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Stochastic Dynamics of Systems with Memory (Stochastic Analysis Seminar, University of Oxford)

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STOCHASTIC DYNAMICS
OF
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Some Basics

- SFDE:

$$\left.\begin{array}{r} dx(t) = H(t, x(t), x_t)\, dt + G(t, x(t), x_t)\, dW(t),\ t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{array}\right\}$$

  $H : [0, T] \times M_2 \to \mathbf{R}^d, \quad G : [0, T] \times M_2 \to \mathbf{R}^{d \times m}$.

  W m-dimensional Brownian motion.

  segment $x_t(s) := x(t + s)\, t \geq 0,\ s \in J := [-r, 0]$.

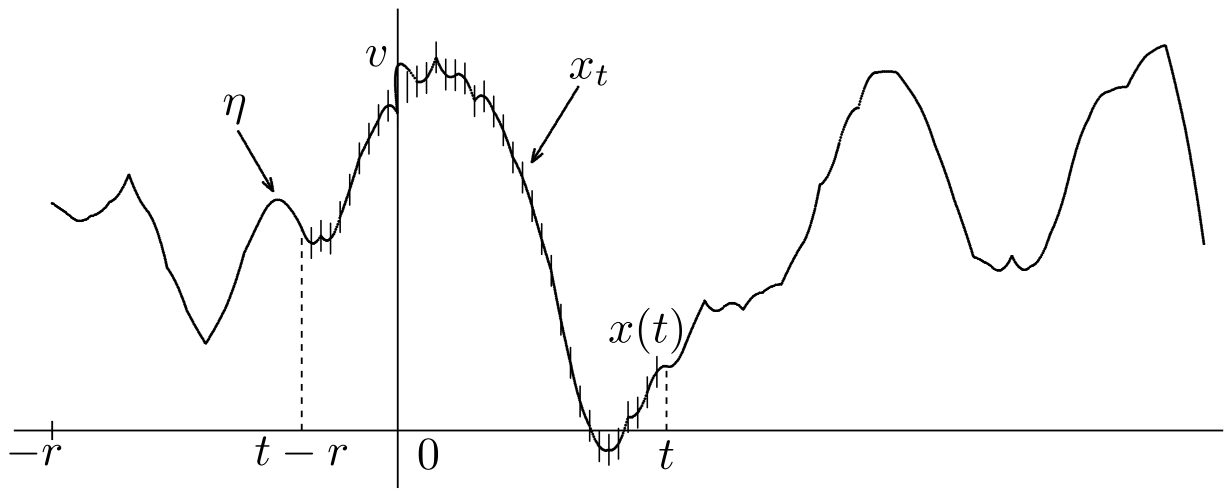
  initial path $\eta : [-r, 0] \to \mathbf{R}^d$.

- Existence of unique solution under Lipschitz + linear growth conditions on H, G.

- Markov process $(x(t), x_t)$ in state space $M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ $\big($or $C([-r, 0], \mathbf{R}^d)\big)$.

- Can compute generator as second order differential operator on state space.

- x_t is not a semimartingale; but Itô's formula holds for tame and quasi-tame functions on state space. ([Mo], Pitman Books 1984)



Stable Manifolds

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.

- *Local Stable Manifold Theorem* for SFDE's: Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proof: Ruelle-Oseledec multiplicative ergodic theory+ perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a P-preserving (ergodic) semigroup on $(\Omega, \mathcal{F}, P)$.

$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ-algebra.

Definition.

$k =$ non-negative integer, $\epsilon \in (0,1]$. A $C^{k,\epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \to E$ such that:

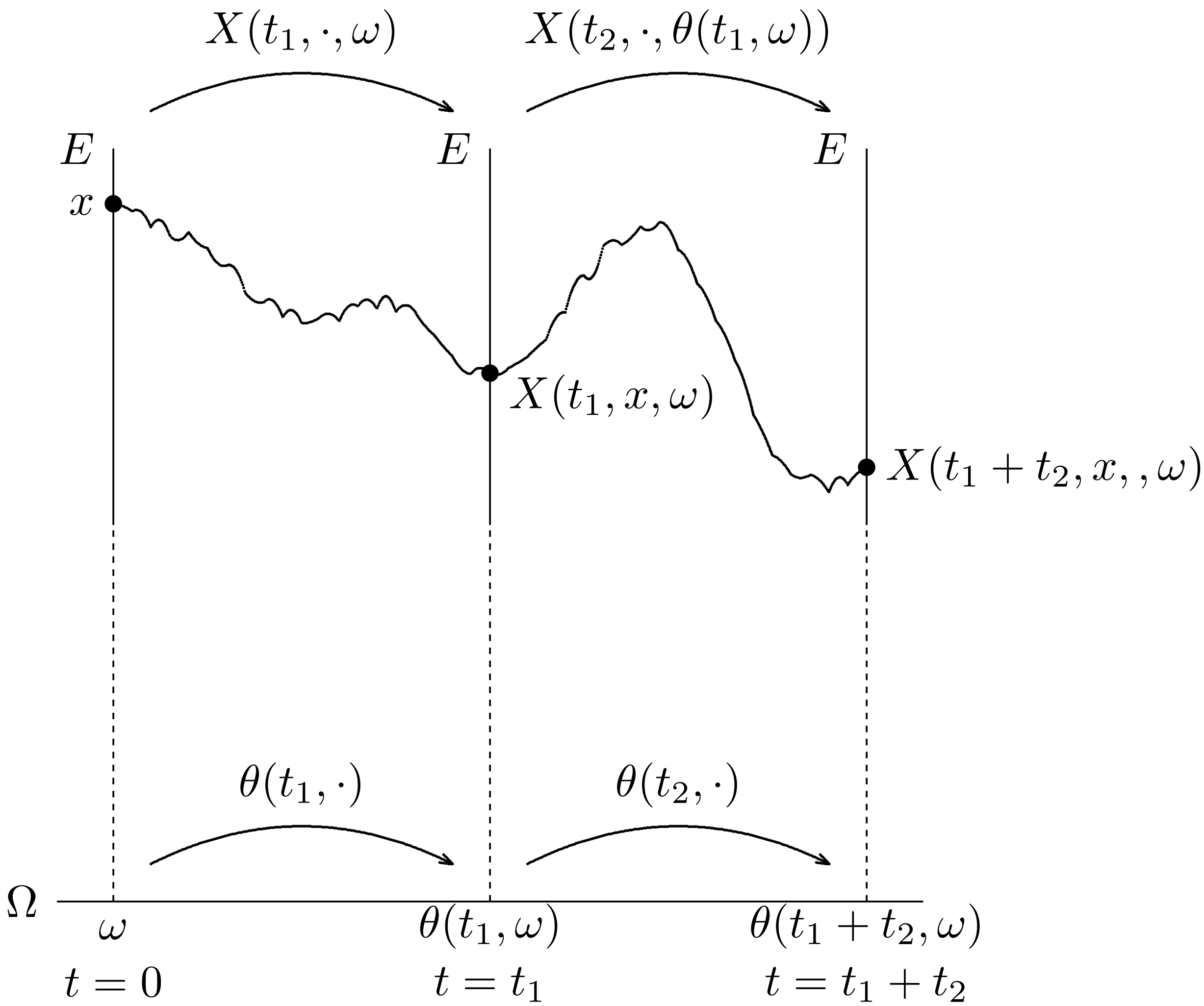
(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t,x) \mapsto X(t,x,\omega) \in E$ is continuous; for fixed $(t,\omega) \in$

$\mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t,x,\omega) \in E$ is $C^{k,\epsilon}$ $\big(D^k X(t,x,\omega)$ is C^ϵ in $x\big)$.

(ii) $X(t_1+t_2,\cdot,\omega) = X(t_2,\cdot,\theta(t_1,\omega)) \circ X(t_1,\cdot,\omega)$ for all $t_1,t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

(iii) $X(0,x,\omega) = x$ for all $x \in E, \omega \in \Omega$.

Cocycle Property



Vertical solid lines represent random fibers: copies of E.

Definition

A random variable $Y : \Omega \to E$ is a *stationary point* for the cocycle (X, θ) if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \qquad (1)$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X, θ) along a stationary random point Y: Get an $L(E)$-valued cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$. (Follows from cocycle property of X and chain rule.)

Theorem 1. *(Oseledec-Ruelle)*

Let $T : \mathbf{R}^+ \times \Omega \to L(E)$ be strongly measurable, such that (T, θ) is an $L(E)$-valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\|_{L(E)} < \infty,$$
$$E \sup_{0 \le t \le 1} \log^+ \|T(1 - t, \theta(t, \cdot))\|_{L(E)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ *such that* $\theta(t,\cdot)(\Omega_0) \subseteq \Omega_0$ *for all* $t \in \mathbf{R}^+$, *and for each* $\omega \in \Omega_0$,

$$\lim_{t\to\infty} [T(t,\omega)^* \circ T(t,\omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ *is self-adjoint with a non-random spectrum*

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the λ_i*'s are distinct. Each* e^{λ_i} *(with* $\lambda_i > -\infty$*) has a fixed finite non-random multiplicity* m_i *and eigen-space* $F_i(\omega)$, *with* $m_i := dim F_i(\omega)$. *Define*

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \ i > 1, \ E_\infty := ker\Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t\to\infty} \frac{1}{t} \log \|T(t,\omega)x\| = \begin{cases} \lambda_i & \textit{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega), \\ -\infty & \textit{if} \quad x \in E_\infty(\omega), \end{cases}$$

and

$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \geq 0, \ i \geq 1$.

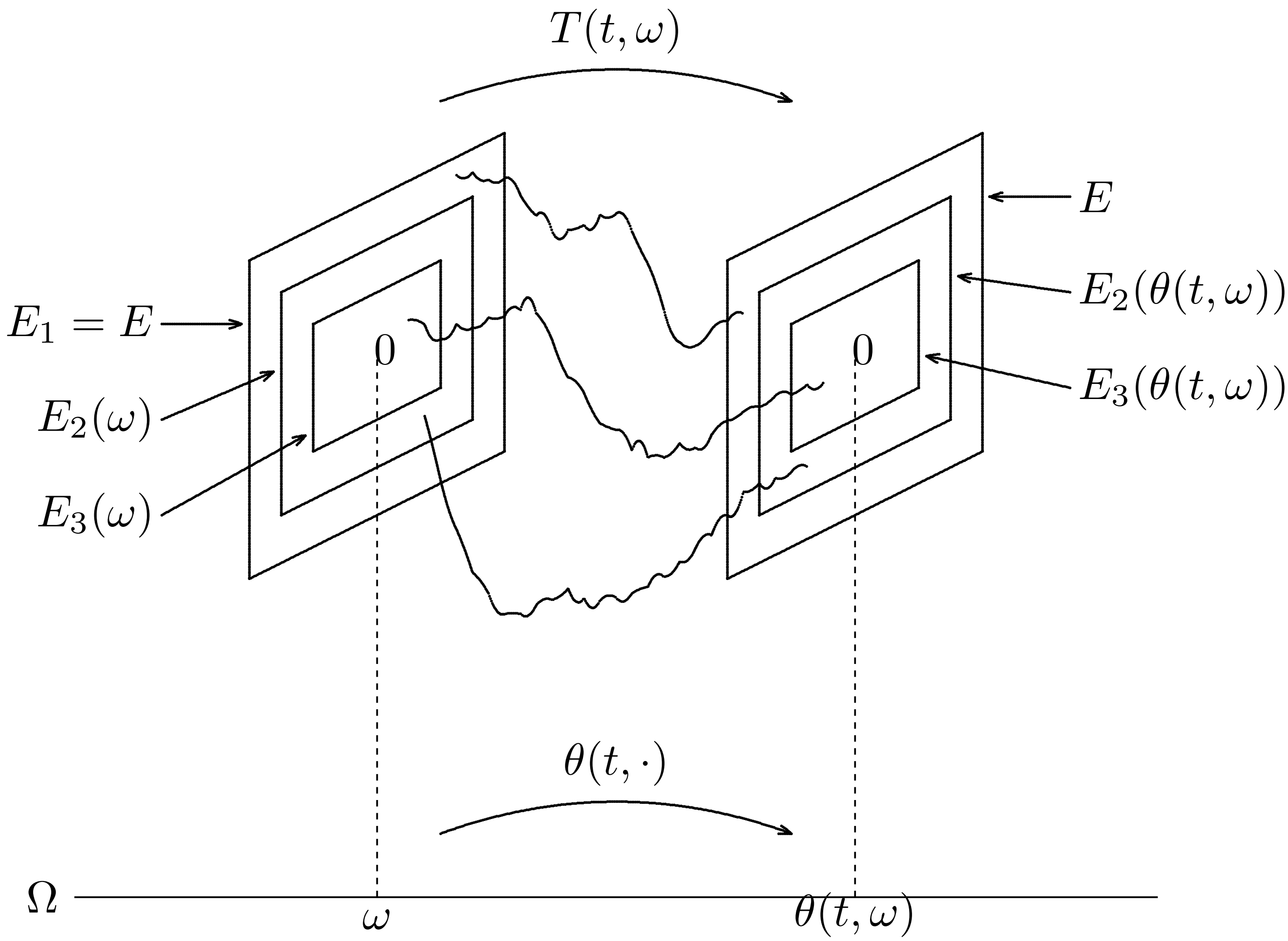
Proof.

Based on discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). $\square$

Lyapunov Spectrum:

$\{\lambda_1, \lambda_2, \lambda_3, \cdots\} :=$ *Lyapunov spectrum* of (T, θ).

Spectral Theorem



Definition

A stationary point $Y(\omega)$ of (X,θ) is *hyperbolic* if the linearized cocycle $(DX(t,Y(\omega),\omega),\theta(t,\omega))$ has a non-vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be s.t. $\lambda_{i_0} < 0 < \lambda_{i_0-1}$.

Assume $X(t,\cdot,\omega)$ *locally compact* and

$$E \log^+ \sup_{0\leq t_1,t_2\leq r} \|D_2X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(E)} < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimensional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{x \in E : \lim_{t\to\infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(x)\| \leq \lambda_i\},$$

$i \geq 1$ and all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Let $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ be the unstable and stable subspaces associated with the linearized cocycle (DX, θ) ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

and the invariance

$$DX(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),$$

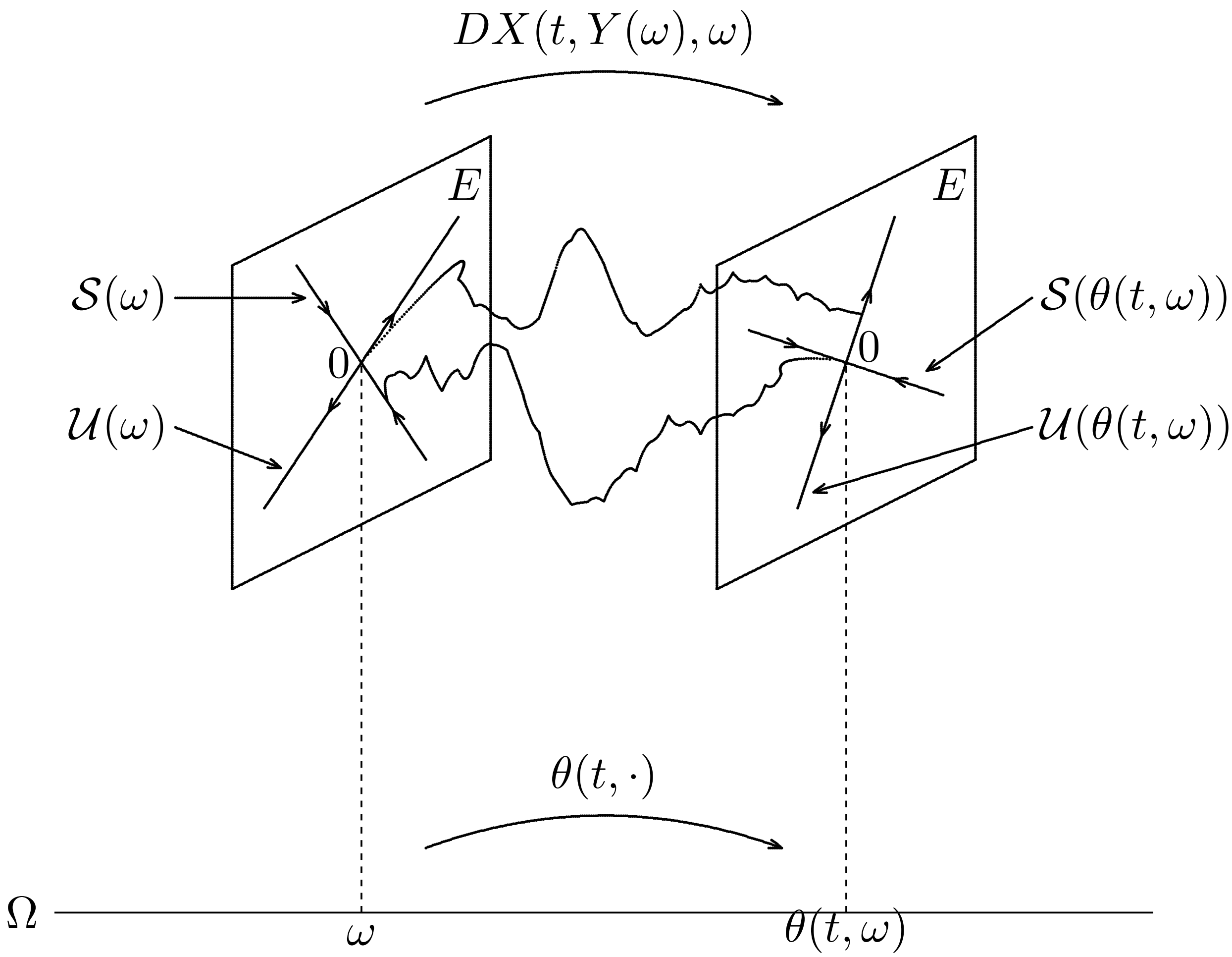
$$DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$, with *exponential dichotomies*

$$\|DX(t, Y(\omega), \omega)(x)\| \leq \|x\| e^{-\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, x \in \mathcal{S}(\omega),$$

$$\|DX(t, Y(\omega), \omega)(x)\| \geq \|x\| e^{\delta_2 t} \quad \text{for all} \quad t \geq \tau_2^*, x \in \mathcal{U}(\omega),$$

with $\tau_i^* = \tau_i^*(x, \omega) > 0, i = 1, 2,$ random times and $\delta_i > 0, i = 1, 2,$ fixed.



$DX(t, Y(\omega), \omega)$
E
E
$\mathcal{S}(\omega)$
0
$\mathcal{U}(\omega)$
$\mathcal{S}(\theta(t, \omega))$
0
$\mathcal{U}(\theta(t, \omega))$
$\theta(t, \cdot)$
Ω
ω
$\theta(t, \omega)$

Nonlinear SFDE's

"Regular" Itô SFDE with finite memory:

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + \sum_{i=1}^{m} G_i(x(t))\,dW_i(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \quad (I)$$

Solution segment $x_t(s) := x(t+s),\ t \geq 0, s \in [-r, 0]$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space $(\Omega, \mathcal{F}, P)$. $\bar{\mathcal{F}} := P-$completion of $\mathcal{F}$.

State space M_2, Hilbert with usual norm $\|\cdot\|$.

Can allow for "smooth memory" in diffusion coefficient.

$H : M_2 \to \mathbf{R}^d$, $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \to L(\mathbf{R}^m, \mathbf{R}^d)$, $C_b^{k+1,\delta}$; $G := (G_1, \cdots, G_m)$, $k \geq 1$.

$B((v,\eta),\rho)$ open ball, radius ρ, center $(v,\eta) \in M_2$;

$\bar{B}((v,\eta),\rho)$ closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t,(v,\eta),\cdot) = (x(t), x_t)$. X is $C^{k,\epsilon}$ for any $\epsilon \in (0,\delta)$, takes bounded sets into relatively compact sets in M_2. (X,θ) is a perfect cocycle on M_2 ([M-S.4]).

SDDE

$$dx(t) = x(t-1)dW(t)$$

does not admit a (regular) stochastic semiflow.

Idea of Proof.

Construction and regularity of the cocycle (X, θ): SFDE is equivalent to the neutral integral equation:

$$\zeta(t, x(t,\omega), \omega) = v + \int_0^t F(u, \zeta(u, x(u,\omega), \omega), x(u,\omega), x_u(\cdot,\omega), \omega)\, du,$$

$0 \leq t \leq T, (v, \eta) \in M_2$;

$F : [0, \infty) \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ is given by

$$F(t, z, v, \eta, \omega) := \{D\psi(t, z, \omega)\}^{-1} H(v, \eta)$$

$t \geq 0, z, v \in \mathbf{R}^d, \eta \in L^2([-r, 0], \mathbf{R}^d), \omega \in \Omega.$

ψ is the $C^{k+1,\epsilon}$ $\big(0 < \epsilon < \delta\big)$ stochastic flow of the sde

$$\left.\begin{aligned} d\psi(t) &= G(\psi(t))\, dW(t), \quad t \geq 0 \\ \psi(0) &= x \in \mathbf{R}^d \end{aligned}\right\}$$

$\zeta : [0,\infty) \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is defined by

$$\zeta(t,x,\omega) := \psi(t,\cdot,\omega)^{-1}(x), \quad t \geq 0,\, x \in \mathbf{R}^d,\, \omega \in \Omega.$$

Read existence and properties of cocycle from integral equation. $\square$

Example

Consider the affine linear sfde

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\, dt + G\, dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I')$$

where $H : M_2 \to \mathbf{R}^d$ is a continuous linear map, G is a fixed $(d \times p)$-matrix, and W is p-dimensional Brownian motion. Assume that the linear deterministic $(d \times d)$-matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t)\, dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d)\times L^2([-r,0],L(\mathbf{R}^d)) \to L(\mathbf{R}^d)\times L^2([-r,0],L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^{0} T_{-u}(I,0)G\,dW(u) \tag{2}$$

where I is the identity $(d\times d)$-matrix. Integration by parts and

$$W(t,\theta(t_1,\omega)) = W(t+t_1,\omega) - W(t_1,\omega), \quad t, t_1 \in \mathbf{R}, \tag{3}$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See ([Mo],(1984), Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is "hyperbolic", one can show that a stationary point of (I') exists.

For general white-noise an invariant measure on M_2 for the one-point motion gives a stationary point by enlarging probability space. Conversely, let $Y : \Omega \to M_2$ be a stationary random point independent of the Brownian motion $W(t)$, $t \geq 0$. Let $\rho := P \circ Y^{-1}$ be the distribution of Y. By independence of Y and W, ρ is an invariant measure for the one-point motion

Theorem 2. *([M-S], 2000) (The Stable Manifold Theorem)*

Assume smoothness hypotheses on H and G. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i <$

$\cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon}$ ($\epsilon \in (0,\delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $(v,\eta) \in \bar{B}(Y(\omega),\rho_1(\omega))$ *such that*

$$\|X(nr,(v,\eta),\omega) - Y(\theta(nr,\omega))\| \le \beta_1(\omega)\, e^{(\lambda_{i_0}+\epsilon_1)nr}$$

for all integers $n \ge 0$. *Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega) - Y(\theta(t,\omega))\| \le \lambda_{i_0}$$

for all $(v,\eta) \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. *In particular, codim* $\tilde{\mathcal{S}}(\omega) =$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b)
$$\limsup_{t\to\infty} \frac{1}{t} \log \left[\sup\left\{ \frac{\|X(t,(v_1,\eta_1),\omega) - X(t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|} : (v_1,\eta_1) \ne (v_2,\eta_2),\ (v_1,\eta_1),\ (v_2,\eta_2) \in \tilde{\mathcal{S}}(\omega) \right\} \right] \le \lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$. *Also*

$$DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v, \eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a unique "history" process* $y(\cdot, \omega) : \{-nr : n \geq 0\} \to M_2$ *such that* $y(0, \omega) = (v, \eta)$ *and for each integer* $n \geq 1$, *one has* $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n-1)r, \omega)$ *and*

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0 - 1} - \epsilon_2) nr}.$$

Furthermore, for each $(v, \eta) \in \tilde{\mathcal{U}}(\omega)$, *there is a unique continuous-time "history" process also denoted by* $y(\cdot, \omega) : (-\infty, 0] \to M_2$ *such that* $y(0, \omega) = (v, \eta)$,

$X(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y(-t,\omega) - Y(\theta(-t,\omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semi-flow* DX *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) *Let* $y(\cdot, (v_i, \eta_i), \omega), i = 1, 2$, *be the history processes associated with* $(v_i, \eta_i) = y(0, (v_i, \eta_i), \omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1, 2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \Bigg[\sup \Bigg\{ \frac{\|y(-t,(v_1,\eta_1),\omega) - y(-t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} : \\ (v_1,\eta_1) \neq (v_2,\eta_2),\ (v_i,\eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1,2 \Bigg\} \Bigg] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t, \cdot, \theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega)))$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DX(t, \cdot, \theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t, \cdot, \theta(-t,\omega))|\mathcal{U}(\theta(-t,\omega)) : \mathcal{U}(\theta(-t,\omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

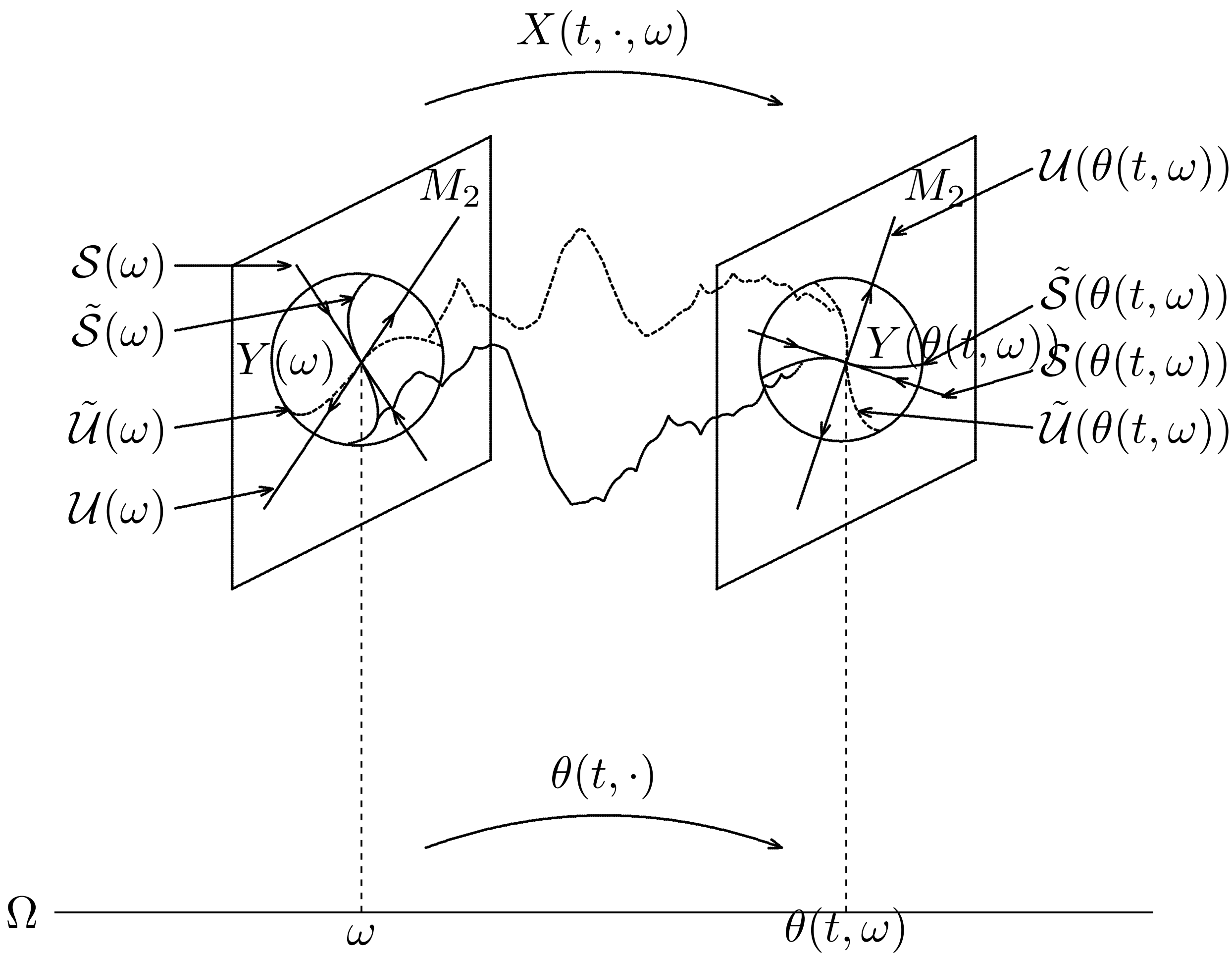
(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G *are* C_b^∞. *Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞.

Figure summarizes essential features of Stable Manifold Theorem:

Stable Manifold Theorem



$t > \tau_1(\omega)$

A picture is worth a 1000 words!

Outline of Proof of Theorem 2

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- Linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. By stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where $D =$ spatial (Fréchet) derivatives.
- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than

the delay r ([M-S.4]), and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.

- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of "perfect versions" of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions give invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- Establish continuous-time integrability estimates on the spatial derivatives of the nonlinear cocycle X in a neighborhood of the stationary point Y. Estimates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.

- Introduce the auxiliary perfect cocycle

  $$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)),\ t \in \mathbf{R}^+, \omega \in \Omega.$$

  Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr,\cdot,\omega),\theta(nr,\omega))$ near 0 and hence (by translation) for $X(nr,\cdot,\omega)$

near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.

- Final key step: Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X. Use arguments underlying the proofs of Theorems 4.1 and

5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.

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