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Abstract. We compute the genus class group of a torsion-free module over a reduced Witt ring of finite stability index. This is applied to modules locally isomorphic to odd degree extensions of formally real fields.

A *ring-order* [7] is a reduced, noetherian ring R of dimension 1 such that its integral closure $\tilde{R}$ is finitely generated over R. A simple example is the Witt ring of a Pythagorean field with a finite square class group. For a ring-order R, Weigand and Guralnick [6] have defined the genus class group, $\mathrm{Genus}(M)$, of a torsion-free, finitely generated R-module M which consists of the modules locally isomorphic to M. They show:

$$1 \to K_M \to (\tilde{R}/\mathcal{C})^* \to \mathrm{Genus}(M) \to \mathrm{Pic}(\tilde{R}) \to 1 \tag{0.1}$$

is exact, where $\mathcal{C}$ is the conductor $(R : \tilde{R})$ and K_M can be explicitly described. Our goal is to show (0.1) also holds for reduced Witt rings of finite stability index (which need not be noetherian). This generalizes previous work on projective modules over Witt rings [5] and includes a wider class of odd degree extensions, since, if F is a Pythagorean field and K/F is an odd degree extension, then WK is always a torsion-free WF-module (but rarely a projective WF-module). The specific class of extensions considered here is: let F be a formally real field with finite stability index, K an odd degree extension of F and suppose $[K^*/\sum K^{*2} : F^*/\sum F^{*2}]$ is finite. In this case, our result applies to $(WK)_{red}$ as a $(WF)_{red}$-module. Moreover, the result brings out more clearly the crucial role played by the conductor $\mathcal{C}$.

For any ring A, A^* denotes the group of units of A. All Witt rings are assumed to be real, that is, have orderings. If W is a Witt ring then X_W (or just X if the choice of ring is understood) is the space of orderings on W. This is given by the Harrison topology with a basis of clopen sets of the form

$$H(a_1, \dots, a_n) = \{\alpha \in X : a_i >_\alpha 0 \quad \text{for all } i\},$$

where the a_i are in the associated group G ($G = F^*/F^{*2}$ in the field case). We will sometimes identify an ordering α with its character on G writing, for example, $\alpha(a) = 1$

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instead of $a >_\alpha 0$. The signature of a form q with respect to α will be written $\hat{q}(\alpha)$. Give $\mathbb{Z}$ the discrete topology. Then $\hat{q} : W \to \mathbb{Z}$ is continuous. The ring of continuous functions from W to $\mathbb{Z}$ is denoted $C(W, \mathbb{Z})$.

R will always denote a reduced Witt ring. R may be an abstract Witt ring, in the sense of Marshall [9], although the only case of interest is that of the Witt ring of a formally real Pythagorean field F. For a Witt ring W there are several conditions equivalent to be being reduced that we will use. Namely, W is reduced iff W is torsion-free iff 2 is a non-zero-divisor iff two forms with the same signature at each ordering are necessarily equal [9, 4.20,4.13,4.12]. This last property allows us to identify R with its image in $C(X, \mathbb{Z})$, that is, we will identify a form q with its total signature $\hat{q}$. Using this identification, the total quotient ring, TQ, of R is $C(X, \mathbb{Q})$, where again $\mathbb{Q}$ is given the discrete topology [4, 1.2]. The integral closure $\tilde{R}$ of R in TQ is $C(X, \mathbb{Z})$ [4, 1.13].

For a clopen set $Y \subset X$ let $e_Y : X \to \mathbb{Z}$ denote the characteristic function of Y, that is, $e_Y(\alpha) = 1$ if $\alpha \in Y$ and $e_Y(\alpha) = 0$ if $\alpha \notin Y$. Note $e_Y \in C(X, \mathbb{Z}) = \tilde{R}$. We have that $\mathrm{Pic}(\tilde{R}) = 1$ [4, 1.1] so that our goal is the short exact sequence:

$$1 \to K_M \to (\tilde{R}/\mathcal{C})^* \to \mathrm{Genus}(M) \to 1, \tag{0.2}$$

where K_M contains the image of $\tilde{R}^* \to (\tilde{R}/\mathcal{C})^*$.

E_n will denote the group of exponent 2 and order 2^n. Let $t_1, \ldots, t_n$ be a set of generators for E_n. E_∞ will be the countably generated group of exponent 2, with generators $t_1, \ldots, t_n, \ldots$. $R[E_n]$ is the group ring extension of R by E_n; it is also a reduced Witt ring. If R_1 and R_2 are reduced Witt rings the fiber product is:

$$R_1 \sqcap R_2 = \{(q_1, q_2) \in R_1 \oplus R_2 : \dim(q_1) \equiv \dim(q_2) \pmod 2\}.$$

$R_1 \sqcap R_2$ is also a reduced Witt ring.

The abelian group $C(X, R)/R$ is 2-primary torsion [9, 7.14]. The *stability index* of R, $st(R)$, is the least $k \geq 0$ such that $2^k C(X, \mathbb{Z}) \subset R$ (equivalently, the least k with 2^k in the conductor $\mathcal{C}$). If there is no such k then $st(R)$ is infinite.

A subgroup $T \subset G$ is a *fan* if (i) $-1 \notin T$, (ii) if $a, b \in T$ then every element represented by the binary form $\langle a, b \rangle$ is in T, and (iii) every subgroup $S \subset G$ of index 2, containign T but not -1, is an ordering. Suppose the index $[G : T]$ is finite, say n. Let X/T denote the set of orderings that contain T. Then $|X/T|$ equals the number of subgroups of index 2 of G/T that do not contain $-T$, namely $|X/T| = 2^{n-1}$. We have [9, 7.16] that:

$$st(R) = sup\{k : \text{there exists a fan } T \text{ with } |X/T| = 2^k\}.$$

1. Basics.

We begin by justifying the assumption that R has finite stability index. For ring-orders A, the integral closure is finitely generated over A. The first use of this in [6] is to get a non-zero-divisor in the conductor $\mathcal{C}$. But if X is infinite then $\tilde{R}$ is not finitely generated over R and $\mathcal{C}$ may be 0. The assumption that the stability index is finite gives a power of 2, a non-zero-divisor, in the conductor. This assumption may not be necessary but (0.2) can fail if the stability index is infinite. We verify these assertions.

Lemma 1.1.

(1) *If X is infinite then $\tilde{R}$ is not a finitely generated R-module.*
(2) *If $R = \mathbb{Z}[E_\infty]$ then $\mathcal{C} = 0$ and (0.2) fails.*

Proof. (1) Suppose instead that $\tilde{R}$ is generated by $f_1, \dots, f_n$ over R. Find a disjoint clopen cover $\{Y_j : 1 \le j \le t\}$ of X such that each $f_i | Y_j$ is constant, say $k_{ij} \in \mathbb{Z}$. Since X is infinite, some Y_j, say Y_1, is infinite. Pick distinct $\alpha, \beta \in Y_1$ and a $b \in F$ such that $\alpha(b) = 1$ and $\beta(b) = -1$. Then $Z = H(b) \cap Y_1$ is a proper clopen subset of Y_1. Let $e_j = e_{Y_j}$. We have:

$$f_i = \sum_j k_{ij} e_j,$$

so that $\tilde{R}$ is generated by $e_1, \dots, e_t$. Write $e_Z = \sum r_i e_i$, where each $r_i \in R$. Plugging in α gives $1 = \hat{r}_1(\alpha)$ while plugging in β gives $0 = \hat{r}_1(\beta)$. But $r_1 \in R$ so that its signatures are either all even or all odd, a contradiction.

(2) Let $q \in \mathcal{C}$. Say q has all of its entries in $span\{-1, t_1, \dots, t_k\}$, where the t_i's are a subset of the generators of E_∞. Let $\alpha \in X$ and let $\epsilon_i = \alpha(t_i)$ for $1 \le i \le k$. Choose any $m \ge k$ and set

$$Z = H(\epsilon_1 t_1, \dots, \epsilon_k t_k, t_{k+1}, \dots t_m).$$

Let T be the fan spanned by $\{\epsilon_1 t_1, \dots, \epsilon_k t_k, t_{m+1}, \dots\}$ so that $|X/T| = 2^{m-k}$. Then there is a unique ordering β in $Z \cap (X/T)$, namely the ordering with $\beta(t_i) = \epsilon_i$ for $1 \le i \le k$ and $\beta(t_i) = 1$ for $i > k$. Now $qe_Z \in R$ so by the easy half of the Representation Theorem [8, 7.2] we get $\hat{q}(\beta) \equiv 0 \pmod{2^{m-k}}$. Since $\alpha(t_i) = \beta(t_i)$ for $1 \le i \le k$ and all entries of q are combinations of these t_i, we have $\hat{q}(\alpha) = \hat{q}(\beta)$. Thus $\hat{q}(\alpha) \equiv 0 \pmod{2^{m-k}}$. But m was arbitrary so $\hat{q}(\alpha) = 0$. And α was arbitrary so $q = 0$.

Lastly, suppose (0.2) holds. K_M contains the image of $\tilde{R}^*$ in $(\tilde{R}/\mathcal{C})^*$ so that $\mathcal{C} = 0$ gives $K_M = \tilde{R}^*$. Thus each Genus$(M) = 1$. But Genus$(R) = \text{Pic}(R) \ne 1$ by [4, 1.17]. Thus (0.2) fails. $\square$

We need some technical results on localizing R and $\mathcal{C}$. Recall [9, 4.18] that the maximal ideals of R are the fundamental ideal IR of even dimensional forms and all

$$P(\alpha, p) = \{q \in R : \hat{q}(\alpha) \equiv 0 \pmod{p}\},$$

where $\alpha \in X$ and p is an odd prime. For a maximal ideal m of R, the notation $\tilde{R}_m$ means the integral closure of R_m.

Lemma 1.2. *Let m be a maximal ideal of R.*

(1) *If $m = P(\alpha, p)$ for some $\alpha \in X$ and odd prime p, then $\tilde{R}_m \cong \mathbb{Z}_{(p)}$ and $R_m = \tilde{R}_m$.*
(2) *If $m = IR$ then $\tilde{R}_m = C(X, \mathbb{Z}_{(2)})$.*

Proof. (1) Map $R_m \to \mathbb{Z}_{(p)}$ by sending φ/q to $\hat{\varphi}(\alpha)/\hat{q}(\alpha)$. This is clearly a well-defined surjective homomorphism. Suppose $\hat{\varphi}(\alpha) = 0$. By the Regularity Theorem of [1] there exists a $\psi \in R$ and a positive integer k such that $\hat{\psi}(\beta) = 2^k$ if $\hat{\varphi}(\beta) = 0$ and $\hat{\psi}(\beta) = 0$ if $\hat{\varphi}(\beta) \ne 0$. Then $\psi \notin m$, $\psi\varphi = 0$ and $\varphi/q = 0/\psi = 0$. Thus the map is injective as well. Hence $R_m \cong \mathbb{Z}_{(p)}$.

$\mathbb{Z}_{(p)}$ is an integral domain so R_m is also. The total quotient ring of R_m, call it $K(m)$, consists of all fractions, with a non-zero denominator, of elements of R_m . The isomorphism of the first part extends to an isomorphism of $K(m)$ onto $\mathbb{Q}$. Since $\mathbb{Z}_{(p)}$ is integrally closed in $\mathbb{Q}$, R_m is integrally closed in $K(m)$ and $R_m = \tilde{R}_m$.

(2) $R_{IR} \subset C(X, \mathbb{Z}_{(2)}) \subset C(X, \mathbb{Q})$ and $C(X, \mathbb{Z}_{(2)})$ is integrally closed in $C(X, \mathbb{Q})$ since $\mathbb{Z}_{(2)}$ is integrally closed in $\mathbb{Q}$. So we need only check that $C(X, \mathbb{Z}_{(2)})$ is integral over R_{IR}. But any $f \in C(X, \mathbb{Z}_{(2)})$ is a finite $\mathbb{Z}_{(2)}$-combination of e_Y's and $e_Y \in C(X, \mathbb{Z})$ is integral over R. Thus f is integral over R_{IR}. $\square$

The following is standard when $\tilde{R}$ is finitely generated over R but false in general.

Lemma 1.3. *For all maximal ideals m of R*

$$(\tilde{R}_m)^* \to (\tilde{R}_m/\mathcal{C}(R_m))^*$$

is surjective, where $\mathcal{C}(R_m)$ denotes $(R_m : \tilde{R}_m)$.

Proof. If $m \neq IR$ then $m = \mathcal{P}(\alpha, p)$ for some $\alpha \in X$ and odd prime p. Then $R_m = \tilde{R}_m$ by (1.2). So $\mathcal{C}(R_m) = \tilde{R}_m$ and the result is vacuous. Suppose then that $m = IR$. If $R = \mathbb{Z}$ then again $R_m = \mathbb{Z}_{(2)} = \tilde{R}_m$ and the result is vacuous. So we may assume $|X| \geq 2$. If $f \in \mathcal{C}(R_m)$ then $f \in R_m$ so that f can be written as h/k where $k(\alpha)$ is odd for all $\alpha \in X$. We first claim that if $h/k \in \mathcal{C}(R_m)$ then $h(\alpha) \in 2\mathbb{Z}$ for all $\alpha \in X$. Choose any clopen subset Y with $Y \neq \emptyset, X$. We have $e_Y(h/k) = \varphi/q$ for some $\varphi, q \in R$ with q odd dimensional. For $\alpha \notin Y$ we get $\hat{\varphi}(\alpha) = 0$. In particular, φ is even dimensional. For $\alpha \in Y$ we get $h(\alpha)/k(\alpha) = \hat{\varphi}(\alpha)/\hat{q}(\alpha)$. So $\hat{q}(\alpha)h(\alpha) = k(\alpha)\hat{\varphi}(\alpha)$, q odd and φ even dimensional gives $h(\alpha) \in 2\mathbb{Z}$, proving the claim.

Now $\tilde{R}_m = C(X, \mathbb{Z}_{(2)})$ by (1.2). Suppose $f + \mathcal{C}(R_m) \in (\tilde{R}_m/\mathcal{C}(R_m))^*$. Then for some $g \in C(X, \mathbb{Z}_{(2)})$ we have $fg - 1 \in \mathcal{C}(R_m)$. Write $fg - 1 = h/k$ where all $k(\alpha)$ are odd and all $h(\alpha)$ are even by the claim. Pick $\alpha \in X$ and write $f(\alpha) = a/b$ and $g(\alpha) = c/d$ where $a, b, c, d \in \mathbb{Z}$ and b, d are odd. Then

$$k(\alpha)(ac - bd) = h(\alpha)bd \in 2\mathbb{Z}.$$

This implies a is odd. Thus $f(\alpha) \in (\mathbb{Z}_{(2)})^*$ for all $\alpha \in X$. So $f \in (\tilde{R}_m)^*$. $\square$

Proposition 1.4. *If R has finite stability index then for every maximal ideal m we have $(R_m : \tilde{R}_m) = \mathcal{C}_m$.*

Proof. If $m \neq IR$ then again $m = \mathcal{P}(\alpha, p)$ for some $\alpha \in X$ and odd prime p, so that $R_m = \tilde{R}_m$ by (1.2). Thus $(R_m : \tilde{R}_m) = R_m$. The stability index is finite so there is a 2-power in $\mathcal{C}$. Hence $\mathcal{C}_m = R_m$ also. So suppose $m = IR$.

Let $\varphi/q \in (R_{IR} : \tilde{R}_{IR})$. Then $\varphi \in (R_{IR} : \tilde{R})$ and we want to show $\varphi \in \mathcal{C}$. Pick an $\alpha \in X$ with $\hat{\varphi}(\alpha) \neq 0$. Pick any fan T with $\alpha \in X/T$. X/T is finite since the stability index is. There exists a clopen $Y \subset X$ such that $Y \cap (X/T) = \{\alpha\}$ since X/T is finite. Now $\varphi e_Y = \psi/p$ for some forms $\psi, p \in R$ with p odd dimensional. Thus $p\varphi e_Y \in R$. By the easy half of the Representation Theorem [8,7.2] we have:

$$\begin{aligned}\sum_{\beta\in X/T}(p\varphi e_Y)(\beta)&\equiv 0 \pmod{|X/T|}\\ \hat{p}(\alpha)\hat{\varphi}(\alpha)&\equiv 0 \pmod{|X/T|}\\ \hat{\varphi}(\alpha)&\equiv 0 \pmod{|X/T|},\end{aligned}$$

since $\hat{p}(\alpha)$ is odd. Hence either $\hat{\varphi}(\alpha)=0$ or $\hat{\varphi}\equiv 0 \pmod{|X/T|}$ for any fan T with $\alpha\in(X/T)$.

We now show $\varphi\in\mathcal{C}$. Pick any clopen subset Z and any fan S. Again, X/S is finite since the stability index is. For each $\alpha\in X/S$ we have $\hat{\varphi}(\alpha)\equiv 0 \pmod{|X/S|}$. Thus, by the non-trivial half of the Representation Theorem, we have $\varphi e_Z\in R$. So $\varphi\in\mathcal{C}$ since any $f\in\tilde{R}$ is a finite $\mathbb{Z}$-combination of e_Z's. $\square$

2. Projective modules.

We need results about projective modules over both $C(X,\mathbb{Z})$ and $C(X,\mathbb{Z}_{(2)})$. So for this section, let X be any topological space that is compact, Hausdorff and totally disconnected. Let D be a PID, given the discrete topology. We will consider modules over $C(X,D)$.

For each $\alpha\in X$ and prime $p\in D$ set

$$\mathcal{P}(\alpha,p)=\{f\in C(X,D): f(\alpha)\equiv 0 \pmod{p}\}.$$

A module M over a ring A is *torsion-free* if for every regular $a\in A$ (that is, a is a non-zero-divisor) and every non-zero $m\in M$ we have $am\neq 0$.

Lemma 2.1. *For X and D as above*

1. *The maximal ideals of $C(X,D)$ are the $\mathcal{P}(\alpha,p)$, over all $\alpha\in X$ and primes $p\in D$.*
2. *Let $m=\mathcal{P}(\alpha,p)$ be a maximal ideal of $C(X,D)$. Then $C(X,D)_m\cong D_{(p)}$.*
3. *If M is a finitely generated torsion-free $C(X,D)$-module then M is projective.*

Proof. (1) This is [10, 3.1.2,3.2.1] for $D=\mathbb{Z}$ and the proof works for any PID D.

(2) Say $m=\mathcal{P}(\alpha,p)$. The map $\varphi: C(X,D)_m\to D_{(p)}$ sending f/g to $f(\alpha)/g(\alpha)$ is clearly a surjective homomorphism. If $\varphi(f/g)=0$ then $f(\alpha)=0$. Let $Y=f^{-1}(0)$ and set $h=e_Y$. Then $fh=0$, $h(\alpha)=1$ and so $h\notin m$. Then $f/g=0/h=0$ and φ is injective.

(3) Let m be a maximal ideal of $C(X,D)$. Then M_m is a torsion-free $C(X,D)_m$-module. By (2) $C(X,D)_m$ is a PID, so M_m is free. Thus M is projective. $\square$

The following is presumably well-known.

Lemma 2.2. *If A is a reduced ring and $V_1,\dots,V_t$ is a disjoint clopen cover of Spec(A) then there exist orthogonal idempotents $e_1,\dots,e_t$ of A such that $A\cong\oplus Ae_i$ and, for each i, Spec(Ae_i) is homeomorphic to V_i.*

Proof. Let $V=V_1$. Then V closed means $V=V(I)$ for some ideal I and V open means $V=V(J)'$ for some ideal J (here Y' means the complement of Y). Then

$$I\cap J=\bigcap_{P\in\mathrm{Spec}(A)}P=\mathrm{nil}(A)=0$$

and $V \cap V' = \emptyset$ implies $I + J = A$. Thus $A = I \oplus J$. Write $1 = e + f$ with $e \in I$ and $f \in J$. Then $I = Ae$, $J = Af$ and, as $J \cong A/I$, $\operatorname{Spec}(J) \cong V(I) = V_1$. Similarly decompose Af. $\square$

Proposition 2.3. *Let P be a finitely generated projective module over $\tilde{S} = C(X, D)$. Then there exist orthogonal idempotents $e_1, \dots, e_t$ such that:*

(1) *$P = \oplus P e_i$.*
(2) *For each i, $\tilde{S} e_i \cong C(Y_i, D)$ for some clopen $Y_i \subset X$. The set $\{Y_i : 1 \le i \le t\}$ is a disjoint clopen cover of X.*
(3) *Each Pe_i is a free $\tilde{S}e_i$-module.*

Further, if P is faithful then P contains a copy of $\tilde{S}$ as a direct summand.

Proof. The rank map $r : \operatorname{Spec}(\tilde{S}) \to \mathbb{N}$ given by $r(I) = \operatorname{rank}(P_I)$ is continuous and, as P is finitely generated, bounded. Let $V_1, \dots, V_t$ be a clopen cover of $\operatorname{Spec}(\tilde{S})$ such that $r|V_i$ is constant for each i. Let $e_1, \dots, e_t$ be the associated idempotents, as in (2.2). Then $\tilde{S} \cong \oplus \tilde{S} e_i$ and $P \cong \oplus P e_i$.

Now $e_i(\alpha)^2 = e_i(\alpha)$ for all $\alpha \in X$ so, as D is a domain, $e_i(\alpha) = 0$ or 1. Set $Y_i = e_i^{-1}(1)$. Then $\{Y_i\}$ is disjoint clopen cover of X. Map $\tilde{S} e_i \to C(Y_i, D)$ by $fe_i \mapsto fe_i|Y_i$. This is easily seen to be an isomorphism.

Each Pe_i is a projective $\tilde{S}e_i$-module of constant rank. hence Pe_i is $\tilde{S}e_i$-free by [4,1.1]. Lastly, if P is faithful then no $Pe_i = 0$. So each Pe_i contains at least one copy of $\tilde{S}e_i$ as a direct summand. Thus P contains a copy of $\tilde{S}$ as a direct summand. $\square$

We recall determinants. Let A be a ring and let P be a finitely generated projective A-module. Suppose there are idempotents e_i such that $A = \oplus A e_i$, $P = \oplus P e_i$ and each Pe_i is a projective Ae_i-module of constant rank r_i. This holds if A is noetherian or if $A = C(X, D)$ by (2.3). The *determinant* of P is the class in $\operatorname{Pic}(A)$ of the rank-one projective module:

$$\det(P) = \bigoplus_i \wedge^{r_i} P e_i.$$

If φ is an endomorphism of P then the induced endomorphism of $\det(P)$ is multiplication by a unique element of A, called $\det \varphi$. When $P = A^k$ is free then φ is given by a matrix and $\det \varphi$ is the usual matrix determinant.

We extend, in one direction, Wiegand's Lifting Theorem [12,1.1].

Proposition 2.4. *Let $I \subset \tilde{S} = C(X, D)$ be an ideal and let P be a finitely generated projective $C(X, D)$-module. Suppose φ is an $\tilde{S}/I$-automorphism of P/IP with determinant 1. Then there exists an automorphism ψ of P such that ψ induces φ.*

Proof. We may assume P is free by (2.3) as $\operatorname{Aut}(\oplus P e_i) = \oplus \operatorname{Aut}(P e_i)$. Let $\bar{M}$ be a matrix for φ. Let M be any lifting to $C(X, D)$. Suppose $M = (m_{ij})$ with $\det M = 1 + z$, for some $z \in I$. There is a clopen cover $Y_1, \dots, Y_t$ such that all m_{ij} and z are constant on each Y_k. Call these constants $m_{ij}(k)$ and $z(k)$. Then $M(k)$ gives an endomorphism of a free D-module with determinant 1 modulo $z(k)$. By the Lifting Theorem for D [12,1.1] we can find a matrix $N(k) = (n_{ij}(k))$ over D such that all $n_{ij}(k) \in z(k)D$ and $\det(M(k) + N(k)) = 1$. Set $n_{ij} = \sum_k n_{ij}(k) e_{Y_k}$ and for the $p_{ij}(k) \in D$ that satisfy $n_{ij}(k) = p_{ij}(k) z(k)$, set

$p_{ij} = \sum_k p_{ij}(k)e_{Y_k}$. Then for all $\alpha \in X$ we have $n_{ij}(\alpha) = p_{ij}(\alpha)z(\alpha)$ and so each $n_{ij} \in I$. And we still have $\det(M+N) = 1$. $\square$

If, in the set-up of (2.4), $\det\varphi \neq 1$ we can still lift to an endomorphism (not necessarily an automorphism) of P.

Corollary 2.5. *Let $\tilde{S} = C(X, D)$ and let P be a faithful, finitely generated projective $\tilde{S}$-module. Let I be an ideal of $\hat{S}$.*

1. *Given any $\bar{x} \in (\tilde{S}/I)^*$ there exists an automorphism of P/IP of determinant $\bar{x}$.*
2. *If φ is an automorphism of P/IP then there exists an endomorphism of P that induces φ.*

Proof. (1) We can write $P = \tilde{S} \oplus N$, for some $\tilde{S}$-module N by (2.3). Let $x \in \tilde{S}$ lift $\bar{x}$. Let μ be the endomorphism of P that is multiplication by x in the first coordinate and the identity in the second. Then μ induces say $\bar{\mu} : P/IP \to P/IP$ of determinant $\bar{x}$. As $\bar{x}$ is a unit, $\bar{\mu}$ is an automorphism.

(2) Let $\det\varphi = \bar{x} \in (\tilde{S}/I)^*$. Then $\varphi\bar{\mu}^{-1}$is an automorphism of determinant 1, where $\bar{\mu}$ is the automorphism constructed in the first paragraph. Lift this to an automorphism ψ of P by (2.4). Then $\psi\mu$ is an endomorphism of P that induces φ. $\square$

3. The construction.

Here we simply follow the development in [12], substituting results from the first two sections for the results on noetherian rings used by Wiegand. For the reader's convenience we supply most of the details.

We continue with the notation of the last section : X is a compact, Hausdorff, totally disconnected topological space, D is a PID with the discrete topology and $\tilde{S} = C(X, D)$. Further, let QD be the quotient field of D and set $TQ = C(X, QD)$. Let S be any ring with total quotient ring TQ and integral closure, in TQ, $\tilde{S}$. The two cases of $S \subset \tilde{S} \subset TQ$ that we need are $R \subset C(X, \mathbb{Z}) \subset C(X, \mathbb{Q})$ and $R_{IR} \subset C(X, \mathbb{Z}_{(2)}) \subset C(X, \mathbb{Q})$. Let $\mathcal{C}$ be the conductor $(S : \tilde{S})$.

Let M be a finitely generated, torsion-free S-module. Let $T(M)$ be the torsion submodule of $\tilde{S} \otimes M$ (all tensor products are over S unless otherwise specified). Define $\tilde{S}M$ to be $\tilde{S} \otimes M/T(M)$. Note that the map $M \to \tilde{S}M$ is injective since being torsion-free is equivalent to $M \to TQ \otimes M$ being injective. By construction $\tilde{S}M$ is a finitely generated, torsion-free $\tilde{S}$-module hence projective by (2.1).

Lemma 3.1. *Suppose M is a finitely generated, torsion-free S-module. If M is faithful and $\mathcal{C}$ contains a regular element then $\tilde{S}M$ is faithful.*

Proof. Suppose there exists a $y \in \tilde{S}$ such that $y\tilde{S}M = 0$, that is, $y(\tilde{S} \otimes M) \subset T(M)$. Since M is finitely generated we can find a regular $z \in \tilde{S}$ such that $zy(\tilde{S} \otimes M) = 0$. Let $w \in \mathcal{C}$ be regular. Then $wzy \in S$ and $0 = wzy(1 \otimes M) = 1 \otimes wzyM$. Now M is torsion-free so $M \to TQ \otimes M$ is injective. Hence $wzyM = 0$. Then M faithful implies $wzy = 0$ and hence that $y = 0$ since wz is regular. $\square$

We can write M in terms of the *standard pull-back* :

$$\begin{array}{ccc} M & \longrightarrow & \tilde{S}M \\ \Big\downarrow & & \Big\downarrow \\ M/\mathcal{C}M & \longrightarrow & \tilde{S}M/\mathcal{C}M. \end{array}$$

Suppose now that M is faithful. For any $x \in (\tilde{S}/\mathcal{C})^*$ choose an automorphism φ of $\tilde{S}M/\mathcal{C}M$ with $\det\varphi = x$, which is possible by (2.5). Define M^x by the pull-back:

$$\begin{array}{ccccc} M^x & & & & \tilde{S}M \\ \Big\downarrow & & & & \Big\downarrow \\ M/\mathcal{C}M & \longrightarrow & \tilde{S}M/\mathcal{C}M & \xrightarrow{\varphi} & \tilde{S}M/\mathcal{C}M. \end{array}$$

We will let $\pi : \tilde{S}^* \to (\tilde{S}/\mathcal{C})^*$ be the natural projection.

Proposition 3.2. *Let M be a faithful, finitely generated torsion-free $\tilde{S}$-module.*

1. *M^x is well-defined, that is, it does not depend on the choice of the automorphism φ of determinant x.*
2. *$(M^x)^y \cong M^{xy}$.*
3. *$M^x \cong M$ iff there exists an automorphism $\theta : \tilde{S}M/\mathcal{C}M \to \tilde{S}M/\mathcal{C}M$ of determinant xu, where $u \in \pi(\tilde{S}^*)$, such that $\theta(M/\mathcal{C}M) \subset M/\mathcal{C}M$. In particular, if $x \in \pi(\tilde{S}^*)$ then $M^x \cong M$.*

Proof. These results are part of [12,2.2]. The proofs are diagram chases, which continue to be valid here, and liftings which follow from (2.4) and (2.5). □

We now revert to our original set-up : R is a reduced Witt ring, X is its space of orderings and $\tilde{R} = C(X, \mathbb{Z})$.

Proposition 3.3. *Let R be a reduced Witt ring with finite stability index. Let M be a faithful, finitely generated, torsion-free R-module. Then for all $x \in (\tilde{R}/\mathcal{C})^*$, we have $(M^x)_m \cong M_m$ for each maximal ideal m of R.*

Proof. Now $\mathcal{C}$ contains regular element, namely 2^s where s is the stability index. So $\tilde{R}M$ is faithful and hence contains a copy of $\tilde{R}$ as a direct summand by (2.3). So we can write $\tilde{R}M/\mathcal{C}M \cong (\tilde{R}/\mathcal{C}) \oplus N$, for some $(\tilde{R}/\mathcal{C})$-module. Let φ be multiplication by $x = f + \mathcal{C} \in (\tilde{R}/\mathcal{C})^*$ in the first coordinate and the identity on the second coordinate. Let m be a maximal ideal of R and set $y = f + \mathcal{C}_m \in (\tilde{R}_m/\mathcal{C}_m)^*$. Now $\mathcal{C}_m = (R_m : \tilde{R}_m)$ by (1.4) so $\tilde{R}_m M_m/\mathcal{C}_m M_m \cong \tilde{R}_m/\mathcal{C}_m \oplus N_m$. Let ψ be multiplication in the first coordinate by y and the identity on the second coordinate. If j is the map $\tilde{R}M/\mathcal{C}M \to \tilde{R}_m M_m/\mathcal{C}_m M_m$ then $\psi j = j\varphi$.

Define $\alpha : (M^x)_m \to (M_m)^y$ by:

$$\frac{1}{q}(g \otimes m_1 + T(M), m_2 + \mathcal{C}M) \mapsto \left(\frac{g}{q} \otimes m_1 + T(M_m), \frac{m_2}{q} + \mathcal{C}_m M_m\right),$$

where $q \in R \setminus m$, $g \in \tilde{R}$ and $m_1, m_2 \in M$. We check the image is indeed in $(M_m)^y$.

$$\begin{aligned}\psi((1 \otimes \frac{m_2}{q} + T(M_m)) + \mathcal{C}_m M_m) &= \frac{1}{q} \cdot \psi((1 \otimes m_2 + T(M_m)) + \mathcal{C}_m M_m) \\ &= \frac{1}{q} \cdot \psi j((1 \otimes m_2 + T(M)) + \mathcal{C} M) \\ &= \frac{1}{q} \cdot j\varphi((1 \otimes m_2 + T(M)) + \mathcal{C} M) \\ &= \frac{1}{q} \cdot j(g \otimes m_1 + T(M)) \\ &= \frac{g}{q} \otimes m_1 + T(M_m),\end{aligned}$$

where the computation of φ follows since we began with $(g \otimes m_1 + T(M), m_2 + \mathcal{C}M) \in M^x$. Thus α is well-defined and can easily be check to be an isomorphism. Thus $(M^x)_m \cong (M_m)^y$. Now $\pi : (\tilde{R}_m)^* \to (\tilde{R}_m/\mathcal{C}_m)^*$ is surjective by (1.34) hence, by (3.2) applied to $S = R_m$, $(M_m)^y \cong M_m$. So $(M^x)_m \cong M_m$. $\square$

Proposition 3.4. *Let R have finite stability index. Let M, N be faithful, finitely generated torsion-free R-modules. If $\tilde{R}M \cong \tilde{R}N$ and $M_{IR} \cong N_{IR}$ then $N \cong M^x$ for some $x \in (\tilde{R}/\mathcal{C})^*$.*

Proof. Set $S = R \setminus IR$. Each $s \in S$ is a unit modulo $\mathcal{C}$ as some $2^k \in \mathcal{C}$ and $s \notin IR$ implies $(2^k, s) = R$. So

$$S^{-1}M/\mathcal{C}S^{-1}M = M/\mathcal{C}M \quad \text{and} \quad S^{-1}(\tilde{R}/\mathcal{C}) = \tilde{R}/\mathcal{C}.$$

We are given isomorphisms $\alpha : S^{-1}M \to S^{-1}N$ and $\beta : \tilde{R}M \to \tilde{R}N$. Now α induces an isomorphism $\alpha_1 : M/\mathcal{C}M \to N/\mathcal{C}N$ and hence an isomorphism

$$\alpha_2 = 1_{S^{-1}\tilde{R}} \otimes \alpha : S^{-1}\tilde{R} \otimes_{S^{-1}R} S^{-1}M \to S^{-1}\tilde{R} \otimes_{S^{-1}R} S^{-1}N.$$

Now $S^{-1}\tilde{R} \otimes_{S^{-1}R} S^{-1}M \cong S^{-1}(\tilde{R} \otimes_R M)$. Hence α_2 maps $S^{-1}(\tilde{R} \otimes M)$ to $S^{-1}(\tilde{R} \otimes N)$. Since α_2 takes torsion to torsion it induces an isomorphism $\alpha_3 : S^{-1}(\tilde{R}M) \to S^{-1}(\tilde{R}N)$. This in turn gives an isomorphism $\alpha_4 : \tilde{R}M/\mathcal{C}M \to \tilde{R}N/\mathcal{C}N$ upon modding out by $\mathcal{C}S^{-1}\tilde{R}M$ and its image.

Further, β also induces an isomorphism $\beta_1 : \tilde{R}M/\mathcal{C}M \to \tilde{R}N/\mathcal{C}N$. Set $\varphi = \beta_1^{-1}\alpha_4$, an automorphism of $\tilde{R}M/\mathcal{C}M$. Let $x = \det \varphi \in (\tilde{R}/\mathcal{C})^*$. Then the pull-back of M^x and the standard pull-back of N are isomorphic via β, β_1, α_1. Thus $N \cong M^x$. $\square$

Theorem 3.5. *Let R be a reduced Witt ring with finite stability index. Let M, N be faithful, finitely generated torsion-free R-modules. The following are equivalent:*

(1) *$M^x \cong N$ for some $x \in (\tilde{R}/\mathcal{C})^*$.*
(2) *$M_m \cong N_m$ for all maximal ideals m of R.*
(3) *$M_{IR} \cong N_{IR}$.*

Proof. $(1 \to 2)$ is (3.3) and $(2 \to 3)$ is clear so we show $(3 \to 1)$. Given that $M_{IR} \cong N_{IR}$ we need to show $\tilde{R}M \cong \tilde{R}N$ by (3.4). To simplify the notation, set $P = \tilde{R}M$ and $Q = \tilde{R}N$. We can find a disjoint clopen cover $\{Y_i\}$ of X such that the rank functions (from $\mathrm{Spec}(R)$ to $\mathbb{N}$) of both M and N are constant on each Y_i. Then there is a set of orthogonal idempotents $\{e_i\}$ such that

$$P = \bigoplus_{i=1}^{t} Pe_i \quad \text{and} \quad Q = \bigoplus_{i=1}^{t} Qe_i,$$

where each Pe_i and Qe_i is a free $\tilde{R}e_i$-module (see the proof of (2.3)). Let $r_i(P)$ denote the rank (over $\tilde{R}e_i$) of Pe_i; similarly define $r_i(Q)$. Clearly $P \cong Q$ iff $r_i(P) = r_i(Q)$ for all i.

Let $\alpha \in X$ and let $\tilde{m} = \mathcal{P}(\alpha, 2)$. Write (as in (2.3)) $\tilde{R}e_i$ as $C(Y_i, \mathbb{Z})$. If $\alpha \in Y_i$ then for $j \neq i$ we have $(Pe_j)_{\tilde{m}} = 0$ since $e_j e_i = 0$ and $e_i \notin \tilde{m}$. Thus

$$P_{\tilde{m}} = (Pe_i)_{\tilde{m}} \cong (C(Y_i, \mathbb{Z})^{r_i(P)})_{\tilde{m}} \cong (\mathbb{Z}_{(2)})^{r_i(P)},$$

by (2.1). Hence $r_i(P) = r_i(Q)$ for all i iff for all $\alpha \in X$, $P_{\tilde{m}} \cong Q_{\tilde{m}}$ where $\tilde{m} = \mathcal{P}(\alpha, 2)$.

Let A be a finitely generated projective $\tilde{R}$-module. The map

$$\begin{gathered} \varphi : \tilde{R}_{\tilde{m}} \otimes_{R_{IR}} A_{\tilde{m}} \to (\tilde{R} \otimes_R A)_{\tilde{m}} \\ \frac{f}{g} \otimes \frac{a}{x} \mapsto \frac{f \otimes a}{gx}, \end{gathered}$$

is a well-defined isomorphism (with inverse $(f \otimes a)/g \mapsto (f/g) \otimes a$). Here $f \in \tilde{R}$, $g, x \in \tilde{R} \backslash \tilde{m}$ and $a \in A$. Next, let *tor* denote the torsion submodule of $(\tilde{R} \otimes P)_{\tilde{m}}$. The projection

$$\psi : (\tilde{R} \otimes A)_{\tilde{m}} \to (\tilde{R}A)_{\tilde{m}} = (\tilde{R} \otimes A/T(A))_{\tilde{m}}$$

has kernel *tor*. Namely, if $\sum (f_i \otimes a_i)/g_i \in tor$ then there exists a regular $z \in \tilde{R}_{\tilde{m}}$ with $z \sum (f_i \otimes a_i)/g_i = 0$. Thus $zh \sum f_i \otimes a_i = 0$ for some $h \notin \tilde{m}$. This implies $h \sum f_i \otimes a_i$ is torsion in $\tilde{R} \otimes A$. Hence

$$\sum \frac{f_i \otimes a_i}{g_i} = \sum \frac{hf_i \otimes a_i}{hg_i} \in \ker(\psi).$$

The reverse inclusion is similar. We thus have for all $\tilde{m} = \mathcal{P}(\alpha, 2)$:

$$P_{\tilde{m}} = (\tilde{R}M)_{\tilde{m}} \cong (\tilde{R} \otimes M)_{\tilde{m}}/tor \cong (\tilde{R}_{\tilde{m}} \otimes_{R_{IR}} M_{IR})/tor.$$

So $M_{IR} \cong N_{IR}$ implies $P_{\tilde{m}} \cong Q_{\tilde{m}}$ for all $\tilde{m} = \mathcal{P}(\alpha, 2)$ which, by the first part of the proof, implies $P \cong Q$. $\square$

Definition. For an R-module M that is faithful, finitely generated and torsion-free, the *genus* $\mathrm{Genus}(M)$ is the set of isomorphism classes of R-modules N, also faithful, finitely generated and torsion-free, with $M_m \cong N_m$ for all maximal ideals m of R. Δ_M is the set of $x \in (\tilde{R}/\mathcal{C})^*$ such that there exists an automorphism θ of $\tilde{R}M/\mathcal{C}M$ of determinant x such that $\theta(M\mathcal{C}M) \subset M/\mathcal{C}M$. Δ_m is a subgroup of $(\tilde{R}/\mathcal{C})^*$.

Theorem 3.6. *Let R be a reduced Witt ring of finite stability index. Let M be a faithful, finitely generated torsion-free R-module. Then the following is exact:*

$$1 \to \pi(\tilde{R}^*)\Delta_M \to (\tilde{R}/\mathcal{C})^* \to \mathit{Genus}(M) \to 1,$$

where $\pi : \tilde{R}^* \to (\tilde{R}/\mathcal{C})^*$ is the natural projection.

Proof. Combine (3.5) and (3.2)(3). □

Note that Genus(M) inherits the structure of a group from the sequence of (3.6). To write the operation explicitly: if $A, B \in$ Genus(M) then $A + B = M^{xy}$ where $A = M^x$ and $B = M^y$. This is well-defined since if $A = M^z$ and $B = M^w$ also then by (3.2)(2) we have $M = M^{x^{-1}z}$ and $M = M^{yw^{-1}}$. Hence $M^{xy} = M^{zw}$.

Lemma 3.7. *Let R have finite stability index. Let M, N be faithful, finitely generated torsion-free R-modules.*

(1) *If $x \in (\tilde{R}/\mathcal{C})^*$ then $(M \oplus N)^x \cong M^x \oplus N$.*
(2) $\Delta_M \Delta_N \subset \Delta_{M \oplus N}$.

Proof. (1) It is easy to check that:

$$\frac{\tilde{R}(M \oplus N)}{\mathcal{C}(M \oplus N)} \cong \frac{\tilde{R}M}{\mathcal{C}M} \oplus \frac{\tilde{R}N}{\mathcal{C}N}.$$

Given $x \in (\tilde{R}/\mathcal{C})^*$ pick an automorphism φ of $\tilde{R}M/\mathcal{C}M$ with determinant x. Then $\psi = \varphi \oplus 1$ is an automorphism of $\tilde{R}(M \oplus N)/\mathcal{C}(M \oplus N)$ with determinant x. Thus $(M \oplus N)^x \cong M^x \oplus N$.

(2) If $x \in \Delta_M$ then let θ be the automorphism of $\tilde{R}M/\mathcal{C}M$ with determinant x and satisfying $\theta(M/\mathcal{C}M) \subset M/\mathcal{C}M$. Then $\theta \oplus 1$ has determinant x also and takes $(M \oplus N)/\mathcal{C}(M \oplus N)$ into itself. Thus $x \in \Delta_{M \oplus N}$. Similarly for $x \in \Delta_N$. □

For ring-orders A one has $\Delta_M \Delta_N = \Delta_{M \oplus N}$ by [13, 1.7]. We have been unable to decide if this holds for our Witt rings. For this reason, we are unable to extend the various cancellation results of [6],[12] and [13].

Corollary 3.8. *Let R have finite stability index. Let M be a finitely generated torsion-free R-module that contains R as a direct summand. Then there is a projection $\mathit{Pic}(R) \twoheadrightarrow \mathit{Genus}(M)$.*

Proof. Pic$(R) =$ Genus(R) and we have $\Delta_R \subset \Delta_M$ by (3.7). So the sequence of (3.6) gives the projection:

$$\mathrm{Pic}(R) \cong \frac{(\tilde{R}/\mathcal{C})^*}{\pi(\tilde{R}^*)\Delta_R} \to \frac{(\tilde{R}/\mathcal{C})^*}{\pi(\tilde{R}^*)\Delta_M} \cong \mathrm{Genus}(M).$$

□

When $M = R^n$ (3.8) gives back half of [5, 2.8] which says, in this notation, that Genus$(R^n) = $ Pic(R).

Corollary 3.8. *Suppose R has stability index $st(R) \leq 2$. Let M, N be faithful, finitely generated torsion-free modules. The following are equivalent:*
(1) $M \cong N$.
(2) $M_m \cong N_m$ *for all maximal ideals* m *of* R.
(3) $M_{IR} \cong N_{IR}$.

Proof. We have $4 \in \mathcal{C}$ so each element $x \in (\tilde{R}/\mathcal{C})^*$ is represented by a function $f \in \tilde{R} = C(X, \mathbb{Z})$ with values in $\{\pm 1\}$. Thus $x \in \pi(\tilde{R}^*)$. Then (3.6) gives Genus(M) is trivial. Hence (2) implies (1). Clearly (1) implies (2) and the equivalence of (2) and (3) is (3.5). $\square$

The equivalence of (1) and (2) fails if $st(R) \geq 3$ since in this case $\mathrm{Pic}(R) = \mathrm{Genus}(R)$ is non-trivial [5,2.5]. Also (3.9) is a partial generalization of [4, 1.17] (the case of $M = R$) and [5, 2.5] (the case of M free).

4. Examples.

We begin with a simple example to illustrate Wiegand's construction of M^x and the failure, in general, of cancellation.

Example. Let $R = \mathbb{Z}[E_3]$. Here $\tilde{R} = \mathbb{Z}^8$, $\mathcal{C} = 8\tilde{R}$ and $\tilde{R}/\mathcal{C} = (\mathbb{Z}/8\mathbb{Z})^8$. We have $(\tilde{R}/\mathcal{C})^* = \{\pm 1, \pm 3\}^8$ and $\pi(R^*) = \{\pm 1\}^8$. Let $M = R$ so that $\tilde{R}M = \tilde{R}$. We want to compute Δ_M.

Let $x \in (\tilde{R}/\mathcal{C})^*$; write $x = y + \mathcal{C}$ for some $y \in \tilde{R}$. The only automorphism of $\tilde{R}M/\mathcal{C}M = \tilde{R}/\mathcal{C}$ of determinant x is multiplication by x, call it μ. Then $x \in \Delta_M$ iff $\mu(M/\mathcal{C}M) \subset M/\mathcal{C}M$ iff $x(1+\mathcal{C}) \in M/\mathcal{C}$ iff $y+\mathcal{C} \in M/\mathcal{C}$ iff $y \in M = R$. Now for any $q \in R$, $q+\mathcal{C} \in (\tilde{R}/\mathcal{C})^*$ iff every signature of q is odd iff q is odd-dimensional.Hence $\Delta_M = (R \setminus IR)/\mathcal{C}$.

We next compute $\pi(R^*)\Delta_M$. If $z_1, z_2 \in E_3$ are independent then $\langle z_1, z_2, z_1 z_2\rangle$ has signature 3 at the two orderings with z_1, z_2 positive, and signature -1 at the other six orderings. Indeed, any element of $(\tilde{R}/\mathcal{C})^*$ with two coordinates 3 and the rest -1 comes from an element of R and hence is in Δ_M. So $\pi(R^*)\Delta_M$ has index at most two in $(\tilde{R}/\mathcal{C})^*$. Now none of the elements $(\pm 3, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ of $\tilde{R}$ are in R [3, p. 380] so $y = (3, 1, 1, 1, 1, 1, 1, 1) \notin \pi(R^*)\Delta_M$. Thus $\pi(R^*)\Delta_M$ has index two in $(\tilde{R}/\mathcal{C})^*$ and y represents the non-trivial coset. Thus:

$$R^y = \{(r + \mathcal{C}, z) \in (R/\mathcal{C}) \oplus \tilde{R} : \hat{r}(\alpha_1) = 3z_1, \hat{r}(\alpha_i) = z_i \quad \text{for} \quad i \geq 2\},$$

is locally isomorphic, but not isomorphic, to R. We know $\mathrm{Pic}(R) = \mathrm{Genus}(M)$ consists of R and $I = (\langle 1, 1, t_1\rangle, \langle 1, t_2, t_2 t_3\rangle)$, where t_1, t_2, t_3 generate E_3, by [3, p. 380] again. So $R^y \cong I$.

Now let $\rho_1 = \langle\langle t_1, t_2, t_3\rangle\rangle$, the 3-fold Pfister form $\langle 1, t_1\rangle \otimes \langle 1, t_2\rangle \otimes \langle 1, t_3\rangle$. Let α_i, for $1 \leq i \leq 8$, be the orderings on R and let α_1 be the ordering with t_1, t_2, t_3 all positive. Let e_i be the characteristic function for α_i, that is, $e_i(\alpha_j) = \delta_{ij}$. Let ρ_i be the 3-fold Pfister form with $8e_i = \rho_i$. Let $N = R/\mathrm{ann}(\rho_1)$; N is torsion-free, but not faithful. Now $\tilde{R}N = e_1 \otimes \mathbb{Z} \cong \mathbb{Z}$ since if $i \geq 2$ we have:

$$8(e_i \otimes (1 + \mathrm{ann}(\rho_1)) = 1 \otimes (\rho_i + \mathrm{ann}(\rho_1)) = 0,$$

and so $e_i \otimes N \subset T(N)$. Then $\tilde{R}N/\mathcal{C}N \cong \mathbb{Z}/8\mathbb{Z}$. Let φ be the automorphism of $\tilde{R}N/\mathcal{C}N$ that, as a map of $\mathbb{Z}/8\mathbb{Z}$, is multiplication by 3. Then $\det\varphi = (3,1,1,1,1,1,1,1)$. Note that $\varphi(N/\mathcal{C}N) \subset N/\mathcal{C}N$. So $\det\varphi \in \Delta_N$.

Let $A = R \oplus N$. A is torsion-free and faithful. By [13, 1.7] we have $\Delta_A = \Delta_R\Delta_N = (\tilde{R}/\mathcal{C})^*$. We thus obtain from (3.6) and (3.7)

$$R \oplus N \cong (R \oplus N)^y \cong R^y \oplus N \cong I \oplus N.$$

Since R is not isomorphic to I we see that cancellation fails here. For other examples of the failure of cancellation, when R is noetherian, see [12, 2.3]

We next consider odd degree extensions. For a Witt ring R, R_t denotes the torsion ideal of R and R_{red} is R/R_t, which is again a Witt ring. For a field F we let $\sum F^{*2}$ denote the set of non-zero sums of squares in F.

Lemma 4.1. *Let K/F be an odd degree extension of formally real fields. Let $R = (WF)_{red}$ and $M = (WK)_{red}$. Then M is a faithful, torsion-free R-module. M is finitely generated over R, iff $F^*/\sum F^{*2}$ has finite index in $K^*/\sum K^{*2}$.*

Proof. Let $i_* : WF \to WK$ be induced by the inclusion $F \subset K$. M is an R-module via $(q + W_tF)(\varphi + W_tK) = i_*(q)\varphi + W_tK$, where $q \in WF$ and $\varphi \in WK$. If $i_*(q) \in W_tK$ and $\alpha \in X_F$ then α extends to, say, $\beta \in X_K$ [11, III,4.3] and $0 = \widehat{i_*(q)}(\beta) = \hat{q}(\alpha)$. So $q \in W_tF$ and M is a faithful R-module. If $q + W_tF$ is regular then $\hat{q}(\alpha) \neq 0$ for all $\alpha \in X_F$ and so $i_*(q)\varphi \in W_tK$ implies $\varphi \in W_tK$. So M is torsion-free. Lastly, $F^* \cap \sum K^{*2} = \sum F^{*2}$ since if $a \in F^* \cap \sum K^{*2}$ then $n\langle 1\rangle \perp \langle -a\rangle$ is isotropic over K, for some n, and hence isotropic over F by Springer's Theorem [11, II, 5.3]. Thus:

$$\frac{F^*}{\sum F^{*2}} \cong \frac{F^*\sum K^{*2}}{\sum K^{*2}} \hookrightarrow \frac{K^*}{\sum K^{*2}}.$$

The group associated to R, G_R, is $F^*/\sum F^{*2}$. Similarly, $G_M = K^*/\sum K^{*2}$. So M is finitely generated over R iff G_M is finitely generated over G_R. □

Lemma 4.2. *Let K/F be an odd degree extension of formally real fields. Let $R = (WF)_{red}$ and $M = (WK)_{red}$. Suppose M is finitely generated over R. Let $\alpha \in X_F$, p an odd prime and $m = P(\alpha, p) \subset R$. Then $M_m \cong \mathbb{Z}_{(p)}^r$, where r is the number of extensions of α to K.*

Proof. Let $\beta_1, \ldots, \beta_r$ be the extensions of α to K. We map M_m to $\mathbb{Z}_{(p)}^r$ by:

$$\frac{\psi}{s} \mapsto \left(\frac{\hat{\psi}(\beta_1)}{\hat{s}(\alpha)}, \ldots, \frac{\hat{\psi}(\beta_r)}{\hat{s}(\alpha)}\right),$$

where $\psi = \varphi + W_tK$ and $s = q + W_tF$ for some $\varphi \in WK$ and $q \in WF$. Suppose ψ/s is sent to 0. Then $\hat{\varphi}(\beta_i) = 0$ for each i. Set $Z = \{\beta \in X_K : \hat{\varphi}(\beta) \neq 0\}$. Z is clopen. Let $\epsilon : X_K \to X_F$ be the restriction map. Then $\epsilon(Z)$ is clopen by the Open Mapping Theorem of [2]. Thus there exist a $u \in WF$ and positive integer k such that $\hat{u}(\gamma)$ is 2^k for $\gamma \notin \epsilon(Z)$

and 0 for $\gamma \in \epsilon(Z)$. No β_i is in Z so α is not in $\epsilon(Z)$. Thus $u + W_tF \notin P(\alpha, p)$ and $u\varphi = 0$. Hence $\psi/s = 0/u$ and the map is injective.

To show the map is surjective, let $\bar{x} = (a_1/b, \ldots, a_r/b) \in \mathbb{Z}^r_{(p)}$. Pick clopen sets $Y_1, \ldots, Y_r$ in X_K such that for every i we have $Y_i \cap \{\beta_1, \ldots, \beta_r\} = \{\beta_i\}$. There is a positive integer n such that for each i there exists a $v_i \in WK$ with $\hat{v}_i$ equal to 2^n on Y_i and 0 off Y_i. Then:

$$\frac{\sum a_i v_i + W_tK}{2^n b + W_tF} \mapsto \bar{x}.$$

□

Corollary 4.3. *Let K and L be odd degree extensions of formally real F of finite stability index. Let $R = (WF)_{red}$, $M = (WK)_{red}$ and $N = (WL)_{red}$. Suppose M and N are finitely generated R-modules. If $M_{IR} \cong N_{IR}$ as R-modules then*

(1) *Each $\alpha \in X_F$ has the same number of extensions to K as to L.*
(2) *$M^x \cong N$ for some $x \in (\tilde{R}/\mathcal{C})^*$.*
(3) *If K/F is separable then Genus(M) is a quotient of Pic(R).*

Proof. (2) follows from (3.5) and (1) follows from (3.5) and (4.2). When K/F is separable then K is a simple extension of F and so $i_* : WF \to WK$ is a split monomorphism. In particular, $WK \cong WF \oplus Q$, for some WF-module Q. Thus R is a direct summand of M. Apply (3.8). □

We remark that if $M_{IR} \cong N_{IR}$ as rings then it is simple to show $M \cong N$. Thus the significance of (4.3) comes from having the weaker condition that M_{IR} and N_{IR} be isomorphic as R-modules.

Corollary 4.4. *Let K and L be odd degree extensions of formally real F with $st(F) \le 2$. Let $R = (WF)_{red}$, $M = (WK)_{red}$ and $N = (WL)_{red}$. Suppose M and N are finitely generated R-modules. If $M_{IR} \cong N_{IR}$ as R-modules then $M \cong N$.*

Proof. Combine (3.9) with (4.1). □

We close with some computations of $\mathcal{C}_R$, the conductor of R, and $\tilde{R}/\mathcal{C}_R$ for a reduced Witt ring R.

Lemma 4.5. *Let $R = S[E_1]$, where $E_1 = \{1, t\}$ is the group of order 2 and S is a reduced Witt ring. If $s_1 + s_2t \in \mathcal{C}_R$, for some $s_i \in S$, then $s_1, s_2 \in \mathcal{C}_S$.*

Proof. Each ordering α on S has two extensions to R, one with $t > 0$,denoted α^+, and one with $t < 0$, denoted α^-. Let $B \subset X_S$ be clopen. Let $A = B^+ \cup B^-$, where $B^+ = \{\alpha^+ : \alpha \in B\}$ and similarly for B^-. Then $(s_1 + s_2t)e_A = q_1 + q_2t$ for some $q_1, q_2 \in S$. Evaluate this at α^+ and α^- for some $\alpha \in B$ to get:

$$\hat{s}_1(\alpha) + \hat{s}_2(\alpha) = \hat{q}_1(\alpha) + \hat{q}_2(\alpha)$$
$$\hat{s}_1(\alpha) - \hat{s}_2(\alpha) = \hat{q}_1(\alpha) - \hat{q}_2(\alpha).$$

Thus for each $\alpha \in B$ and for $i = 1, 2$ we have $\hat{s}_i(\alpha) = \hat{q}_i(\alpha)$. Evaluate now at β^+ and β^- for some $\beta \notin B$ to get:

$$\hat{q}_1(\beta) + \hat{q}_2(\beta) = 0 = \hat{q}_1(\beta) - \hat{q}_2(\beta).$$

Thus $\hat{q}_i(\beta) = 0$ for all $\beta \notin B$ and $i = 1, 2$. This implies $s_1 e_B = q_1$ and $s_2 e_B = q_2$. Since B was arbitrary, we have $s_1, s_2 \in \mathcal{C}_S$. $\square$

Proposition 4.6. *Let $R = S[E_1]$ with $E_1 = \{1, t\}$. Then:*

1. $\mathcal{C}_R = (\langle 1, t\rangle, \langle 1, -t\rangle)\mathcal{C}_S$.
2. $\tilde{R}/\mathcal{C}_R \cong (\tilde{S}/2\mathcal{C}_S) \oplus (\tilde{S}/2\mathcal{C}_S)$.

Proof. (1) Let $I = (\langle 1, t\rangle, \langle 1, -t\rangle)\mathcal{C}_S$. Let $s \in \mathcal{C}_S$ and set $\varphi = \langle 1, t\rangle s$. We will show $\varphi \in \mathcal{C}_R$ (the proof for $\langle 1, -t\rangle s$ is the same). Let $A \subset X_R$ be clopen. Write $A = B^+ \cup C^-$ for some clopen subsets B, C of X_S. Suppose $se_B = q \in S$. Then $\varphi e_A = \varphi e_{B^+} = 2se_B = 2q \in S \subset R$. Hence $\varphi \in \mathcal{C}_R$.

Now suppose $s_1 + s_2 t \in \mathcal{C}_R$. Let $A = H(t)$, the set of orderings on R with $t > 0$. Then $(s_1 + s_2 t)e_A = q_1 + q_2 t$ for some $q_1, q_2 \in S$. Evaluating at α^+ and α^- for any $\alpha \in X_S$ gives:

$$\begin{aligned} \hat{s}_1(\alpha) + \hat{s}_2(\alpha) &= \hat{q}_1(\alpha) + \hat{q}_2(\alpha) \\ 0 &= \hat{q}_1(\alpha) - \hat{q}_2(\alpha). \end{aligned}$$

Thus $q_1 = q_2$ and $s_1 + s_2 = 2q_1$. Hence:

$$s_1 + s_2 t = -2q_1 + \langle 1, t\rangle s_2.$$

Now $\langle 1, t\rangle q_1 = (s_1 + s_2 t)e_A \in \mathcal{C}_R$. By (4.5) $q_1 \in \mathcal{C}_S$ and so $2q_1 \in I$. Again by (4.5) we have $s_2 \in \mathcal{C}_S$ and so $\langle 1, t\rangle s_2 \in I$. So $s_1 + s_2 t \in I$.

(2) First $\tilde{R} \cong \tilde{S} \oplus \tilde{S}$ via $f \mapsto (f|_{H(t)}, f|_{H(-t)})$. Also, by (1), $\mathcal{C}_R = 2\tilde{R}\mathcal{C}_S$. The result follows. $\square$

Proposition 4.7. *If $R = R_1 \sqcap R_2$, the fiber product of R_1 and R_2, then*

$$\mathcal{C}_R = (\mathcal{C}_{R_1} \cap IR_1) \oplus (\mathcal{C}_{R_2} \cap IR_2).$$

Proof. Suppose $s \in \mathcal{C}_{R_1} \cap IR_1$. Let $A \subset X_R$ be clopen. Write $A = A_1 \cup A_2$ with $A_i \subset X_{R_i}$ for $i = 1, 2$. Then $(s, 0)e_A = (se_{A_1}, 0) \in IR_1 \times 0 \subset R$. This shows that $(\mathcal{C}_{R_1} \cap IR_1) \times 0 \subset \mathcal{C}_R$. Similarly $0 \times (\mathcal{C}_{R_2} \cap IR_2) \subset \mathcal{C}_R$. Hence $(\mathcal{C}_{R_1} \cap IR_1) \oplus (\mathcal{C}_{R_2} \cap IR_2) \subset \mathcal{C}_R$.

Now suppose $(s_1, s_2) \in \mathcal{C}_R$. Let $B \subset X_{R_1}$ be clopen. We have $(s_1 e_B, 0) = (s_1, s_2)e_B \in R$. Then $s_1 e_B \in IR_1$ for every clopen $B \subset X_{R_1}$. In particular, $s_1 \in \mathcal{C}_{R_1}$ and, taking $B = X_{R_1}$, $s_1 \in IR_1$. So $s_1 \in \mathcal{C}_{R_1} \cap IR_1$ and the proof that $s_2 \in \mathcal{C}_{R_2} \cap IR_2$ is similar. $\square$

Corollary 4.8. *Suppose $R = R_1 \sqcap R_2$s finitely generated.*

1. *If neither R_i is $\mathbb{Z}$ then $\tilde{R}/\mathcal{C}_R \cong (\tilde{R}_1/\mathcal{C}_{R_1}) \oplus (\tilde{R}_2/\mathcal{C}_2)$.*
2. *If $R_1 = \mathbb{Z}$ and $R_2 \neq \mathbb{Z}$ then $\tilde{R}/\mathcal{C}_R \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \tilde{R}_2/\mathcal{C}_{R_2}$.*
3. *$R_1 = R_2 = \mathbb{Z}$ then $\mathcal{C}_R \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.*

Proof. $\tilde{R} \cong \tilde{R}_1 \oplus \tilde{R}_2$. If $R_i \neq \mathbb{Z}$ then $\mathcal{C}_{R_i} \subset IR_i$ since R_i is either a group ring (use (4.6)) or a product (use (4.7)). Thus if neither R_i is $\mathbb{Z}$ then $\mathcal{C}_r = \mathcal{C}_{R_1} \oplus \mathcal{C}_{R_2}$, by (4.7), and the result follows. If $R_i = \mathbb{Z}$ then $\mathcal{C}_{R_i} = \mathbb{Z}$ and $\mathcal{C}_{R_i} \cap IR_i = (2)$. Apply (4.7). $\square$

(4.6) and (4.8) give an inductive method of computing $\tilde{R}/\mathcal{C}$ for any finitely generated reduced Witt ring R. As an example:

$$R = (\mathbb{Z}[E_3] \sqcap \mathbb{Z}[E_2] \sqcap \mathbb{Z})[E_1]$$
$$\tilde{R}/\mathcal{C} = (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^8 \oplus (\mathbb{Z}/16\mathbb{Z})^{16}.$$

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