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Abstract

Biases in measurement of dynamics of time series from calculation of half-life received more attention lately. In particular, this issue amplifies the controversy surrounding the purchasing power parity doctrine. Cross-sectional and temporal aggregations along with mis-specified models were identified before as sources of this bias. We identified a few other sources of bias, namely, sampling error, wrong approximations, and structural breaks in time series. These sources should receive adequate attention for a sound measure of half-life.

Keywords: Impulse response function; Structural break

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1. Introduction

The empirical evidence of the high persistence of the deviation of the real exchange rates from its long run equilibrium warrants a simple measure that captures this slow transitional dynamics[1]. Consequently, economists borrowed the concept of half-life from the natural sciences. The "half-life" in natural sciences has been defined as the time required for the amount of radioactivity to decrease by one-half. In real exchange rate literature, following the same spirit, we define half-life as the time required for the effects of a unit innovation to dissipate to one-half. The half-life is also used in economics as a simple measure of the dynamics of time series such as the income level and the price level.

However, in empirical studies of half-lives, controversies surrounding the accuracy of half-life estimates often arise as some over-estimate and others under-estimate compared with what would be commonly expected (for a detailed discussion see Murray and Papell, 2002; Taylor, 2001). Efforts have been made to explore the sources of differences in half-life estimates: Basker and Hernandez-Murillo (2004), Choi *et al.* (2005), Chen and Devereux (2003), and Imbs *et al.* (2005) investigated cross-sectional aggregations as a contributing factor[2]; Chambers (2004) investigated temporal aggregation as a contributing factor for biased estimates and Taylor (2001) investigated temporal aggregation and mis-specified linear models as contributing factors. The expositions of the latter two articles are within the context of the autoregressive process of order one, AR(1).

[1] The rate of convergence of the real exchange rate has been estimated to be roughly 15% (Froot and Rogoff, 1995; Cheung and Lai, 2000a).

[2] Chen and Engel (2005), however, showed that the cross-sectional aggregation bias might not be large enough to explain the PPP puzzle.

In this paper, we explore other sources for the differences in half-life estimates in an effort to add feasible explanations to the puzzles of the purchasing power parity (PPP). These sources are the sensitivity of the half-life formula, an inappropriate formula commonly used for the half-life, and mis-specified models attributable to structural breaks. We found from our simulations that the half-life formula commonly used is very sensitive to the sampling error even if the autoregressive process is AR(1). The formula for half-life can be quite inaccurate when time series is of higher order (for example, AR(2)) or a mixed process (for example, ARMA(1,1)). Moreover, when there exists a structural break in time series and we do not take into account this issue, we over-estimate half-lives.

The present paper consists of five sections. In section 2, we discuss how sensitive the commonly-used half-life formula obtained from AR(1) model is. Biases resulting from using the half-life formula for higher order autoregressive processes and mixed processes are discussed in section 3. Effects of structural breaks on half-life calculations are discussed in section 4. At the end some concluding remarks are made.

2. Sensitivity of the half-life formula

Following the cumulative impulse response analysis of Campbell and Mankiw (1987), researchers define the moving average (MA) coefficients of the MA representation of the process as impulse responses. More specifically, for a linear process $y_t = \sum_{j=0}^{\infty} y_j e_{t-j}$ where $y_0 = 1$ and the e_t 's are independent identically distributed random variables, the half-life, denoted by *h*, is such that $y_h = 1/2$, that is the lag where the impulse response y_j becomes half of the initial impulse response.

However, unlike radioactive material, the impulse response does not always decay monotonically. If the y_j is not a monotonically decreasing function of the lag *j*, the half-life is not well-defined (Cheung and Lai, 2000b; Choi *et al*., 2005).

The often-used formula for the half-life of a (stationary) time series in the econometric literature is $h = -\log 2 / \log r_1$ where r_1 is the autocorrelation of y_t at lag 1, that is $r_1 = corr(y_t, y_{t-1})$. This formula is valid only when $r_1 > 0$, and correct if y_t is an AR(1) satisfying

$$y_t = r_1 y_{t-1} + e_t \,. \tag{1}$$

This is because for the AR(1), $y_j = r_1^j$. If $r_1 < 0$ for an AR(1) process, the impulse response y_j oscillates between positive and negative values, and the half-life is not well-defined.

Given a sample of size *n*, the half-life of an AR(1) process is usually estimated by

$$\hat{h} = -\frac{\log 2}{\log \hat{r}_1} \;, \tag{2}$$

where $\hat{r}_1$ is the least squares estimator of r_1. From the first order Taylor series expansion, we obtain

$$\hat{h} - h \approx \frac{(\log 2)\hat{d}}{r_1(\log r_1)^2}, \tag{3}$$

where $\hat{d} = \hat{r}_1 - r_1$. It is well known that $Var(\hat{r}_1) \approx (1 - r_1^2)/n$. Therefore

$$Var(\hat{h} - h) \approx \left\{ \frac{(\log 2)}{r_1(\log r_1)^2} \right\}^2 \frac{1 - r_1^2}{n} \,. \tag{4}$$

Using this, we tabulate the (approximate) coefficient of variation (CV) of $\hat{h}$ in Table 1 for $0.8 \le r_1 \le 0.95$ and for the sample size *n*=100 and 200. Within the range of r_1 the CV varies 33% to 64% for the sample of size 100, which amounts to 100 years of annual data. More specifically, for the AR(1) process with $r_1 = 0.9$, the half-life is 6.58. With *n*=100 the CV of $\hat{h}$ is 46% and the standard error of $\hat{h}$ is 3.02. Therefore an estimate of the half-life of 3.6 years or less is as likely as that of the half-life of 9.6 years or more for annual data. This illustrates that half-life estimates are very sensitive to the sampling error.

3. Precision of the approximate formula

More often than not, the process of interest is not just an AR(1) process. Rather it is a higher order AR process or a mixed process such as an autoregressive moving-average (ARMA) process. For such models the aforementioned half-life formula serves as an approximation, and the quality of this approximation needs investigation.

For an autoregressive process of order *p*, AR(*p*), y_t satisfying

$$y_t = \sum_{j=1}^{p} f_j y_{t-j} + e_t , \tag{5}$$

the impulse response y_j satisfies the linear difference equation

$$y_j - f_1 y_{j-1} - \cdots - f_{p-1} y_{j-(p-1)} - f_p = 0 , \tag{6}$$

and the half-life *h* is obtained by solving $y_h = 1/2$. It is well known that the impulse response y_j is obtained from the roots of the auxiliary equation

$$m^p - f_1 m^{p-1} - \cdots - f_{p-1} m - f_p = 0 . \tag{7}$$

As the y_j does not necessarily decay monotonically, the half-life is not always well-defined. Often employed practice in economics literature is to approximate the half-life based on the formula,

$$h = -\frac{\log 2}{\log(1+b)} \tag{8}$$

without regard to existence of the well-defined half-life, by obtaining the 'convergence speed' b from the following error correction representation of the AR(p) model

$$\Delta y_t = b y_{t-1} + \sum_{j=1}^{p-1} f_j^* \Delta y_{t-j} + e_t, \tag{9}$$

where $b = \sum_{j=1}^{p} f_j - 1$ and $f_j^* = -\sum_{k=j+1}^{p} f_k$. We note that for an AR(1) process $b = r_1 - 1$ and the formula in (8) is equivalent to that in (2).

For ease of exposition, we assess the quality of this approximation using the following AR(2) process

$$y_t = f_1 y_{t-1} + f_2 y_{t-2} + e_t. \tag{10}$$

It is well known that the impulse response of this process is

$$y_j = \begin{cases} (1+j)(f_1/2)^j & \text{if} \quad f_1^2 + 4f_2 = 0 \\ c_1 l_1^j + c_2 l_2^j & \text{if} \quad f_1^2 + 4f_2 \neq 0 \end{cases}, \tag{11}$$

where $l_1 = \frac{f_1 + \sqrt{f_1^2 + 4f_2}}{2}$, $l_2 = \frac{f_1 - \sqrt{f_1^2 + 4f_2}}{2}$, $c_1 = l_1/(l_1 - l_2)$, and $c_2 = l_2/(l_2 - l_1)$.

For the AR(2) process to be stationary, it is well known (see p. 60 of Box, Jenkins, and Reinsel, 1994) that the AR coefficients f_1 and f_2 lie in the triangular region

$$f_2 + f_1 < 1,\ f_2 - f_1 < 1,\ -1 < f_2 < 1. \tag{12}$$

Within this triangular region, the impulse response y_j decreases monotonically only in the region for $f_1 > 0$ and $f_1^2 + 4f_2 > 0$. Therefore, in the other region the half-life is not well-defined. However, the approximate formula yields a 'half-life' as long as $f_2 + f_1 > 0$. Even in the region where the half-life is well defined, the approximate formula can be quite inaccurate. The region where the difference between the half-life by (11) and the approximate half-life of (8) is more than 3 is shaded in Fig. 1.

When an AR(1) process at a higher frequency is aggregated and observed at a lower frequency, this observed process becomes an ARMA (1,1) process

$$y_t = fy_{t-1} + e_t - qe_{t-1}, \tag{13}$$

see Wei (1996) and Chambers (2004). The impulse response y_j is obtained by

$$y_j = (f - q)f^{j-1} \tag{14}$$

and the exact half-life h is

$$h = -\frac{\log 2}{\log f} - \frac{\log(f - q)}{\log f} + 1 \tag{15}$$

by solving $(f - q)f^{h-1} = 1/2$, provided $f > q$ and $f > 0$. Since the lag one autocorrelation of the ARMA(1,1) process is

$$r_1 = \frac{(1 - qf)(f - q)}{(1 - 2qf + q^2)}, \tag{16}$$

the approximate formula (based on an AR(1) model) yields a half-life of

$$-\log 2/\log\left\{\frac{(1 - qf)(f - q)}{(1 - 2qf + q^2)}\right\}. \tag{17}$$

Also one could consider an approximated model of AR(2) instead of an AR(1) model. In such case the approximate formula based on an AR(2) model yields a half-life of

$$-\log 2/\log\left\{\frac{(1-qf)(f-q)}{1-(f+1)q+q^2}\right\}, \tag{18}$$

where the proof is given in Appendix.

In order to illustrate inaccuracies of the approximate formulas, the region where the difference between the half-life by (15) and the approximate half-lives (17) or (18) is more than 3 is shaded in Fig. 2., even when the parameters are known. Although not pursued here, the inaccuracy is worse when models are estimated.

4. The effect of structural breaks

It is well known that the Dickey-Fuller unit root test lacks the power, when a true process is trend stationary with structural breaks, see Perron (1989). This implies that the LSE (of the Dickey-Fuller type) estimator of r_1 in (1), or b in (9) is over-estimated. Macro economic data, such as price indices and exchange rates, often go through structural breaks in the trend (or level) so that the analysis without such breaks incorporated yields over-estimated half-lives.

To assess the effect of a structural break in the trend (at a single point of time) on the estimation of half-lives, we conduct a small Monte Carlo experiment. We generated 10,000 replications of a series $\{y_t\}$ of length $T=100$ defined by

$$y_t = gD_t + ay_{t-1} + e_t \tag{19}$$

where $D_t = t - T_0$ if $t > T_0$, and 0 otherwise, representing a structural break in trend at T_0. For simplicity, we assume $T_0 = 50$ and the innovations e_t are i.i.d. $N(0,1)$. For various values of a and g, we take $a = 0.6, 0.8, 0.9$ and $g = 0.1, 0.2, 0.4$. For $a = 0.6, 0.8, 0.9$, the corresponding half-lives are 1.36, 3.11, and 6.58.

As an estimation of half-life where a structural break is not considered, we computed the half-life, $\hat{h}_1 = -\ln(2)/\ln(\hat{a}_1)$ based on the following model

$$y_t = m_1 + g_1 t + a_1 y_{t-1} + \tilde{e}_t \,. \qquad (20)$$

And, as the calculation of half-life where a structural break is considered, we computed the half-life, $\hat{h}_2 = -\ln(2)/\ln(\hat{a}_2)$, based on the following model,

$$y_t = g_2 D_t + a_2 y_{t-1} + \tilde{e}_t \,. \qquad (21)$$

We assume T_0 is known so that the comparison is not affected by the estimation of break point, T_0.

In Table 2, we compare the results of the estimation from the models of (20) and (21). From the fourth and the sixth columns, it is observed, as in Andrews (1993) and Murray and Papell (2005), that all the estimators $\hat{a}_2$ are biased downward. Therefore the half-life estimators, $\hat{h}_2$, are all under-estimated even though the structural breaks are considered.

From the third and fifth columns, we see that estimators of $\hat{a}_1$ are biased upward except for (a, g) =(0.8, 0.1), (0.9, 0.1) and (0.9, 0.2), and all the estimators of half-life, $\hat{h}_1$, are over-estimated except for (a, g) =(0.9, 0.1). Also from the last two columns it is observed that all mean squared errors (MSEs), of $\hat{h}_1$ are larger than those of $\hat{h}_2$. In the cases for (a, g) =(0.8, 0.1), (0.9, 0.1) and (0.9, 0.2), $\hat{h}_1$ has larger MSE than $\hat{h}_2$ does, although the corresponding $\hat{a}_1$ is less biased than $\hat{a}_2$. This can be explained from Fig. 3 which shows the distributions of $\hat{a}_1$, $\hat{a}_2$, $\hat{h}_1$, and $\hat{h}_2$ when (a, g) =(0.9, 0.2). The distribution of $\hat{a}_1$ has a higher concentration near one than that of $\hat{a}_2$ which makes the right tail of $\hat{h}_1$ longer than that of $\hat{h}_2$.

These over-estimation phenomenon are not surprising results because $\hat{a}_1$'s are ready to converge to one as sample size becomes larger regardless of the value of a, see Perron (1989). Therefore when structural breaks are in doubt, it is desirable that the model with the breaks is considered.

5. Conclusions

Researchers identified a number of sources of bias in half-life estimation, namely, cross-sectional aggregation, temporal aggregation, and mis-specified models. However, we identified a few other sources of instability of the conventional half-life estimation. We found that even for AR(1) process, the sampling bias cannot be ignored. For higher order or mixed time series process, the biases resulting from the use of conventional formula is quite large. The presence of structural breaks in time series creates additional noise in half-life calculation. Thus, a more appropriate calculation of half-life requires adequate attention paid to these issues.

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Appendix. Proof of (18)

Assume we consider an AR(2) as approximate model for ARMA(1,1). By a property of the partial autocorrelation function, we can find the coefficients (f_{12}, f_{22}) of the AR(2) using

$$\begin{bmatrix} f_{12} \\ f_{22} \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

Then we obtain

$$f_{12} + f_{22} = \frac{1}{1 - r_1^2} (r_1 - r_1 r_2 + r_2 - r_1^2)$$

$$= (1 + f) \frac{r_1}{1 + r_1}$$

$$= \frac{(1 - qf)(f - q)}{1 - (f + 1)q + q^2}$$

since $r_2 = f r_1$ and $r_1 = (1 - qf)(f - q) / \{1 - 2qf + q^2\}$. Therefore, we can deduce

$$\frac{-\log 2}{\log(f_{12} + f_{22})} = -\log 2 / \log \left\{ \frac{(1 - qf)(f - q)}{1 - (f + 1)q + q^2} \right\}.$$

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Table 1

Approximate standard errors and coefficient of variations of the half-life estimates for selected AR(1) process with sample sizes 100 and 200

r_1	h	$n=100$		$n=200$	
		$\sqrt{Var(\hat{h}-h)}$	CV (%)	$\sqrt{Var(\hat{h}-h)}$	CV (%)
0.800	3.1063	1.0440	33.61	0.7382	23.77
0.805	3.1955	1.0857	33.98	0.7677	24.02
0.810	3.2894	1.1302	34.36	0.7991	24.29
0.815	3.3884	1.1777	34.76	0.8327	24.58
0.820	3.4928	1.2285	35.17	0.8687	24.87
0.825	3.6032	1.2830	35.61	0.9072	25.18
0.830	3.7200	1.3416	36.07	0.9487	25.50
0.835	3.8439	1.4047	36.54	0.9933	25.84
0.840	3.9755	1.4728	37.05	1.0415	26.20
0.845	4.1156	1.5465	37.58	1.0935	26.57
0.850	4.2650	1.6264	38.13	1.1500	26.96
0.855	4.4247	1.7133	38.72	1.2115	27.38
0.860	4.5958	1.8081	39.34	1.2785	27.82
0.865	4.7795	1.9117	40.00	1.3518	28.28
0.870	4.9773	2.0255	40.69	1.4322	28.78
0.875	5.1909	2.1508	41.43	1.5209	29.30
0.880	5.4223	2.2894	42.22	1.6189	29.86
0.885	5.6737	2.4433	43.06	1.7277	30.45
0.890	5.9480	2.6149	43.96	1.8490	31.09
0.895	6.2484	2.8073	44.93	1.9851	31.77
0.900	6.5788	3.0242	45.97	2.1384	32.50
0.905	6.9439	3.2700	47.09	2.3122	33.30
0.910	7.3496	3.5506	48.31	2.5106	34.16
0.915	7.8030	3.8732	49.64	2.7387	35.10
0.920	8.3130	4.2471	51.09	3.0032	36.13
0.925	8.8909	4.6846	52.69	3.3125	37.26
0.930	9.5513	5.2017	54.46	3.6782	38.51
0.935	10.3133	5.8205	56.44	4.1157	39.91
0.940	11.2023	6.5711	58.66	4.6465	41.48
0.945	12.2528	7.4965	61.18	5.3008	43.26
0.950	13.5134	8.6593	64.08	6.1231	45.31

Fig. 1. The region where the difference between the half-life by (11) and the approximate formula of (8) is more than 3

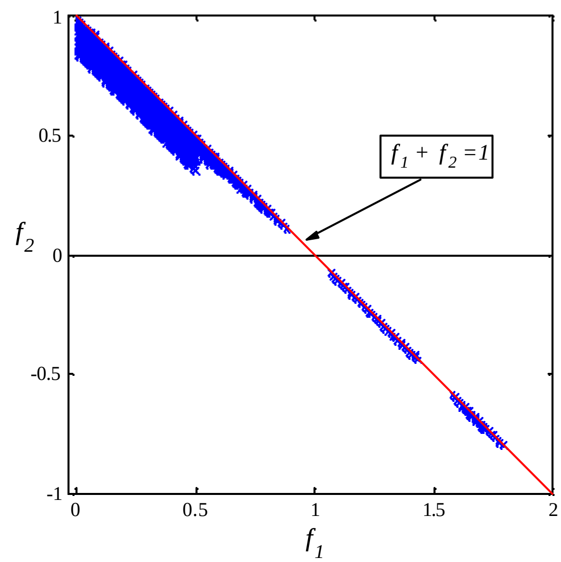


Fig. 2. The region where the difference between the half-life by (15) and the approximate formula by (17): AR(1) or (18): AR(2) is more than 3

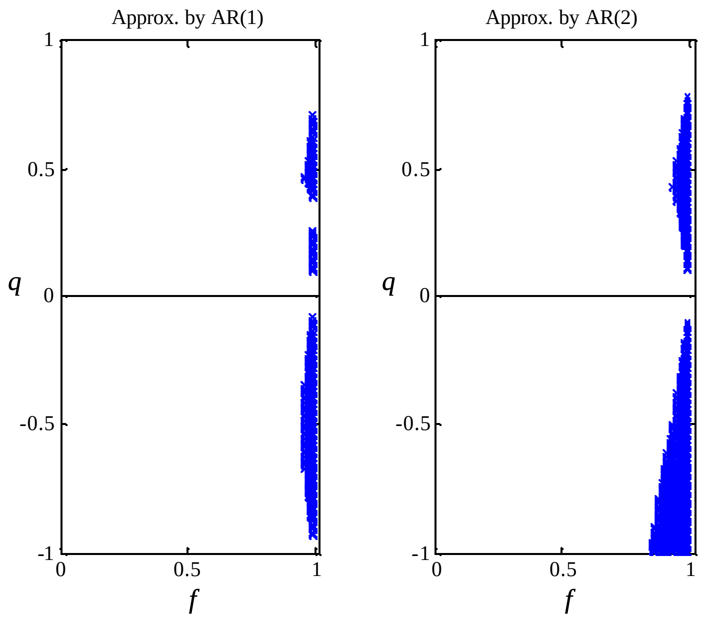


Table 2

Averages and standard deviations of the estimated half-lives and other statistics depending on the structural change

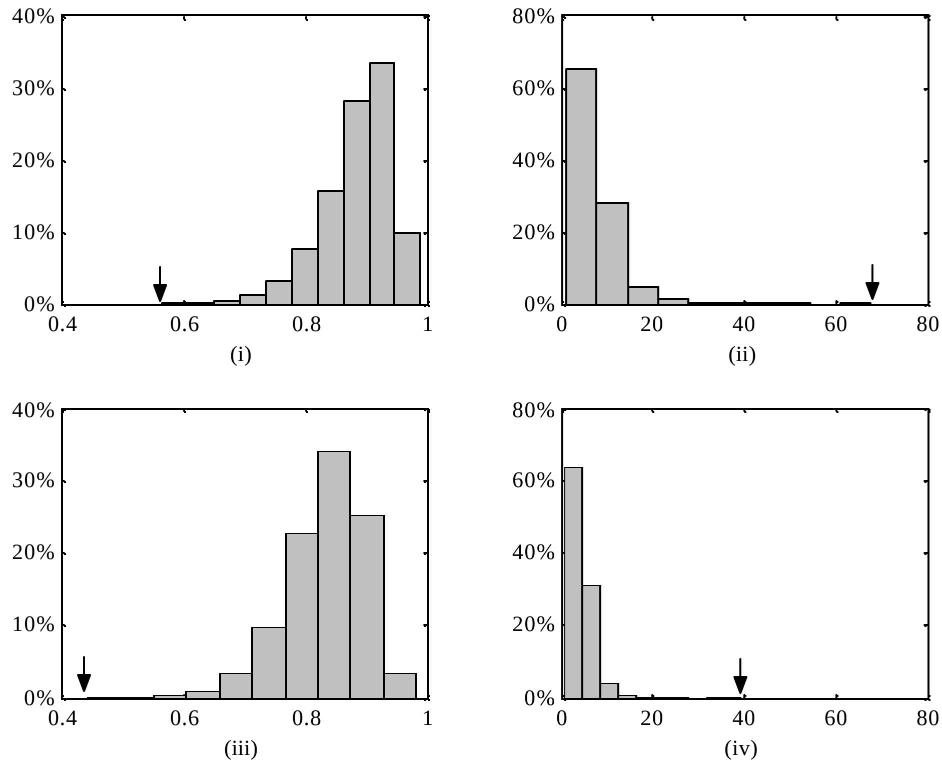
a	g	$\hat{a}_1$	$\hat{a}_2$	$\hat{h}_1$	$\hat{h}_2$	MSE_1	MSE_2
0.6	0.1	0.67 (0.08)	0.55 (0.09)	1.83 (0.54)	1.23 (0.33)	0.52 (0.86)	0.13 (0.18)
	0.2	0.81 (0.05)	0.55 (0.09)	3.58 (1.02)	1.23 (0.33)	5.99 (5.72)	0.12 (0.18)
	0.4	0.93 (0.02)	0.55 (0.09)	10.65 (2.89)	1.21 (0.32)	94.68 (70.95)	0.13 (0.16)
0.8	0.1	0.78 (0.07)	0.74 (0.07)	3.22 (1.25)	2.57 (0.89)	1.58 (3.78)	1.09 (1.54)
	0.2	0.86 (0.05)	0.74 (0.07)	5.24 (2.20)	2.56 (0.87)	9.41 (23.91)	1.06 (1.46)
	0.4 (11)	0.94 (0.02)	0.74 (0.07)	13.51 (5.80)	2.51 (0.90)	141.94 (217.81)	1.17 (1.98)
0.9	0.1 (1)	0.85 (0.06)	0.83 (0.06)	5.29 (2.90)	4.54 (2.23)	10.10 (41.04)	9.12 (22.82)
	0.2 (10)	0.89 (0.06)	0.83 (0.06)	7.52 (4.72)	4.54 (2.26)	23.13 (100.53)	9.28 (21.26)
	0.4 (175)	0.95 (0.03)	0.83 (0.07)	16.03 (9.27)	4.59 (3.08)	175.23 (377.13)	13.43 (64.79)

Note: 1. $\hat{a}_1$ and $\hat{a}_2$ are the estimators of a in the model (19) by the estimated models (20) and (21), respectively.

2. $\hat{h}_j = -\ln(2)/\ln(\hat{a}_j)$ for $j = 1, 2$ denotes the estimator of half-life (not adjusted to integers) and $MSE_j = (\hat{h}_j - h_0)^2$ for $j = 1, 2$, where h_0 is the true half-life, 1.36, 3.11, and 6.58 corresponding to the a =0.6, 0.8, and, 0.9, respectively.

3. The parentheses in the second column denote the number of the cases where $\hat{a}_1$ >=1 or $\hat{a}_2$ >=1. We do not consider these cases in the results because the corresponding half-lives cannot be calculated. The parentheses in the other columns denote the corresponding standard deviations.

Fig. 3. Histograms for $\hat{a}_1$, $\hat{h}_1$, $\hat{a}_2$, and $\hat{h}_2$ which correspond to (i), (ii), (iii), and (iv), respectively, when $(a, g) = (0.9, 0.2)$



Note: 1. The y-axis in each histogram denotes the relative frequency.

2. The mark of arrows denotes an existing range of the histograms.