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On Linearizability of Strict Feedforward Systems

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Abstract— In this paper we address the problem of linearizability of systems in strict feedforward form. We provide an algorithm, along with explicit transformations, that linearizes a system by change of coordinates when some easily checkable conditions are met. Those conditions turn out to be necessary and sufficient, that is, if one fails the system is not linearizable. We revisit type I and type II classes of linearizable strict feedforward systems provided by Krstic in [6] and illustrate our algorithm by various examples mostly taken from [5], [6].

I. INTRODUCTION

THE pioneering work on strict feedforward systems can be traced back to the papers of Teel [15], [16]. Since then, it has been followed by a growing literature [9], [3], [4], [13], [8], [1], [10], [14], [5], [7], [6], [12]. Recently, Krstic [5], [6] addressed the problem of linearizability of strict feedforward systems, and provided two classes (type I and type II) that are linearizable by change of coordinates. By providing linearizing changes of coordinates in some examples, Krstic mentioned that *there is no systematic way of finding those changes of coordinates*.

The objective of this paper, inspired by those of Krstic, is to show that there is indeed a systematic way of finding the linearizing coordinates of *any* strict feedforward system that is linearizable (type I and type II do not exhaust all linearizable strict feedforward systems). We will provide an algorithm, along with necessary and sufficient conditions, for a system in strict feedforward form to be brought into a linear one. The maximal number of steps required by the algorithm is $\frac{n(n-1)}{2}$, where n is the dimension of the system.

Throughout the paper, linearizability means state-linearizability, that is, bringing a control system to a linear one via a change of coordinates (defined by a diffeomorphism) in the state space. This problem was solved in the early eighties: necessary and sufficient geometric conditions to linearize a control system via change of coordinates have been expressed in an invariant form in terms of Lie brackets of vector fields defining the system (see Theorem II.2). Unfortunately, those conditions do not provide a way of finding the change of coordinates explicitly except for solving a system of partial differential equations. For strict feedforward systems, however, finding linearizing coordinates is much simpler: each of the $\frac{n(n-1)}{2}$ steps of our algorithm involves elementary operations, composing, differentiating, and integrating functions only but not solving differential equations.

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II. MAIN RESULTS

Consider *smooth (analytic)* single-input control systems

$$\Sigma \; : \; \dot{x} = f(x) + g(x)u,$$

either locally in a neighborhood $X \times U$ of $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ or globally on $\mathbb{R}^n \times \mathbb{R}$, in *strict feedforward form* (SFF), i.e,

$$(\text{SFF})\begin{cases} f_j(x) = f_j(x_{j+1}, \ldots, x_n), & 1 \le j \le n-1 \\ f_n(x) = 0, \\ g_j(x) = g_j(x_{j+1}, \ldots, x_n), & 1 \le j \le n-1 \\ g_n \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}. \end{cases}$$

We say that the (SFF)-system is *control-normalized*, and we denote it by (SFFcn) if $g(x) = (0, \ldots, 0, 1)^\top$.

Algorithm I provides a constructive proof of our first result asserting that any (SFF)-system can be brought into a control-normalized (SFFcn)-form. Algorithm II gives a constructive procedure to linearize any (SFFcn)-system that can be linearized. Those two algorithms form our main result:

Theorem II.1 (i) *Consider a system Σ in (SFF)-form. There exists a change of coordinates $z = \phi(x)$ that transforms Σ into a control-normalized strict feedforward system*

$$\bar{\Sigma} \; : \; \dot{z} = \bar{f}(z) + \bar{g}(z)u, \quad z \in \mathbb{R}^n,$$

that is, such that

$$(\text{SFFcn})\begin{cases} \bar{f}_j(z) = \bar{f}_j(z_{j+1}, \ldots, z_n), & 1 \le j \le n-1 \\ \bar{f}_n(z) = 0, \\ \bar{g}(z) = (0, \ldots, 0, 1)^\top. \end{cases}$$

(ii) *Any (SFFcn)-system that satisfies the conditions (S1) and (S2) of Theorem II.2 below can be transformed into a linear controllable system by a diffeomorphism $\sigma(z)$.*

(iii) *The components of the normalizing diffeomorphism $\phi(x)$ of* (i) *and those of the linearizing diffeomorphism $z = \sigma(z)$ of* (ii) *can be calculated via elementary operations, composing, differentiating, and integrating the components of the (SFF)-system and (SFFcn)-system: respectively $(n-1)$ steps for ϕ and $\frac{(n-1)(n-2)}{2}$ steps for σ, thus a total of $\frac{n(n-1)}{2}$).*

We want to point out that the results stated above are global.

A. *Algorithm I.* Algorithm I proves Theorem II.1 (i) and defines the diffeormorphism $\phi(x)$ explicitly using $n-1$ steps. The existence of $\phi(x)$ is guaranteed by the "flow box" theorem that in our case assures global rectification of g.

Step 1. The system $\Sigma = \Sigma^0$, in the original coordinates $x = (x_1, \ldots, x_n)^\top$, is in the form $\Sigma^0 \; : \; \dot{x} = f(x) + g(x)u$.

Without loss of generality we assume $g_n = 1$ (if not, replace x_n by x_n/g_n). The first step is to apply the change of coordinates $\boldsymbol{x_1} = \phi_1(x)$ whose components are given by

$$\boldsymbol{x_{1j}} = \phi_{1j}(x) = x_j, \; j \neq n-1$$

$$\boldsymbol{x_{1n-1}} = \phi_{1n-1}(x) = x_{n-1} - \int_0^{x_n} g_{n-1}(s)\mathrm{d}s$$

that annihilates the component g^1_{n-1} of the new control vector field g^1 and takes Σ^0 into $\Sigma^1 : \dot{\boldsymbol{x}}_1 = f^1(\boldsymbol{x_1}) + g^1(\boldsymbol{x_1})u$, with

$$f^1(\boldsymbol{x_1}) = \left(f^1_1(\boldsymbol{x_1}), f^1_2(\boldsymbol{x_1}), \ldots, f^1_{n-1}(\boldsymbol{x_1}), 0\right)^\top,$$
$$g^1(\boldsymbol{x_1}) = \left(g^1_1(\boldsymbol{x_1}), g^1_2(\boldsymbol{x_1}), \ldots, g^1_{n-2}(\boldsymbol{x_1}), 0, 1\right)^\top,$$

where $f^1_{n-1}(\boldsymbol{x_1}) = f_{n-1}(\boldsymbol{x_{1n}})$ and for any $1 \leq j \leq n-2$

$$f^1_j(\boldsymbol{x_1}) = f^1_j(\boldsymbol{x_{1j+1}}, \ldots, \boldsymbol{x_{1n}}) = f_j \circ \phi_1^{-1}(\boldsymbol{x_1}),$$
$$g^1_j(\boldsymbol{x_1}) = g^1_j(\boldsymbol{x_{1j+1}}, \ldots, \boldsymbol{x_{1n}}) = g_j \circ \phi_1^{-1}(\boldsymbol{x_1}).$$

Step 2. Apply the change of coordinates $\boldsymbol{x_2} = \phi_2(\boldsymbol{x_1})$, whose components are given by

$$\boldsymbol{x_{2j}} = \phi_{2j}(\boldsymbol{x_1}) = \boldsymbol{x_{1j}}, \; j \neq n-2$$

$$\boldsymbol{x_{2n-2}} = \phi_{2n-2}(\boldsymbol{x_1}) = \boldsymbol{x_{1n-2}} - \int_0^{\boldsymbol{x_{1n}}} g^1_{n-2}(\boldsymbol{x_{1n-1}}, s)\mathrm{d}s$$

to take Σ^1 into $\Sigma^2 : \dot{\boldsymbol{x}}_2 = f^2(\boldsymbol{x_2}) + g^2(\boldsymbol{x_2})u$, with

$$f^2(\boldsymbol{x_2}) = \left(f^2_1(\boldsymbol{x_2}), f^2_2(\boldsymbol{x_2}), \ldots, f^2_{n-1}(\boldsymbol{x_2}), 0\right)^\top,$$
$$g^2(\boldsymbol{x_2}) = \left(g^2_1(\boldsymbol{x_2}), g^2_2(\boldsymbol{x_2}), \ldots, g^2_{n-3}(\boldsymbol{x_2}), 0, 0, 1\right)^\top,$$

where for any $1 \leq j \leq n-1$ we have

$$f^2_j(\boldsymbol{x_2}) = f^2_j(\boldsymbol{x_{2j+1}}, \ldots, \boldsymbol{x_{2n}}) = f^1_j \circ \phi_2^{-1}(\boldsymbol{x_2}),$$
$$g^2_j(\boldsymbol{x_2}) = g^2_j(\boldsymbol{x_{2j+1}}, \ldots, \boldsymbol{x_{2n}}) = g^1_j \circ \phi_2^{-1}(\boldsymbol{x_2}).$$

Notice that the components f^2_j and g^2_j of Σ^2 are, respectively, those f^1_j and g^1_j of Σ^1 re-expressed in the new coordinates, except for g^2_{n-2} (annihilated) and f^2_{n-2} transformed as:

$$f^2_{n-2}(\boldsymbol{x_2}) = f^1_{n-2}(\boldsymbol{x_1}) + \frac{\partial \phi_{2n-2}}{\partial \boldsymbol{x_{1n-1}}}(\boldsymbol{x_1}) f^1_{n-1}(\boldsymbol{x_{1n}}) \Big|_{\boldsymbol{x_1} = \phi_2^{-1}(\boldsymbol{x_2})}.$$

Remark that the inverses of $\boldsymbol{x_1} = \phi_1(x)$ and $\boldsymbol{x_2} = \phi_2(\boldsymbol{x_1})$ are easily computable and given, respectively, by

$$x_j = \phi_{1j}^{-1}(\boldsymbol{x_1}) = \boldsymbol{x_{1j}}, \; j \neq n-1$$

$$x_{n-1} = \phi_{1n-1}^{-1}(x) = \boldsymbol{x_{1n-1}} + \int_0^{\boldsymbol{x_{1n}}} g_{n-1}(s)\mathrm{d}s$$

and

$$\boldsymbol{x_{1j}} = \phi_{1j}^{-1}(\boldsymbol{x_2}) = \boldsymbol{x_{2j}}, \; j \neq n-2$$

$$\boldsymbol{x_{1n-2}} = \phi_{1n-2}^{-1}(\boldsymbol{x_2}) = \boldsymbol{x_{2n-2}} + \int_0^{\boldsymbol{x_{2n}}} g^1_{n-2}(\boldsymbol{x_{2n-1}}, s)\mathrm{d}s.$$

Step k. Assume, after applying a change of coordinates $\boldsymbol{x_{k-1}} = \phi_{k-1} \circ \cdots \circ \phi_1(x)$, that Σ^0 has been brought into

$$\Sigma^{k-1} \; : \; \dot{\boldsymbol{x}}_{k-1} = f^{k-1}(\boldsymbol{x_{k-1}}) + g^{k-1}(\boldsymbol{x_{k-1}})u,$$

with

$$f^{k-1}(\boldsymbol{x_{k-1}}) = \left(f^{k-1}_1(\boldsymbol{x_{k-1}}), .., f^{k-1}_{n-1}(\boldsymbol{x_{k-1}}), 0\right)^\top,$$
$$g^{k-1}(\boldsymbol{x_{k-1}}) = \left(g^{k-1}_1(\boldsymbol{x_{k-1}}), .., g^{k-1}_{n-k}(\boldsymbol{x_{k-1}}), 0, \ldots, 0, 1\right)^\top,$$

where for any $1 \leq j \leq n-1$ we have

$$f^{k-1}_j(\boldsymbol{x_{k-1}}) = f^{k-1}_j(\boldsymbol{x_{k-1j+1}}, \ldots, \boldsymbol{x_{k-1n}})$$
$$g^{k-1}_j(\boldsymbol{x_{k-1}}) = g^{k-1}_j(\boldsymbol{x_{k-1j+1}}, \ldots, \boldsymbol{x_{k-1n}}).$$

We then apply the change of coordinates $\boldsymbol{x_k} = \phi_k(\boldsymbol{x_{k-1}})$, whose components $\phi_{kj}(\boldsymbol{x_{k-1}}), j = 1, \ldots, n$, are given by

$$\boldsymbol{x_{kj}} = \boldsymbol{x_{k-1j}}, \; j \neq n-k$$

$$\boldsymbol{x_{kn-k}} = \boldsymbol{x_{k-1n-k}} - \int_0^{\boldsymbol{x_{k-1n}}} g^{k-1}_{n-k}(\boldsymbol{x_{k-1n-k+1}}, .., \boldsymbol{x_{k-1n-1}}, s)\mathrm{d}s$$

to annihilate g^{k-1}_{n-k}. The system Σ^{k-1} is then transformed into

$$\Sigma^k \; : \; \dot{\boldsymbol{x}}_k = f^k(\boldsymbol{x_k}) + g^k(\boldsymbol{x_k})u,$$

with

$$f^k(\boldsymbol{x_k}) = \left(f^k_1(\boldsymbol{x_k}), \ldots, f^k_{n-k}(\boldsymbol{x_k}), \ldots, f^k_{n-1}(\boldsymbol{x_k}), 0\right)^\top,$$
$$g^k(\boldsymbol{x_k}) = \left(g^k_1(\boldsymbol{x_k}), \ldots, g^k_{n-k-1}(\boldsymbol{x_k}), 0, \ldots, 0, 1\right)^\top,$$

where for any $1 \leq j \leq n-1$ we have

$$f^k_j(\boldsymbol{x_k}) = f^k_j(\boldsymbol{x_{kj+1}}, \ldots, \boldsymbol{x_{kn}}) = f^{k-1}_j \circ \phi_k^{-1}(\boldsymbol{x_k})$$
$$g^k_j(\boldsymbol{x_k}) = g^k_j(\boldsymbol{x_{kj+1}}, \ldots, \boldsymbol{x_{kn}}) = g^{k-1}_j \circ \phi_k^{-1}(\boldsymbol{x_k}).$$

Once again the components f^k_j and g^k_j of Σ^k are those of Σ^{k-1} re-expressed in the new coordinates $\boldsymbol{x_k} = \phi_k(\boldsymbol{x_{k-1}})$, except for g^k_{n-k} (annihilated) and f^k_{n-k} transformed as:

$$f^k_{n-k}(\boldsymbol{x_k}) = f^{k-1}_{n-k}(\boldsymbol{x_{k-1n-k+1}}, \ldots, \boldsymbol{x_{k-1n}})$$

$$+ \sum_{l=n-k+1}^{n-1} \frac{\partial \phi_{kn-k}}{\partial \boldsymbol{x_{k-1l}}}(\boldsymbol{x_{k-1}}) f^{k-1}_l(\boldsymbol{x_{k-1}}) \Big|_{\boldsymbol{x_{k-1}} = \phi_k^{-1}(\boldsymbol{x_k})},$$

where the inverse $\boldsymbol{x_{k-1}} = \phi_k^{-1}(\boldsymbol{x_k})$ is defined by

$$\boldsymbol{x_{k-1j}} = \boldsymbol{x_{kj}}, \; j \neq n-k$$

$$\boldsymbol{x_{k-1n-k}} = \boldsymbol{x_{kn-k}} + \int_0^{\boldsymbol{x_{kn}}} g^{k-1}_{n-k}(\boldsymbol{x_{kn-k+1}}, .., \boldsymbol{x_{kn-1}}, s)\mathrm{d}s.$$

This completes Algorithm I and shows that any (SFF)-system is equivalent to a (SFFcn)-system by change of coordinates $z = \phi(x) = \phi_{n-2} \circ \cdots \circ \phi_1(x)$. $\blacksquare$

In Algorithm II below we will assume that the system has been reduced to a control-normalized (SFFcn) via Algorithm I, and we will provide a sequence of changes of coordinates whose composition linearizes the system (SFFcn) provided some necessary and sufficient conditions are satisfied. Before, let us recall (see e.g. [2], [11]) the following

Theorem II.2 *A control-affine system $\Sigma : \dot{x} = f(x) + ug(x)$ is locally equivalent, via a change of coordinates $z = \psi(x)$ to a linear controllable system $\dot{z} = Az + bu$ if and only if*
(S1) $\dim \mathrm{span}\{ad_f^q g(x), \; 0 \leq q \leq n-1\} = n$
(S2) $[ad_f^q g, ad_f^r g] = 0, \; 0 \leq q < r \leq n.$

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B. *Algorithm II.* Consider a (SFF)-system and apply Algorithm I to bring it into a control-normalized (SFFcn)-form

$$\bar{\Sigma} \ : \ \dot{z} = \bar{f}(z) + \bar{g}(z)u, \quad z \in \mathbb{R}^n,$$

that is, such that

$$(\text{SFFcn})\begin{cases} \bar{f}_j(z) = \bar{f}_j(z_{j+1}, \ldots, z_n), & 1 \leq j \leq n-1 \\ \bar{f}_n(z) = 0, \\ \bar{g}(z) = (0, \ldots, 0, 1)^\top. \end{cases}$$

Step 1. Consider condition (S2) of Theorem II.2 for $q = 0$, $r = 1$ and denote it by $(\mathcal{L}_n)$. Then

$$(\mathcal{L}_n) \Longrightarrow \frac{\partial^2 \bar{f}_j}{\partial z_n^2} \equiv 0, \quad \text{for all } 1 \leq j \leq n-1.$$

If the condition $(\mathcal{L}_n)$ fails to be satisfied, that is, there exists $1 \leq j \leq n-1$ such that $\frac{\partial^2 \bar{f}_j}{\partial z_n^2} \neq 0$, then the system is NOT linearizable by change of coordinates and the algorithm stops. Otherwise, as we will show, the system can be simplified (annihilation of all, but f_{n-1}, terms containing z_n) in at most $n-1$ substeps. To start with, notice that the condition $(\mathcal{L}_n)$ implies, in particular, that $f_{n-1}(z) = \lambda_{n-1} z_n$ (recall $f(0) = 0$) and, replacing z_{n-1} by z_{n-1}/λ_{n-1}, we can assume $\lambda_{n-1} = 1$.

Substep 1. Due to $(\mathcal{L}_n)$, decompose $\bar{f}_{n-2}$ uniquely as

$$\bar{f}_{n-2}(z_{n-1}, z_n) = \tilde{f}_{n-2}(z_{n-1}) + z_n \theta_{\mathbf{1}n-2}(z_{n-1}).$$

Then apply the change of coordinates $z_\mathbf{1}^1 = \sigma_\mathbf{1}^1(z)$, whose components are given by

$$z_{\mathbf{1}j}^1 = \sigma_{\mathbf{1}j}^1(z) = z_j, \ j \neq n-2$$

$$z_{\mathbf{1}n-2}^1 = \sigma_{\mathbf{1}n-2}^1(z) = z_{n-2} - \int_0^{z_{n-1}} \theta_{\mathbf{1}n-2}(s)\mathrm{d}s$$

to take $\bar{\Sigma}$ into the form

$$\bar{\Sigma}_\mathbf{1}^1 \ : \ \dot{z}_\mathbf{1}^1 = \bar{f}_\mathbf{1}^1(z_\mathbf{1}^1) + \bar{g}_\mathbf{1}^1(z_\mathbf{1}^1)u, \quad z_\mathbf{1}^1 \in \mathbb{R}^n,$$

with $\bar{g}_\mathbf{1}^1(z_\mathbf{1}^1) = (0, \ldots, 0, 1)^\top$ and

$$\bar{f}_\mathbf{1}^1(z_\mathbf{1}^1) = \left(\bar{f}_{\mathbf{1}1}^1(z_\mathbf{1}^1), \ldots, \bar{f}_{\mathbf{1}n-2}^1(z_\mathbf{1}^1), z_{\mathbf{1}n}^1, 0\right)^\top.$$

Moreover, for any $1 \leq j \leq n-3$, we have

$$\bar{f}_{\mathbf{1}j}^1(z_\mathbf{1}^1) = \bar{f}_{\mathbf{1}j}^1(z_{\mathbf{1}j+1}^1, \ldots, z_{\mathbf{1}n}^1) = \bar{f}_j \circ \eta_\mathbf{1}^1(z_\mathbf{1}^1)$$

and

$$\bar{f}_{\mathbf{1}n-2}^1(z_\mathbf{1}^1) = \bar{f}_{n-2}(z) + z_n \frac{\partial \sigma_{\mathbf{1}n-2}^1}{\partial z_{n-1}}\Big|_{z = \eta_\mathbf{1}^1(z_\mathbf{1}^1)},$$

where $z = \eta_\mathbf{1}^1(z_\mathbf{1}^1)$ is the inverse of $z_\mathbf{1}^1 = \sigma_\mathbf{1}^1(z)$, whose components are given by

$$z_j = \eta_{\mathbf{1}j}^1(z_\mathbf{1}^1) = z_{\mathbf{1}j}^1, \ j \neq n-2$$

$$z_{n-2} = \eta_{\mathbf{1}n-2}^1(z_\mathbf{1}^1) = z_{\mathbf{1}n-2}^1 + \int_0^{z_{\mathbf{1}n-1}^1} \theta_{\mathbf{1}n-2}(s)\mathrm{d}s.$$

Notice that the component $\bar{f}_{\mathbf{1}n-2}^1(z_\mathbf{1}^1)$ depends exclusively on the variable $z_{\mathbf{1}n-1}^1$. Before we proceed to the next steps, let us explain the notation here for a better understanding of the upcoming changes of coordinates.

Throughout this algorithm, the *bold subscript* will refer to the corresponding step of the algorithm while the *upperscript* refers to the corresponding substep (as outlined above).

Let us assume, after applying a change of coordinates $z_\mathbf{1}^{k-1} = \sigma_\mathbf{1}^{k-1} \circ \cdots \circ \sigma_\mathbf{1}^1(z)$, that $\bar{\Sigma}$ has been brought into

$$\bar{\Sigma}_\mathbf{1}^{k-1} \ : \ \dot{z}_\mathbf{1}^{k-1} = \bar{f}_\mathbf{1}^{k-1}(z_\mathbf{1}^{k-1}) + \bar{g}_\mathbf{1}^{k-1}(z_\mathbf{1}^{k-1})u,$$

with $\bar{g}_\mathbf{1}^{k-1}(z_\mathbf{1}^{k-1}) = (0, \ldots, 0, 1)^\top$ and

$$\bar{f}_\mathbf{1}^{k-1}(z_\mathbf{1}^{k-1}) = \left(\bar{f}_{\mathbf{1}1}^{k-1}(z_\mathbf{1}^{k-1}), \ldots, \bar{f}_{\mathbf{1}n-2}^{k-1}(z_\mathbf{1}^{k-1}), z_{\mathbf{1}n}^{k-1}, 0\right)^\top,$$

where, for any $1 \leq j \leq n-2$, we have

$$\bar{f}_{\mathbf{1}j}^{k-1}(z_\mathbf{1}^{k-1}) = \begin{cases} \bar{f}_{\mathbf{1}j}^{k-1}(z_{\mathbf{1}j+1}^{k-1}, \ldots, z_{\mathbf{1}n}^{k-1}), & 1 \leq j \leq n-k-1, \\ \bar{f}_{\mathbf{1}j}^{k-1}(z_{\mathbf{1}j+1}^{k-1}, \ldots, z_{\mathbf{1}n-1}^{k-1}), & n-k \leq j \leq n-2. \end{cases}$$

The last components $\bar{f}_{\mathbf{1}n-2}^{k-1}, \ldots, \bar{f}_{\mathbf{1}n-k}^{k-1}$ do not depend on the variable $z_{\mathbf{1}n}^{k-1}$ and the remaining components are affine in $z_{\mathbf{1}n}^{k-1}$ because of $(\mathcal{L}_n)$.

Substep k. Decompose $\bar{f}_{\mathbf{1}n-k-1}^{k-1}$ uniquely as follows

$$\bar{f}_{\mathbf{1}n-k-1}^{k-1}(z_\mathbf{1}^{k-1}) = \tilde{f}_{\mathbf{1}n-k-1}^{k-1}(z_{\mathbf{1}n-k}^{k-1}, \ldots, z_{\mathbf{1}n-1}^{k-1})$$
$$+ z_{\mathbf{1}n}^{k-1}\theta_{\mathbf{1}n-k-1}(z_{\mathbf{1}n-k}^{k-1}, \ldots, z_{\mathbf{1}n-1}^{k-1})$$

and apply $z_\mathbf{1}^k = \sigma_\mathbf{1}^k(z_\mathbf{1}^{k-1})$, whose components are given by

$$z_{\mathbf{1}j}^k = z_{\mathbf{1}j}^{k-1}, \ j \neq n-k-1$$

$$z_{\mathbf{1}n-k-1}^k = z_{\mathbf{1}n-k-1}^{k-1} - \int_0^{z_{\mathbf{1}n-1}^{k-1}} \theta_{\mathbf{1}n-k-1}(z_{\mathbf{1}n-k}^{k-1}, .., z_{\mathbf{1}n-2}^{k-1}, s)\mathrm{d}s$$

to cancel the terms $z_{\mathbf{1}n}^{k-1}\theta_{\mathbf{1}n-k-1}(z_{\mathbf{1}n-k}^{k-1}, \ldots, z_{\mathbf{1}n-1}^{k-1})$.

This change of coordinates takes $\bar{\Sigma}_\mathbf{1}^{k-1}$ into

$$\bar{\Sigma}_\mathbf{1}^k \ : \ \dot{z}_\mathbf{1}^k = \bar{f}_\mathbf{1}^k(z_\mathbf{1}^k) + \bar{g}_\mathbf{1}^k(z_\mathbf{1}^k)u,$$

with $\bar{g}_\mathbf{1}^k(z_\mathbf{1}^k) = (0, \ldots, 0, 1)^\top$ and

$$\bar{f}_\mathbf{1}^k(z_\mathbf{1}^k) = \left(\bar{f}_{\mathbf{1}1}^k(z_\mathbf{1}^k), \ldots, \bar{f}_{\mathbf{1}n-2}^k(z_\mathbf{1}^k), z_{\mathbf{1}n}^k, 0\right)^\top,$$

where, for any $1 \leq j \leq n-2$, we have

$$\bar{f}_{\mathbf{1}j}^k(z_\mathbf{1}^k) = \begin{cases} \bar{f}_{\mathbf{1}j}^k(z_{\mathbf{1}j+1}^k, \ldots, z_{\mathbf{1}n}^{k-1}), & 1 \leq j \leq n-k-2, \\ \bar{f}_{\mathbf{1}j}^k(z_{\mathbf{1}j+1}^k, \ldots, z_{\mathbf{1}n-1}^k), & n-k-1 \leq j \leq n-2. \end{cases}$$

Moreover, $\bar{f}_{\mathbf{1}j}^k(z_\mathbf{1}^k) = \bar{f}_{\mathbf{1}j}^{k-1} \circ \eta_\mathbf{1}^k(z_\mathbf{1}^k)$, for $j \neq n-k-1$ and

$$f_{\mathbf{1}n-k-1}^k(z_\mathbf{1}^k) = f_{\mathbf{1}n-k-1}^{k-1}(z_{\mathbf{1}n-k}^{k-1}, \ldots, z_{\mathbf{1}n}^{k-1})$$

$$+ \sum_{l=n-k}^{n-1} \frac{\partial \sigma_{\mathbf{1}n-k-1}^k}{\partial z_{\mathbf{1}l}^{k-1}}(z_\mathbf{1}^{k-1})\bar{f}_{\mathbf{1}l}^{k-1}(z_\mathbf{1}^{k-1})\Big|_{z_\mathbf{1}^{k-1}=\eta_\mathbf{1}^k(z_\mathbf{1}^k)},$$

where $z_\mathbf{1}^{k-1} = \eta_\mathbf{1}^k(z_\mathbf{1}^k)$ stands for the inverse of the diffeomorphism $z_\mathbf{1}^k = \sigma_\mathbf{1}^k(z_\mathbf{1}^{k-1})$ whose components are

$$\eta_{\mathbf{1}j}^k(z_\mathbf{1}^k) = z_{\mathbf{1}j}^k, \ j \neq n-k-1$$

$$\eta_{\mathbf{1}n-k-1}^k(z_\mathbf{1}^k) = z_{\mathbf{1}n-k-1}^k + \int_0^{z_{\mathbf{1}n-1}^k} \theta_{\mathbf{1}n-k-1}(z_{\mathbf{1}n-k}^k, .., z_{\mathbf{1}n-2}^k, s)\mathrm{d}s.$$

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Applying successively the changes of coordinates $z_1^1 = \sigma_1^1(z)$, $z_1^2 = \sigma_1^2(z_1^1)$, ..., $z_1^{n-2} = \sigma_1^{n-2}(z_1^{n-3})$ whose composition is denoted by $\sigma_1 \triangleq \sigma_1^{n-2} \circ \cdots \circ \sigma_1^1$, we transform the system $\bar{\Sigma}$ into

$$\bar{\Sigma}_1 \; : \; \dot{z}_1 = \bar{f}_1(z_1) + \bar{g}_1(z_1)u, \quad z_1 \in \mathbb{R}^n,$$

where $\bar{g}_1(z_1) = (0, \ldots, 0, 1)^\top$ and

$$\bar{f}_1(z_1) = \left(\bar{f}_{11}(z_1), \ldots, \bar{f}_{1n-2}(z_1), z_{1n}, 0 \right)^\top,$$

with

$$\bar{f}_{1j}(z_1) = \bar{f}_{1j}(z_{1j+1}, \ldots, z_{1n-1}), \; 1 \le j \le n-2.$$

We have constructed new coordinates in which none of the first $n-2$ components of the new system depends on z_{1n}.
Step 2. Consider condition (S2) of Theorem II.2 for $q = 1$, $r = 2$ and denote it by $(\mathcal{L}_{n-1})$. Then

$$(\mathcal{L}_{n-1}) \Longrightarrow \frac{\partial^2 \bar{f}_{1j}}{\partial z_{1n-1}^2} \equiv 0 \quad \text{for all } 1 \le j \le n-2.$$

If condition $(\mathcal{L}_{n-1})$ fails to be satisfied, then the system is NOT linearizable by change of coordinates and the algorithm stops. Otherwise, the system can be reduced (annihilation of terms containing the variable z_{1n-1}) using at most $n-2$ substeps since the condition $(\mathcal{L}_{n-1})$ means that each component is affine with respect to the variable z_{1n-1}.

The substeps follow the same line as those of *step 1* if we omit the last component of the dynamics $\dot{z}_{1n} = u$, i.e., if we view the system as defined in $\mathbb{R}^{n-1}$ with new control $u = z_{1n}$.

Repeating the process detailed in **Step 1** recursively (as long as (S2) holds, thus giving rise to the algorithmic conditions $(\mathcal{L}_{n-j})$ of type $(\mathcal{L}_n)$), the system can be brought into

$$\bar{\Sigma}_{k-1} \; : \; \dot{z}_{k-1} = \bar{f}_{k-1}(z_{k-1}) + \bar{g}_{k-1}(z_{k-1})u, \; z_{k-1} \in \mathbb{R}^n,$$

where $\bar{g}_{k-1}(z_{k-1}) = (0, \ldots, 0, 1)^\top$ and

$$\bar{f}_{k-1}(z_{k-1}) = \left(\bar{f}_{k-11}(z_{k-1}), \ldots, \bar{f}_{k-1n-1}(z_{k-1}), 0 \right)^\top,$$

with

$$\bar{f}_{k-1j}(z_{k-1}) = \bar{f}_{k-1j}(z_{k-1j+1}, .., z_{k-1n-k+1}), \, 1 \le j \le n-k$$

$$\bar{f}_{k-1j}(z_{k-1}) = z_{k-1j} \text{ for any } n-k+1 \le j \le n-1.$$

Step k. (General Step.) Consider condition (S2) of Theorem II.2 for $q = k-1$, $r = k$ and denote it by $(\mathcal{L}_{n-k+1})$. Then

$$(\mathcal{L}_{n-k+1}) \Longrightarrow \frac{\partial^2 \bar{f}_{k-1j}}{\partial z_{k-1n-k+1}^2} \equiv 0 \quad \text{for all } 1 \le j \le n-k.$$

If condition $(\mathcal{L}_{n-k+1})$ fails to be satisfied, then the system is NOT linearizable by change of coordinates and the algorithm stops. Otherwise, the system can be reduced using at most $n-k$ substeps. In the condition $(\mathcal{L}_{n-k+1})$, the term $\partial z_{k-1n-k+1}^2$ refers to the second derivative with respect to the variable $z_{k-1n-k+1}$.

To begin with, notice that condition $(\mathcal{L}_{n-k+1})$ implies, in particular, that $\bar{f}_{k-1n-k}(z_{k-1}) = z_{k-1n-k+1}$.

Substep 1. Decompose the $(n - k - 1)^{st}$ component $\bar{f}_{k-1n-k-1}(z_{k-1n-k}, z_{k-1n-k+1})$ uniquely as:

$$\bar{f}_{k-1n-k-1}(\cdot) \;=\; \tilde{f}_{k-1n-k-1}(z_{k-1n-k})$$
$$+ z_{k-1n-k+1}\theta_{k-1n-k-1}(z_{k-1n-k})$$

and apply the change of coordinates $z_k^1 = \sigma_k^1(z_{k-1})$:

$$z_{kj}^1 = z_{k-1j}, \qquad j \ne n-k-1$$

$$z_{kn-k-1}^1 = z_{k-1n-k-1} - \int_0^{z_{k-1n-k}} \theta_{k-1n-k-1}(s)\mathrm{d}s$$

to annihilate the terms $z_{k-1n-k+1}\theta_{k-1n-k-1}(z_{k-1n-k})$ in the component $\bar{f}_{k-1n-k-1}$.
Substep 2. Decompose $\bar{f}_{kn-k-2}^1(z_{kn-k-1}^1, \ldots, z_{kn-k+1}^1)$ (obtained after change of coordinates $z_k^1 = \sigma_k^1(z_{k-1})$) as

$$\bar{f}_{kn-k-2}^1(\cdot) \;=\; \tilde{f}_{kn-k-2}^1(z_{kn-k-1}^1, z_{kn-k}^1)$$
$$+ z_{kn-k+1}^1 \theta_{kn-k-2}^1(z_{kn-k-1}^1, z_{kn-k}^1).$$

Then, apply the change of coordinates $z_k^2 = \sigma_k^2(z_{k-1}^1)$:

$$z_{kj}^2 = z_{kj}^1, \qquad j \ne n-k-2$$

$$z_{kn-k-2}^2 = z_{kn-k-2}^1 - \int_0^{z_{kn-k}^1} \theta_{kn-k-2}(z_{kn-k-1}^1, s)\mathrm{d}s$$

to annihilate the terms $z_{kn-k+1}^1 \theta_{kn-k-2}^1(z_{kn-k-1}^1, z_{kn-k}^1)$ in the component $\bar{f}_{kn-k-2}^1$.

Because, on one hand side, the changes of coordinates $z_k^1 = \sigma_k^1(z_k), z_k^2 = \sigma_k^2(z_k^1), \ldots,$ are affine in their corresponding variable $z_{kn-k+1}, z_{kn-k+1}^1, \ldots,$ and, on the other, the first $n - k$ components of the system are independent of those variables, it follows that the condition $(\mathcal{L}_{n-k+1})$ remains invariant after each change of coordinates.

Thus, the algorithm can be carried out for all components following a similar line as in the precedent substeps. ∎

Counting the steps. Starting with $\Sigma : \dot{x} = f(x) + g(x)u$, Algorithm I uses $n - 1$ steps to normalize the components of the control vector g, and hence puts the system into control-normalized (SFFcn)-form. Assuming that each of the conditions $(\mathcal{L}_j)$ is satisfied, so that the system is linearizable, we need up to $n - 2$ changes of coordinates to cancel terms containing z_n in the first $n - 2$ components, then $n - 3$ changes of coordinates to cancel terms containing z_{n-1} in the first $n - 3$ components, and so on. Finally, one change of coordinates is needed to cancel terms containing z_3 in the first component. The Algorithms I & II involve a maximum

$$1 + 2 + \cdots + (n - 1) = \frac{n(n - 1)}{2} \text{ changes of coordinates.}$$

The composition of all changes of coordinates provides the linearizing change of coordinates $z = \Phi(x)$. Because the linear system $\dot{z} = Az + bu$, with (A, b) a controllable pair, is stabilizable by a suitable choice of $u = Kz$ (so as $\mathbb{A} \triangleq A + bK$ is Hurwitz), it follows obviously that the closed-loop system $\dot{x} = f(x) + g(x)K\Phi(x)$ is stabilizable. The closed forms solutions of a linearizable (SFF)-form Σ are obtained by $x(t, 0, x^0) = \Phi^{-1} \circ e^{t\mathbb{A}} \circ \Phi(x^0)$.

III. Illustrative Examples

We will provide examples to illustrate the algorithm. We first start with some examples worked out by Krstic [5].

Example III.1 We consider the system from [5] (see [6]).

$$\dot{x}_1 = x_2 + \left(\tfrac{1}{2}x_2 - \tfrac{1}{12}x_3x_4\right)u, \qquad \dot{x}_3 = x_4 + x_4u$$
$$\dot{x}_2 = x_3 + \tfrac{1}{2}x_3u, \qquad\qquad\qquad \dot{x}_4 = u.$$

This is a 4-dimensional system in (SFF)-form. If linearizable, we will need a maximum of 6 steps to achieve linearization.

First apply the change of coordinates

$$\boldsymbol{x_{11}} = x_1, \qquad \boldsymbol{x_{13}} = x_3 - \tfrac{1}{2}x_4^2,$$
$$\boldsymbol{x_{12}} = x_2, \qquad \boldsymbol{x_{14}} = x_4$$

to transform the system into

$$\begin{cases} \dot{\boldsymbol{x}}_{11} = \boldsymbol{x_{12}} + \tfrac{1}{24}\left(12\boldsymbol{x_{12}} - 2\boldsymbol{x_{13}x_{14}} - \boldsymbol{x_{14}^3}\right)u, \ \dot{\boldsymbol{x}}_{13} = \boldsymbol{x_{14}}, \\ \dot{\boldsymbol{x}}_{12} = \boldsymbol{x_{13}} + \tfrac{1}{2}\boldsymbol{x_{14}^2} + \tfrac{1}{2}\left(\boldsymbol{x_{13}} + \tfrac{1}{2}\boldsymbol{x_{14}^2}\right)u, \qquad \dot{\boldsymbol{x}}_{14} = u. \end{cases}$$

Next, the change of coordinates

$$\boldsymbol{x_{21}} = \boldsymbol{x_{11}}, \qquad \boldsymbol{x_{22}} = \boldsymbol{x_{12}} - \tfrac{1}{2}\left(\boldsymbol{x_{13}x_{14}} + \tfrac{1}{6}\boldsymbol{x_{14}^3}\right)$$
$$\boldsymbol{x_{23}} = \boldsymbol{x_{13}}, \qquad \boldsymbol{x_{24}} = \boldsymbol{x_{14}}$$

transforms the system into

$$\begin{cases} \dot{\boldsymbol{x}}_{21} = \boldsymbol{x_{22}} + \tfrac{1}{12}\left(6\boldsymbol{x_{23}x_{24}} + \boldsymbol{x_{24}^3}\right), \quad \dot{\boldsymbol{x}}_{23} = \boldsymbol{x_{24}} \\ \qquad\quad + \tfrac{1}{6}\left(3\boldsymbol{x_{22}} + \boldsymbol{x_{23}x_{24}}\right)u, \\ \dot{\boldsymbol{x}}_{22} = \boldsymbol{x_{23}}, \qquad\qquad\qquad\qquad \dot{\boldsymbol{x}}_{24} = u. \end{cases}$$

Finally, we apply the change of coordinates

$$\boldsymbol{x_{31}} = \boldsymbol{x_{21}} - \tfrac{1}{6}\left(3\boldsymbol{x_{22}x_{24}} + \tfrac{1}{2}\boldsymbol{x_{23}x_{24}^2}\right), \quad \boldsymbol{x_{33}} = \boldsymbol{x_{23}}$$
$$\boldsymbol{x_{32}} = \boldsymbol{x_{22}}, \qquad\qquad\qquad\qquad\qquad \boldsymbol{x_{34}} = \boldsymbol{x_{24}}$$

to transform the system into a linear one

$$\dot{\boldsymbol{x}}_{31} = \boldsymbol{x_{32}}, \ \ \dot{\boldsymbol{x}}_{33} = \boldsymbol{x_{34}}, \ \ \dot{\boldsymbol{x}}_{32} = \boldsymbol{x_{33}}, \ \ \dot{\boldsymbol{x}}_{34} = u.$$

The linearizing diffeomorphism is obtained by composition:

$$\boldsymbol{x_{31}} = x_1 - \tfrac{1}{24}\left(12x_2x_4 - 4x_3x_4^2 + x_4^4\right), \ \boldsymbol{x_{33}} = x_3 - \tfrac{1}{2}x_4^2$$
$$\boldsymbol{x_{32}} = x_2 - \tfrac{1}{2}\left(x_3x_4 - \tfrac{1}{3}x_4^3\right), \qquad\qquad \boldsymbol{x_{34}} = x_4. \qquad \blacktriangle$$

Example III.2 Consider the type I linearizable strict feed-forward systems from [5] (with $x_{n+1} = u$):

$$\begin{cases} \dot{x}_1 = x_2 + \sum_{i=2}^{n-1} \pi_i(x_i)x_{i+1} + \pi_n(x_n)u, \\ \dot{x}_j = x_{j+1}, \qquad\qquad\qquad\qquad\quad 2 \le j \le n. \end{cases}$$

The first step is to normalize $g = (\pi(x_n), 0, \dots, 0, 1)^\top$. The change of coordinates $\boldsymbol{x_1} = \phi(x)$ defined by

$$\boldsymbol{x_{11}} = \phi_{11}(x) = x_1 - \int_0^{x_n} \pi_n(s)\mathrm{d}s,$$
$$\boldsymbol{x_{1j}} = \phi_{1j}(x) = x_{2j}, \quad 2 \le j \le n$$

brings the system into the form

$$\begin{cases} \dot{\boldsymbol{x}}_{11} = \boldsymbol{x_{12}} + \sum_{i=2}^{n-1} \pi_i(\boldsymbol{x_{1i}})\boldsymbol{x_{1i+1}}, \\ \dot{\boldsymbol{x}}_{1j} = \boldsymbol{x_{1j+1}}, \qquad\qquad\qquad 2 \le j \le n+1. \end{cases}$$

The system is control-normalized (replace $\boldsymbol{x_1}$ by z)

$$\bar{\Sigma} \ : \ \dot{z} = \bar{f}(z) + \bar{g}(z)u, \quad z \in \mathbb{R}^n,$$

where $\bar{g} = (0, \dots, 0, 1)^\top$ and

$$\bar{f}(z) = \left(z_2 + \sum_{i=2}^{n-1} \pi_i(z_i)z_{i+1}, z_3, \dots, z_n, 0\right)^\top.$$

Obviously, the condition

$$(\mathcal{L}_n) \Longrightarrow \frac{\partial^2 \bar{f}_j}{\partial z_n^2} \equiv 0, \quad \text{for all } 1 \le j \le n-1$$

is satisfied, hence the first step of algorithm II applies.

Indeed, the first component $\bar{f}_1$ decomposes uniquely as

$$\bar{f}_1(z_2, \dots, z_n) = \tilde{f}_1(z_2, \dots, z_{n-1}) + z_n\theta_{11}(z_2, \dots, z_{n-1})$$
$$= z_2 + \sum_{i=2}^{n-2} \pi_i(z_i)z_{i+1} + z_n\pi_{n-1}(z_{n-1}).$$

Since only the first component is nonlinear (hence to be normalized), there are no multiple substeps. We will then drop the upperscripts that would correspond to those substeps. The diffeomorphism $z_1 = \sigma_1(z)$, whose components are

$$\boldsymbol{z_{11}} = \sigma_{11}(z) = z_1 - \int_0^{z_{n-1}} \pi_{n-1}(s)\mathrm{d}s,$$
$$\boldsymbol{z_{1j}} = \sigma_{1j}(z) = z_j, \ 2 \le j \le n$$

takes the system into the form

$$\Sigma_1 \ : \ \dot{\boldsymbol{z}}_1 = \bar{f}_1(\boldsymbol{z_1}) + \bar{g}_1(\boldsymbol{z_1})u, \quad \boldsymbol{z_1} \in \mathbb{R}^n,$$

where $\bar{g}_1(\boldsymbol{z_1}) = (0, \dots, 0, 1)^\top$ and

$$\bar{f}_1(\boldsymbol{z_1}) = \left(\boldsymbol{z_{12}} + \sum_{i=2}^{n-2} \pi_i(\boldsymbol{z_{1i}})\boldsymbol{z_{1i+1}}, \boldsymbol{z_{13}}, \dots, \boldsymbol{z_{1n}}, 0\right)^\top.$$

The condition

$$(\mathcal{L}_{n-1}) \Longrightarrow \frac{\partial^2 \bar{f}_{1j}}{\partial z_{1n-1}^2} \equiv 0, \quad \text{for all } 1 \le j \le n-2$$

holds, and $\boldsymbol{z_2} = \sigma_2(z)$, whose components are given by

$$\boldsymbol{z_{21}} = \sigma_{21}(\boldsymbol{z_1}) = \boldsymbol{z_{11}} - \int_0^{\boldsymbol{z_{1n-2}}} \pi_{n-2}(s)\mathrm{d}s,$$
$$\boldsymbol{z_{2j}} = \sigma_{2j}(z) = \boldsymbol{z_{1j}}, \ 2 \le j \le n$$

allows to cancel the terms $\boldsymbol{z_{1n-1}}\pi_{n-2}(\boldsymbol{z_{1n-2}})$.

We can thus define recursively changes of coordinates

$$\boldsymbol{z_0} \triangleq z, \quad \boldsymbol{z_k} = \sigma_k(\boldsymbol{z_{k-1}}), \quad k = 1, 2, \dots, n-2, \text{ where}$$
$$\boldsymbol{z_{k1}} = \sigma_{k1}(\boldsymbol{z_{k-1}}) = \boldsymbol{z_{k-11}} - \int_0^{\boldsymbol{z_{k-1n-k}}} \pi_{n-k}(s)\mathrm{d}s$$
$$\boldsymbol{z_{kj}} = \sigma_{kj}(\boldsymbol{z_{k-1}}) = \boldsymbol{z_{k-1j}}, \quad \text{for } j = 2, \dots, n.$$

It follows that the composition

$$y = \boldsymbol{z_{n-2}} = \sigma_{n-2} \circ \cdots \circ \sigma_2 \circ \sigma_1 \circ \phi_1(x)$$
$$\left(x \xrightarrow{\phi_1} \boldsymbol{x_1} \triangleq z \xrightarrow{\sigma_1} \boldsymbol{z_1} \xrightarrow{\sigma_2} \boldsymbol{z_2} \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n-2}} \boldsymbol{z_{n-2}}\right)$$

linearizes the system. It is straightforward that (see [5])

$$y_1 = x_1 - \sum_{k=2}^{n} \int_0^{x_k} \pi_k(s)\mathrm{d}s, \ y_j = x_j, \ j = 2, \dots, n. \quad \blacktriangle$$

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We now consider a 3-dimensional system of type II:

$$\begin{cases} \dot{x}_1 &=& x_2 + \phi_1(x_2, x_3)u, \\ \dot{x}_2 &=& x_3 + \phi_2(x_3)u, \\ \dot{x}_3 &=& u, \end{cases} \qquad \text{(III.1)}$$

where $\phi_2(0) = 0$ and $\phi_3(0) = 0$. Theorem III.3 below gives necessary and sufficient conditions for its linearizability.

Theorem III.3 *System (III.1) is linearizable if and only if*

$$\phi_1(x_2, x_3) = x_2\gamma_1(x_3) + \theta_1(x_3)$$

with

$$\gamma_1(x_3) = \frac{d}{dx_3}\left(\frac{1}{x_3}\int_0^{x_3}\phi_2(s)\mathrm{ds}\right).$$

Proof. *Sufficiency.* The sufficiency is straightforward and can be deduced from [5] (see Theorem 3), where

$$\mu_1(x_3) = \frac{1}{x_3}\int_0^{x_3}\phi_2(s)\mathrm{ds}, \quad \text{and} \quad \theta_1(x_3) = \phi_1(0, x_3).$$

Necessity. First, we transform the system into a (SFFcn)-form following Algorithm I. The change of coordinates

$$\begin{array}{rcl} \boldsymbol{x_{11}} &=& x_1 \\ \boldsymbol{x_{12}} &=& x_2 - \int_0^{x_3}\phi_2(s)\mathrm{ds} \\ \boldsymbol{x_{13}} &=& x_3 \end{array}$$

transforms the system into

$$\begin{cases} \dot{\boldsymbol{x}}_{11} &=& \boldsymbol{x_{12}} + \int_0^{\boldsymbol{x_{13}}}\phi_2(s)\mathrm{ds} + \hat{\phi}_1(\boldsymbol{x_{12}}, \boldsymbol{x_{13}})u \\ \dot{\boldsymbol{x}}_{12} &=& \boldsymbol{x_{13}} \\ \dot{\boldsymbol{x}}_{13} &=& u, \end{cases}$$

where $\hat{\phi}_1(\boldsymbol{x_{12}}, \boldsymbol{x_{13}}) = \phi_1\left(\boldsymbol{x_{12}} + \int_0^{\boldsymbol{x_{13}}}\phi_2(s)\mathrm{ds}, \boldsymbol{x_{13}}\right)$.

Next, we apply the change of coordinates

$$\begin{array}{rcl} z_1 = \boldsymbol{x_{21}} &=& \boldsymbol{x_{11}} - \int_0^{\boldsymbol{x_{13}}}\hat{\phi}_1(\boldsymbol{x_{12}}, s)\mathrm{ds} \\ z_2 = \boldsymbol{x_{22}} &=& \boldsymbol{x_{12}} \\ z_3 = \boldsymbol{x_{23}} &=& \boldsymbol{x_{13}} \end{array}$$

to take the system into the form

$$\begin{cases} \dot{z}_1 &=& z_2 + \bar{f}_1(z_2, z_3) \\ \dot{z}_2 &=& z_3 \\ \dot{z}_3 &=& u, \end{cases}$$

where $\bar{f}_1(z_2, z_3) = \int_0^{z_3}\phi_2(s)\mathrm{ds} - z_3\int_0^{z_3}\frac{\partial\hat{\phi}_1}{\partial z_2}(z_2, s)\mathrm{ds}$. According to Algorithm II, in order for the system to be linearizable, condition $(\mathcal{L}_3) \implies \frac{\partial^2\bar{f}_1}{\partial z_3^2} \equiv 0$ should hold.

Since $\bar{f}_1(z_2, 0) = 0$ and $\frac{\partial\bar{f}_1}{\partial z_3}(z_2, 0) = 0$, then $(\mathcal{L}_3)$ yields

$$\bar{f}_1(z_2, z_3) = \int_0^{z_3}\phi_2(s)\mathrm{ds} - z_3\int_0^{z_3}\frac{\partial\hat{\phi}_1}{\partial z_2}(z_2, s)\mathrm{ds} \equiv 0.$$

It follows that $\frac{\partial\hat{\phi}_1}{\partial z_2}(z_2, z_3)$ is a function of z_3 alone, and hence $\hat{\phi}_1(z_2, z_3)$ is affine in z_2. Because

$$\hat{\phi}_1(z_2, z_3) = \phi_1(z_2 + \int_0^{z_3}\phi_2(s)\mathrm{ds}, z_3),$$

we deduce that $\phi_1(z_2, z_3) = z_2\gamma_1(z_3) + \theta_1(z_3)$ is also affine in z_2, and it is straightforward to see that γ_1 satisfies

$$\int_0^{z_3}\phi_2(s)\mathrm{ds} - z_3\int_0^{z_3}\gamma_1(s)\mathrm{ds} \equiv 0.$$

This implies that γ_1 is defined like in the theorem and this achieves the proof of the necessity. ∎

A type II example considered by Krstic [5] is

$$\begin{cases} \dot{x}_1 &=& x_2 + (\frac{\mathbf{1}}{\mathbf{2}}x_2 + x_3\sin x_3)u \\ \dot{x}_2 &=& x_3 + x_3u \\ \dot{x}_3 &=& u \end{cases}$$

and he pointed out that the key restriction for linearizability is the **boldfaced** $\frac{\mathbf{1}}{\mathbf{2}}$. This is justified by Theorem III.3 with

$$\phi_1(x_2, x_3) = \frac{1}{2}x_2 + x_3\sin x_3 \quad \text{and} \quad \phi_2(x_3) = x_3.$$

The condition of Theorem III.3 implies indeed that

$$\phi_1(x_2, x_3) = x_2\gamma_1(x_3) + \theta_1(x_3)$$

where

$$\gamma_1(x_3) = \frac{d}{dx_3}\left(\frac{1}{x_3}\int_0^{x_3}\phi_2(s)\mathrm{ds}\right) = \frac{d}{dx_3}\left(\frac{1}{x_3}\int_0^{x_3}s\mathrm{ds}\right) = \frac{1}{2}.$$

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