

THE STABLE MANIFOLD THEOREM FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

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We formulate a *Local Stable Manifold Theorem* for stochastic (ordinary) differential equations in Euclidean space that are driven by multidimensional Brownian motion. The result is stated for Stratonovich stochastic differential equations, but it is also true for stochastic differential equations of Itô-type under somewhat weaker conditions. In fact, the theorem holds for general stochastic differential equations driven by Kunita-type spatial semimartingales with stationary ergodic increments. Details of the proof of the theorem will appear elsewhere. The proof uses Ruelle-Oseledec multiplicative ergodic theory. For Stratonovich equations, the stable and unstable manifolds are dynamically characterized using forward and backward solutions of the anticipating stochastic differential equation.

Consider the stochastic differential equation

$$dx(t) = h(x(t)) dt + \sum_{i=1}^m g_i(x(t)) \circ dW_i(t), \quad (I)$$

where $h, g_i : \mathbf{R}^d \rightarrow \mathbf{R}^d, 1 \leq i \leq m$, are vector fields on \mathbf{R}^d . Suppose that for some $k \geq 1, \delta \in (0, 1)$, h is $C_b^{k, \delta}$, viz. h has all derivatives $D^j h, 1 \leq j \leq k$, continuous and globally bounded, and $D^k h$ is Hölder continuous with exponent δ . Suppose also that $g_i, 1 \leq i \leq m$, are globally bounded and are in $C_b^{k+1, \delta}$. Let $W := (W_1, \dots, W_m)$ be

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m -dimensional Brownian motion on Wiener space (Ω, \mathcal{F}, P) . Denote by $\theta : \mathbf{R} \times \Omega \rightarrow \Omega$ the canonical Brownian shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

Let $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \rightarrow \mathbf{R}^d$ be the stochastic flow generated by (I). It is known that ϕ satisfies the perfect cocycle property

$$\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),$$

for all $s, t \in \mathbf{R}$ and all $\omega \in \Omega$.

Definition.

Say that the stochastic differential equation (I) has a *stationary trajectory* if there exists an \mathcal{F} -measurable random variable $Y : \Omega \rightarrow \mathbf{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. In the sequel, we will always refer to the stationary trajectory (1) by $\phi(t, Y)$.

By the Oseledec Theorem, the linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ admits a non-random finite Lyapunov spectrum defined by

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v)|, \quad v \in \mathbf{R}^d,$$

and taking values in $\{\lambda_i\}_{i=1}^p$ with non-random (finite) multiplicities q_i , $1 \leq i \leq p$, and $\sum_{i=1}^p q_i = d$.

Definition.

A stationary trajectory $Y(\theta(t, \omega))$ of (I) is said to be *hyperbolic* if $E \log^+ |Y(\cdot)| < \infty$, and if the linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ has a Lyapunov spectrum $\{\lambda_p < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$ which *does not contain* 0.

Let $\{U(\omega), S(\omega) : \omega \in \Omega\}$ denote the unstable and stable subspaces for the linearized cocycle $(D_2\phi(t, Y(\cdot), \cdot), \theta(t, \cdot))$.

It can be shown that the (possibly anticipating) stationary trajectory $\phi(t, Y(\omega), \omega)$ is a *solution* of the anticipating Stratonovich stochastic differential equation

$$\left. \begin{aligned} d\phi(t, Y) &= h(\phi(t, Y)) dt + \sum_{i=1}^m g_i(\phi(t, Y)) \circ dW_i(t), \quad t > 0 \\ \phi(0, Y) &= Y. \end{aligned} \right\} \quad (II)$$

Furthermore, we can linearize the stochastic differential equation (I) along the stationary trajectory and then match the solution of the linearized equation with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. That is to say, the (possibly non-adapted) process $D_2\phi(t, Y(\omega), \omega)$, $t \geq 0$, satisfies the associated Stratonovich linearized stochastic differential equation

$$\left. \begin{aligned} dD_2\phi(t, Y) &= Dh(\phi(t, Y))D_2\phi(t, Y) dt + \sum_{i=1}^m Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t > 0 \\ D_2\phi(0, Y) &= I. \end{aligned} \right\} \quad (III)$$

In (III), the symbols D_2, D denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories $\phi(t, Y), D_2\phi(t, Y, \cdot)$, $t < 0$, satisfy the backward Stratonovich stochastic differential equations corresponding to (II) and (III) above. Equation (II) and its backward counterpart provide dynamic characterizations of the stable and unstable manifolds in the theorem below.

For any $\rho \in \mathbf{R}^+$ and $x \in \mathbf{R}^d$ denote by $B(x, \rho)$ the open ball in \mathbf{R}^d , center x and radius ρ . Denote by $\bar{B}(x, \rho)$ the corresponding closed ball. Let $\mathcal{C}(\mathbf{R}^d)$ be the class of all non-empty compact subsets of \mathbf{R}^d . Give $\mathcal{C}(\mathbf{R}^d)$ the Hausdorff metric d^* :

$$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$$

where $A_1, A_2 \in \mathcal{C}(\mathbf{R}^d)$, and $d(x, A_i) := \inf\{|x - y| : y \in A_i\}$, $x \in \mathbf{R}^d$, $i = 1, 2$. Let $\mathcal{B}(\mathcal{C}(\mathbf{R}^d))$ be the Borel σ -algebra on $\mathcal{C}(\mathbf{R}^d)$ with respect to the metric d^* .

We now state the local stable manifold theorem for the stochastic differential equation (I) around a hyperbolic stationary solution. Details of the proof are given in [M-S].

Theorem. (*The Stable Manifold Theorem*)

Assume that the SDE (I) satisfies the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$. Suppose the linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ has a Lyapunov spectrum $\{\lambda_p < \dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^ \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,*

(ii) \mathcal{F} -measurable random variables $\rho_i, \beta_i : \Omega^ \rightarrow [0, \infty)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:*

There are $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0}$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$ is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega)$ and is non-random.

$$(b) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_1, x_2 \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ such that

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega).$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0-1} + \epsilon_2)n}$$

for all integers $n \geq 0$. Also

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}.$$

for all $x \in \tilde{\mathcal{U}}(\omega)$. Furthermore, $\mathcal{U}(\omega)$ is the tangent space to $\tilde{\mathcal{U}}(\omega)$ at $Y(\omega)$. In particular, $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$ and is non-random.

$$(e) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_1, x_2 \in \tilde{\mathcal{U}}(\omega) \right\} \right] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\phi(-t, \cdot, \omega)(\tilde{\mathcal{U}}(\omega)) \subseteq \tilde{\mathcal{U}}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega).$$

Also

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0.$$

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.

$$\mathbf{R}^d = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

(h) The mappings

$$\begin{aligned} \Omega &\rightarrow \mathcal{C}(\mathbf{R}^d), & \Omega &\rightarrow \mathcal{C}(\mathbf{R}^d), \\ \omega &\mapsto \tilde{\mathcal{S}}(\omega) & \omega &\mapsto \tilde{\mathcal{U}}(\omega) \end{aligned}$$

are $(\mathcal{F}, \mathcal{B}(\mathcal{C}(\mathbf{R}^d)))$ -measurable.

Assume, in addition, that $h, g_i, 1 \leq i \leq m$ have bounded derivatives of all orders.

Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ .

Remarks.

(i) The stable manifold theorem still holds for the Stratonovich SDE (I) if the bounded condition on the g_i 's is replaced by the somewhat weaker requirement that the functions

$$\mathbf{R}^d \ni x \mapsto \sum_{l=1}^m \frac{\partial g_l^i(x)}{\partial x_j} g_l^j(x) \in \mathbf{R}$$

are in $C_b^{k, \delta}$ for each $1 \leq i, j \leq d$ and some $k \geq 1, \delta \in (0, 1)$.

(ii) The stable manifold theorem holds for the Itô SDE

$$dx(t) = h(x(t)) dt + \sum_{i=1}^m g_i(x(t)) dW_i(t), \quad (II)$$

where $h, g_i : \mathbf{R}^d \rightarrow \mathbf{R}^d, 1 \leq i \leq m$, are in $C_b^{k,\delta}$.

Reference

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