an induced G-action on C with which R becomes a G-homeomorphism, and we say the G-homeomorphism R is a *G-section* to the G-flow. A discrete *G-tower* (X', T') over a G-homeomorphism (X, T) is a discrete tower over (X, T), together with a G-action $(x, j) \mapsto (xg, j)$ (in the notation above) induced by the G action $x \mapsto xg$ for (X, T).

The standard theory carries over to the G setting. We call two G-homeomorphisms *G-flow equivalent* if they are conjugate to G-sections of the same G-flow. G-sections of two G-flows are G-flow equivalent if and only if the flows are G-equivalent. In the case that T_1 and T_2 are G-homeomorphisms of zero dimensional compact metrizable spaces, T_1 and T_2 are G-flow equivalent if and only if there is a third G-homeomorphism T such that there are discrete G-towers T_1' and T_2' over T which are G-conjugate respectively to T_1 and T_2.

2.4. **Skew products.** Let $T : X \to X$ be a homeomorphism, with X zero dimensional. Let τ be a continuous map from X into the finite group G. Define a homeomorphism $S\colon X \times G \to X \times G$ by the rule $(x, h) \mapsto (T(x), \tau(x)h)$. With the natural right G-action on $X \times G$, $g : (x, h) \mapsto (x, hg)$, S is a G-homeomorphism. S is $T \ltimes_\tau G$, the *skew product* over T built from the *skewing function* τ.

Conversely, suppose $S : X \to X$ is a G-homeomorphism, with X zero dimensional. Let $q : X \to \overline{X}$ be the map onto the quotient space of G-orbits, and let T be the homeomorphism induced by S on $\overline{X}$. Because X is zero dimensional and the G action is free, we can find a closed open subset C of X such that $\{Cg\colon g \in G\}$ is a partition of X. Using the homeomorphism $q|C$, identify $\overline{X}$ with C. Using the maps $Cg \to C \times G$ $(xg \mapsto (x, g))$, identify X with $C \times G$. In this notation, q is the standard projection $C \times G \to C$, and the G action on $C \times G$ is $h\colon (x, g) \mapsto (x, gh)$. To display the skew product structure, define $\tau\colon C \to G$ by setting $\tau(x) = g$ if $S(x) \in Cg$. It follows for $x \in C$ that $S\colon (x, e) \mapsto (T(x), \tau(x)e)$. Because S commutes with the G action, we conclude that for any (x, g) we have $S\colon (x, g) \to (T(x), \tau(x)g)$. So, up to G-conjugacy, every G-homeomorphism of a zero dimensional space is a skew product.

Finally, suppose we have a G-homeomorphism T. The given G-action induces a natural G action on the mapping torus Y of T, with respect to which the natural flow on Y is a G-flow, and T is conjugate to the obvious G-section of this flow.

2.5. **Cocycles.** Let $T : X \to X$ be a homeomorphism. We may regard a continuous skewing function $\tau \colon X \to G$ as defining a *cocycle* for T. We say two such skewing functions τ and ρ are *cohomologous* if there is another continuous function h from X into G such that for all x in X, $\tau(x) = [h(x)]^{-1}\rho(x)h(Tx)$. Such a function h is called a *transfer function.* It is an easy exercise to verify that two skew products $T_1 \ltimes_\tau G$ and $T_2 \ltimes_\rho G$ are G-conjugate if and only if there is a topological conjugacy ϕ of T_1 and T_2 such that $\tau \circ \phi$ is cohomologous to ρ.

2.6. **Shifts of finite type and matrices over $\mathbb{Z}_+$.** Here we give minimal background for shifts of finite type (SFTs). See the texts [23, 24] for an introduction to SFTs.

In this subsection, all matrices will be $\mathbb{N} \times \mathbb{N}$ with entries in $\mathbb{Z}_+$ and (except for the identity matrix I) with all but finitely many entries equal to zero. (In particular, det $(I - A)$ is well defined as a limit of the determinants of the principal $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ submatrices.) Given such a matrix A, let $\mathcal{G}_A$ be the directed graph with vertex set $\mathbb{N}$ and with exactly $A(i, j)$ edges from i to j. Let $\mathcal{E}$ be the edge set and define Σ_A to be the subset of $\mathbb{Z}^{\mathcal{E}}$ realized by bi-infinite paths in $\mathcal{G}_A$. With the natural topology, Σ_A is a zero dimensional compact metrizable space. Let $\sigma_A : \Sigma_A \to \Sigma_A$ be the *shift map*, $(\sigma_A(s))_i = s_{i+1}$. The homeomorphism σ_A is the edge SFT induced by A. Every SFT is topologically conjugate to some edge SFT.

Matrices A and B over a semiring $\mathcal{R}$ are strong shift equivalent (SSE) over $\mathcal{R}$ if they are connected by a string of elementary moves of the following sort: there are R and S over $\mathcal{R}$ such that $A = RS$ and $B = SR$. A fundamental result in symbolic dynamics is that σ_A is topologically conjugate to σ_B if and only if A is SSE over $\mathbb{Z}_+$ to B [42]. Refined computable invariants of SSE are known, but it is still not known even if SSE over $\mathbb{Z}_+$ is decidable.

If $A = (A_{ij})$ and

$$B = \begin{pmatrix} 0 & A_{11} & \cdots & A_{1n} \\ 1 & 0 & \cdots & 0 \\ 0 & A_{21} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1} & \cdots & A_{nn} \end{pmatrix},$$

then we say A and B are connected by a Parry-Sullivan move or a PS move.

It follows from the Parry-Sullivan result described above that SFTs σ_A and σ_B are flow equivalent if and only if the matrices A and B can be connected by SSE and Parry-Sullivan moves [32]. (The Parry-Sullivan moves allow for building the discrete towers.)

An SFT σ_A is *irreducible* if for any edges e and f which appear in points of Σ_A, there is a path in $\mathcal{G}_A$ beginning with e and ending with f. When σ_A and σ_B are irreducible and nontrivial (not just a single periodic orbit), they are flow equivalent if and only if the matrices $I - A$ and $I - B$ are SL($\mathbb{Z}$)-equivalent. This equivalence is determined by two simple invariants: the Parry-Sullivan number $\det(I - A)$ and the isomorphism class of the Bowen-Franks group $\mathrm{cok}(I-A)$ [32, 4, 15]. The Huang classification of reducible SFTs up to flow equivalence is much more complicated. (Huang's original arguments are developed in [17, 18, 19, 20] and an almost complete unpublished manuscript, "The K-web invariant and flow equivalence of reducible shifts of finite type." A complete alternate development is contained in [6, 8].) In this paper, we only address G-flow equivalence of irreducible SFTs.

2.7. **Skew products, G-SFTs and matrices over $\mathbb{Z}_+G$.** By a G-SFT we mean an SFT together with a free G-action with which it commutes. (Usually "G-SFT" is not restricted to free actions [11, 12, 13]; we adopt the restriction only for this paper, where we only consider free actions.) In this subsection, we'll consider presentations of G-SFTs.

Let A be an $\mathbb{N}\times\mathbb{N}$ matrix with entries in $\mathbb{Z}_+G$ and with all but finitely many entries equal to zero. Such a matrix A determines a weighted directed graph $\mathcal{G}_A$ as follows. As an unweighted graph, it is the graph $\mathcal{G}_{\alpha(A)}$. Recall α is the augmentation map (2.1). If $A(i,j) = \sum n_g g$ then exactly n_g of the edges from i to j are weighted g. Let $\ell(e)$ denote the weight on an edge e. Define a locally constant function $\tau_A \colon \Sigma_{\alpha(A)} \to G$ by the rule $x \mapsto \ell(x_0)$. This function then defines a skew product over $\sigma_{\alpha(A)}$. This skew product can be presented as an edge SFT with the graph $\mathcal{G}$ constructed as follows. Let the vertex set of $\mathcal{G}$ be the product of G and the vertex set of $\mathcal{G}_{\alpha(A)}$. For each edge e from i to j in $\mathcal{G}_{\alpha(A)}$, for each g in G draw an edge from (g,i) to $(\ell(e)g,j)$. We write $S_A = \sigma_{\alpha(A)} \ltimes \tau_A$. To make S_A a G-SFT, for each pair of vertices v, v' of $\mathcal{G}$, we choose an ordering of the edges from v to v', and then let g in G act by the one block map given by the unique automorphism of $\mathcal{G}$ which acts on the vertex set $\mathcal{G}$ by $(h,j) \mapsto (hg,j)$ and which is order-preserving on edges.

It is not difficult to see that for any locally constant function into G from a SFT σ, there is a matrix A over $\mathbb{Z}_+G$ and a topological conjugacy from σ to $\sigma_{\alpha(A)}$ which takes the given function to τ_A, and therefore any G skew product over an SFT can be presented as some S_A. Moreover, a G-SFT can be presented as a skew product (Sec. 2.4–our assumption of freeness is necessary for this), and it is not difficult to see that the base map for this skew product must be SFT in order for the skew product to be SFT. Thus all G-SFTs are G-conjugate to those arising by this construction of S_A.

Proposition 2.7.1. [30] *Let G be a finite group. The following are equivalent for matrices A and B over $\mathbb{Z}_+G$ and their associated skew product systems S_A and S_B.*

1. *A and B are SSE over $\mathbb{Z}_+G$.*
2. *There is a topological conjugacy $\varphi\colon \sigma_{\alpha(A)} \to \sigma_{\alpha(B)}$ such that $\tau_A \sim \tau_B \circ \varphi$.*
3. *The G-SFTs S_A and S_B are G-conjugate.*

Proof. We will prove (2) $\Longrightarrow$ (1). As shown by Parry [28], the given conjugacy φ can be given as a string of state splittings from $\alpha(A)$ to some C followed by the reversal of a string of state splittings from $\alpha(B)$ to C. The SSE's over $\mathbb{Z}_+$ which give the splittings are easily adapted to SSE's over $\mathbb{Z}_+G$ which reflect the corresponding lifting of edge labelings (we give an example following the proof). In this way, we produce $\mathbb{Z}_+G$ matrices A', B' such that $\alpha(A') = C = \alpha(B')$, the skewing functions derived from A' and B' are the functions lifted from the skewing functions defined from A and B, and they are cohomologous. If h is a continuous transfer function giving the cohomology of these functions, then in fact $h(x)$ is determined by the initial vertex of x_0 ([29, Lemma 9.1] proves this for irreducible SFTs, and the essential ideas of that proof can be extracted to prove the general case). Therefore there is a diagonal matrix D with $D(i,i) = g_i \in G$, such that $DA'D^{-1} = B'$. The $\mathbb{Z}_+G$ strong shift equivalence from A' to $DA'D^{-1}$ is given by the pair $(A'D^{-1}, D)$. □

Above, in restricting to $\mathbb{Z}G$ with G finite, we have not given the most general statement of Parry's results.

Example 2.7.2. Here is the example promised in the preceding proof. Let $A = \begin{pmatrix} g & h \\ j & k+\ell \end{pmatrix}$ over some $\mathbb{Z}_+G$. Consider the row splitting of

$\alpha(A)$ defined by the elementary SSE

$$\alpha(A) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} .$$

Then the $\mathbb{Z}_+G$ SSE which captures the label lifting is simply

$$A = \begin{pmatrix} g & h \\ j & k+\ell \end{pmatrix} = \begin{pmatrix} g & 0 & h \\ 0 & j & k+\ell \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\begin{pmatrix} g & 0 & h \\ g & 0 & h \\ 0 & j & k+\ell \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 & h \\ 0 & j & k+\ell \end{pmatrix} .$$

Remark 2.7.3. The equivalence of (1) and (2) in Prop. 2.7.1, established by Parry following the related innovation of Parry and Tuncel for Markov chains [28, 33], is a key step to a proper algebraic approach to G-SFTs. Otherwise, the facts and constructions above are at most minor variations of well known results (see e.g. [13, Sec. 3.2],[11, 1, 28, 29]). We also remark that [21] gives a realization result for G-SFTs which employs the positive K-theory technique introduced in [22].

3. Positive equivalence

Below, we allow a square matrix to be $n \times n$ or $\mathbb{N} \times \mathbb{N}$. Infinite matrices A, B are nonzero in only finitely many entries. Thus infinite matrices $I-A, I-B$ equal the infinite identity matrix except in finitely many entries.

Definition 3.1. A square matrix M over $\mathbb{Z}$ or $\mathbb{Z}G$ is *irreducible* if its entries are nonnegative (i.e. in $\mathbb{Z}_+$ or $\mathbb{Z}_+G$) and for each index pair (i, j) there is an $k > 0$ with $M^k(i, j) > 0$. The matrix M is *essentially irreducible* if it has a unique principal submatrix that is irreducible and that is contained in no larger irreducible principal submatrix. Such a submatrix is called the *irreducible core* of M.

We consider matrices over $\mathbb{Z}G$. A *basic elementary matrix* is a matrix of the form $E_{ij}(x)$, which denotes a matrix equal to the identity except for perhaps the off-diagonal ij entry (so, $i \neq j$), which is equal to x. Suppose $g \in G$, $E = E_{ij}(g)$ and A is a square matrix over $\mathbb{Z}_+G$

such that g is a summand of $A(i,j)$. Then we say that each of the equivalences

$$(E,I)\colon (I-A) \to E(I-A)\,, \qquad (E^{-1},I)\colon E(I-A) \to (I-A)\,,$$
$$(I,E)\colon (I-A) \to (I-A)E\,, \qquad (I,E^{-1})\colon (I-A)E \to (I-A)$$

is a *basic positive equivalence* over $\mathbb{Z}G$. Here the equivalences (E,I) and (I,E) are *forward* basic equivalences while (E^{-1},I) and (I,E^{-1}) are *backward* basic equivalences.

Definition 3.2. An equivalence $(U,V)\,:\,(I-A)\to(I-B)$ is a *positive equivalence* if it is a composition of basic positive equivalences. When there exists a positive equivalence from $I-A$ to $I-B$, we write $I-A \overset{+}{\sim} I-B$.

The effect of a basic positive equivalence on the induced graph is discussed in detail in [7, page 278] when G is trivial. Our situation is entirely analogous. Suppose $(E,I)\colon (I-A)\to(I-A')$ is a basic forward positive equivalence, $E=E_{ij}(g)$. Then A and A' agree except perhaps in row i, where

$$A'(i,k) = A(i,k) + gA(j,k) \qquad \text{if } k \neq j\,, \qquad\qquad \text{and}$$
$$A'(i,j) = A(i,j) + gA(j,j) - g\,.$$

Consequently the weighted graph $\mathcal{G}'$ associated to A' is constructed from the weighted graph $\mathcal{G}$ for A as follows. An edge e from i to j with weight g is deleted from $\mathcal{G}$. For each $\mathcal{G}$-edge f beginning at j, add an additional edge (called $[ef]$) from i to k with weight gh (where h is the weight of f and k is the terminal vertex of f). See Figure 1. (There, G-labels are suppressed for simplicity. If the labels of the edges e,f',f'' are g,h',h'', then the labels of the new edges $[ef'],[ef'']$ are gh',gh''.)

FIGURE 1. A basic positive equivalence.

The correspondence of the graphs $\mathcal{G},\mathcal{G}'$ induces a bijection of σ_A-orbits and $\sigma_{A'}$-orbits,

$$\dots bef'eegf'ef''d\dots \;\leftrightarrow\; \dots b[ef']eegf'[ef'']d\dots$$

This bijection of orbits does not arise from a bijection of points for the SFTs, but it does correspond to a homeomorphism of their mapping tori (after changing time by a factor of 2 over the clopen sets $\{x : x_0 = [ef]\}$, the new flow is conjugate to the old one), which lifts to a G-equivariant equivalence of the mapping tori flows for the respective skew products.

The bijection of orbits above respects finiteness of orbits and the induced homeomorphism of mapping tori above respects respects density of orbits. Consequently, positive equivalence respects essential irreducibility and nontriviality (infinite number of orbits). Positive equivalence need not respect the size of the irreducible core of a presenting matrix.

Theorem 3.3. *Let G be a finite group, and let A and B be square matrices over $\mathbb{Z}_+G$. Then $I - A \overset{+}{\sim} I - B$ if and only if S_A and S_B are G-flow equivalent.*

Proof. We explained above that $I - A \overset{+}{\sim} I - B$ implies the G-flow equivalence of S_A and S_B. Now suppose S_A and S_B are G-flow equivalent.

First suppose S_A and S_B are G-conjugate. Then by Proposition 2.7.1, B and A are SSE over $\mathbb{Z}_+G$. In the polynomial setting of [9], the G-weighted SFTs defined by A, B can be presented by polynomial matrices $I - tA$, $I - tB$, and any SSE over $\mathbb{Z}_+G$ from A to B gives rise to a composition of polynomial positive equivalences via the *polynomial strong shift equivalence equations* [9, Theorem 7.2]. These equivalences, after setting the variable t equal to 1, produce a positive equivalence from $I - A$ to $I - B$.

In the polynomial setting of [9], a matrix $I - tA$ as above can be positively equivalent to a matrix $I - B(t)$, where the entries of $B(t)$ may involve higher powers of the variable t. A matrix $B(t)$ over $t\mathbb{Z}_+G[t]$ presents a discrete G-tower whose base is obtained by setting every t^m to t, and up to G-conjugacy every discrete G-tower over a G-SFT arises in this way. Changing t^n to t^m does not change the image under $t \mapsto 1$. □

Remark 3.4. There is a more complicated way to handle the preceding proof, along the lines of [7, pp. 296-297] (which was the case $\mathbb{Z}G = \mathbb{Z}$). One can provide a decomposition of a state-splitting SSE move into positive equivalences, and provide a separate decomposition for an SSE which for some i corresponds to multiplying row i by g and column i by g^{-1}. Such moves generate SSE over $\mathbb{Z}_+G$. Lastly one can decompose a PS move into a finite string of basic positive equivalences.

4. The weight class

Suppose A is a matrix over $\mathbb{Z}_+G$, with τ_A the associated labeling of edges. The *weight* of a path e of edges $e_1e_2\cdots e_k$ from vertex i to j is defined to be $\tau(e) = \tau_A(e_1)\tau_A(e_2)\cdots\tau_A(e_k)$. (So, g is the weight of some path from i to j if and only if $\pi_g(A^n(i,j)) > 0$ for some $n \in \mathbb{N}$.)

Definition 4.1. Suppose G is a finite group, A is an essentially irreducible matrix over $\mathbb{Z}_+G$ and i is a vertex indexing a row of the irreducible core of A. Then $W_i(A)$ is the subgroup of G which is the set of weights of paths from i to i, and the weight class of A, $W(A)$, is the conjugacy class of $W_i(A)$ in G. A member of $W(A)$ is a *weights group* for A; if $W(A)$ contains one element, then it is *the* weights group for A.

Let us verify two implicit claims of the definition. First, $W_i(A)$ is a group because it is a semigroup and G is finite. Second, we check given $i \neq j$ that W_i and W_j are conjugate subgroups in G. Appealing to irreducibility, let x be the weight of some path from i to j and let y be the weight of some path from j to i. Because G is finite, we may assume $y = x^{-1}$ (if necessary after replacing y with $y(xy)^k$ for suitable k). Then $xW_jx^{-1} = W_i$, because

$$W_i \supset xW_jx^{-1} \supset x(x^{-1}W_ix)x^{-1} = W_i \ .$$

If G is abelian, then there is only one group in the weight class of A, and it is the union of the $W_i(A)$. If G is not abelian, then $\cup_i W_i(A)$ can generate a group strictly containing each $W_i(A)$, and this larger group will not be the right group for our analysis.

Proposition 4.2. *Suppose A is an irreducible matrix over $\mathbb{Z}_+G$, and there is a positive $\mathbb{Z}_+G$ equivalence from $I-A$ to $I-B$. Then $W(A) = W(B)$.*

Proof. From the description in Section 3, it is clear that when there is a basic positive equivalence from $I - A$ to $I - B$, there must be a vertex i, indexing a row in the irreducible core of A and also in the irreducible core of B, such that $W_i(A)$ and $W_i(B)$ are equal. □

Example 4.3. Suppose G is any nontrivial finite group. Let g be an element of G not equal to the identity e. In the ring $\mathbb{Z}G$, the formal element e is the multiplicative identity 1. Consider the matrices over $\mathbb{Z}_+G$,

$$A = \begin{pmatrix} g & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

The weight class $W(B)$ is trivial while $W(A)$ is not, so by Proposition 4.2 there cannot be a positive $\mathbb{Z}G$-equivalence from $I - A$ to $I - B$. However, there is an $\mathrm{E}(\mathbb{Z}G)$-equivalence:

$$\begin{pmatrix}1 & -g\\ 0 & 1\end{pmatrix}(I-A) = \begin{pmatrix}1 & -g\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-g & -1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix} = I-B\,. \quad \square$$

Example 4.3 shows that positive $\mathbb{Z}G$-equivalence of nontrivial irreducible S_A does not follow from $E(\mathbb{Z}G)$-equivalence. This issue is clarified in the positive K-theory framework [6, Sec. 8].

We will use the next lemma to pass from a matrix A over $\mathbb{Z}_+G$ to a matrix over $\mathbb{Z}_+H$, when H is in the weight class. The lemma is modeled on the Parry-Schmidt argument [31] for presentations of Markov chains. Recall α denotes the augmentation map (2.1).

Proposition 4.4. *Suppose A is an irreducible matrix over $\mathbb{Z}_+G$, and H is a group in the weight class of A. Then there is a diagonal matrix D over $\mathbb{Z}_+G$ with each diagonal entry in G (i.e., $\alpha(D) = \mathrm{Id}$) such that every entry of DAD^{-1} lies in $\mathbb{Z}_+H$.*

Proof. First consider $H = W_\ell(A)$, where ℓ is some vertex of A. For each j pick a path from ℓ to j and let the j-th diagonal element d_j of D be the G-weight of this path. Let b_j be the G-weight of a path from j back to ℓ. Now, if $A(i,j)$ has h as a summand, then $d_ihd_j^{-1}$ is the corresponding summand in $DAD^{-1}(i,j)$. Write

$$d_ihd_j^{-1} = (d_ihb_j)(d_jb_j)^{-1}. \tag{4.5}$$

Let k be the order of d_jb_j in G. Then the right hand side of (4.5) is $(d_ihb_j)(d_jb_j)^{k-1}$, a product of weights from ℓ to ℓ.

Finally, if $H = gW_\ell(A)g^{-1}$, then replace D above with gD. $\square$

The following example is extracted from an example of Derek Holt analyzed by Parry [29, Sec. 10], and shows that cohomology over G does not imply cohomology over a group in the weight class.

Example 4.6. Let $G = S_4$, the group of permutations of $\{1,2,3,4\}$. Define permutations $a = (12)(34), b = (13)(24), c = (14)(23)$. Let H be the subgroup $\{e,a,b,c\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Consider two 1×1 matrices over $\mathbb{Z}_+H$, $A = (a+b)$ and $B = (a+c)$. Let d be the transposition (12), so $dad^{-1} = a$ and $dbd^{-1} = c$. Then $dAd^{-1} = B$ and there is a positive $\mathbb{Z}G$ equivalence from $I - A$ to $I - B$. On the other hand, if we consider A and B as matrices over $\mathbb{Z}_+G$, we see that $\{H\}$ is the weight class of A and B (H is a normal subgroup of G), but the matrices $I - A$ and $I - B$ are not even $\mathrm{SL}(\mathbb{Z}H)$ equivalent: the determinant is defined for matrices over the commutative ring $\mathbb{Z}H$, and $\det(I - A) \neq \det(I - B)$. $\square$

Fortunately, the passage from G to the weight class is no worse than indicated by the previous example.

Theorem 4.7. *Let A and B be essentially irreducible matrices over $\mathbb{Z}_+H$, such that H is a weights group for A and B, and H is a subgroup of the finite group G. Then there is a positive $\mathbb{Z}_+G$ equivalence from $I - A$ to $I - B$ if and only if there exists an element γ of G such that*

- *$\gamma H \gamma^{-1} = H$, and*
- *there is a positive $\mathbb{Z}_+H$ equivalence from $I - A$ to $I - \gamma^{-1}B\gamma$.*

Proof. We will prove the nontrivial direction ("only if"). The assumed positive $\mathbb{Z}G$-equivalence from $I - A$ to $I - B$ involves time changes as well as conjugacies, and we refine the discussion of the proof of Proposition 2.7.1 to incorporate these time changes; they can be captured by including with the splittings from A to C a set of Parry-Sullivan moves, which can like the splittings be mirrored in the positive equivalence framework using only matrices over $\mathbb{Z}_+H$. Thus, as in the proof of Proposition 2.7.1, we end up with $\alpha(A') = C = \alpha(B')$; a diagonal D with $D(i,i) = g_i \in G$ such that $DA'D^{-1} = B'$; a positive $\mathbb{Z}_+H$ equivalence from $I - A$ to $I - A'$; and another from $I - B$ to $I - B'$.

Let τ_A and τ_B denote the edge-labeling functions defined by A' and B'. Then for any path $e = e_1e_2\cdots e_k$ of edges from vertex i to vertex j, we have

$$g_i\tau_A(e)(g_j)^{-1} = \tau_B(e)\ . \tag{4.8}$$

Because H is a weights group for A and B and all entries of A and B are in $\mathbb{Z}_+H$, it holds for each pair of vertices i,j in the irreducible core that every element of H arises as $\tau_A(e)$ for some path e from i to j. Because the right side of (4.8) lies in H, we have $g_iH(g_j)^{-1} \subset H$. We conclude for every i,j that $g_iH(g_j)^{-1} = H$. Let $\gamma = g_1$. For each j,

$$H = g_jH\gamma^{-1} = g_j\gamma^{-1}(\gamma H\gamma^{-1}) = g_j\gamma^{-1}H$$

and therefore for some $h_j \in H$ we have $g_j = h_j\gamma$. Now $D = (D')\gamma I$, where $D'(j,j) = h_j$, and therefore

$$\gamma^{-1}B'\gamma = (\gamma^{-1}D'\gamma)A'(\gamma^{-1}D'\gamma)^{-1}\ .$$

The entries of $\gamma^{-1}D'\gamma$ lie in $\mathbb{Z}_+H$. We have now a $\mathbb{Z}_+H$ SSE from A' to $\gamma^{-1}B'\gamma$. The $\mathbb{Z}_+H$ SSE from B' to B yields a $\mathbb{Z}_+H$ SSE from $\gamma^{-1}B'\gamma$ to $\gamma^{-1}B\gamma$, by replacement of each elementary SSE (R,S) with $\gamma^{-1}R\gamma, \gamma^{-1}S\gamma$. Thus we have a $\mathbb{Z}_+H$ SSE from A' to $\gamma^{-1}B\gamma$, and there is a positive $\mathbb{Z}_+H$ equivalence from $I - A'$ to $I - \gamma^{-1}B\gamma$, and by composition from $I - A$ to $I - \gamma^{-1}B\gamma$. ☐

Theorem 4.7 is reminiscent of a similar reduction of Parry and Schmidt in their extension of Livšic theory to nonabelian cocycles [29, Theorems 6.4, 9.5], [36]. They were particularly concerned with deducing cohomology of certain G-valued functions given conjugate weights on each periodic orbit. This is a much stronger assumption than we use, and yields a correspondingly stronger conclusion.

5. Equivalence through very positive matrices

In this section we give the heart of the proofs of our main results. Throughout this section k denotes a positive integer greater than 1 and all matrices will be $k \times k$. Let $\mathfrak{M}_+$ denote the set of $k \times k$ very positive matrices over $\mathbb{Z}G$ ("very positive" was defined in (2.1)). We say an equivalence (U, V) is a *basic elementary equivalence* if one of U, V is I and the other has the form $E_{ij}(g)$ or $E_{ij}(-g)$.

Definition 5.1. An equivalence $(U, V) : B \to B'$ is a *positive equivalence through* $\mathfrak{M}_+$ if it can be given as a composition of basic elementary equivalences over $\mathbb{Z}G$,

$$B = B_0 \to B_1 \to B_2 \cdots \to B_n = B' ,$$

such that every B_i is in $\mathfrak{M}_+$.

Lemma 5.2. *Suppose $(U, V) : A - I \to A' - I$ is a positive equivalence through $\mathfrak{M}_+$. Then $(U, V) : I - A \to I - A'$ is a positive equivalence.*

Proof. It suffices to consider the case that (U, V) is a basic elementary equivalence, and this case is clear. □

The lemma explains our interest in the following theorem.

Theorem 5.3. *Suppose U and V are in $E(k, \mathbb{Z}G)$ and $UBV = B'$, with B and B' matrices in $\mathfrak{M}_+$. Suppose also that there are matrices X and Y in $E(k, \mathbb{Z}G)$ such that $XBY = D$, where D has block diagonal form $I_2 \oplus F$.*

Then $(U, V) : B \to B'$ is a positive equivalence through $\mathfrak{M}_+$.

The rest of this section is devoted to the proof of Theorem 5.3, which generalizes the arguments of [7, Sec. 5]. We begin with a definition.

Definition 5.4. A *signed transposition matrix* is the matrix of a transposition, but with one of the off-diagonal 1's replaced by -1. A *signed permutation matrix* is any product of signed transposition matrices.

It is not difficult to verify that the matrix of any even permutation is a signed permutation matrix.

Recall that $\alpha(A)$ is the matrix obtained by applying the augmentation map α to A entrywise.

Lemma 5.5. *Suppose $B \in \mathfrak{M}_+$ and $E = E_{ij}(g)$ or $E = E_{ij}(-g)$ where $g \in G$. Suppose the ith row of $\alpha(EB)$ is not the zero row. Then in $\mathrm{E}(k, \mathbb{Z}G)$ there is a nonnegative matrix Q and a signed permutation matrix S such that $(SE, Q)\colon B \to SEBQ$ is a positive equivalence through $\mathfrak{M}_+$.*

Proof. If $E(i,j) = g$, then let $Q = I = S$. Now, suppose $E(i,j) = -g$. Select l such that $\alpha\Big(B(i,l) - gB(j,l)\Big) \neq 0$, and set $x = B(i,l) - gB(j,l)$, that is $x = (EB)(i,l)$. Let $y = \sum_{h\in G} h \in \mathbb{Z}G$. Then $xy = \sum(\sum x_f)h$, where $x = \sum x_f f$, with all sums over G. Thus all coefficients $(xy)_h$ of xy are the same nonzero number.

Case I: xy $\gg$ 0. Here we may repeatedly add y times column l of B to the other columns, until we have a matrix B' with $B'(i,m) \gg B'(j,m)$ for all $m = 1, \dots, k$. This B' is BQ for some Q which is a product of nonnegative basic elementary matrices, and $(E,Q)\colon B \to EBQ$ is the composition of positive equivalences through $\mathfrak{M}_+$, $(I,Q)\colon B \to BQ$ followed by $(E,I)\colon BQ \to EBQ$. Let $S = I$.

Case II: xy $\ll$ 0. For concreteness of notation, let $(i,j) = (1,2)$. Let M_l denote (in this proof only) row l of a matrix M. We can choose a suitable Q, in the manner of Case I, to obtain Q nonnegative such that $(BQ)_2 \gg (BQ)_1$ and $(gBQ)_2 \gg (BQ)_1$ and $(I,Q)\colon B \to BQ$ is a positive equivalence in $\mathfrak{M}_+$. For simplicity of notation, we now write BQ as B and we restrict what we write to rows 1 and 2, e.g.

$$E = \begin{pmatrix} 1 & -g \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} .$$

Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} (SE)B &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & g \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} B_2 \\ gB_2 - B_1 \end{pmatrix} \gg 0 . \end{aligned}$$

Write SE as the product

$$SE = E_1E_2E_3E_4 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} .$$

Write the equivalence $SE : B \to SEB$ as the composition of left multiplications by E_1, E_2, E_3, E_4:

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \to \begin{pmatrix} B_1 \\ B_2 - B_1 \end{pmatrix} \to \begin{pmatrix} B_2 \\ B_2 - B_1 \end{pmatrix} \to \begin{pmatrix} B_2 \\ B_2 - B_1 + gB_2 \end{pmatrix} \to \begin{pmatrix} B_2 \\ gB_2 - B_1 \end{pmatrix} .$$

This finishes the proof. □

Lemma 5.6. *Suppose B is a $k \times k$ matrix over $\mathbb{Z}G$ and $\alpha(B)$ has rank at least 2. Suppose $U \in \mathrm{E}(k, \mathbb{Z}G)$, and no row of $\alpha(B)$ or $\alpha(UB)$ is the zero row. Then U is the product of basic elementary matrices, $U = E_n \cdots E_1$, such that for $1 \leq j \leq n$ the matrix $\alpha(E_j E_{j-1} \cdots E_1 B)$ does not have a zero row.*

Proof. Without loss of generality, assume $U \neq I$. The proof is clear for $k = 2$, since $\alpha(B)$ will have full rank. Let $k \geq 3$. (The reader may wish to work through the proof for $k = 3$ on a first reading.)

Let $\mathcal{E}(i)$ denote the set of $\mathbb{Z}G$ matrices which equal I both on the diagonal and outside of row i. Let $\mathcal{U}$ be the set of factorizations $U = U_n \cdots U_1$ such that for $1 \leq h \leq n$, the matrix U_h is not the identity and there is an index i_h such that $U_h \in \mathcal{E}(i_h)$. Given such a factorization $U = U_n \cdots U_1$, let

$$z = \#\{h \colon 1 \leq h \leq n \text{ and row } i_h \text{ of } \alpha(U_h \cdots U_1 B) \text{ is the zero row}\}.$$

Step 1. We will produce an element of $\mathcal{U}$ for which $z = 0$.

By induction, it suffices to begin with a factorization $U = U_n \cdots U_1$ from $\mathcal{U}$ for which $z > 0$, and produce another factorization from $\mathcal{U}$ with reduced z. Pick s minimal such that row i_s of $\alpha(U_s \cdots U_1 B)$ is zero, and let t be minimal such that $t > s$ and $i_t = i_s$. (This t exists because row i_s of $\alpha(UB)$ is nonzero.) We will change the factorization by replacing the subword $U_t \cdots U_s$ with a suitable word $U'_T \cdots U'_s$, to be defined recursively; T will either be t or $t-1$.

First pick $j_s \neq i_s$ such that row j_s of $\alpha(U_{s-1} \cdots U_1 B)$ is nonzero ($U_{s-1} \cdots U_1 B$ just denotes B in the case that $s = 1$). Choose F_s an elementary matrix which acts to add a multiple of row j_s to row i_s, such that (to avoid re-indexing) $F_s^{-1} U_s \neq I$. Define $U'_s = F_s^{-1} U_s \in \mathcal{E}(i_s)$. Now $U_t \cdots U_s = U_t \cdots U_{s+1} F_s U'_s$ and row i_s of $\alpha(U'_s U_{s-1} \cdots U_1 B)$ is not zero.

Now we give the recursive step. Suppose $s < m = r + 1 \leq t$ and we have produced $U_t \cdots U_{r+1} F_{m-1} U'_r \cdots U'_s = U_t \cdots U_s$ (and consequently, $F_{m-1} U'_r \cdots U'_s = U_{m-1} \cdots U_s$) such that there is a nonzero integer c_{m-1} and an index $j_{m-1} \neq i_s$, such that $F_{m-1}(i_s, j_{m-1}) = c_{m-1}$ and otherwise $F_{m-1} = I$. We will replace $U_m F_{m-1}$ with new terms. There are three cases.

Case 1: $\mathbf{m < t}$ and $\mathbf{j_{m-1} \neq i_m}$. Set $F_m = F_{m-1}$ and $U'_{r+1} = F_m^{-1} U_m F_m$. For example, if $k = 3$ and $(i_s, i_m, j_{m-1}) = (1, 2, 3)$, then we would have for some a, b, c that

$$U'_{r+1} = F_m^{-1} U_m F_m = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ a & 1 & ac+b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & ac+b \\ 0 & 0 & 1 \end{pmatrix} .$$

Now $U'_{r+1} \in \mathcal{E}(i_m)$, and $F_m U'_{r+1} = U_m F_{m-1}$, and row i_m of $U_m \cdots U_1 B$ equals row i_m of $U'_{r+1} U'_r \cdots U'_s U_{s-1} \cdots U_1 B$.

Case 2: $\mathbf{m < t}$ and $\mathbf{j_{m-1} = i_m}$. Choose an index j_m such that $j_m \notin \{i_m, i_s\}$ and row j_m of $\alpha(U'_r \cdots U'_s U_{s-1} \cdots U_1 B)$ is not zero. This is possible because rows i_s and j_{m-1} of $\alpha(U'_r \cdots U'_s U_{s-1} \cdots U_1 B)$ are linearly dependent, (since row i_s of $F_m U'_r \cdots U'_s U_{s-1} \cdots U_1 B$ equals row i_s of $U_m \cdots U_1 B$ which is the zero row under α) and $\operatorname{rank}(\alpha(B)) \geq 2$. Pick F_m with $F_m(i_s, j_m) = 1$ and otherwise $F_m = I$. Set $U'_{r+1} = F_m^{-1} F_{m-1}$ and $U'_{r+2} = F_m^{-1} U_m F_m$. Now

- $F_m U'_{r+2} U'_{r+1} = F_m (F_m^{-1} U_m F_m)(F_m^{-1} F_{m-1}) = U_m F_{m-1}$,
- $U'_{r+1} \in \mathcal{E}(i_s)$ and row i_s of $\alpha(U'_{r+1} \cdots U'_s U_{s-1} \cdots U_1 B)$ is not zero,
- $U'_{r+2} \in \mathcal{E}(i_m)$ and row i_m of $U'_{r+2} \cdots U'_s U_{s-1} \cdots U_1 B$ equals row i_m of $U_m \cdots U_1 B$.

Case 3: $\mathbf{m = t}$. If $U_t F_{t-1} \neq I$, then set $U'_T = U'_{r+1} = U_t F_{t-1} \in \mathcal{E}(i_s)$: row i_s is the same in the matrices $U'_T \cdots U'_s U_{s-1} \cdots U_1 B$ and $U_m \cdots U_1 B$. If $U_t F_{t-1} = I$, then simply delete $U_t F_{t-1}$, so $U'_T = U'_r$.

The new factorization has z reduced. This concludes Step 1.

Step 2. Suppose we have the factorization from $\mathcal{U}$ with $z = 0$, $U = U_n \cdots U_1$, with $U_h \in \mathcal{E}(i_h)$. For $1 \leq h \leq n$, we will replace U_h with a suitable product of elementary matrices in $\mathcal{E}(i_h)$. The argument will be clear from the case $h = 1$. For notational simplicity, suppose $i_1 = 1$. Write U_1 as a product $U_1 = E_{k_1} \cdots E_1$ of basic elementary matrices which agree with I outside row 1. Now, choose a row $l > 1$ of B such that row l of $\alpha(B)$ is not a rational multiple of row 1 of $\alpha(U_1 B)$ (such a row l exists because $\operatorname{rank}(\alpha(B)) > 1$). Let E_0 be the elementary matrix which adds row l to row 1: if $s > 0$, then $\alpha((E_0)^s B)$ has row 1 not zero. Choose a nonnegative integer m large enough that for $1 \leq j \leq k_1$, row 1 of $\alpha([E_j \cdots E_1 (E_0)^m] B)$ is nonzero. Then for $0 \leq s \leq m$,

$$\begin{aligned} [E_0^{-s}][E_k \cdots E_1 (E_0)^m] B &= [E_0^{m-s}][E_k \cdots E_1] B \\ &= [E_0^{m-s}] U_1 B \end{aligned}$$

and therefore row 1 of $\alpha([E_0^{-s}][E_{k_1}\cdots E_1(E_0)^m]B)$ cannot be zero. (Since each E_i, for $i=0,\dots,k_1$, affects only row 1, they all commute with each other.) Thus the factorization $U_1=(E_0)^{-m}E_{k_1}\cdots E_1(E_0)^m$ has the required properties. □

Lemma 5.7. *Suppose B and B' are in $\mathfrak{M}_+$; $\alpha(B)$ and $\alpha(B')$ have rank ≥ 2; U and W are in $\mathrm{E}(k,\mathbb{Z}G)$; the matrix $\alpha(UB)$ has at least one strictly positive entry; and $UB=B'W$. Then the equivalence $(U,W^{-1})\colon B\to B'$ is a positive equivalence through $\mathfrak{M}_+$.*

Proof. We divide the proof into four steps.

Step 1: Reduction to the case $\alpha(UB)$ has all entries positive. Consider an entry $\alpha\Big((UB)(i,j)\Big)>0$. We can repeatedly add column j to other columns until row i of $\alpha(UB)$ has all entries strictly positive. This corresponds to multiplying from the right by a nonnegative matrix Q in $\mathrm{E}(k,\mathbb{Z})\subset E(k,\mathbb{Z}G)$, giving $UBQ=B'WQ$. Then we can repeatedly add row i of UBQ to other rows until all entries of $\alpha(UBQ)$ are positive. This corresponds to multiplying from the left by a matrix P in $\mathrm{E}(k,\mathbb{Z})$, resulting in a matrix $(PU)(BQ)=(PB')(WQ)$ whose augmentation has all entries positive. Also, there are positive equivalences in $\mathfrak{M}_+$ given by

$$(I,Q)\colon B\to BQ\ ,\qquad (P,I)\colon B'\to PB'\ .$$

Therefore, after replacing (U,B,B',W) with (PU,BQ,PB',WQ), we may assume without loss of generality that $\alpha(UB)$ has all entries positive.

Step 2: Factoring U and B $\to$ SUBQ through $\mathfrak{M}_+$. By Lemma 5.6, we can write U as a product of basic elementary matrices, $U=E_l\cdots E_1$, such that for $1\leq j\leq l$, the matrix $\alpha(E_j\cdots E_1B)$ has no zero row. By Lemma 5.5 and Step 1, given the pair (E_1,B), there is a nonnegative Q_1 in $\mathrm{E}(k,\mathbb{Z}G)$ and a signed permutation matrix S_1 such that

$$(S_1E_1,Q_1)\colon B\to S_1E_1BQ_1$$

is a positive equivalence in $\mathfrak{M}_+$. We observe that

$$UBQ_1=S_1^{-1}[S_1E_lS_1^{-1}]\cdots[S_1E_2S_1^{-1}][S_1E_1]BQ_1\ .$$

Now, for $2\leq j\leq l$, the matrix $S_1E_jS_1^{-1}$ is again a basic elementary matrix E'_j, and the matrix $\alpha(E'_j\cdots E'_2(S_1E_1BQ_1))$ has no zero rows.

Again using Lemma 5.5, for the pair $([S_1E_2S_1^{-1}],[S_1E_1BQ_1])$ choose a signed permutation matrix S_2 and nonnegative Q_2 producing a positive equivalence in $\mathfrak{M}_+$

$$(S_2[S_1E_2S_1^{-1}],Q_2)\colon S_1E_1BQ_1\to S_2[S_1E_2S_1^{-1}]S_1E_1BQ_1Q_2$$

so that we get a positive equivalence in $\mathfrak{M}_+$

$$([S_2S_1E_2S_1^{-1}][S_1E_1], Q_1Q_2) \colon B \to [S_2S_1E_2E_1BQ_1Q_2]$$

and we observe that

$$UBQ_1Q_2 = S_1^{-1}S_2^{-1}[S_2S_1E_lS_1^{-1}S_2^{-1}]\cdots[S_2S_1E_3S_1^{-1}S_2^{-1}][S_2S_1E_2S_1^{-1}][S_1E_1]BQ_1Q_2 .$$

Continue this, to obtain a signed permutation matrix $S = S_l \cdots S_1$ and nonnegative $Q = Q_1 \cdots Q_l$ such that

$$\begin{aligned} UBQ &= S^{-1}[S_l \cdots S_1E_lS_1^{-1} \cdots S_{l-1}^{-1}]\cdots[S_2S_1E_2S_1^{-1}][S_1E_1]BQ \\ &= S^{-1}(SUBQ) \end{aligned}$$

and $(SU, Q) \colon B \to SUBQ$ is a positive equivalence in $\mathfrak{M}_+$.

Step 3: Realizing the permutation. We continue from Step 2. It remains to show that

$$(S, I) \colon UBQ \to SUBQ$$

is a positive equivalence in $\mathfrak{M}_+$. Since S is a product of signed transposition matrices, it may be described as a permutation matrix in which some rows have been multiplied by -1. Since UBQ and $SUBQ$ are strictly positive, it must be that S is a permutation matrix. Also, $\det(S) = 1$, so if $S \neq I$ then S is the matrix of a permutation which is a product of 3-cycles. So it is enough to realize the positive equivalence in $\mathfrak{M}_+$ in the case that S is the matrix of a 3-cycle. For this we write the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

as the following product $C_0C_1 \cdots C_5$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} .$$

For $0 \leq i \leq 5$, the matrix $C_iC_{i+1} \cdots C_5$ is nonnegative. Therefore the equivalence $(C, I) \colon A \to CA$ is a positive equivalence through $\mathfrak{M}_+$ whenever $A \in \mathfrak{M}_+$.

Step 4. Conclusion. We now have several positive equivalences through $\mathfrak{M}_+$, namely $(SU, Q) : B \to SUBQ$, $(S^{-1}, I) : SUBQ \to UBQ$, and $(I, Q^{-1}) : UBQ \to UB$. By composition, (U, I) is a positive equivalence through $\mathfrak{M}_+$ from B to $UB = B'W$. By a similar argument (invoking corollaries to Lemmas 5.5 and 5.6 for multiplications on the right), we can show (I, W) is a positive equivalence through $\mathfrak{M}_+$ from B to $B'W$. This proves Lemma 5.7. □

Proof of Theorem 5.3. We will use Lemma 5.7 twice: first to give a positive equivalence from B to itself, and then to give another from B to B'. The inverse of the first followed by the second will equal (U, V) and thus establish that (U, V) is a positive equivalence.

Notation: For a 2×2 matrix H and $m \in \mathbb{N}$ let $L = L_m(H) = \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix} H \oplus I_{k-2}$. For a matrix Q let $Q\{12; *\}$ denote the submatrix consisting of the first two rows of Q.

By assumption, there are matrices X and Y in $E(k, \mathbb{Z}G)$ such that $XBY = D$, where D has block diagonal form $I_2 \oplus F$.

Step 1. We will show that for a suitable 2×2 matrix H and integer m large enough the self equivalence $(X^{-1}LX, YL^{-1}Y^{-1}) : B \to B$ is a positive equivalence. The matrix $\alpha(XBY)\{12; *\} = \alpha(D)\{12; *\}$ has rank two, so $\alpha(XB)\{12; *\}$ has rank two, and thus there exists an $H \in SL(2, \mathbb{Z})$ such that the first row R of $H[\alpha(XB)\{12; *\}]$ has both a positive and a negative entry.

Let C be the first column of $\alpha(X^{-1}) = \alpha(X)^{-1}$. Since C is not the zero vector the $k \times k$ matrix CR has a positive and a negative entry.

Now, if m is sufficiently large, then the corresponding entries of $\alpha(X^{-1}LXB)$ and CR will have the same sign provided the corresponding entry of CR is not zero.

We now apply Lemma 5.7 to see that $(X^{-1}LX, YL^{-1}Y^{-1})$ is a positive equivalence from B to itself.

Step 2. For large enough m the entries of $\alpha(UX^{-1}LXB)$ agree in sign with the corresponding nonzero entries of $\alpha(U)CR$. Since $\alpha(U)$ is nonsingular, the matrix $\alpha(U)CR$ is nonzero and so contains positive and negative entries, because R does. Thus, by Lemma 5.7 $(UX^{-1}LX, YL^{-1}Y^{-1}V)$ is a positive equivalence form B to B'. This concludes the proof. □

6. The main results

Given an $n \times n$ matrix A, we define $(I - A)_\infty$ to be the $\mathbb{N} \times \mathbb{N}$ matrix equal to $I - A$ in its $n \times n$ upper left-hand corner and equal to the infinite identity outside this block. The next theorem is our central result.

Theorem 6.1. *Let G be a finite group, and let A and B be nontrivial essentially irreducible matrices over $\mathbb{Z}_+G$ such that $W(A) = W(B) = G$. If $(U, V) : (I - A)_\infty \to (I - B)_\infty$ is an $E(\mathbb{Z}G)$-equivalence, then it is a positive $\mathbb{Z}G$ equivalence.*

Proof. First, we may assume that that A and B have a common size k with only zero entries outside the upper left $k-2 \times k-2$ corner (expanding a matrix A to a larger matrix with zero entries does not affect $(I-A)_\infty$), and consequently the 2×2 identity matrix is a summand of $I-A$ and of $I-B$. By Lemma 6.6 (which we defer to the end of this section), after replacing $I-A$ and $I-B$ with matrices positively equivalent over $\mathbb{Z}G$, we may assume that $A-I$ is very positive and likewise that $B-I \gg 0$. By Lemma 5.2, $(U,V): I-A \to I-B$ is a positive equivalence if $(U,V): A-I \to B-I$ is a positive equivalence through $\mathfrak{M}_+$ (Definition 5.1). By Theorem 5.3, $(U,V): A-I \to B-I$ is indeed a positive equivalence through $\mathfrak{M}_+$. □

Remark 6.2. Note, in Theorem 6.1 we not only showed a positive equivalence exists, in addition we showed every equivalence is a positive equivalence. In the case G is trivial, this additional information proves [7, Sec. 7] surjectivity of a certain homomorphism to $\mathrm{Aut}(\mathrm{cok}(I-A))$ from the mapping class group of the mapping torus of an irreducible nontrivial SFT S_A. (For this homomorphism, the action of a basic flow equivalence is multiplication by the corresponding basic elementary matrix.) In the case G is nontrivial, our map goes from an equivariant mapping class group to the $\mathbb{Z}G$ module $\mathrm{cok}(I-A)$, and from Theorem 6.1 we similarly know the range in $\mathrm{Aut}(\mathrm{cok}(I-A))$ is the set of automorphisms induced by $\mathrm{E}(\mathbb{Z}G)$ self equivalences of $(I-A)$.

Remark 6.3. Suppose in Theorem 6.1 that $(I-A)_\infty$, $(I-B)_\infty$, U and V equal I outside their upper left $n \times n$ corners. Then the proof of Theorem 6.1 shows that the factorization of (U,V) into basic positive equivalences can be achieved using only matrices which equal I outside their upper left $(n+2) \times (n+2)$ corners.

Theorem 6.4 (Classification Theorem). *Let G be a finite group, and let A and B be essentially irreducible nontrivial matrices over $\mathbb{Z}_+G$. For S_A and S_B to be G-flow equivalent, it is necessary that $W(A) = W(B)$. Now suppose $W(A) = W(B)$ and H is a group in this weight class. Let $\overline{A}$ and $\overline{B}$ be matrices over $\mathbb{Z}H$ which are positively $\mathbb{Z}G$ equivalent to A and B, respectively. ($\overline{A}$ and $\overline{B}$ exist by Proposition 4.4). Then the following are equivalent:*

1. *S_A and S_B are G-flow equivalent.*
2. *There exists $\gamma \in G$ such that $\gamma H \gamma^{-1} = H$ and there is an $E(\mathbb{Z}H)$ equivalence from $(I-\overline{A})_\infty$ to $(I-\gamma\overline{B}\gamma^{-1})_\infty$.*

Proof. The necessity of $W(A) = W(B)$ was Proposition 4.2. The implication (1) $\implies$ (2) follows from Theorems 3.3 and 4.7. The implication (2) $\implies$ (1) follows from Theorem 6.1. □

Theorem 6.4 reduces the G-flow equivalence classification to the problem of classifying matrices up to $E(\mathbb{Z}G)$ equivalence, which we discuss in Sections 8 and 9. The positivity constraints on the matrices $I - A$ we study does not lead to a smaller $E(\mathbb{Z}G)$ equivalence problem, because for any finitely supported B over $\mathbb{Z}G$ there is an $E(\mathbb{Z}G)$ equivalence from $I - B$ to a matrix $I - A$ where A is essentially irreducible and nontrivial with weight class $\{G\}$ (Proposition 8.8). We extract now one consequence of Theorem 6.4 and the algebra. $\mathrm{SK}_1(\mathbb{Z}G)$ is discussed in Section 8.

Theorem 6.5. *Suppose G is a finite abelian group and A is a square irreducible matrix over $\mathbb{Z}_+G$ such that $I-A$ is injective (i.e. $\det(I-A)$ is not a zero divisor in $\mathbb{Z}G$). Then the following hold.*

1. *The number of distinct G-flow equivalence classes defined by matrices B such that $\det(I - B) = \det(I - A)$ is finite.*
2. *If $\mathrm{SK}_1(\mathbb{Z}G)$ is trivial and $\det(I - B) = \det(I - A)$, then A and B determine the same G-flow equivalence class if and only if they have the same weight class and the $\mathbb{Z}G$ modules $\mathrm{cok}(I - A)$ and $\mathrm{cok}(I - B)$ are isomorphic.*

Proof. (1) When $I - A$ is injective, $\mathrm{cok}(I - A)$ is finite. Therefore (crudely) only finitely many isomorphism classes of cokernel module are possible. The conclusion now follows from the Classification Theorem 6.4, Corollary 9.9, and the finiteness of $\mathrm{SK}_1(\mathbb{Z}G)$ [27]. (2) This follows from the Classification Theorem 6.4 and Proposition 9.5. □

We finish this section with the (somewhat tedious) proof for the reduction to very positive matrices.

Lemma 6.6 (Very Positive Presentation). *Let A be an essentially irreducible $m \times m$ matrix over $\mathbb{Z}_+G$, $m \geq 2$, such that $W_i(A) = G$ for $1 \leq i \leq m$ and $\alpha(A)$ has more than one cycle. Then there is a positive equivalence over $\mathbb{Z}G$ from $I - A$ to a matrix $I - B$ such that $\pi_g((B - I)(i, j)) > 0$ for every g in G and every entry index (i, j), i.e., $B - I \gg 0$.*

Proof. We sequentially adjust the matrix A without renaming it each time. Let A be $n \times n$, where n changes as A does. We relabel so that the irreducible core submatrix is in the upper left-hand corner.

Step 0: Diagonalizing cycles. First we describe a certain cycle-shortening construction. Let $i = i_0, i_1, \ldots, i_k = i$ be a finite sequence of indices corresponding to a cycle (cyclic path of edges) $\mathcal{C} = e_1e_2 \cdots e_k$ with weight g, where e_t runs from i_{t-1} to i_t and has weight g_t (so $g_1g_2 \cdots g_k = g$). We require that some intermediate index i_r is not i.

We will construct a positive equivalence $(I-A) \to (I-B)$ for which we claim $A(i,i) \geq B(i,i) + g$ and also $A(t,t) \geq B(t,t)$ for all t. The latter part of this claim will be clear because the construction will be a composition of forward basic positive equivalences.

First suppose the path length k satisfies $k > 2$. Let e denote the edge e_r.

(1) If $r < k-1$, then produce a new matrix $I - A'$ by applying the basic positive equivalence $(E_{i_r,i_{r+1}}(g_{r+1}), I)$. The A-cycle $\mathcal{C}$ gives rise to an A'-cycle $\mathcal{C}'$, which looks like $\mathcal{C}$ except that any $e_s e_{s+1}$ for which $e_s = e$ is replaced by an edge from $i_{s-1} = i_r$ to i_{s+1} with weight $g_s g_{s+1}$. The cycle $\mathcal{C}'$ is still a cycle from i to i, it still passes through an index other than i, and it has the same weight as $\mathcal{C}$.

(2) If we do not have $r < k-1$, then $r = k-1 \geq 2$, and we may similarly apply the basic positive equivalence $(I, E_{i_{r-1},i_r}(g_r))$ to shorten the cycle.

Repeating the moves above, we reach the case of path length $k = 2$. Apply the basic positive equivalence $(E_{i_0,i_1}(g_1), I)$. We have shortened the cycle to a cycle from i to i with the same weight. This completes the proof of the claim.

Given A, let A'' denote the irreducible core of A, its maximal irreducible principal submatrix.

Step 1: Nonzero trace. If A has zero trace, then diagonalize a cycle as in the previous step to achieve nonzero trace.

Step 2: Trim. Suppose row j of A is zero and some entry $A(i,j) \neq 0$. Let $A(i,j) = g_1 + \cdots + g_k$ and set $E = E_{ij}(g_1 + \cdots + g_k)$, so $E = E_{ij}(g_1) \cdots E_{ij}(g_k)$. Then $E(I-A) = (I-B)$ where $B = A$ except for the entry $B(i,j) = 0$, and $(E,I) : (I-A) \to (I-B)$ is a positive equivalence. After if necessary applying such positive equivalences, and analogous equivalences (E,I), we may assume that $A(i,j) = 0$ unless both i and j are indices for A''.

Step 3: Core at least 2×2. Suppose the irreducible core A'' is 1×1, say $A'' = (A(1,1))$. Because there is more than one cycle, we can write $A(1,1) = g + b$ where $g \in G$ and $0 \neq b \in \mathbb{Z}_+G$. Subtract g times row 2 of $(I-A)$ from row 1; then subtract column 2 from column 1. The result of these two positive equivalences is a matrix with $\begin{pmatrix} b & g \\ 1 & 0 \end{pmatrix}$ as the irreducible core.

Step 4: Very positive core diagonal. At this point we have A'' at least 2×2 in size and with an index i such that $A''(i,i) \neq 0$. Pick an index $j \neq i$ for A''. Every element of G is the weight of some cycle from i to i, so it follows by irreducibility of A'' that every element of G is the weight of some cycle from j to j which runs through i. This statement remains true after we diagonalize a cycle from j to j as in Step 0, because i and j must remain in the irreducible core, because the ii and jj entries are nonzero and do not decrease. Consequently we can diagonalize cycles until $A''(t,t) \gg 0$ for every diagonal entry of A''.

Step 5: A=core. If $1 \le t \le m$ and t is not an index for A'': pick an index s for A''; subtract row t of A from row s; then subtract column t from column s. This positive equivalence $I - A \to I - C$ produces C whose irreducible core has an index set enlarged by $\{t\}$. Apply Step 4 again to the tt and ss entries as needed to get all diagonal entries of $C'' \gg 0$. Repeat until $A = A''$ with very positive diagonal.

Step 6: Very positive A. Suppose $i \neq j$, $g \in G$ and $A''(i,j) - g \geq 0$. Following Step 3, $(E_{ij}(g), I) : (I - A) \to (I - C)$ is a basic positive equivalence; $C(i,j) \gg 0$; and $C \geq A$. So, we may apply basic positive equivalences to arrive at A'' on an unchanged index set with $A'' \gg 0$.

□

7. Twistwise flow equivalence

As noted in the Introduction, when $G = \mathbb{Z}/2$ the equivalence relation of G-flow equivalence is called twistwise flow equivalence. Let t denote the generator of $\mathbb{Z}/2$, so $t^2 = 1$. We write $A(t)$ for a matrix over $\mathbb{Z}_+G$ and let $A(1)$ and $A(-1)$ denote the matrices over $\mathbb{Z}$ obtained from setting t to 1 and -1.

Suppose $A(t)$ is given. We define the *ribbon set* R to be a flow on a fiber bundle with fiber $(-1,1)$ over the one-dimension suspension flow (B,ϕ) of $A(1)$, associated to $A(t)$ as follows. We can pass to a higher block presentation so that we may assume $A(t)$ has only 1's, 0's and t's as entries. Then there is an oriented Markov partition $\mathcal{D} = \{D_1, \dots, D_k\}$ on a cross section of F that induces $A(1)$. Let $B_{ij} = \{x \in B \mid x \in \phi_t(y), y \in D_i, \text{ and } \phi_{\tau(y)}(y) \in D_j \text{ for } 0 \le t \le \tau(y)\}$, where τ is the first return time map for $\mathcal{D}$. Let $R_{ij} = B_{ij} \times (-1,1)$. Attach the nonempty R_{ij}'s so that the core is F and the gluing of the edge fibers (end points of the B_{ij} crossed with the fiber $(-1,1)$) are the identity if $A_{ij} = 1$ and multiplication by -1 if $A_{ij} = t$. Call this set R. We place a flow on R that agrees with ϕ on the core B and so that all other orbits are forward asymptotic to B and exit R in reverse time. Two matrices are twistwise flow equivalent if and only if they have topologically equivalent ribbon sets. Ribbon sets are realized naturally as stable bundles of basic sets of Smale flows [37].

We now define the invariants of twistwise flow equivalence established in [37, 38, 39]. Let $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $A(t)$ is $k \times k$ define $A(T)$ to be the $2k \times 2k$ matrix over $\mathbb{Z}_+$ obtained by converting each $a + bt$ entry of $A(t)$ to the block $aI + bT = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. The determinants of the three matrices $I - A(1), I - A(-1), I - A(T)$ were established as invariants of twistwise flow equivalence, as were the isomorphism classes of their cokernel groups. (We remark that the group $\text{cok}(I - A(T))$ is isomorphic to the group obtained from the $\mathbb{Z}\mathbb{Z}/2$ module $\text{cok}(I - A(t))$ by forgetting the module structure.) The orientability of the ribbon set was determined by checking the diagonal entries of $A^i(t)$ for $i = 1, \dots, k$ for t's. The ribbon set is orientable if t appears in none of these entries, and is nonorientable otherwise. Orientability is an invariant independent of the others; in the setting of this paper, orientability is triviality of the weight class.

From the results of this paper, it is easy to see that the previously known invariants were not complete. For example, none of those invariants distinguish a matrix and its transpose, so Example 8.6 and Proposition 8.8 can be used to produce a pair which agree on the previously known invariants but are not twistwise flow equivalent. The methods and results of this paper are also useful for establishing twistwise flow equivalence when it holds.

Example 7.1. Let $A = \begin{pmatrix} 0 & t \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$, and $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $E(I - A) = I - B$, so by Theorem 6.4, A and B are twistwise flow equivalent. This answers a question in [39, page 9]. Here E does not give a basic positive equivalence. However, following the philosophy of the proof of Theorem 6.1, if we let $Q_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ then (I, Q_1), (I, Q_2), (E, I), (I, Q_2^{-1}), (I, Q_1^{-1}) is a sequence of basic positive equivalences taking $I - A$ to $I - B$. $\square$

In [38] Table 2 lists some 3×3 matrices. Several pairs have identical invariants: counting down, 1 & 5, 2 & 7, 4 & 13, and 17 & 18. It was unknown if they were twistwise flow equivalent. We can now report that simple hand calculations show that the matrices corresponding to these pairs are twistwise flow equivalent. Section 8 includes some additional results on twistwise flow equivalence.

8. E($\mathbb{Z}G$)-EQUIVALENCE

In this section, we'll give some general background on E($\mathbb{Z}G$)-equivalence, with some results and examples for the case $G = \mathbb{Z}/2$. Recall our convention (2.1) that in this paper a ring means a ring with 1.

Let $\mathcal{R}$ be a ring. $\mathrm{E}(n, \mathcal{R})$ denotes the group of $n \times n$ elementary matrices over $\mathcal{R}$, the subgroup of $\mathrm{GL}(n, \mathcal{R})$ generated by basic elementary matrices. Similarly, we let $\mathrm{E}(\mathcal{R})$ denote the subgroup of $\mathrm{GL}(\mathcal{R})$ generated by the basic elementary matrices. The group $\mathrm{GL}(\mathcal{R})/\mathrm{E}(\mathcal{R})$ is the abelian group $\mathrm{K}_1(\mathcal{R})$ studied in algebraic K-theory [34]. When $\mathcal{R}$ is commutative (so $\mathrm{SL}(\mathcal{R})$ can be defined as the group of invertible matrices with determinant 1), the quotient group $\mathrm{SL}(\mathcal{R})/\mathrm{E}(\mathcal{R})$ is denoted $\mathrm{SK}_1(\mathcal{R})$. If G is a finite group, then $\mathrm{SK}_1(\mathbb{Z}G)$ denotes the kernel of the map $\mathrm{K}_1(\mathbb{Z}G) \to \mathrm{K}_1(\mathbb{Q}G)$ (the definitions agree if G is abelian). If G is finite, then $\mathrm{SK}_1(\mathbb{Z}G)$ is finite. If $\mathcal{R}$ is $\mathbb{Z}$, or $\mathcal{R} = \mathbb{Z}G$ with $G = \mathbb{Z}/2$, then every element of $\mathrm{SL}(\mathcal{R})$ is a product of basic elementary matrices, and $\mathrm{SK}_1(\mathcal{R})$ is trivial. In general, though, $\mathrm{SK}_1(\mathbb{Z}G)$ is not trivial when G is a finite group. For example, $\mathrm{SK}_1(\mathbb{Z}G)$ is not trivial if $G = (\mathbb{Z}/p)^n$ with p an odd prime and $n \geq 3$. See [27] for the characterization of the finite abelian G with trivial $\mathrm{SK}_1(\mathbb{Z}G)$ and other background on $\mathrm{SK}_1(\mathbb{Z}G)$.

We will say an $n \times n$ matrix D over $\mathbb{Z}$ is a *Smith normal form* if D is a diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ satisfying the following conditions: d_{i+1} divides d_i whenever $1 < i \leq n$ and $d_{i+1} \neq 0$; $d_{i+1} = 0$ implies $d_i = 0$; and $d_i \geq 0$ if $i > 1$.(Our notation here is slightly unconventional.) It is well known that any $n \times n$ matrix over $\mathbb{Z}$ is $\mathrm{SL}(n, \mathbb{Z})$ equivalent to a unique Smith normal form. Because $\mathrm{E}(n, \mathbb{Z}) = \mathrm{SL}(n, \mathbb{Z})$, the Smith normal form also gives a complete invariant of $\mathrm{E}(n, \mathbb{Z})$-equivalence.

This classification extends to $\mathbb{N} \times \mathbb{N}$ matrices. We will say a Smith normal form is a matrix whose upper left corner is a finite Smith normal form and which otherwise equals the infinite identity matrix. If A is an $n \times n$ square matrix over $\mathbb{Z}$, then $(I - A)_\infty$ is $E(\mathbb{Z})$-equivalent to a unique Smith normal form, and this form is the matrix whose upper left corner is the Smith normal form of $I_n - A$, and which equals I elsewhere. (It is to make this last statement that we reversed the usual order of diagonal elements in our definition of Smith normal form.) So in the $\mathbb{Z}$ case, we have everything: a good normal form; a good algorithm for generating it; a decision procedure for determining whether two matrices are equivalent; an equivalence classification given by the classification of the cokernel group ($\mathbb{Z}$-module) with a little more information (sign of the determinant) to reflect the refinement of $\mathrm{GL}(\mathbb{Z})$ equivalence by $\mathrm{E}(\mathbb{Z})$ equivalence; and immediate stabilization (i.e., if

A and B are $n \times n$ and $(I-A)_\infty$ and $(I-B)_\infty$ are $E(\mathbb{Z})$ equivalent, then $(I-A)$ and $(I-B)$ are $E(n,\mathbb{Z})$ equivalent).

Given G a finite group, the results of this paper obviously lead one to ask similarly for a classification of matrices over $\mathbb{Z}G$ up to E($\mathbb{Z}G$)-equivalence, when the matrices are $n \times n$, or equal I except in finitely many entries. This very natural algebraic problem is far more difficult than in the $\mathbb{Z}$ case. In particular, there is nothing so nice as the Smith normal form; even for $G = \mathbb{Z}/2$, a matrix might not be equivalent over GL($\mathbb{Z}G$) to any triangular matrix (8.7), or to its transpose (8.6). The problem even of GL($\mathbb{Z}G$) equivalence seems not to have been addressed directly in the algebra literature, although there are powerful results [16] in a more general setting which point the way to substantial progress. In the rest of this section, we make no attempt to address the general problem, but we do give some illustrative concrete results and examples in the case $G = \mathbb{Z}/2$.

From here until Proposition 8.8, $G = \mathbb{Z}/2$. We write elements of $\mathbb{Z}G$ in the form $a + tb$, where a, b are integers and $t^2 = 1$. We will use the well known [34, Sec. 2.4] embedding of $\mathbb{Z}G$ into $\mathbb{Z}^2$, $\delta : a + tb \mapsto (a + b, a - b)$. One easily checks that δ is a ring monomorphism whose image is $\{(c, d) : c \equiv d \bmod 2\}$. If we write a matrix over $\mathbb{Z}G$ in the form $A+tB$ (A and B over $\mathbb{Z}$), then applying δ entrywise gives an embedding of matrix rings (also called δ), $A + tB \mapsto (A + B, A - B)$. Under this embedding, the image of SL($\mathbb{Z}G$) is $\{(C, D) \in \mathrm{SL}(\mathbb{Z}) \times \mathrm{SL}(\mathbb{Z}) : C \equiv D \bmod 2\}$. We will say that a matrix M over $\mathbb{Z}G$ is a Smith normal form if $\delta(M) = (C, D)$ where C and D are Smith normal forms for $\mathbb{Z}$. In this case, M is diagonal over $\mathbb{Z}G$ and its diagonal entries satisfy the divisibility and zero conditions we gave above for the $\mathbb{Z}$ form; the nonnegativity condition is replaced by the corresponding nonnegativity of the image under δ. Clearly, M can be $E(\mathbb{Z}G)$ equivalent to at most one Smith normal form.

Theorem 8.1 (Normal Form). *Let $G = \mathbb{Z}/2$. Let M be an $n \times n$ matrix over $\mathbb{Z}G$. Write $M = A + Bt$ with A and B $n \times n$ matrices over $\mathbb{Z}$. If* $\det(A + B)$ *is odd, then M is* E($\mathbb{Z}G$)*-equivalent to a Smith normal form. This is the form corresponding to (C, D), where C and D are the Smith normal forms for $A + B$ and $A - B$.*

The theorem follows immediately from a more general lemma.

Lemma 8.2. *Let $G = \mathbb{Z}/2$. Let M be an $n \times n$ matrix over $\mathbb{Z}G$ and let $(C, D) = \delta(M)$. Suppose the mod-2 rank of C is k. Then M is* E($n, \mathbb{Z}G$)*-equivalent to a matrix M' such that $\delta(M') = (C', D')$ where the bottom right $k \times k$ corners of C' and D' are Smith normal forms* *(equal to the bottom right corners of the Smith normal forms for C and D) and the other entries in the last k rows and columns are zero.*

Proof. Multiplication of M by a matrix in $\mathrm{E}(n, \mathbb{Z}G)$ corresponds to multiplication of (C, D) from the same side by a pair of matrices in $\mathrm{E}(n, \mathbb{Z}) \times \mathrm{E}(n, \mathbb{Z})$ which are equal mod 2. So, an equivalence $M \to UMV$ corresponds to an $\mathrm{E}(n, \mathbb{Z}) = \mathrm{SL}(n, \mathbb{Z})$ equivalence $(C, D) \to (U_1CV_1, U_2DV_2)$ where $U_1 - U_2$ and $V_1 - V_2$ are zero mod 2. We will act on the given pair (C, D) with such equivalences. Note the condition $C \equiv D$ mod 2 persists under this action.

Let (U_1, V_1) be an $\mathrm{E}(n, \mathbb{Z})$ equivalence putting C into the Smith normal form for $\mathbb{Z}$. Apply this along with $(U_2, V_2) = (U_1, V_1)$. The mod-2 rank assumption tells us that the last k diagonal entries of C are now odd and the other entries of C are even. The same is true of D. From here we will use equivalences with $(U_1, V_1) = (I, V_1)$, to achieve the required form for D without disturbing the form for C. That is, we will act on D with the *even elementary matrices*: matrices in $\mathrm{E}(n, \mathbb{Z})$ equal mod 2 to the identity. In particular we may freely add even multiples of rows and columns to other rows and columns.

We claim that such even elementary operations may be used to put D into a form such that the all entries of the last row and column are zero except for the diagonal entry, which is the gcd of the entries of D. Without loss of generality, we suppose $n > 1$.

Step 1. Consider the bottom row of C, row n. The last entry is odd and the rest are even. Pick j such that $D(n, j) = a$ is a nonzero entry of smallest magnitude in row n. Add even multiples of column j to other columns to produce the condition that every entry in row n lies in the interval $[-|a|, |a|]$. If any nonzero entry b of row n now satisfies $|b| < |a|$, then again add even multiples of columns to others until all entries lie in $[-|b|, |b|]$. Continue until there is some nonzero entry a such that all entries of row n lie in the set $\{-|a|, 0, |a|\}$. This number $|a|$ must be the gcd of the original entries of row n. Consequently $|a|$ is odd. Since the entries of row 1 were never changed mod n, the diagonal entry of row n must be a and the others then must be 0.

Step 2. Apply the Step 1 idea to column n, putting it into the form $[0 \cdots 0a]^t$ (a may have decreased).

If row n is no longer in the form $[0 \cdots 0a]$, then re-apply Step 1. Repeat Steps 1 and 2 as needed until both row 1 and column 1 are zero except for the odd entry on the diagonal. Call this "the process".

If $|a|$ is not the gcd of all the matrix entries, then there is some higher row i containing an element not divisible by a. Add twice row i to row n. Now row n has a gcd smaller than $|a|$. Apply "the process" again. The one nonzero entry in row n or column n, on the diagonal, has decreased in magnitude. Finitely many iterations therefore produce the diagonal entry a such that $|a|$ is the gcd of the matrix entries. Finally, if necessary after multiplying the last row and a higher row both by -1 (this corresponds to multiplying by a determinant 1 matrix which equals I mod 2), we can assume $a > 0$. This finishes the proof of the claim.

Repeat this procedure on successive submatrices until a matrix is produced which satisfies the statement of the theorem. If $k = n$, then at the final step there will not be a "higher row" and there will not be freedom to adjust the sign of the diagonal entry—it must equal the sign of the determinant. □

Remark 8.3. The lemma shows that Theorem 8.1 is true under the weaker assumption that at most one entry of the Smith form for $A+B$ is even, because in this case the algorithm of the lemma produces a matrix which is equivalent to $A-B$ and which must be a Smith normal form.

Corollary 8.4. *Let $G = \mathbb{Z}/2$. If $M = A+tB$ where A and B are square integral matrices with* $\det(A+B)$ *odd, then M is* $\mathrm{E}(\mathbb{Z}G)$*-equivalent to its transpose.*

Remark 8.5. Equivalence to the transpose gives rise to an interpretation of G-flow equivalence to the time-reversed flow as in [15]. Because irreducible matrices over $\mathbb{Z}$ are equivalent to diagonal matrices, Franks could include that the mapping torus flows of irreducible shifts of finite type are flow equivalent to their time-reversed flows. For G-flow equivalence with G nontrivial, this holds in some cases (e.g. Cor. 8.4) but not in general, as the next example shows.

Example 8.6. For $G = \mathbb{Z}/2$, there is a matrix M over $\mathbb{Z}G$ which is not $\mathrm{GL}(\mathbb{Z}G)$-equivalent to its transpose.

Proof. We will give a 2×2 example M. (It is not difficult to verify for this example that $M \oplus I$, where I is the infinite identity matrix, is also not equivalent $\mathrm{GL}(\mathbb{Z}G)$-equivalent to its transpose.) Define M, and consequently $\delta(M) = 2(C, D)$, as follows:

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + t \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

To show M is not equivalent to its transpose, we suppose there are $\mathrm{GL}(\mathbb{Z})$ matrices U_1, U_2, V_1, V_2 such that $U_1 \equiv U_2 \bmod 2$, $V_1 \equiv V_2 \bmod 2$, $U_1 C V_1 = C$ and $U_2 D V_2 = D^t$, and then find a contradiction. First consider the equivalence $C = U_1CV_1$:

$$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}\alpha & \beta\\\gamma & \delta\end{pmatrix} = \begin{pmatrix}a\alpha + 2b\gamma & a\beta + 2b\delta\\c\alpha + 2d\gamma & c\beta + 2d\delta\end{pmatrix}\ .$$

We see that a and α must be odd, and then also that c and β must be even, and then because the determinants of U_1 and V_1 are odd that d and δ must be odd. So we have

$$U_1 = \begin{pmatrix}1 & *\\0 & 1\end{pmatrix} \quad \text{mod } 2 \qquad \text{and} \qquad V_1 = \begin{pmatrix}1 & 0\\ * & 1\end{pmatrix} \quad \text{mod } 2\ ,$$

with $*$ indicating an entry which is not specified mod 2. Consequently, mod 2 we have

$$U_2DV_2 = U_1DV_1 = \begin{pmatrix}1 & *\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix} = \begin{pmatrix}* & 1\\0 & 0\end{pmatrix} \neq D^t\ .$$

This contradiction finishes the proof. □

Example 8.7. Let $G = \mathbb{Z}/2$. There is a matrix M over $\mathbb{Z}G$ such that M is not $\mathrm{GL}(\mathbb{Z}G)$-equivalent to a triangular matrix. In particular, M is not equivalent to a Smith normal form.

Proof. We will give a 2×2 example M. (It is not difficult to verify for this example that $M \oplus I$, where I is the infinite identity matrix, is also not equivalent to a triangular matrix.) Set

$$M = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} + t\begin{pmatrix}1 & -1\\-1 & 0\end{pmatrix}$$

so that $\delta(M) = 2(I, D)$ where $D = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$. Suppose M' is upper triangular and $\mathrm{GL}(\mathbb{Z}G)$-equivalent to M. Then $\delta(M') = 2(C', D')$ for some matrices C', D' over $\mathbb{Z}$ which are upper triangular. Here C' must be $\mathrm{GL}(\mathbb{Z})$-equivalent to I, so its diagonal entries must be ± 1. Let U be a matrix which acts to add a multiple of row 2 to row 1, such that UC' is diagonal. Let W be a diagonal matrix with diagonal entries from $\{1, -1\}$ such that $WUC' = I$. Note WUD' is upper triangular. Replace (C', D') with (WUC', WUD'). At this point we have $2(I, D)$ equivalent to some $2(I, D'')$ where D'' is upper triangular. So, there are $\mathrm{GL}(2, \mathbb{Z})$ matrices $U_1 \equiv U_2$ mod 2 and $V_1 \equiv V_2$ mod 2 such that $U_1(2I)V_1 = 2I$ and $U_2(2D'')V_2 = 2D$. Now V_1 must equal $(U_1)^{-1}$ and therefore mod 2 we have D similar to a triangular matrix. This is impossible because the characteristic polynomial of D considered over the field $\mathbb{Z}/2$ is irreducible. □

We finish with the proposition mentioned in Section 6.

Proposition 8.8. *Suppose G is a finite group, and B is a finitely supported $\mathbb{N} \times \mathbb{N}$ matrix over $\mathbb{Z}G$. Then $I - B$ is $E(\mathbb{Z}G)$ equivalent to some matrix $(I - A)_\infty$ over $\mathbb{Z}_+G$, where A has weight class $\{G\}$ and $A \gg 0$.*

Proof. Suppose B is zero outside its upper left $n \times n$ corner. Let y denote the sum of all elements in G and let m be a positive integer. Subtract my times row $n + 1$ from the rows $1, 2, \dots, n$. Then add column $n + 1$ to the columns $1, 2, \dots, n$. Finally, add row 1 to row $n + 1$. If m is sufficiently large, we get a matrix $I - C$ for which C is zero except in the upper left $(n+1) \times (n+1)$ corner, where every entry of C is greater than y. Let A be the upper left $(n+1) \times (n+1)$ corner of C. □

9. E($\mathbb{Z}G$)-equivalence of injective matrices

Recall, if C is an $n \times n$ matrix, then C_∞ denotes the $\mathbb{N} \times \mathbb{N}$ matrix whose upper left corner is C and which otherwise is equal to the infinite identity matrix. We begin with an easy application of a theorem of Fitting [14]. Recall our convention (2.1) that in this paper a ring means a ring with 1.

Lemma 9.1. *Suppose $\mathcal{R}$ is a ring, and C and D are injective square matrices over $\mathcal{R}$. Then the following are equivalent.*

(1) *There exist $V \in E(\mathcal{R})$ and $U \in \mathrm{GL}(\mathcal{R})$ such that $UC_\infty V = D_\infty$.*
(2) *The $\mathcal{R}$-modules $\mathrm{cok}(C)$ and $\mathrm{cok}(D)$ are isomorphic.*

Proof. We will prove the nontrivial implication, which is (2) $\Longrightarrow$ (1). Let matrices act on row vectors. Suppose C and D are $m \times m$ and $n \times n$, respectively. Let e.g. I_n denote the $n \times n$ identity matrix. Because C and D have isomorphic cokernels, there is an invertible matrix V_1 such that the matrices

$$\begin{pmatrix} C & 0 \\ 0 & I_m \end{pmatrix} V_1 \quad \text{and} \quad \begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix} \tag{9.2}$$

have the same image. For this claim we refer to Warfield's modern (and English) presentation [41, p.1816] of Fitting's result; it is evident from the proof that the matrix V_1 can be chosen from $\mathrm{E}(m + n, \mathcal{R})$.

Because the displayed matrices are injective with equal image, obviously [14, 41] there exists an invertible matrix U_1 such that

$$U_1 \begin{pmatrix} C & 0 \\ 0 & I_m \end{pmatrix} V_1 = \begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix} .$$

Finally, let E be a matrix in $\mathrm{E}(m+n,\mathcal{R})$ such that

$$E^{-1}\begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix} E = \begin{pmatrix} D & 0 \\ 0 & I_n \end{pmatrix}$$

and set $U = (EU_1)_\infty$ and $V = (V_1E^{-1})_\infty$. □

Remark 9.3. [41, p.1823] For a certain finite group G (the generalized quaternion group of order 32), Swan [40, p.57] found an ideal P, not free as a $\mathbb{Z}G$ module, but still with module isomorphisms $\mathbb{Z}G \oplus \mathbb{Z}G \cong P \oplus P \cong \mathbb{Z}G \oplus P$. This yields 2×2 matrices over $\mathbb{Z}G$ with isomorphic cokernels but nonisomorphic kernels. Therefore Lemma 9.1 would be false without the hypothesis of injectivity.

Remark 9.4. An imperfection of Fitting's general result is that the size of the identity summands in (9.2) depends on the matrices C, D. However, if d is a positive integer in the stable range (defined below) of the ring $\mathcal{R}$, then those summands I_m, I_n can be chosen with $m = n = d$, and under some additional conditions on $\mathcal{R}$ (for example if $\mathcal{R}$ is commutative) this bound can be lowered to $d-1$ [41, pp.1822-1823]. When G is a finite group, the Krull dimension (see [26, Ch. 6] for the definition for a not necessarily commutative ring) of the Noetherian ring $\mathbb{Z}G$ is 1 [26, Prop. 6.5.5, p. 211], and consequently 2 is in (and is then easily seen to be the minimum integer in) the stable range of $\mathbb{Z}G$ [26, Thm. 6.7.3, p. 220].

To define stable range, say a row vector $(a_1,\dots,a_n)$ over $\mathcal{R}$ is a *right unimodular row* if there are elements $x_i \in \mathcal{R}$, $1 \le i \le n$, with $\sum_i a_i x_i = 1$. The *stable range* of $\mathcal{R}$ is the set of positive integers d such that for any right unimodular row $(a_1,\dots,a_n)$ with $n > d$, there exist elements $b_i \in \mathcal{R}$, $1 \le i \le n-1$, such that the row $(a_1 + a_n b_1, \dots, a_{n-1} + a_n b_{n-1})$ is again right unimodular.

We pause to isolate for later use a particularly simple statement.

Proposition 9.5. *Suppose $\mathcal{R}$ is a commutative ring; $\mathrm{SK}_1(\mathcal{R})$ is trivial; C and D are finite square matrices over $\mathcal{R}$; and C is injective. Then the following are equivalent.*

1. *There exist U, V in $E(\mathcal{R})$ such that $UC_\infty V = D_\infty$.*
2. $\det(C) = \det(D)$ *and the $\mathcal{R}$-modules* $\mathrm{cok}(C)$ *and* $\mathrm{cok}(D)$ *are isomorphic.*

Proof. We check the nontrivial implication, (2) $\Longrightarrow$ (1). By Lemma 9.1, we have matrices U, V such that $UC_\infty V = D_\infty$ with $V \in \mathrm{E}(\mathcal{R})$. Because $\det(V) = 1$ and $\det(D) = \det(C) \neq 0$, we have also $\det(U) = 1$. Because $\mathrm{SK}_1(\mathcal{R})$ is trivial, it follows that U and V are in $E(\mathcal{R})$. □

Now we want to observe that injective matrices with a given cokernel isomorphism class are classified up to elementary equivalence by a quotient of K_1.

Proposition 9.6. *Let $\mathcal{R}$ be a ring. Let $\mathcal{C}$ be the set of all square injective matrices over $\mathcal{R}$ with cokernel module isomorphic to that of a given square injective matrix over $\mathcal{R}$. Let $\mathcal{E}(\mathcal{C})$ be the partition of $\mathcal{C}$ such that C and D are in the same element of $\mathcal{E}(\mathcal{C})$ if C_∞ and D_∞ are $\mathrm{E}(\mathcal{R})$ equivalent. Then there is a subgroup H of $\mathrm{K}_1(\mathcal{R})$ such that the following hold.*

(1) *For any C and D in $\mathcal{C}$, if (U,V) is a $\mathrm{GL}(\mathcal{R})$ equivalence from C_∞ to D_∞, i.e. $UC_\infty V = D_\infty$, then there exists an elementary equivalence from C_∞ to D_∞ if and only if $[UV] \in H$.*
(2) *For any $C \in \mathcal{C}$, the map $\mathrm{GL}(\mathcal{R}) \to \mathcal{C}$ defined by $U \mapsto UC_\infty$ induces a well defined bijection $(\mathrm{K}_1(\mathcal{R}))/H \to \mathcal{E}(\mathcal{C})$.*
(3) *If $\mathcal{R}$ is commutative, or if $\mathcal{R} = \mathbb{Z}G$ with G finite, then $H \subset \mathrm{SK}_1(\mathcal{R})$*

Proof. We write $C \sim D$ if there is an elementary equivalence from C to D. If U is in $\mathrm{GL}(n,\mathcal{R})$, then it is well known that the matrix $\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}$ is in $\mathrm{E}(2n,\mathcal{R})$, and therefore that for any $n \times n$ matrix C over $\mathcal{R}$

$$\begin{pmatrix} UC & 0 \\ 0 & I \end{pmatrix} \sim \begin{pmatrix} U^{-1} & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} UC & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} \sim \begin{pmatrix} CU & 0 \\ 0 & I \end{pmatrix} .$$

We will use this simple fact repeatedly. From here, we suppress the subscript ∞ and consider all matrices infinite. U and V will denote elements of $\mathrm{GL}(\mathcal{R})$. Note, if $C \sim D$, then $CU \sim DU$ and $UC \sim UD$, and in particular $U(CV) \sim U(VC)$. Also, $U(VC) \sim U(CV) = (UC)V \sim V(UC)$. Thus $UVC \sim VUC$ and similarly $CUV \sim CVU$.

Choose a matrix C in $\mathcal{C}$ and define H_C to be the set of U in $\mathrm{GL}(\mathcal{R})$ such that $UC \sim C$ (or equivalently $CU \sim C$). If $UC \sim C$ and $VC \sim C$ then $U(VC) \sim U(C) \sim C$, and similarly $U^{-1}(C) \sim U^{-1}(UC) = C$. Therefore H_C is a group. We claim $UCV \sim C$ if and only if $UV \in H_C$. If $UV \in H_C$, then $UCV \sim UVC \sim C$. Conversely, if $UCV \sim C$, then $C \sim UCV \sim UVC$ and thus $UV \in H_C$.

Next suppose that D is another element of $\mathcal{C}$. We claim $H_C = H_D$. Suppose $UDV \sim D$. By Lemma 9.1 there are X, Y in $\mathrm{GL}(\mathcal{R})$ such that $D = XCY$. Thus $XCY \sim UXCYV \sim XUCVY$, so $UCV \sim C$ and $UV \in H_C$. Similarly, $UV \in H_C$ implies $UDV \sim D$. Thus shows the group H_C does not depend on the choice of C from $\mathcal{C}$.

Notice H_C contains the commutator of $\mathrm{GL}(\mathcal{R})$, since $UVCU^{-1}V^{-1} \sim VUCU^{-1}V^{-1} \sim C$. The commutator is the kernel of the map $\mathrm{GL}(\mathcal{R}) \to K_1(\mathcal{R})$. Define H as the image of H_C in $K_1(\mathcal{R})$. It follows that $[U] \in H \iff U \in H_C$. This proves (1). It then follows that in (2) we have a well defined injection $(\mathrm{K}_1(\mathcal{R}))/H \to \mathcal{E}(\mathcal{C})$, which is surjective by Lemma 9.1.

To prove (3), suppose $[U] \in H$. Perhaps after passing to another representative of $[U]$, we have $E \in \mathrm{E}(\mathcal{R})$ such that $UC = CE$. If $\mathcal{R}$ is commutative, the injectivity of C forces $\det(U) = 1$, i.e., $[U] \in \mathrm{SK}_1(\mathcal{R})$. Suppose now that $\mathcal{R} = \mathbb{Z}G$ with G finite. Let $\overline{U}, \overline{C}, \overline{E}$ denote the images of U, C, E under the map induced by the inclusion $\mathbb{Z}G \to \mathbb{Q}G$. The injectivity of C implies that $\overline{C}$ is invertible. Now $(\overline{C})^{-1}\overline{U}\overline{C} = \overline{E}$, which implies that $\overline{U}$ and $\overline{E}$ are $\mathrm{E}(\mathbb{Q}G)$ equivalent. In other words, $[U]$ is in the kernel of the induced map $\mathrm{K}_1(\mathbb{Z}G) \to \mathrm{K}_1(\mathbb{Q}G)$, and $[U] \in \mathrm{SK}_1(\mathcal{R})$. □

Remark 9.7. In the case of $\mathbb{Z}G$ with G not abelian, we thank Jonathan Rosenberg [35] for the statement and proof of part (3) of Proposition 9.6,

Remark 9.8. In Proposition 9.6, if $\mathcal{C}$ contains an element of $\mathrm{GL}(\mathcal{R})$, then clearly the group H is trivial. We do not know whether it is possible for H to be nontrivial.

Our main interest in the next result is the case $\mathcal{R} = \mathbb{Z}G$, where G is finite (so, $\mathrm{SK}_1(\mathbb{Z}G)$ is finite [27]) and abelian. In this case, it is straightforward to check whether a square matrix C over $\mathbb{Z}G$ is injective (examine the matrix for the regular representation or equivalently check whether $\det(C)$ is a zero divisor in $\mathbb{Z}G$). Also in this case, the $\mathbb{Z}G$ module $\mathrm{cok}(C)$ is finite if and only if C is injective.

Corollary 9.9. *Suppose $\mathcal{R}$ is a commutative ring and $\mathrm{SK}_1(\mathcal{R})$ is finite. Suppose $D_1, \dots, D_k$ are finite square matrices over $\mathcal{R}$ such that*

1. *The modules $\mathrm{cok}(D_i)$ are isomorphic;*
2. *the determinants $\det(D_i)$ are equal and are not equal to a zero divisor in $\mathcal{R}$; and*
3. *for $i \neq j$, there is no $\mathrm{E}(\mathcal{R})$ equivalence from $(D_i)_\infty$ to $(D_j)_\infty$.*

Then $k \leq |\mathrm{SK}_1(\mathcal{R})|$. □

Remark 9.10. Let G be a subgroup of the positive reals under multiplication, and let A be a finite square matrix A with entries in $\mathbb{Z}_+G$, with $\alpha(A)$ irreducible. Then A presents the shift of finite type $\sigma_{\alpha(A)}$ together with an invariant Markov measure, μ_A [25, 33]. Let B be another such matrix, and (after the normalizations described in [25]), suppose that $I-A$ and $I-B$ have equal determinant, and that G is the common group of weights over periodic cycles for μ_A and μ_B. Then [25, 31, 33] there is a measure preserving topological conjugacy $(\sigma_{\alpha(A)}, \mu_A) \to (\sigma_{\alpha(B)}, \mu_B)$ if and only if A and B are strong shift equivalent over $\mathbb{Z}_+G$, if and only if (by [9, Theorem 7.2]) there is a positive equivalence of polynomial matrices from $I-tA$ to $I-tB$. In this case (after setting $t=1$), we get matrices U, V in $\mathrm{E}(\mathbb{Z}G)$ such that $U(I-A)_\infty V = (I-B)_\infty$. (In fact, this elementary equivalence class of $I-A$ is also an invariant of *stochastic flow equivalence* [2].) Bill Parry [30] has asked if the cokernel module of $(I-A)$ is a complete invariant of equivalence over $\mathbb{Z}G$ when $\det(I-A)$ is nonzero. The next result, which follows immediately from Proposition 9.5, answers this question in the affirmative.

Proposition 9.11. *Suppose* $\mathcal{R} = \mathbb{Z}G$ *where* $G \cong \mathbb{Z}^n$ *and* A, B *are finite square matrices over* $\mathcal{R}$ *and* $\det(I-A)$ *is nonzero. The following are equivalent.*

(1) *There exist* U, V *in* $\mathrm{E}(\mathcal{R})$ *such that* $U(I-A)_\infty V = (I-B)_\infty$.
(2) $\det(I-A) = \det(I-B)$ *and the* $\mathcal{R}$*-modules* $\mathrm{cok}(I-A)$ *and* $\mathrm{cok}(I-B)$ *are isomorphic.*

Proof. For any commutative ring $\mathcal{R}$, the units group $\mathcal{R}^*$ of $\mathcal{R}$ is a direct summand of $K_1(\mathcal{R})$. The projection from $K_1(\mathcal{R})$ to $\mathcal{R}^*$ is given by det, and the complementary summand is $\mathrm{SK}_1(\mathcal{R})$. Let $G = \mathbb{Z}^n$, and let $\mathcal{U}$ denote the set of "obvious" units of $\mathbb{Z}G$, $\mathcal{U} = \{\pm g : g \in G\}$. Then the det map on $K_1(\mathbb{Z}G)$ is an isomorphism to $(\mathbb{Z}G)^*$, and moreover $(\mathbb{Z}G)^* = \mathcal{U}$ [3]. (The statement of the relevant Corollary in [3, p. 63] shows $K_1(\mathbb{Z}G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^n$. Because $\mathcal{U} \cong \mathbb{Z}/2 \oplus \mathbb{Z}^n$, it follows here that det is injective. That $(\mathbb{Z}G)^* = \mathcal{U}$ follows from the construction of the isomorphism proving the Corollary.) Because $\mathrm{SK}_1(\mathbb{Z}G)$ is trivial, Prop. 9.11 follows from Prop. 9.5. □

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