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THE SPIN L-FUNCTION OF QUASI-SPLIT D_4

WEE TECK GAN AND JOSEPH HUNDLEY

1. Introduction

The purpose of this paper is to construct a Rankin-Selberg integral for the Spin L-function $L(\underline{s}, \pi, V_{\mathrm{Spin},E})$ of a generic cuspidal representation π of a *quasi-split* adjoint group G_E of type D_4. Let us formulate our results more precisely.

1.1. **Étale cubic algebras.** Let F be a number field with adele ring $\mathbb{A}$ and absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. An étale cubic algebra is an F-algebra E such that $E \otimes_F \overline{F} \cong \overline{F}^3$. More concretely, an étale cubic F-algebra is of the form:

$$E = \begin{cases} F \times F \times F; \\ F \times K, \text{ where } K \text{ is a quadratic field extension of } F; \\ \text{ a cubic field.} \end{cases}$$

Since the split algebra $F \times F \times F$ has automorphism group S_3 (the symmetric group on 3 letters), the isomorphism classes of étale cubic algebras E over F are naturally classified by the set of conjugacy classes of homomorphisms

$$\rho_E : \mathrm{Gal}(\overline{F}/F) \longrightarrow S_3.$$

By composing the homomorphism ρ_E with the sign character of S_3, we obtain a quadratic character (possibly trivial) of $\mathrm{Gal}(\overline{F}/F)$ which corresponds to an étale quadratic algebra K_E. We call K_E the *discriminant algebra* of E. To be concrete,

$$K_E = \begin{cases} F \times F, \text{ if } E = F^3 \text{ or a cyclic cubic field;} \\ K, \text{ if } E = F \times K; \\ \text{the unique quadratic subfield in the Galois closure of } E \text{ otherwise.} \end{cases}$$

We shall let χ_{K_E} denote the quadratic idele class character associated to K_E.

1.2. **Twisted form of S_3.** Fix an étale cubic F-algebra E. Then, via the associated homomorphism ρ_E, $\mathrm{Gal}(\overline{F}/F)$ acts on S_3 (by inner automorphisms) and thus defines a twisted form Γ_E of the finite constant group scheme S_3. For any commutative F-algebra A, we have

$$\Gamma_E(A) = \mathrm{Aut}_A(E \otimes_F A).$$

1.3. **Quasi-split groups of type D_4.** Because S_3 is the outer automorphism group of $PGSO_8$ (the split adjoint group of type D_4), associated to E is a quasi-split adjoint group G_E of type D_4. The outer automorphism group of G_E is precisely the finite group scheme Γ_E.

The Langlands dual group of G_E is the simply-connected complex Lie group

$$G_E^\vee = Spin_8(\mathbb{C}).$$

This has 3 irreducible 8-dimensional representations V_{μ_i}, whose highest weights μ_i ($i = 1, 2$ or 3) are the three fundamental weights associated to the three satellite vertices in the Dynkin diagram of type D_4. These 3 representations are permuted by the outer automorphism group S_3, and the sum

$$V_{Spin} = V_{\mu_1} \oplus V_{\mu_2} \oplus V_{\mu_3}$$

extends to a faithful irreducible representation of $Spin_8(\mathbb{C}) \rtimes S_3$. In fact, there are two such extensions which differ from each other by twisting with the sign character of S_3. So we shall need to specify the extension precisely later (in (2.8)).

The L-group ${}^L G_E$ is a certain semidirect product of $Spin_8(\mathbb{C})$ with $\mathrm{Gal}(\overline{F}/F)$. More precisely, the action of $\mathrm{Gal}(\overline{F}/F)$ on $Spin_8(\mathbb{C})$ is via the homomorphism ρ_E. Thus there is a natural map

$${}^L G_E \longrightarrow Spin_8(\mathbb{C}) \rtimes S_3,$$

whose restriction to $\mathrm{Gal}(\overline{F}/F)$ is ρ_E. Via this map, we may view V_{Spin} as a representation of ${}^L G_E$. We denote this representation by $V_{\mathrm{Spin},E}$ and call this the **Spin representation** of ${}^L G_E$. Note that $V_{\mathrm{Spin},E}$ is reducible unless E is a field.

1.4. **The Spin L-function.** Now let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $G_E(\mathbb{A})$. For almost all v, the representation π_v is unramifed and gives rise to a semisimple conjugacy class $t_{\pi_v} \in {}^L G_E$ (its Satake parameter). We may thus define the partial Spin L-function associated to π:

$$L^S(s, \pi, V_{\mathrm{Spin},E}) = \prod_{v \notin S} \frac{1}{\det(1 - q_v^{-s} t_{\pi_v} | V_{\mathrm{Spin},E})}$$

where S is a finite set of places, including the archimedean ones, such that π_v is unramified for all $v \notin S$. This Euler product converges when $Re(s)$ is sufficiently large.

In fact, since the representation $V_{\mathrm{Spin},E}$ is reducible when E is not a field, it is natural to define a refinement of the above L-function by introducing a different variable for each irreducible constituent of $V_{\mathrm{Spin},E}$. More precisely, we have:

- If $E = F \times F \times F$, we set $\underline{s} = (s_1, s_2, s_3)$ and

$$L^S(\underline{s}, \pi, V_{\mathrm{Spin},E}) = \prod_{v \notin S} \frac{1}{\det(1 - q_v^{-s_1} t_{\pi_v} | V_{\mu_1}) \cdot \det(1 - q_v^{-s_2} t_{\pi_v} | V_{\mu_2}) \cdot \det(1 - q_v^{-s_3} t_{\pi_v} | V_{\mu_3})}.$$

- If $E = F \times K$, we may assume without loss of generality that the two fundamental weights μ_2 and μ_3 are permuted by the Galois action, so that V_{μ_1} and $V_{\mu_2} \oplus V_{\mu_3}$ are the two irreducible constituents of $V_{\mathrm{Spin},E}$. Setting $\underline{s} = (s_1, s_{23}, s_{23})$, we then define

$$L^S(\underline{s}, \pi, V_{\mathrm{Spin},E}) = \prod_{v \notin S} \frac{1}{det(1 - q_v^{-s_1} t_{\pi_v} | V_{\mu_1}) \cdot det(1 - q_v^{-s_{23}} t_{\pi_v} | V_{\mu_2} \oplus V_{\mu_3})}.$$

- if E is a field, then $\underline{s} = (s, s, s)$ and $L^S(\underline{s}, \pi, V_{\mathrm{Spin},E})$ is as originally defined.

1.5. **The refined zeta function of** E**.** Likewise, since E is the product of fields E_i, the zeta function $\zeta_E(s)$ decomposes as the product of zeta functions of these fields. It is natural to define a refinement $\zeta_E(\underline{s})$, with $\underline{s}$ defined as above in the three different cases. Thus, for example,

$$\zeta_{F \times K}(\underline{s}) = \zeta_F(s_1) \cdot \zeta_K(s_{23}).$$

In addition, we define another refinement of the zeta function of E:

$$\hat{\zeta}_E(\underline{s}) = \begin{cases} \zeta(s_1+s_2-s_3)\cdot\zeta(s_2+s_3-s_1)\cdot\zeta(s_3+s_1-s_2), \text{ if } E = F^3; \\ \zeta(2s_{23}-s_1)\cdot\zeta_K(s_1), \text{ if } E = F\times K; \\ \zeta_E(s), \text{ if } E \text{ is a field.} \end{cases}$$

This may seem somewhat artificial at first, but note that if one sets all the variables s_i equal to s, one recovers $\zeta_E(s)$. In fact, this refined function arises naturally in the normalizing factor of the Eisenstein series in our Rankin-Selberg integral. In any case, one can simply regard it as a shorthand for the rather convoluted function it represents.

1.6. **The Rankin-Selberg integral.** The goal of this paper is to construct a Rankin-Selberg integral for $L(\underline{s},\pi,V_{\mathrm{Spin},E})$ when π is globally generic, i.e. possesses a non-zero Whittaker-Fourier coefficient. The Rankin-Selberg integral we consider is of Shimura type. Thus it involves the integral of a cusp form, an Eisenstein series and a "theta" function:

$$Z_E(\varphi,\Phi,f,\underline{s}) = \int_{G_E(F)\backslash G_E(\mathbb{A})} \varphi(g)\cdot\overline{\theta(f)(g)}\cdot E(\Phi,\underline{s},g)\,dg.$$

To explain the various notations,

- φ is a cusp form in π;
- $\theta(f)$ is a vector in the minimal representation Π_E of $G_E(\mathbb{A})$, which is the analog of the Weil representation of a metaplectic group;
- $E(\Phi,\underline{s},g)$ is an Eisenstein series on $G_E(\mathbb{A})$ associated to a standard section $\Phi_{\underline{s}}$ of a certain principal series $I_{Q_E}(\underline{s})$ of G_E. This principal series $I_{Q_E}(\underline{s})$ is induced from a character on the (not-necessarily maximal) parabolic subgroup Q_E whose Levi factor is of semisimple type A_1 corresponding to the middle vertex of the Dynkin diagram of type D_4.

Our main theorem is then:

1.7. **<u>MAIN THEOREM</u>:** The global zeta integral admits an Euler product. Moreover, we have:

$$Z_E(\varphi,\Phi,f,\underline{s}) = \frac{L^S(\underline{s},\pi,V_{\mathrm{Spin},E})}{\hat{\zeta}_E^S(\underline{s}+\underline{1})\cdot\zeta_E^S(2\underline{s})\cdot\zeta^S(|\underline{s}|)}\cdot\left(\prod_{v\in S} Z_{E,v}(\varphi_v,\Phi_v,f_v,\underline{s})\right)$$

where $\underline{1}=(1,..,1)$ and $|\underline{s}|$ denotes the sum of the components of $\underline{s}$ (so for example, $|\underline{s}|=3s$ if E is a field). Moreover, each local factor for $v\in S$ admits meromorphic continuation as a function of $\underline{s}$. Thus, the Spin L-function $L^S(\underline{s},\pi,V_{\mathrm{Spin},E})$ can be meromorphically continued.

Our investigation is inspired by the recent paper [GH] of the second author with D. Ginzburg. There, they considered the case when the group G_E is split and constructed a multi-variable Rankin-Selberg integral which is inherently asymmetric: it gives the L-functions associated to two of the degree 8 representations (say V_{μ_1} and V_{μ_2}), with one of them obtained two times (say V_{μ_1}). The Rankin-Selberg integral we consider here is motivated by the S_3-symmetry of D_4, and has the advantage that it extends in a self evident fashion to all the quasi-split forms. More importantly, when E is a field, the Spin L-function is one which cannot be analyzed by the Langlands-Shahidi method.

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2. **Preliminaries**

We begin by establishing some notations and introducing some background results.

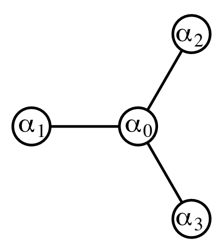
2.1. D_4 **root system.** We fix a Borel subgroup B_E of G_E and a Levi subgroup T_E which is a maximal torus, both defined over F. Let V_E denote the unipotent radical of B_E. Over the algebraic closure $\overline{F}$, the choice of (T_E, B_E) gives a set Δ of simple absolute roots for the set Ψ of absolute roots of T_E in G_E. Using the standard realization of the root system of type D_4, we label these simple roots by

$$\begin{cases} \alpha_1 = \epsilon_1 - \epsilon_2 \\ \alpha_0 = \epsilon_2 - \epsilon_3 \\ \alpha_2 = \epsilon_3 - \epsilon_4 \\ \alpha_3 = \epsilon_3 + \epsilon_4, \end{cases}$$

where $\{\epsilon_i\}$ is the standard basis of $\mathbb{R}^4$. In particular, α_0 is the branch (or middle) vertex in the Dynkin diagram of type D_4. We let

$$\beta_0 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_0$$

denote the highest root in Ψ. Here is the Dynkin diagram:



For each absolute root γ, we let U_γ be the associated root subgroup; it may not be defined over F. Indeed, the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ acts naturally on Ψ and U_γ is defined over F iff γ is fixed by the Galois action. We also let w_γ denote the element of the Weyl group corresponding to the reflection in γ.

If E is a cubic field, then $\mathrm{Gal}(\overline{F}/F)$ permutes the roots α_1, α_2 and α_3 transitively. If $E = F \times K$ with K a quadratic field, then without loss of generality, we assume that α_1 is fixed, whereas α_2 and α_3 are exchanged by the Galois action. If E is the split algebra, the Galois action on Ψ is trivial.

Let ω_i be the fundamental weight so that

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij},$$

and let μ_i be the fundamental coweight so that

$$\langle \alpha_j, \mu_i \rangle = \delta_{ij}.$$

In terms of the basis $\{\epsilon_i^*\}$ dual to $\{\epsilon_i\}$, the fundamental coweights are given by:

$$\begin{cases} \mu_1 = \epsilon_1^*, \\ \mu_0 = \epsilon_1^* + \epsilon_2^* \\ \mu_2 = (\epsilon_1^* + \epsilon_2^* + \epsilon_3^* - \epsilon_4^*)/2 \\ \mu_3 = (\epsilon_1^* + \epsilon_2^* + \epsilon_3^* + \epsilon_4^*)/2. \end{cases}$$

Because G_E is adjoint, the set $\{\mu_i\}$ is a basis for the cocharacter group

$$X_*(T_E) = \mathrm{Hom}_{\overline{F}}(\mathbb{G}_m, T_E).$$

2.2. G_2 **root system.** We have already fixed the pair (T_E, B_E). If we further fix a Chevalley-Steinberg system of épinglage relative to this pair, then we have a compatible system of isomorphisms $U_\gamma \cong \mathbb{G}_a$ defined over $\overline{F}$ which are permuted by $\mathrm{Gal}(\overline{F}/F)$. This gives a splitting of the outer automorphism group

$$\Gamma_E \hookrightarrow Aut(G_E).$$

The subgroup scheme of G_E fixed pointwise by Γ_E is independent of the choice of the épinglage relative to (T_E, B_E), and is isomorphic to the split exceptional group of type G_2.

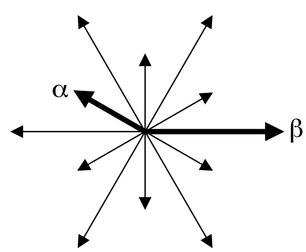
Observe that $B = G_2 \cap B_E$ is a Borel subgroup of G_2 and $T = T_E \cap G_2$ is a maximal split torus of G_2. The torus T is such that

$$X_*(T) = \langle \mu_0, \mu_1 + \mu_2 + \mu_3 \rangle.$$

Via the adjoint action of T on G_E, we obtain the root system Ψ_{G_2} of G_2, so that

$$\Psi_{G_2} = \Psi|_T.$$

Here is a diagram of the root system of type G_2.



We denote the short simple root of this G_2 root system by α and the long simple root by β. Then

$$\beta = \alpha_0|_T \quad \text{and} \quad \alpha = \alpha_1|_T = \alpha_2|_T = \alpha_3|_T.$$

Thus, the short root spaces have dimension 3, whereas the long root spaces have dimension 1. For each root $\gamma \in \Psi_{G_2}$, the associated root subgroup U_γ is defined over F and the Chevalley-Steinberg system of épinglage gives isomorphisms:

$$U_\gamma \cong \begin{cases} \mathrm{Res}_{E/F}\,\mathbb{G}_a, & \text{if } \gamma \text{ is short;} \\ \mathbb{G}_a, & \text{if } \gamma \text{ is long.} \end{cases}$$

When E is a cubic field, T is in fact the maximal F-split torus of G_E and Ψ_{G_2} is the relative root system of G_E.

For each $\gamma \in \Psi_{G_2}$, we shall also let N_γ denote the root subgroup of G_2 corresponding to γ. In particular,

$$N_\gamma = U_\gamma \cap G_2.$$

Because the highest root β_0 of the D_4-root system restricts to that of the G_2-root system, we shall let β_0 denote the highest root of the G_2-root system also. This should not cause confusion.

2.3. **Two parabolic subgroups.** The G_2 root system gives rise to 2 parabolic subgroups of G_E. One of these is a maximal parabolic $P_E = M_E N_E$ known as the Heisenberg parabolic. Its unipotent radical N_E is a Heisenberg group and its Levi subgroup M_E is spanned by the 3 satellite vertices in the Dynkin diagram. The other parabolic $Q_E = L_E U_E$ is a not-necessarily-maximal parabolic; its Levi subgroup L_E is spanned by the branch vertex α_0 and its unipotent radical U_E is a 3-step unipotent group. We shall need to examine the structure of these 2 parabolics more carefully.

2.4. **The Heisenberg parabolic P_E.** Let us begin with the Heisenberg parabolic $P_E = M_E N_E$. Its unipotent radical is a Heisenberg group with center $Z = U_{\beta_0}$. Moreover,

$$N_E/Z = U_\beta \times U_{\beta+\alpha} \times U_{\beta+2\alpha} \times U_{\beta+3\alpha} \cong F \times E \times E \times F.$$

Note that $P_E \cap G_2$ is the Heisenberg maximal parabolic $P = MN$ of G_2, with

$$M = G_2 \cap M_E \cong GL_2 \quad \text{and} \quad N = G_2 \cap N_E.$$

Let $\Omega_E(F)$ denote the minimal non-zero $M_E(F)$-orbit on

$$\overline{N}_E(F)/\overline{Z}(F) = U_{-\beta} \times U_{-\beta-\alpha} \times U_{-\beta-2\alpha} \times U_{-\beta-3\alpha} \cong F \times E \times E \times F.$$

It is the orbit of a highest weight vector and its Zariski closure is a cone. A non-zero element (a, x, y, d) lies in Ω_E iff

$$\begin{cases} xy = ad \\ x^\# = ay \\ y^\# = dx. \end{cases}$$

Here

$$\# : E \longrightarrow E$$

is the canonical quadratic map with the property that $x \cdot x^\# = \mathbb{N}(x)$, where $\mathbb{N}(x)$ denotes the norm of x. So, for example, the element $a \cdot (1, x, y, d)$, with $a \in F^\times$, lies in $\Omega_E(F)$ iff

$$y = x^\# \quad \text{and} \quad d = \mathbb{N}(x).$$

Observe that there is a natural map $\overline{N}_E/\overline{Z} \longrightarrow \overline{N}/\overline{Z}$ given by

$$(a, x, y, d) \mapsto (a, Tr(x), Tr(y), d)$$

where $Tr(x)$ denotes the trace of x. Given any element $\chi \in \overline{N}/\overline{Z}$, we let $\Omega_{E,\chi}$ denote the fiber of this map over χ. For example, if E is a field and $\chi = (1, 0, 0, 0)$, then $\Omega_{E,\chi} = \{(1, 0, 0, 0)\}$ since the only $x \in E$ with $\mathbb{N}(x) = 0$ is $x = 0$.

2.5. **The 3-step parabolic Q_E.** Now we come to the parabolic Q_E. The unipotent radical U_E has a filtration

$$\{1\} \subset U_E^{(1)} \subset U_E^{(2)} \subset U_E$$

such that

$$U_E^{(1)} = U_{\beta_0} \times U_{\beta_0 - \beta}$$

is the center of U_E. Further,

$$U_E^{(2)} = [U_E, U_E] = U_{\beta_0} \times U_{\beta_0 - \beta} \times U_{2\alpha+\beta}$$

is the commutator subgroup of U_E and is abelian. In particular, U_E is a 3-step unipotent group; hence we call Q_E the 3-step parabolic. Note that $Q = G_2 \cap Q_E = L \cdot U$ is the non-Heisenberg maximal parabolic of G_2, with

$$L = G_2 \cap L_E \cong GL_2 \text{ and } \quad U = G_2 \cap U_E.$$

It will be necessary to have a more explicit description of L_E. By examining the root datum of L_E, one can show that

$$L_E \cong (GL_2 \times \mathrm{Res}_{E/F}\, \mathbb{G}_m)/\Delta\mathbb{G}_m.$$

In terms of this isomorphism, the simple absolute roots of G_E are given on a torus element

$$t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times (c_1, c_2, c_3) \in T_E(\overline{F}) \subset L_E(\overline{F})$$

by:

$$\alpha_0(t) = \frac{a}{b} \quad \text{and} \quad \alpha_i(t) = \frac{b}{c_i}$$

for $i = 1, 2$ or 3.

From the above description, one sees that the center Z_{L_E} of the Levi subgroup L_E is such that

$$X_*(Z_{L_E}) = \langle \mu_1, \mu_2, \mu_3 \rangle.$$

Moreover, the group $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$ can be described as follows. For $1 \leq i \leq 3$, let us define $\chi_i \in X^*(L_E)$ by

$$\chi_i(g, c_1, c_2, c_3) = \frac{det(g) \cdot c_i}{c_1 c_2 c_3}.$$

Then the elements χ_i form a $\mathbb{Z}$-basis of $X^*(L_E)$. On restriction to T_E, we have:

$$\chi_i = \beta_0 - \alpha_0 - \alpha_i \quad \text{for } i = 1, 2 \text{ or } 3.$$

Now taking into account the Galois action, we have:

$$\mathrm{Hom}_F(L_E, \mathbb{G}_m) = \begin{cases} \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2 \oplus \mathbb{Z}\chi_3, & \text{if } E = F \times F \times F; \\ \mathbb{Z}\chi_1 \oplus \mathbb{Z} \cdot (\chi_2 \oplus \chi_3), & \text{if } E = F \times K; \\ \mathbb{Z} \cdot (\chi_1 + \chi_2 + \chi_3), & \text{if } E \text{ is a field.} \end{cases}$$

Observe that the character

$$\nu_{L_E} = \chi_1 + \chi_2 + \chi_3$$

always belong to $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$. Moreover, the modulus character of Q_E is

$$\delta_{Q_E} = 3 \cdot \nu_{L_E}.$$

2.6. **Eisenstein series.** With the above description of $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$, we can now define an unramified character of $L_E(F_v)$ or $L_E(\mathbb{A})$:

$$\chi_{\underline{s}} = \begin{cases} |\chi_1|^{1+s_1} \cdot |\chi_2|^{1+s_2} \cdot |\chi_3|^{1+s_3}, \text{ if } E = F^3; \\ |\chi_1|^{1+s_1} \cdot |\chi_2 + \chi_3|^{1+s_{23}}, \text{ if } E = F \times K; \\ |\chi_1 + \chi_2 + \chi_3|^{1+s}, \text{ if } E \text{ is a field.} \end{cases}$$

We can then consider the degenerate principal series

$$I_{Q_E}(\underline{s}) = Ind_{Q_E}^{G_E} \chi_{\underline{s}} \quad \text{(unnormalized induction)}.$$

The associated Eisenstein series is given by

$$E(\Phi, \underline{s}, g) = \sum_{\gamma \in Q_E(F) \backslash G_E(F)} \Phi_s(\gamma g)$$

for a standard section $\Phi_s \in I_{Q_E}(\underline{s})$. This sum converges absolutely at $\underline{s} = (s_1, s_2, s_3)$ if $s_i + s_{i+1} \gg s_{i+2}$ for $i = 1, 2, 3$ and where the subscripts are taken modulo 3. We let Ω_R denote the region consisting of those $\underline{s}$ satisfying $s_i + s_{i+1} - s_{i+2} \geq R$, for $i = 1, 2, 3$.

This Eisenstein series is one of the ingredients in our Rankin-Selberg integral.

2.7. **Geomeric description of** $\mathrm{Spin}_8(\mathbb{C})$**.** At this point, we should describe precisely what we mean by the representation V_{Spin} of $\mathrm{Spin}_8(\mathbb{C}) \rtimes S_3$. For this, it is necessary to give a geometric description of the split group Spin_8. This (very beautiful) description works over an arbitrary field k and can be found, for example, in the paper [Gr].

Let $\mathbb{O}$ be the split octonion algebra over k. Then $\mathbb{O}$ comes equipped with

- an anti-involution $x \mapsto \bar{x}$;
- a (quadratic) norm form $N : \mathbb{O} \longrightarrow k$ such that $N(x) = x \cdot \bar{x} = \bar{x} \cdot x$;
- a (linear) trace form $Tr : \mathbb{O} \longrightarrow k$ such that $Tr(x) = x + \bar{x}$.

In particular, we may consider the special orthogonal group $SO(\mathbb{O}, N)$.

Though the multiplication in $\mathbb{O}$ is not associative, the symbol $Tr(xyz)$ is well-defined and satisfies

$$Tr(xyz) = Tr(yzx) = Tr(zxy).$$

Now the group Spin_8 over k can be described by:

$$\mathrm{Spin}_8 = \{(g_1, g_2, g_3) \in SO(\mathbb{O}, N)^3 : Tr((g_1x)(g_2y)(g_3z)) = Tr(xyz) \quad \text{for all } x, y, z \in \mathbb{O}\}.$$

How does S_3 act on Spin_8? A naive guess is that it acts on an element (g_1, g_2, g_3) by permuting its three components. However, this is not the case. By the above, it is true that cyclic permutations of (g_1, g_2, g_3) do preserve the group Spin_8 so that the 3-cycles in S_3 act in the natural way. A transposition, on the other hand, acts as follows:

$$\sigma : (g_1, g_2, g_3) \mapsto (\hat{g}_2, \hat{g}_1, \hat{g}_3)$$

where

$$\hat{g} : x \mapsto \overline{g(\bar{x})}.$$

2.8. **The representation** V_{Spin}. By the above description of Spin_8, one sees that Spin_8 acts on $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$. The action preserves each factor of $\mathbb{O}$ and gives the three irreducible representations of dimension 8. We can now describe the extension of this action to the semidirect product $\mathrm{Spin}_8 \rtimes S_3$. Not surprisingly, the 3-cycles in S_3 acts on $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$ by cyclic permutation. The action of the transposition σ, however, is given by:

$$\sigma : (x, y, z) \mapsto (\bar{y}, \bar{x}, \bar{z}).$$

It is easy to show that this does define an action of $\mathrm{Spin}_8 \rtimes S_3$. *This representation* V_{Spin} *is what we use in the definition of the Spin L-function.*

We remark that one could consider the twist of V_{Spin} by the sign representation of S_3 and thus obtain a different L-function. Though we stated our main theorem in the introduction for $L(\underline{s}, \pi, V_{\mathrm{Spin},E})$, a slight twist of our Rankin-Selberg integral will give an analogous statement for $L(\underline{s}, \pi, V_{\mathrm{Spin},E} \otimes (\mathrm{sign}))$.

3. **Minimal Representation**

Now we come to another ingredient in our Rankin-Selberg integral. For each local field F_v, the group $G_E(F_v)$ has a so-called minimal representation Π_{E_v}, first studied by Kazhdan [K]. It is the analog of the Weil representation of the metaplectic group.

3.1. **Local minimal representation.** To describe this minimal representation, let $r : S_3 \to \mathrm{GL}_2(\mathbb{C})$ denote the 2-dimensional irreducible representation of S_3. Then the composite $r \circ \rho_{E,v}$ is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}_v/F_v)$. By Jacquet-Langlands, this is the L-parameter of an irreducible admissible representation $r_{E,v}$ of $\mathrm{GL}_2(F_v)$ given by the following table:

E_v	$r_{E,v}$
F_v^3	$\pi(1,1)$
$F_v \times K_v$	$\pi(1, \chi_{K_v})$
Galois field	$\pi(\chi_{E_v}, \chi_{E_v}^{-1})$
non-Galois field	monomial supercuspidal

Here, χ_{K_v} and χ_{E_v} denote the characters of $F_v^\times$ associated to K_v and E_v by local class field theory, and $\pi(\mu_1, \mu_2)$ denotes the representation of $\mathrm{GL}_2(F_v)$ unitarily induced from the character $\mu_1 \times \mu_2$ of the diagonal torus. The central character of $r_{E,v}$ is precisely the quadratic character $\chi_{K_{E,v}}$, and we define a representation of

$$L_E(F_v) \cong (\mathrm{GL}_2(F_v) \times E_v^\times)/\Delta F_v^\times$$

by:

$$\sigma_{E,v} = r_{E,v} \boxtimes (\chi_{K_{E,v}} \circ \mathbb{N}_{E_v/F_v}).$$

Here and elsewhere, $\boxtimes$ refers to "outer tensor product".

Now $\Pi_{E,v}$ is the unique irreducible submodule of the induced representation

$$Ind_{Q_E}^{G_E} \delta_{Q_{E_v}}^{1/6} \sigma_{E,v} \qquad \text{(unnormalized induction)}.$$

If E_v is unramified, then Π_{E_v} is a spherical representation.

3.2. **Automorphic realization.** Let $\Pi_E = \otimes_v \Pi_{E_v}$ be the global minimal representation of $G_E(\mathbb{A})$. It is known (cf. [GGJ]) that there is a unique G_E-equivariant embedding

$$\theta : \Pi_E \longrightarrow \mathcal{A}(G_E)$$

which is constructed using residues of Eisenstein series. We need to know some properties of this minimal representation.

3.3. **Fourier coefficients.** Fix a non-trivial unitary character ψ_v of F_v. Using ψ_v, the Killing form and the exponential map, the unitary characters of $N_E(F_v)$ can be parametrized by elements of $\overline{N}_E(F_v)/\overline{Z}(F_v)$. Let χ_v be a non-trivial unitary character of $N_E(F_v)$. Then the key property of Π_{E_v} is that:

$$\dim \mathrm{Hom}_{N_E(F_v)}(\Pi_{E_v}, \mathbb{C}_{\chi_v}) = \begin{cases} 1, & \text{if } \chi_v \in \Omega_E(F_v); \\ 0, & \text{if not.} \end{cases}$$

Here $\Omega_E(F_v)$ is as introduced in (2.4) and parametrizes the minimal non-trivial $M_E(F_v)$-orbit of characters of $N_E(F_v)$. An element in $\Omega_E(F_v)$ is the character χ_0 such that

$$\chi_0|_{U_{\beta_0-\alpha_0}} = \psi_v$$

and $\chi_0|_{U_\gamma}$ is trivial for any other root subgroup $U_\gamma \subset N_E$. In terms of the description of Ω_E given in (2.4),

$$\chi_0 = (0,0,0,1).$$

These local results on $\mathrm{Hom}_{N_E(F_v)}(\Pi_{E_v}, \mathbb{C}_{\chi_v})$ imply:

Proposition 3.3.1. *(i) For $f \in \Pi_E$, the Fourier expansion of $\theta(f)$ along N_E is supported on the (Zariski) closure of the minimal non-trivial $M_E(F)$-orbit $\Omega_E(F)$ of unitary characters of $N_E(\mathbb{A})$ trivial on $N_E(F)$.*

(ii) For $\chi \in \Omega_E(F)$, the Fourier coefficient $\theta(f)_{N_E,\chi}$ has an Euler product expansion in terms of the local functionals. Namely, if $f = \otimes_v f_v$, then

$$\theta(f)_{N_E,\chi}(g) = \prod_v L_{\chi_v}(g_v \cdot f_v),$$

where each L_{χ_v} is a non-zero element of the 1-dimensional space $\mathrm{Hom}_{N_E(F_v)}(\Pi_v, \mathbb{C}_{\chi_v})$, normalized for almost all v by the requirement that $L_{\chi_v}(f_v^0) = 1$ if f_v^0 is the normalized spherical vector in $\Pi_{E,v}$.

3.4. **An explicit formula.** Finally, we come to the only new result in this section. We have fixed a Chevalley-Steinberg system of épinglage for G_E over F. For almost all finite places v of F, this system of épinglage gives G_E an $\mathcal{O}_{F_v}$-structure such that $G_E(\mathcal{O}_{F_v})$ is a hyperspecial maximal compact subgroup of $G_E(F_v)$.

Fix such a non-archimedean place v, and let L_{χ_0} be a non-zero element of the 1-dimensional space $\mathrm{Hom}_{N_E}(\Pi_{E_v}, \mathbb{C}_{\chi_0})$. We have the following explicit formula:

Proposition 3.4.1. *Assume that v is a finite place such that $G_E(\mathcal{O}_{F_v})$ is hyperspecial (so that E_v is unramified) and let f_0 be a non-zero $G_E(\mathcal{O}_{F_v})$-spherical vector in Π_{E_v}. For $t \in T_E(F_v)$, let us set*

$$k_0(t) = ord(\beta_0(t)/\alpha_0(t)).$$

If χ_0 has conductor zero (with respect to the $\mathcal{O}_{F_v}$-structure on $N_E(F_v)$), then

$$L_{\chi_0}(tf_0) = 0 \quad \textit{if } k_0(t) < 0.$$

However, L_{χ_0} is non-zero on f_0, so that we may normalize L_{χ_0} by setting $L_{\chi_0}(f_0) = 1$. Then, if $k_0(t) \geq 0$, we have:

(i) If $E_v = F_v^3$, then

$$L_{\chi_0}(t \cdot f_0) = |\beta_0(t)| \cdot (k_0(t) + 1)\,.$$

(ii) If $E_v = F_v \times K_v$, then

$$L_{\chi_0}(t \cdot f_0) = \chi_{K_v}(\beta_0(t)) \cdot |\beta_0(t)| \cdot \epsilon_2(k_0(t) + 1)$$

where $\epsilon_2(\cdot)$ is the Legendre symbol $\left(\frac{\cdot}{2}\right)$ (with $\epsilon_2(0) = 0$).

(iii) If E is the unramified cubic field extension of F_v, then

$$L_{\chi_0}(t \cdot f_0) = |\beta_0(t)| \cdot \epsilon_3(k_0(t) + 1),$$

where $\epsilon_3(\cdot)$ is the Legendre symbol $\left(\frac{\cdot}{3}\right)$ (with $\epsilon_3(0) = 0$).

Proof. In the split case, this is a result of Kazhdan-Polishchuk [KP]. We give a different proof which covers the non-split case as well.

Choose a representative in $G_E(\mathcal{O}_{F_v})$ of the Weyl group element w_α; we denote this representative by w_α also. Then w_α normalizes N_E and it will be more convenient to replace χ_0 and L_{χ_0} by $\chi_0' = w_\alpha \chi_0$ and $L' = w_\alpha L_{\chi_0}$. Let us give an explicit construction of L'.

By definition, the linear functional L' factors through the quotient $(\Pi_{E_v})_{N_E,\chi_0'}$ which is 1-dimensional. The root subgroup $U_\alpha(F_v)$ normalizes $N_E(F_v)$ and fixes the character χ_0'. It thus acts on $(\Pi_{E_v})_{N_E,\chi_0'}$ and this action is in fact trivial. Thus, we see that L' factors as:

$$\Pi_{E_v} \longrightarrow (\Pi_{E_v})_{U_E} \xrightarrow{l_{\psi_v}} \mathbb{C}$$

where

$$l_{\psi_v} \in \mathrm{Hom}_{U_\beta}((\Pi_{E_v})_{U_E}, \mathbb{C}_{\psi_v}).$$

Now we recall that $\Pi_{E,v}$ is the unique irreducible submodule of

$$Ind_{Q_E}^{G_E} \delta_{Q_{E_v}}^{1/6} \sigma_{E,v}.$$

By Frobenius reciprocity, there is a natural Q_{E_v}-equivariant map

$$p : \Pi_{E_v} \longrightarrow \delta_{Q_{E_v}}^{1/6} \otimes \sigma_{E,v}$$

which is simply given

$$p(f) = f(1) \quad \text{ for } f \in \Pi_{E_v}.$$

Thus L' is the composite of p with the unique Whittaker functional l_{ψ_v} of $\sigma_{E,v}$.

Now take a torus element

$$t = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, x \right) \in T_E(F_v) \subset L_E(F_v) = (GL_2(F_v) \times E_v^\times)/\Delta F_v^\times.$$

Then

$$L'(t \cdot f_0) = l_{\psi_v}(f_0(t)) = \delta_{Q_{E_v}}(t)^{1/6} \cdot l_{\psi_v}(\sigma_{E,v}(t)(f_0(1)))$$

$$= \frac{|ab|^{3/2}}{|\mathbb{N}(x)|} \cdot \chi_{K_E}(\mathbb{N}(x)) \cdot W_{\psi_v} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where W_{ψ_v} is the normalized spherical Whittaker function of the representation $r_{E,v}$ of $GL_2(F_v)$. The formula for W_{ψ_v} on a torus element is well-known:

$$|a/b|^{-1/2} \cdot W_{\psi_v} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \chi_{K_E}(b) \cdot \epsilon(ord(a/b))$$

where

$$\epsilon(k) := \begin{cases} k+1, \text{ if } E_v = F_v^3; \\ \epsilon_2(k+1), \text{ if } E_v = F_v \times K_v; \\ \epsilon_3(k+1), \text{ if } E_v \text{ is the unramified cubic field.} \end{cases}$$

Thus, we deduce that

$$L'(t \cdot f_0) = \frac{|a^2 b|}{|\mathbb{N}(x)|} \cdot \chi_{K_E}(b \cdot \mathbb{N}(x)) \cdot \epsilon(ord(a/b)) = |\beta_0(t)| \cdot \chi_{K_E}(\beta_0(t)) \cdot \epsilon(ord(\alpha_0(t))).$$

Conjugating back by w_α gives the desired result, since w_α fixes β_0 and sends α_0 to $\beta_0 - \alpha_0$. The proposition is proved. $\square$

For a general vector $f \in \Pi_E$, the above construction of the linear functional L_{χ_0} allows one to obtain the asymptotics of the function $(t,k) \mapsto L_{\chi_0}(tk \cdot f)$ on $T_E(F_v) \times K_v$. This is needed in Section 9; see Lemma 9.2.2. Further, in Lemma 9.5.1, we shall give another description of the functional L_{χ_0} by means of a local Fourier-Jacobi map.

4. **A Rankin-Selberg Integral**

We are now ready to write down our Rankin-Selberg integral. Let π be a globally generic cuspidal representation of $G_E(\mathbb{A})$. For $\varphi \in \pi$, $\Phi_s \in I_{Q_E}(\underline{s})$ and $f \in \Pi_E$, we set

$$Z_E(\varphi, \Phi, f, \underline{s}) = \int_{G_E(F)\backslash G_E(\mathbb{A})} \varphi(g) \cdot \overline{\theta(f)(g)} \cdot E(\Phi, \underline{s}, g)\, dg.$$

The purpose of this section is to unfold this Rankin-Selberg integral of Shimura type.

Theorem 4.0.2. *Let V_E' be the maximal unipotent subgroup of G_E given by*

$$V_E' = w_\alpha V_E w_\alpha^{-1} = N_E \rtimes U_{-\alpha}.$$

It is the unipotent radical of the Borel subgroup $B_E' = w_\alpha B_E w_\alpha^{-1}$, with associated simple roots $-\alpha_i$ ($i = 1,2,3$) and $\beta_0 - \alpha_0$. Let ψ be the generic unitary character of $V_E'(\mathbb{A})$ which is non-trivial on the associated relative simple root subgroups. Then we have:

$$Z_E(\varphi, \Phi, f, \underline{s}) = \int_{V_E'(\mathbb{A})\backslash G_E(\mathbb{A})} \varphi_{V_E',\psi}(g) \cdot \overline{\theta(f)_{N_E,\chi_0}(g)} \cdot \widehat{\Phi_s}(g)\, dg$$

where

$$\widehat{\Phi_s}(g) = \int_{U_{-\alpha}(\mathbb{A})} \Phi_s(ug) \cdot \psi(u)\, du.$$

Proof. Unfolding the Eisenstein series in the range of absolute convergence, we get

$$
\begin{aligned}
&Z_E(\varphi,\Phi,f,\underline{s})\\
&=\int_{Q_E(F)\backslash G_E(\mathbb{A})}\varphi(g)\cdot\overline{\theta(f)(g)}\cdot\Phi_s(g)\,dg\\
&=\int_{Q_E(F)U_E^{(1)}(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\left(\int_{U_E^{(1)}(F)\backslash U_E^{(1)}(\mathbb{A})}\varphi(zg)\cdot\overline{\theta(f)(zg)}\,dz\right)dg\\
&=\int_{Q_E(F)U_E^{(1)}(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\varphi_{U_E^{(1)}}(g)\cdot\overline{\theta(f)_{U_E^{(1)}}(g)}\,dg\\
&\quad+\int_{Q_E(F)U_E^{(1)}(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\left(\sum_{\chi\neq 1}\varphi_{U_E^{(1)},\chi}(g)\cdot\overline{\theta(f)_{U_E^{(1)},\chi}(g)}\right)dg.
\end{aligned}
$$

Lemma 4.0.3. *The first term in the last sum is zero.*

Proof. The argument is easiest when E is a field and is more complicated in the split case; it ultimately relies on the fact that φ is cuspidal.

We shall only give the argument when E is a field. Consider the Fourier expansion of $\theta(f)_{U_E^{(1)}}$ along $N_E(F)\backslash N_E(\mathbb{A})$. Because $\theta(f)_{U_E^{(1)}}$ is left-invariant under $U_{\beta_0-\alpha_0}(\mathbb{A})$, this has the form

$$\theta(f)_{U_E^{(1)}}=\sum_{\chi\in\Omega_E(F):\chi|_{U_{\beta_0-\alpha_0}}=1}\theta(f)_{N_E,\chi}.$$

In other words, the character χ intervening in the above sum is represented by $(a,x,y,0)\in\Omega_E(F)$. But if E is a field, then the only elements of this form in $\Omega_E(F)$ are $(a,0,0,0)$ with $a\in F$.

Consider first the constant term $\theta(f)_{N_E}$ in the Fourier expansion, which corresponds to $a=0$. By [GGJ, Prop. 5.3(iv)], the restriction of $\theta(f)_{N_E}$ to M_E lies in the span of two automorphic characters of $M_E(\mathbb{A})$. Since this is the case for any $f\in\Pi_E$, we see that $\theta(f)_{N_E}$ is left-invariant under $U_\alpha(\mathbb{A})$, and thus under $U_E(\mathbb{A})$.

Now consider the other terms in the Fourier expansion of $\theta(f)_{U_E^{(1)}}$. If ψ_a is the character associated to $(a,0,0,0)\in\Omega_E(F)$ with $a\neq 0$, then $U_\alpha(\mathbb{A})$ normalizes $N_E(\mathbb{A})$ and fixes the character ψ_a. Thus the Fourier coefficient $\theta(f)_{N_E,\psi_a}$ is left-invariant under $U_\alpha(\mathbb{A})$ (we have used the local analog of this fact in the proof of Prop. 3.4.1). Hence $\theta(f)_{N_E,\psi_a}$ is also left-invariant under $U_E(\mathbb{A})$. Hence we have:

$$
\begin{aligned}
&\int_{Q_E(F)U_E^{(1)}(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\varphi_{U_E^{(1)}}(g)\cdot\left(\sum_{a\in F}\overline{\theta(f)_{N_E,\psi_a}(g)}\right)dg\\
&=\int_{L_E(F)U_E(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\left(\sum_{a\in F}\overline{\theta(f)_{N_E,\psi_a}(g)}\right)\cdot\left(\int_{U_E(F)\backslash U_E(\mathbb{A})}\varphi(ug)\,du\right)dg\\
&=0
\end{aligned}
$$

since φ is cuspidal. This proves the lemma when E is a field.

When E is not a field, there will be more terms intervening in the Fourier expansion of $\theta(f)_{U_E^{(1)}}$. Thankfully, there are also more F-rational standard parabolics in G_E. These other terms in the Fourier expansion ultimately lead to the constant terms of φ along these other parabolics. We omit the details.

$\square$

On the other hand, the non-trivial characters of $U_E^{(1)}(\mathbb{A})$ in the second term are permuted transitively by $Q_E(F)$. If we let χ_0 denote the character which is trivial on U_{β_0} and non-trivial on $U_{\beta_0-\alpha_0}$, then the stabilizer of χ_0 in $Q_E(F)$ is the subgroup $A_E V_E$ where A_E is the 3-dimensional torus in T_E such that

$$X_*(A_E) = \langle \mu_1 - \mu_0, \mu_2 - \mu_0, \mu_3 - \mu_0 \rangle.$$

In view of this, we have:

$$\begin{aligned}
&Z_E(\varphi, \Phi, f, \underline{s}) \\
&= \int_{Q_E(F) U_E^{(1)}(\mathbb{A}) \backslash G_E(\mathbb{A})} \Phi_s(g) \cdot \left(\sum_{\gamma \in A_E(F) V_E(F) \backslash Q_E(F)} \varphi_{U_E^{(1)}, \chi_0}(\gamma g) \cdot \overline{\theta(f)_{U_E^{(1)}, \chi_0}(\gamma g)} \right) dg \\
&= \int_{A_E(F) V_E(F) U_E^{(1)}(\mathbb{A}) \backslash G_E(\mathbb{A})} \Phi_s(g) \cdot \varphi_{U_E^{(1)}, \chi_0}(g) \cdot \overline{\theta(f)_{U_E^{(1)}, \chi_0}(g)} \, dg \\
&= \int_{A_E(F) U_\alpha(F) N_E(\mathbb{A}) \backslash G_E(\mathbb{A})} \Phi_s(g) \cdot \left(\int_{N_E(F) U_E^{(1)}(\mathbb{A}) \backslash N_E(\mathbb{A})} \varphi_{U_E^{(1)}, \chi_0}(ng) \cdot \overline{\theta(f)_{U_E^{(1)}, \chi_0}(ng)} \right) dg
\end{aligned}$$

By Prop. 3.3.1(i), we see that

$$\theta(f)_{U_E^{(1)}, \chi_0}(g) = \sum_{x \in E} \theta(f)_{N_E, \psi_x}(g).$$

Here ψ_x corresponds to the element

$$(\mathbb{N}(x), x^{\#}, x, 1) \in \Omega_E(F).$$

Moreover, the group $U_\alpha(F) \cong E$ acts simply transitively on the set $\{\psi_x : x \in E\}$. Note that if $x = 0$, then ψ_0 is simply the trivial extension of χ_0 from $U_E^{(1)}$ to N_E. Thus,

$$\begin{aligned}
Z_E(\varphi, \Phi, f, \underline{s}) &= \int_{A_E(F) U_\alpha(F) N_E(\mathbb{A}) \backslash G_E(\mathbb{A})} \Phi_s(g) \cdot \left(\sum_{x \in E} \varphi_{N_E, \psi_x}(g) \cdot \overline{\theta(f)_{N_E, \psi_x}(g)} \right) dg \\
&= \int_{A_E(F) N_E(\mathbb{A}) \backslash G_E(\mathbb{A})} \Phi_s(g) \cdot \varphi_{N_E, \psi_0}(g) \cdot \overline{\theta(f)_{N_E, \psi_0}(g)} \, dg
\end{aligned}$$

Now consider the Fourier expansion of φ_{N_E, ψ_0} along $U_{-\alpha}$. Because φ is cuspidal, this takes the form:

$$\varphi_{N_E, \psi_0}(g) = \sum_{\gamma \in A_E(F)} \varphi_{V_E', \psi}(\gamma g),$$

where ψ is a generic character of V_E'. Substituting this into the last expression for our Rankin-Selberg integral, and using the fact that $\theta(f)_{N_E, \psi_0}$ is left-invariant under $U_{-\alpha}(\mathbb{A})$, we obtain:

$$
\begin{aligned}
&Z_E(\varphi,\Phi,f,\underline{s})\\
&=\int_{A_E(F)N_E(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\overline{\theta(f)_{N_E,\psi_0}(g)}\cdot\left(\sum_{\gamma\in A_E(F)}\varphi_{V'_E,\psi}(\gamma g)\right)dg\\
&=\int_{N_E(\mathbb{A})\backslash G_E(\mathbb{A})}\Phi_s(g)\cdot\overline{\theta(f)_{N_E,\psi_0}(g)}\cdot\varphi_{V'_E,\psi}(g)\,dg\\
&=\int_{V'_E(\mathbb{A})\backslash G_E(\mathbb{A})}\overline{\theta(f)_{N_E,\psi_0}(g)}\cdot\left(\int_{U_{-\alpha}(\mathbb{A})}\varphi_{V'_E,\psi}(ug)\cdot\Phi_s(ug)\,du\right)dg\\
&=\int_{V'_E(\mathbb{A})\backslash G_E(\mathbb{A})}\overline{\theta(f)_{N_E,\psi_0}(g)}\cdot\varphi_{V'_E,\psi}(g)\cdot\widehat{\Phi_s}(g)\,dg.
\end{aligned}
$$

Theorem 4.0.2 is proved. □

5. **Local Zeta Integral**

After Theorem 4.0.2, we see that

$$Z_E(\varphi,\Phi,f,\underline{s})=\prod_v Z_v(\varphi_v,\Phi_v,f_v,\underline{s})$$

where

$$Z_v(\varphi_v,\Phi_v,f_v,\underline{s})=\int_{V'_E(F_v)\backslash G_E(F_v)}W_{V'_E,\psi}(g\cdot\varphi_v)\cdot\overline{L_{\chi_0}(g\cdot f_v)}\cdot\widehat{\Phi_{s,v}}(g)\,dg.$$

This integral converges when $\underline{s}\in\Omega_R$ (cf. (2.6)) for $R\gg 0$, as we shall show in Prop. 9.1.3. The rest of the paper is devoted to the study of this local zeta integral. In particular, we shall compute the local zeta integral explicitly when all the data involved are unramified. The purpose of this section is to provide explicit formulas for the 3 functions appearing in the local integral in the unramified setting.

Assume henceforth that all the data involved in Z_v are unramified. Then by the Iwasawa decomposition, we have

$$Z_v(\varphi_v,\Phi_v,f_v,s)=\int_{S_{E_v}(F_v)}W_{V'_E,\psi}(t\cdot\varphi_v)\cdot\overline{L_{\chi_0}(t\cdot f_v)}\cdot\widehat{\Phi_{s,v}}(t)\cdot\delta_{B'_E}(t)^{-1}\,dt$$

where $S_{E_v}\subset T_{E_v}$ is the maximal F_v-split torus.

Henceforth, since the setting is entirely local, we shall suppress v from the notations.

5.1. **A change of system of simple roots.** Because we are looking at the Whittaker functional relative to the Borel subgroup B'_E, it is useful to use the corresponding system of simple roots:

$$\alpha'_0=\beta_0-\alpha_0,\qquad \alpha'_i=-\alpha_i,\quad \text{for } i=1,2 \text{ and } 3.$$

For this new system of simple roots, the highest root is still

$$\beta_0=2\alpha'_0+\alpha'_1+\alpha'_2+\alpha'_3$$

and the modulus character of B'_E is given by

$$\delta_{B'_E}=|6(\alpha'_1+\alpha'_2+\alpha'_3)+10\alpha'_0|.$$

The associated fundamental coweights are given by:

$$\mu_0' = \mu_0 \qquad \mu_i' = \mu_0 - \mu_i \quad \text{for } i = 1,\, 2 \text{ and } 3.$$

As above, let $S_E \subset T_E$ denote the maximal F-split torus. We consider the 3 different cases:

- if $E = F \times F \times F$, then $S_E = T_E$. An element of $S_E(F)$ is of the form
$$t = \mu_0'(t_0) \cdot \mu_1'(t_1) \cdot \mu_2'(t_2) \cdot \mu_3'(t_3), \quad \text{with } t_i \in F^\times.$$
We shall sometimes write this element as $t = (t_0, t_1, t_2, t_3)$.
- if $E = F \times K$ with K a quadratic field, then
$$X_*(S_E) = \langle \mu_0', \mu_1', \mu_2' + \mu_3' \rangle.$$
An element of $S_E(F)$ is of the form
$$t = \mu_0'(t_0) \cdot \mu_1'(t_1) \cdot (\mu_2'\mu_3')(t_{23}).$$
We shall write this element as $t = (t_0, t_1, t_{23})$.
- if E is a cubic field, then
$$X_*(S_E) = \langle \mu_0', \mu_1' + \mu_2' + \mu_3' \rangle.$$
An element of $S_E(F)$ is of the form
$$t = \mu_0'(t_0) \cdot (\mu_1'\mu_2'\mu_3')(t_{123}).$$
We shall write this element as $t = (t_0, t_{123})$.

5.2. **Casselman-Shalika formula.** For the value of the Whittaker functional $W_{V_E',\psi}$ on $S_E(F)$, one has the well-known Casselman-Shalika formula. Let us state this precisely since it is slightly more subtle when the group G_E is not split. To the best of our knowledge, the first discussion about the interpretation of the Casselman-Shalika formula in terms of an appropriate dual group in non-split cases is due to Tamir [T]. Our description below is somewhat cleaner than that in [T], since the relative root system involved here is reduced. The description in [T] works for all relative root systems; indeed the case needed in [T] is that of the quasi-split unitary groups whose relative root system is of type BC_n.

For $t \in S_E(F)$, $W_{V_E',\psi}(t \cdot \varphi)$ is zero unless $|t_i| \le 1$ for all entries t_i of t, in which case it depends only on the valuation of the t_i's. If we write $t_i = \varpi^{k_i}$ with $k_i \ge 0$, then we denote the corresponding torus element by

$$t = t(\underline{k}) = \; t(k_0, k_1, k_2, k_3) \text{ or } t(k_0, k_1, k_{23}) \text{ or } t(k_0, k)$$

in the three respective cases.

Now we need to examine the notion of Satake parameter for a general quasi-split group with reduced relative root system. The Satake parameter t_π which was used in the definition of the Spin L-function is a $G_E^\vee$-conjugacy class in the quotient $G_E^\vee \rtimes \mathrm{Gal}(F_{ur}/F)$ of the L-group ${}^L G_E$, where F_{ur} is the maximal unramified extension of F in $\overline{F}$. In fact, t_π lies in the coset $G_E^\vee \cdot \mathrm{Frob}$ of the Frobenius element Frob. Thus we may write

$$t_\pi = (s_\pi, \mathrm{Frob}) \in G_E^\vee \rtimes \mathrm{Gal}(F_{ur}/F)$$

where now s_π is well-defined up to Frob-conjugacy as an element of $G_E^\vee$. Under Frob-conjugation, we may assume further that

$$s_\pi \in H'_E := (G_E^\vee)^{\mathrm{Gal}(F_{ur}/F)},$$

in which case s_π is well-defined up to conjugacy in H'_E. Note that

$$H'_E = \begin{cases} Spin_8(\mathbb{C}), \text{ if } E = F^3; \\ Spin_7(\mathbb{C}), \text{ if } E = F \times K; \\ G_2(\mathbb{C}), \text{ if } E \text{ is a field.} \end{cases}$$

However, the Casselman-Shalika formula is not interpreted in terms of the element t_π or the element s_π. Rather, the group G_E contains a connected F-split subgroup H_E whose root system is equal to the relative root system of G_E and whose maximal split torus is the maximal F-split subtorus S_E of T_E. In our case,

$$H_E = \begin{cases} G_E, \text{ if } E = F^3; \\ SO_7, \text{ if } E = F \times K; \\ G_2, \text{ if } E \text{ is a field,} \end{cases}$$

so that its dual group is

$$H_E^\vee = \begin{cases} \mathrm{Spin}_8(\mathbb{C}), \text{ if } E = F^3; \\ Sp_6(\mathbb{C}), \text{ if } E = F \times K; \\ G_2(\mathbb{C}), \text{ if } E \text{ is a field.} \end{cases}$$

From the diagram of inclusions

$$\begin{tikzcd} S_E \arrow[r] \arrow[d, "\iota"'] & H_E \arrow[d] \\ T_E \arrow[r] & G_E \end{tikzcd}$$

we obtain on the dual side:

$$\begin{tikzcd} T_E^\vee \arrow[r] \arrow[d, "\iota^*"'] & G_E^\vee \\ S_E^\vee \arrow[r] & H_E^\vee \end{tikzcd}$$

where the horizontal arrows are inclusions but ι^* is surjective. Indeed, ι^* restricts to an isogeny

$$(T_E^\vee)^{\mathrm{Gal}(F_{ur}/F)} \longrightarrow S_E^\vee.$$

As explained by Borel in [Bo, §6], the map ι^* induces a bijection

$$\{\text{semisimple Frob-conjugacy classes in } G_E^\vee\} \leftrightarrow \{\text{semisimple conjugacy classes in } H_E^\vee\}$$

We set

$$\overline{s_\pi} = \iota^*(s_\pi).$$

It is the element $\overline{s_\pi}$ which intervenes in the Casselman-Shalika formula.

More precisely, the element $t = t(\underline{k})$ corresponds to an element of $X_*(S_E) \cong X^*(S_E^\vee)$ in the dominant chamber and thus gives rise to an irreducible representation $V_{\underline{k}}$ of $H_E^\vee$. The Casselman-Shalika formula says:

Proposition 5.2.1. *With the notations introduced above,*

$$\delta_{B'_E}(t(\underline{k}))^{-1/2} \cdot W_{V'_E,\psi}(t(\underline{k}) \cdot \varphi) = Tr(\overline{s_\pi}|V_{\underline{k}}).$$

In our unramified computations in the following sections, we shall state more precisely what the representation $V_{\underline{k}}$ is for each given $\underline{k}$.

5.3. **The map ι^*.** For the purpose of calculation, we need to understand the map ι^* more explicitly. When $E = F^3$, there is nothing to do, since ι is the identity map and $\overline{s_\pi} = s_\pi$. We examine the other two cases in turn. Since

$$X_*(T_E^\vee) = X^*(T_E) = \bigoplus_{i=0}^{3} \mathbb{Z}\alpha'_i,$$

the element $s_\pi \in T_E^\vee$ is of the form

$$s_\pi = \prod_{i=0}^{3} \alpha'_i(t_i).$$

Indeed, if s_π is assumed to lie in $(T_E^\vee)^{\mathrm{Gal}(F_{ur}/F)}$, then we further have $t_2 = t_3$ (respectively, $t_1 = t_2 = t_3$) when $E = F \times K$ (respectively when E is a field). On the other hand,

$$X_*(S_E^\vee) = X^*(S_E) \cong \begin{cases} X^*(T_E)/\langle \alpha'_2 - \alpha'_3 \rangle, \text{ if } E = F \times K; \\ X^*(T_E)/\langle \alpha'_1 - \alpha'_2, \alpha'_2 - \alpha'_3 \rangle, \text{ if } E \text{ is a field.} \end{cases}$$

If we let $\overline{\alpha}'_i$ denote the image of α'_i in the quotient lattice above, then a basis of $X_*(S_E^\vee)$ is

$$\begin{cases} \{\overline{\alpha}'_0, \overline{\alpha}'_1, \overline{\alpha}'_2\}, \text{ if } E = F \times K; \\ \{\overline{\alpha}'_0, \overline{\alpha}'_1\}, \text{ if } E \text{ is a field.} \end{cases}$$

Now we have:

- when $E = F \times K$,

$$\iota^* : \alpha_0(t_0) \cdot \alpha'_1(t_1) \cdot \alpha'_2(t_2) \cdot \alpha'_3(t_3) \mapsto \overline{\alpha}'_0(t_0) \cdot \overline{\alpha}'_1(t_1) \cdot \overline{\alpha}'_2(t_2t_3)$$

- when E is a field,

$$\iota^* : \alpha_0(t_0) \cdot \alpha'_1(t_1) \cdot \alpha'_2(t_2) \cdot \alpha'_3(t_3) \mapsto \overline{\alpha}'_0(t_0) \cdot \overline{\alpha}'_1(t_1t_2t_3)$$

5.4. **Formula for $L_{\chi_0}(t \cdot f)$.** On the other hand, Prop. 3.4.1 gives an explicit formula for $L_{\chi_0}(t \cdot f)$. For ease of reference, we restate it here:

- when $E = F^3$,

$$L_{\chi_0}(t(k_0, k_1, k_2, k_3) \cdot f) = |\beta_0(t)| \cdot (ord(\alpha'_0(t)) + 1) = q^{-(2k_0+k_1+k_2+k_3)} \cdot (k_0 + 1).$$

- when $E = F \times K$,

$$L_{\chi_0}(t(k_0, k_1, k_{23}) \cdot f) = (-1)^{k_1} \cdot q^{-(2k_0+k_1+2k_{23})} \cdot \epsilon_2(k_0 + 1).$$

- when E is a field,

$$L_{\chi_0}(t(k_0, k_{123}) \cdot f) = q^{-(2k_0+3k_{123})} \cdot \epsilon_3(k_0 + 1).$$

5.5. **Formula for $\widehat{\Phi_s}(t)$.** Finally, we need a formula for $\widehat{\Phi_s}(t)$. We first note the following simple lemma, whose proof we leave to the reader.

Lemma 5.5.1. *Suppose that Φ is the unramified vector in $Ind_B^{SL_2}\chi_s$ (unnormalized induction), where*

$$\chi_s(\alpha^\vee(t)) = |t|^{1+s}$$

and α is the positive root of SL_2. Then

$$\int_{F_v} \Phi(x_{-\alpha}(r)) \cdot \psi(tr)\, dr = \frac{\zeta(s)}{\zeta(s+1)\cdot\zeta((ord(t)+1)s)}.$$

With this lemma, we can calculate $\widehat{\Phi_s}(t)$. We shall do so assuming that $E = F^3$ is split. The other cases are similarly handled. Let

$$t = \prod_{i=0}^{3} \mu_i'(t_i).$$

Then

$$\begin{aligned}
&\widehat{\Phi_s}(t)\\
=&\int_{F_v^3} \Phi_s(x_{-\alpha_1}(r_1)x_{-\alpha_2}(r_2)x_{-\alpha_3}(r_3)t)\cdot\psi(r_1+r_2+r_3)\,dr_1\,dr_2\,dr_3\\
=&\int_{F_v^3} \Phi_s(t\cdot(\prod_{i=1}^{3} x_{-\alpha_i}(t_i^{-1}r_i))\cdot\psi(r_1+r_2+r_3)\,dr_1\,dr_2\,dr_3\\
=&|t_1|^{s_1+1}|t_2|^{s_2+1}|t_3|^{s_3+1}|t_0|^{s_1+s_2+s_3+3}\cdot\int_{F_v^3}\Phi_s(\prod_{i=1}^{3} x_{-\alpha_i}(t_i^{-1}r_i))\cdot\psi(r_1+r_2+r_3)\,dr_1\,dr_2\,dr_3\\
=&|t_1|^{s_1+2}|t_2|^{s_2+2}|t_3|^{s_3+2}|t_0|^{s_1+s_2+s_3+3}\cdot\\
&\cdot\int_{F_v^3}\Phi_s(\prod_{i=1}^{3} x_{-\alpha_i}(r_i))\cdot\psi(t_1r_1)\cdot\psi(t_2r_2)\cdot\psi(t_3r_3)\,dr_1\,dr_2\,dr_3
\end{aligned}$$

Now, observing that

$$\Phi_s(\alpha_1^\vee(t)) = |t|^{1+s_2+s_3-s_1} \quad \text{and so on,}$$

and appealing to Lemma 5.5.1, we obtain the following proposition.

Proposition 5.5.2. *(i) When $E = F^3$,*

$$\begin{gathered}
\widehat{\Phi_s}(t) =\\
|t_1|^{s_1+2}|t_2|^{s_2+2}|t_3|^{s_3+2}|t_0|^{s_1+s_2+s_3+3}\cdot\\
\frac{\zeta(s_2+s_3-s_1)}{\zeta(s_2+s_3-s_1+1)\cdot\zeta((ord(t_1)+1)(s_2+s_3-s_1))}\cdot\\
\frac{\zeta(s_3+s_1-s_2)}{\zeta(s_3+s_1-s_2+1)\cdot\zeta((ord(t_2)+1)(s_3+s_1-s_2))}\cdot\\
\frac{\zeta(s_1+s_2-s_3)}{\zeta(s_1+s_2-s_3+1)\cdot\zeta((ord(t_3)+1)(s_1+s_2-s_3))}.
\end{gathered}$$

(ii) When $E = F\times K$,

$$\widehat{\Phi_s}(t) =$$

$$|t_1|^{s_1+2}|t_{23}|^{2s_{23}+4}|t_0|^{s_1+2s_{23}+3}.$$

$$\frac{\zeta(2s_{23}-s_1)}{\zeta(2s_{23}-s_1+1)\cdot\zeta((ord(t_1)+1)(2s_{23}-s_1))}.$$

$$\frac{\zeta_K(s_1)}{\zeta_K(s_1+1)\cdot\zeta_K((ord(t_{23})+1)s_1)}.$$

(iii) When E is a field,

$$\widehat{\Phi_s}(t) = |t_{123}|^{3s+6}|t_0|^{3s+3}\cdot\frac{\zeta_E(s)}{\zeta_E(s+1)\cdot\zeta_E((ord(t_{123})+1)s)}.$$

We are now ready to begin the unramified computation, Here is the main local theorem to be proved:

Theorem 5.5.3. *Set*

$$Z^*(\varphi,\Phi,f,s) = \hat{\zeta}_E(\underline{s}+\underline{1})\cdot\zeta(|\underline{s}|)\cdot Z(\varphi,\Phi,f,s).$$

Then

$$Z^*(\varphi,\Phi,f,s) = \frac{L(\underline{s},\pi,V_{E,\mathrm{Spin}})}{\zeta_E(2\underline{s})}.$$

Before diving into the computation, we would like to point out the three main steps in the computation. Hopefully this will make the structure of the computations more transparent.

(i) (Separation of Variables). Both sides of the identities are functions in the vector $\underline{s}$, but on the RHS, one clearly has a separation of variables. Thus, our first step is to prove that the LHS also has separation of variables. This is the most complicated step and uses the results of [BKW] regarding the decomposition of tensor products of representations of classical groups. Obviously, this step is not necessary if E is a field.

(ii) (Replacing s_π by $\overline{s_\pi}$). Observe that the LHS is expressed in terms of $\overline{s_\pi}$ via the Casselman-Shalika formula, but the RHS is in terms of s_π. Thus, the second step is to interpret the RHS in terms of $\overline{s_\pi}$. Obviously, this step is not necessary if $E = F^3$.

(iii) (Comparison) The final step is the comparision of the two sides. This also requires knowledge of decomposition of the tensor product of two representations. However, one of the representations will be fairly small, and so one can appeal to a more direct technique, such as Brauer's method, as opposed to using [BKW].

6. **Unramified Computation: E a field.**

As might be expected, once the subtleties of non-split groups are understood, the computation there turns out to be simpler than the split case. Thus we begin with the case when E is a field.

6.1. **Zeta integral side.** Recall that we have the two versions of Satake parameter:

$$s_\pi \in H'_E \cong G_2(\mathbb{C}) \quad \text{and} \quad \overline{s_\pi} \in H_E^\vee \cong G_2(\mathbb{C}).$$

If we set

$$x = q^{-3s},$$

then the LHS of Theorem 5.5.3 is equal to:

$$(1-x)^{-1} \cdot \sum_{k_0,k} \epsilon_3(k_0+1) \cdot Tr(\overline{s_\pi}|V_{k_0,k}^{H_E^\vee}) \cdot x^{k_0+k} \cdot \frac{1-x^{k+1}}{1-x}$$

$$= (1-x)^{-1} \cdot \sum_{k_0,k} \sum_{l=0}^{k} \epsilon_3(k_0+1) \cdot Tr(\overline{s_\pi}|V_{k_0,k}^{H_E^\vee}) \cdot x^{l+k_0+k}.$$

Here, $V_{k_0,k}$ is the irreducible representation of $H_E^\vee$ with highest weight (k_0,k), where $(1,0)$ stands for the fundamental weight attached to the 7-dim representation of $H_E^\vee$ and $(0,1)$ that of the adjoint representation.

6.2. ***L*-function side.** On the other hand, to explicate the other side of the main theorem, which involves the Spin L-function, we consider the restriction of $V_{E,\mathrm{Spin}}$ to the subgroup $H'_E \times \mathrm{Gal}(\overline{F}/F)$. This decomposes as:

$$V_{E,\mathrm{Spin}} \cong (V_{1,0}^{H'_E} \oplus \mathbf{1}) \boxtimes (\mathbf{1} \oplus \chi_E \oplus \chi_E^{-1})$$

where $V_{1,0}^{H'_E}$ is the 7-dimensional fundamental representation of $H'_E \cong G_2(\mathbb{C})$. Since $t_\pi = (s_\pi, \mathrm{Frob})$ with $s_\pi \in H'_E$, we see that

$$\det(1 - q^{-s} t_\pi | V_{E,\mathrm{Spin}}) = (1-q^{-3s}) \cdot \det(1 - q^{-3s} s_\pi | V_{1,0}^{H'_E}).$$

Thus the RHS of the identity in Theorem 5.5.3 is equal to

$$\begin{aligned}
&\frac{L(s,\pi,V_{E,\mathrm{Spin}})}{\zeta_E(2s)\cdot\zeta(3s)} \\
=&(1-q^{-6s}) \cdot (1-q^{-3s})^{-1} \cdot det(1 - q^{-3s} s_\pi^3 | V_{1,0}^{H'_E})^{-1} \\
=&(1-x)^{-1} \cdot (1-x^2) \cdot \sum_n x^n \cdot Tr(s_\pi^3 | Sym^n(V_{1,0}^{H'_E})) \\
=&(1-x)^{-1} \cdot \sum_n Tr(s_\pi^3 | V_{n,0}^{H'_E}) \cdot x^n.
\end{aligned}$$

In the last equality above, we have used the fact that

$$Sym^n(V_{1,0}) \cong Sym^{n-2}(V_{1,0}) \oplus V_{n,0}.$$

For this fact, see the table on [Br, Pg. 13].

6.3. **Comparison.** We can now attempt to compare the two sides. Unfortunately, one side of the identity is expressed in terms of $\overline{s_\pi}$ while the other in terms of s_π. Thus, we need to express the trace of s_π^3 on $V_{n,0}^{H'_E}$ in terms of the trace of $\overline{s_\pi}$ on some representation of $H_E^\vee$. Using the explicit description of the map ι^* in (5.3) and the Weyl character formula, we obtain:

Lemma 6.3.1.

$$Tr(s_\pi|V_{n,0}^{H'_E}) = \frac{Tr(\overline{s_\pi}|V_{2,n}^{H_E^\vee})}{Tr(\overline{s_\pi}|V_{2,0}^{H_E^\vee})}.$$

If we cancel the factor $(1-x)^{-1}$ from both sides of the desired identity and compare the coefficient of x^n, we see that to complete the proof of Theorem 5.5.3 in this case, we need to prove:

Proposition 6.3.2.

$$\frac{Tr(\overline{s_\pi}|V_{2,n})}{Tr(\overline{s_\pi}|V_{2,0})} = \sum_{k+k_0\le n\le 2k+k_0} \epsilon_3(k_0+1)\cdot Tr(\overline{s_\pi}|V_{k_0,k}).$$

The rest of this section is devoted to the proof of this proposition.

6.4. **Reduction to SL_3.** The long root subgroups of $G_2(\mathbb{C})$ generate a subgroup of G_2 isomorphic to $SL_3(\mathbb{C})$, sharing the same maximal torus $S_E^\vee$. The roots $\{\beta, 3\alpha+\beta\}$ of G_2 form a system of simple roots for SL_3, whose fundamental weights are

$$\begin{cases} \omega_1 = 2\alpha+\beta \\ \omega_2 = \alpha+\beta. \end{cases}$$

We write $W_{a,b}$ for the irreducible representation of $SL_3(\mathbb{C})$ with highest weight $a\omega_1+b\omega_2$ and let $\chi_{a,b}$ be its character. Henceforth, we shall use the coordinates given by the weights ω_1 and ω_2 of SL_3.

Using the Weyl character formula, one checks that

$$Tr(s|V_{a,b}) = \frac{Tr(s|W_{a+b+1,b}) - Tr(s|W_{b,a+b+1})}{Tr(s|W_{1,0}) - Tr(s|W_{0,1})}.$$

Thus the identity of Prop. 6.3.2 is:

$$(6.4.1) \qquad (\chi_{n+3,n}-\chi_{n,n+3})(\chi_{1,0}-\chi_{0,1})$$
$$= \left(\sum_{k+k_0\le n\le 2k+k_0} \epsilon_3(k_0+1)\cdot(\chi_{k+k_0+1,k}-\chi_{k,k+k_0+1})\right)(\chi_{3,0}-\chi_{0,3}).$$

We use Brauer's method, which for these purposes is conveniently stated as follows. Let ξ be any virtual character of $SL_3(\mathbb{C})$, given by

$$\xi(t) = \sum_\nu m(\nu)t^\nu,$$

with the sum being over the weight lattice, $m(\nu)\in\mathbb{Z}$ for each ν, and $m(\nu)=0$ for almost all ν. Note also that, since χ is a virtual character, $m(w\nu)=m(\nu)$ for any weight ν and any w in the Weyl group. For λ any weight, let

$$A_\lambda(t) = \sum_{w\in W_{SL_3}} sgn(w)t^{w\lambda},$$

and let $\chi_\lambda = A_{\lambda+\rho}/A_\rho$. Then

$$\xi\chi_\lambda = \sum_\nu m(\nu)\chi_{\lambda+\nu}. \tag{6.4.2}$$

Observe that χ_λ, so defined, is equal to the character of the irreducible representation with highest weight λ when λ is dominant, and is zero when $\lambda+\rho$ has a stabilizer in the Weyl group, and that $\chi_\lambda + \chi_\mu = 0$ whenever $\lambda+\rho$ and $\mu+\rho$ are related by a simple reflection.

We apply (6.4.2) to both sides of (6.4.1), and summarize the answer by a picture of the weight lattice where each node τ is labeled with the multiplicity of the corresponding χ_τ. As it turns out all multiplicities are 0,1 or -1, and 0's are omitted. The application to the left hand side, with $\xi = \chi_{1,0} - \chi_{0,1}$, is straightforward, and for $n \geq 1$ yields two hexagons of side 1, centered at $(n+3, n)$ and $(n, n+3)$ (here, and throughout, weights are expressed in terms of the basis of fundamental weights) with alternating signs. See Figure 1.

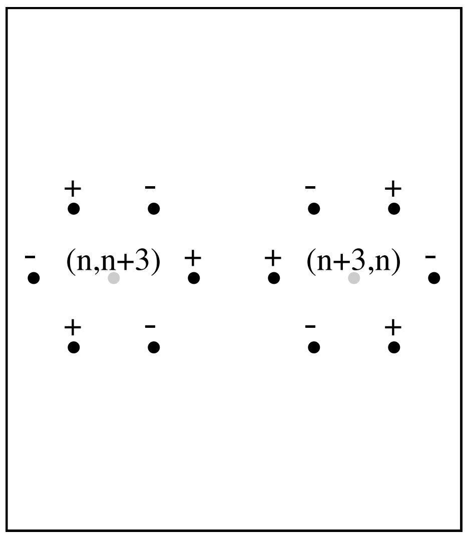


FIGURE 1. the LHS of (6.4.1).

Next we apply (6.4.2) to the right hand side of (6.4.1) with $\xi = \chi_{3,0} - \chi_{0,3}$. The representations $W_{3,0}$ and $W_{0,3}$ have nine weights each, but six of them are common to both, and cancel, leaving a hexagon with sides of length three, and signs alternating. As may be expected, the details are a bit different for the first few values of n. We give an argument which works for $n \geq 6$. The reader may find it an enjoyable diversion to work out the remaining cases. Alternatively, one could use the computer package *LiE* to verify the cases $n < 6$.

Now, we note that

$$\begin{aligned}
&\sum_{k+k_0 \leq n \leq 2k+k_0} \epsilon_3(k_0+1)\cdot(\chi_{k+k_0+1,k} - \chi_{k,k+k_0+1}) \\
&= \sum_{0\leq b<a\leq n+1\leq a+b} \epsilon_3(a-b)\chi_{a,b} - \sum_{0\leq a<b\leq n+1\leq a+b} \epsilon_3(b-a)\chi_{a,b} \\
&= \sum_{0\leq a,b\leq n+1\leq a+b} \epsilon_3(a-b)\chi_{a,b}.
\end{aligned}$$

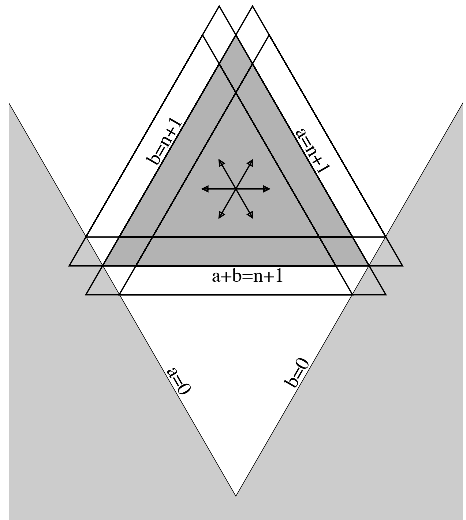


FIGURE 2

The sum is now over the lattice points contained in the triangle with vertices $(n{+}1, 0)$, $(0, n{+}1)$ and $(n + 1, n + 1)$. Let us denote this triangle by Δ_{n+1}.

We now multiply the above sum by ξ, using (6.4.2) on each term. It's clear that χ_τ appears in the product with nonzero multiplicity only if τ lies in one of the six translates of Δ_{n+1} by the weights ν such that $m(\nu) \neq 0$. Setting

$$\epsilon_3(\tau) = \epsilon_3(a - b) \quad \text{ for } \tau = (a, b),$$

observe that

$$\epsilon_3(\tau) = \epsilon_3(\tau + \nu) \quad \text{ for each such } \nu.$$

Figure 2 shows Δ_{n+1}, along with its six translates, visualized as pointing straight up. With things drawn this way, $\chi(\tau)$ is constant on vertical lines, and ρ points straight up. The multiplicity with which $\chi_\tau = \chi_{a,b}$ appears in the product is

$$\epsilon_3(\tau) \sum_{\nu:\tau-\nu\in\Delta_{n+1}} m(\nu).$$

Now, we must analyze the cancellation coming from the overlap of the six translates of this triangle.

This is displayed in Figure 3: we first see how many of the translates with $m(\nu) = 1$ each point lies in, and then how many of the ones with $m(\nu) = -1$. Subtracting, we see that the only weights which contribute lie in six rhombi with side 2, one on each side of each vertex of our triangle. (The precise picture is only valid for n at least 6 but the reader will find that this description holds for n as small as 2.)

We consider the two rhombi near the vertex $(n{+}1, n{+}1)$. As we see in Figure 4, the points on the long diagonals of our two rhombi do not contribute because $\epsilon_3(a - b)$ is zero on these lines. The remaining points give precisely the two hexagons of side 1 we saw before, and the signs match.

Next we turn to the four rhombi at the vertices $(n + 1, 0)$ and $(0, n + 1)$. In Figure 5 one of these rhombi is displayed in grey, while its translate by ρ is shown in black. We see that,

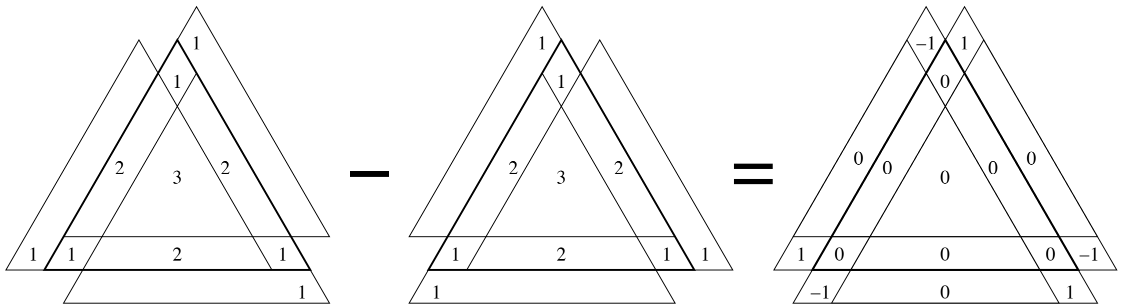


FIGURE 3. Positive and negative translates of Δ_{n+1} cancel almost completely.

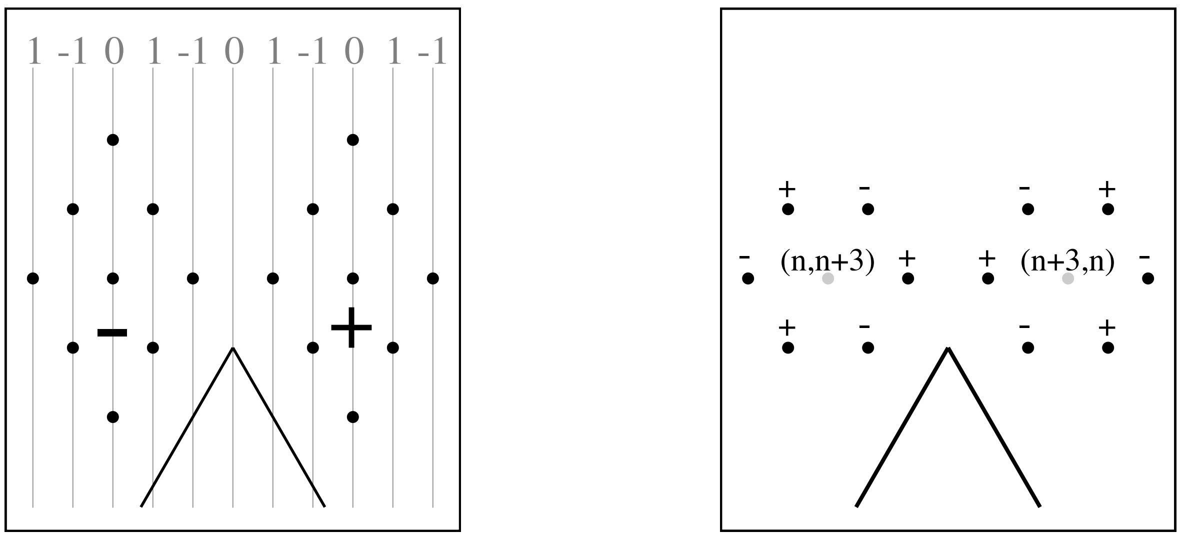


FIGURE 4. grey numbers indicate ϵ_3; black signs indicate $m(\nu)$; cf Fig. 1.

upon translation by ρ, the three points on the short diagonal are taken to points which are stabilized by an elementary reflection, so their contributions vanish. The remaining six points are related by this elementary reflection, so their contributions cancel. This last statement relies on the fact that the values of $\epsilon_3(a-b)$ at corresponding points are equal, which may also be observed from Figure 5.

A similar consideration shows the desired vanishing for the other 3 rhombi. Proposition 6.3.2 is proved.

7. **Unramified Computation:** $E = F \times K$

Now we come to the case when $E = F \times K$. In this case, the element t of $S_E(F)$ is of the form

$$t = \mu_0'(t_0)\mu_1'(t_1)(\mu_2'\mu_3')(t_{23}) = \mu_0'(\varpi^{k_0})\mu_1'(\varpi^{k_1})(\mu_2'\mu_3')(\varpi^{k_{23}}).$$

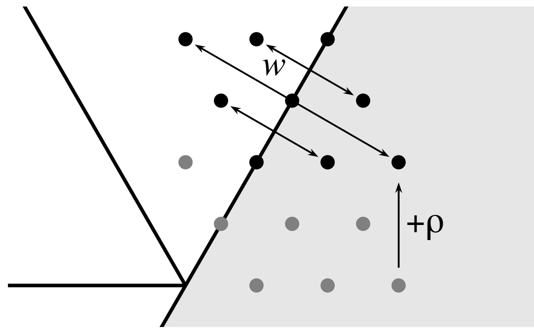


FIGURE 5

We recall the Casselman-Shalika formula:

$$\delta_{B'_E}(t)^{-1/2} W_{V'_E,\psi}(t\cdot\varphi) = Tr(\overline{s_\pi}|V^{H_E^\vee}_{k_1,k_0,k_{23}}).$$

Here, $V^{H_E^\vee}_{k_1,k_0,k_{23}}$ denotes the irreducible representation of $H_E^\vee \cong Sp_6(\mathbb{C})$ with highest weight $k_1\nu_1 + k_0\nu_2 + k_{23}\nu_3$ where ν_1, ν_2, ν_3 are the fundamental weights of $Sp_6(\mathbb{C})$ numbered such that ν_1 corresponds to the standard representation.

7.1. **Zeta integral side.** Let $x = q^{-s_1}$ and $y = q^{-2s_{23}}$. Then the LHS of Theorem 5.5.3 is:

$$(1-xy)^{-1}\cdot\sum_{k_i=0}^{\infty} x^{k_0+k_1}y^{k_0+k_{23}}\epsilon_2(k_0+1)\cdot(-1)^{k_1}\cdot\frac{1-(x^{-1}y)^{k_1+1}}{1-x^{-1}y}\frac{1-x^{2(k_{23}+1)}}{1-x^2}Tr(\overline{s_\pi}|V^{Sp_6}_{k_1,k_0,k_{23}}).$$

Next we have the following crucial lemma, which gives a separation of variables in the above sum:

Lemma 7.1.1. *The LHS of Theorem 5.5.3 is equal to:*

$$\left(\sum_{k=0}^{\infty}(-x)^k Tr(\overline{s_\pi}|V^{Sp_6}_{k,0,0})\right)\left(\sum_{\ell,m=0}^{\infty} y^{\ell+m}(-1)^\ell Tr(\overline{s_\pi}|V^{Sp_6}_{\ell,0,m})\right).$$

So as not to disrupt the flow of the argument, we defer the proof of the lemma to the end of the section.

7.2. **L-function side.** Now we consider the L-function side of the main theorem, which is a priori expressed in terms of $t_\pi \in {}^LG_E$. We know that $V_{E,\mathrm{Spin}}$ is reducible as a representation of ${}^LG_E$: it decomposes as:

$$V_{E,\mathrm{Spin}} = V_{\mu_1}\bigoplus(V_{\mu_2}\oplus V_{\mu_3}).$$

As in the previous section, we consider the restriction of $V_{E,\mathrm{Spin}}$ to the subgroup

$$H'_E \times \mathrm{Gal}(\overline{F}/F) = Spin_7(\mathbb{C})\times\mathrm{Gal}(\overline{F}/F).$$

It is not difficult to see that:

$$V_{\mu_1} \cong (V^{H'_E}_{1,0,0}\boxtimes\chi_K)\bigoplus(\mathbf{1}\boxtimes\mathbf{1}),$$

where $V^{H'_E}_{1,0,0}$ is the 7-dimensional standard representation of $Spin_7$, and

$$V_{\mu_2}\oplus V_{\mu_3} \cong V^{H'_E}_{0,0,1}\boxtimes(\mathbf{1}\oplus\chi_K),$$

where $V^{H'_E}_{0,0,1}$ is the 8-dimensional Spin representation of $Spin_7$. Thus, we obtain:

$$L(\underline{s},\pi,V_{E,\mathrm{Spin}}) = \frac{1}{(1-q^{-s_1})}\cdot\frac{1}{\det(1+q^{-s_1}s_\pi|V^{Spin_7}_{1,0,0})}\cdot\frac{1}{\det(1-q^{-2s_{23}}s_\pi^2|V^{Spin_7}_{0,0,1})}.$$

Next we need to express the above in terms of $\overline{s_\pi}$ rather than s_π. For this, we need to know explicitly the map

$$\iota^* : T_E^\vee \longrightarrow S_E^\vee.$$

The maximal torus $T_E^\vee$ of $Spin_8(\mathbb{C})$ is conveniently parametrized by five complex variables (u,t_1,t_2,t_3,t_4) subject to the relation $t_1t_2t_3t_4 = u^2$. With this parametrization, the map to the maximal torus $S_E^\vee$ of $Sp_6(\mathbb{C})$ is simply

$$(u,t_1,t_2,t_3,t_4)\mapsto(t_1,t_2,t_3).$$

From this, it is not hard to see that

$$\det(1+q^{-s_1}s_\pi|V^{Spin_7}_{1,0,0}) = (1+q^{-s_1})\cdot\det(1+q^{-s_1}\overline{s}_\pi|V^{Sp_6}_{1,0,0}),$$

where $V^{Sp_6}_{1,0,0}$ is the 6-dimensional standard representation of $Sp_6(\mathbb{C})$. Thus, we have:

$$\begin{aligned}&\frac{1}{(1+q^{-s_1})}\cdot\frac{1}{\det(1+q^{-s_1}s_\pi|V^{Spin_7}_{1,0,0})}\\ =&\zeta(2s_1)\cdot\sum_{k=0}^{\infty}Tr(\overline{s}_\pi|Sym^kV^{Sp_6}_{1,0,0})\cdot(-1)^k\cdot q^{-ks_1}\\ =&\zeta(2s_1)\cdot\sum_{k=0}^{\infty}Tr(\overline{s}_\pi|V^{Sp_6}_{k,0,0})\cdot(-1)^k\cdot q^{-ks_1},\end{aligned}$$

where for the last equality, we have used the well-known fact that

$$Sym^n(V^{Sp_6}_{1,0,0}) = V^{Sp_6}_{n,0,0}$$

which can be found in the table on [Br, Pg. 13].

On the other hand, by [Br, Pg. 13] again, we see that

$$det(1-q^{-2s_{23}}s_\pi^2|V^{Spin_7}_{0,0,1})^{-1} = \zeta_K(2s_{23})\sum_{n=0}^{\infty}q^{-2ns_{23}}Tr(s_\pi^2|V^{Spin_7}_{0,0,n}),$$

where $V^{Spin_7}_{0,0,n}$ denotes the irreducible representation of $Spin_7$ with highest weight equal to n times the third fundamental weight. We claim:

Lemma 7.2.1.

$$Tr(s_\pi^2|V^{Spin_7}_{0,0,n}) = \frac{Tr(\overline{s}_\pi|V^{Sp_6}_{1,1,n})}{Tr(\overline{s}_\pi|V^{Sp_6}_{1,1,0})}.$$

Proof. Let us view $s_\pi = (u,t_1,t_2,t_3,t_4)$ as above. The fact that s_π actually lies in the maximal torus of $H'_E \cong Spin_7(\mathbb{C})$ implies that $t_4 = 1$. The coordinates of s_π^2 are then

$(t_1t_2t_3, t_1^2, t_2^2, t_3^2, 1)$. Let $s(t) = t - t^{-1}$. Then the Weyl character formula can be given quite nicely in terms of determinants in this case:

$$Tr(s_\pi^2|V_{0,0,n}^{Spin_7}) = \frac{\begin{vmatrix} s(t_1^{n+5}) & s(t_2^{n+5}) & s(t_3^{n+5}) \\ s(t_1^{n+3}) & s(t_2^{n+3}) & s(t_3^{n+3}) \\ s(t_1^{n+1}) & s(t_2^{n+1}) & s(t_3^{n+1}) \end{vmatrix}}{\begin{vmatrix} s(t_1^5) & s(t_2^5) & s(t_3^5) \\ s(t_1^3) & s(t_2^3) & s(t_3^3) \\ s(t_1^1) & s(t_2^1) & s(t_3^1) \end{vmatrix}}$$

A similar formula holds for the characters of $Sp_6(\mathbb{C})$:

$$Tr(\overline{s}_\pi|V_{k_1,k_2,k_3}^{Sp_6}) = \frac{\begin{vmatrix} s(t_1^{k_1+k_2+k_3+3}) & s(t_2^{k_1+k_2+k_3+3}) & s(t_3^{k_1+k_2+k_3+3}) \\ s(t_1^{k_2+k_3+2}) & s(t_2^{k_2+k_3+2}) & s(t_3^{k_2+k_3+2}) \\ s(t_1^{k_3+1}) & s(t_2^{k_3+1}) & s(t_3^{k_3+1}) \end{vmatrix}}{\begin{vmatrix} s(t_1^3) & s(t_2^3) & s(t_3^3) \\ s(t_1^2) & s(t_2^2) & s(t_3^2) \\ s(t_1^1) & s(t_2^1) & s(t_3^1) \end{vmatrix}}.$$

So we see that

$$Tr(s_\pi^2|V_{0,0,n}^{Spin_7}) = \frac{Tr(\overline{s}_\pi|V_{1,1,n}^{Sp_6})}{Tr(\overline{s}_\pi|V_{1,1,0}^{Sp_6})}$$

as desired. □

7.3. **Comparison.** Summarizing, we have shown that:

$$\frac{L(s_1, \pi, V_{\mu_1})}{\zeta(2s_1)} = \left(\sum_{k=0}^{\infty} (-x)^k Tr(\overline{s}_\pi|V_{k,0,0}^{Sp_6})\right)$$

and

$$\frac{L(s_{23}, \pi, V_{\mu_2} \oplus V_{\mu_3})}{\zeta_K(2s_{23})} = \left(\sum_{n=0}^{\infty} y^n \cdot \frac{Tr(\overline{s}_\pi|V_{1,1,n}^{Sp_6})}{Tr(\overline{s}_\pi|V_{1,1,0}^{Sp_6})}\right).$$

Thus the RHS of the identity in Theorem 5.5.3 is:

$$\left(\sum_{k=0}^{\infty} (-x)^k Tr(\overline{s}_\pi|V_{k,0,0}^{Sp_6})\right) \cdot \left(\sum_{n=0}^{\infty} y^n \cdot \frac{Tr(\overline{s}_\pi|V_{1,1,n}^{Sp_6})}{Tr(\overline{s}_\pi|V_{1,1,0}^{Sp_6})}\right).$$

Comparing this with Lemma 7.1.1, we see that it remains to prove:

Proposition 7.3.1.

$$\frac{Tr(s|V_{1,1,n}^{Sp_6})}{Tr(s|V_{1,1,0}^{Sp_6})} = \sum_{\ell=0}^{n} (-1)^\ell \cdot Tr(s|V_{\ell,0,n-\ell}^{Sp_6}).$$

Proof. This can be proved using Brauer's method, as in Prop. 6.3.2, but is harder to explain in this case since we will be working in 3-dimensional space. We give a proof using the expression of the characters in determinantal form.

The identity to be proven can be restated as

$$\left(\sum_{m=0}^{n}(-1)^{n-m}\begin{vmatrix} s(t_1^{n+3}) & s(t_2^{n+3}) & s(t_3^{n+3}) \\ s(t_1^{m+2}) & s(t_2^{m+2}) & s(t_3^{m+2}) \\ s(t_1^{m+1}) & s(t_2^{m+1}) & s(t_3^{m+1}) \end{vmatrix}\right)\frac{\begin{vmatrix} s(t_1^{5}) & s(t_2^{5}) & s(t_3^{5}) \\ s(t_1^{3}) & s(t_2^{3}) & s(t_3^{3}) \\ s(t_1^{1}) & s(t_2^{1}) & s(t_3^{1}) \end{vmatrix}}{\begin{vmatrix} s(t_1^{3}) & s(t_2^{3}) & s(t_3^{3}) \\ s(t_1^{2}) & s(t_2^{2}) & s(t_3^{2}) \\ s(t_1^{1}) & s(t_2^{1}) & s(t_3^{1}) \end{vmatrix}}$$

$$= \begin{vmatrix} s(t_1^{n+5}) & s(t_2^{n+5}) & s(t_3^{n+5}) \\ s(t_1^{n+3}) & s(t_2^{n+3}) & s(t_3^{n+3}) \\ s(t_1^{n+1}) & s(t_2^{n+1}) & s(t_3^{n+1}) \end{vmatrix}.$$

We first check that

$$\Delta(t) := \frac{\begin{vmatrix} s(t_1^{5}) & s(t_2^{5}) & s(t_3^{5}) \\ s(t_1^{3}) & s(t_2^{3}) & s(t_3^{3}) \\ s(t_1^{1}) & s(t_2^{1}) & s(t_3^{1}) \end{vmatrix}}{\begin{vmatrix} s(t_1^{3}) & s(t_2^{3}) & s(t_3^{3}) \\ s(t_1^{2}) & s(t_2^{2}) & s(t_3^{2}) \\ s(t_1^{1}) & s(t_2^{1}) & s(t_3^{1}) \end{vmatrix}} = (t_1^{-1}+t_2^{-1})(t_1^{-1}+t_3^{-1})(t_2^{-1}+t_3^{-1})(1+t_1t_2)(1+t_1t_3)(1+t_2t_3).$$

Next we note that

$$\begin{vmatrix} s(t_1^{n+3}) & s(t_2^{n+3}) & s(t_3^{n+3}) \\ s(t_1^{m+2}) & s(t_2^{m+2}) & s(t_3^{m+2}) \\ s(t_1^{m+1}) & s(t_2^{m+1}) & s(t_3^{m+1}) \end{vmatrix} = \sum_{\epsilon\in\{\pm1\}^3}\begin{vmatrix} \epsilon_1 t_1^{\epsilon_1(n+3)} & \epsilon_2 t_2^{\epsilon_2(n+3)} & \epsilon_3 t_3^{\epsilon_3(n+3)} \\ \epsilon_1 t_1^{\epsilon_1(m+2)} & \epsilon_2 t_2^{\epsilon_2(m+2)} & \epsilon_3 t_3^{\epsilon_3(m+2)} \\ \epsilon_1 t_1^{\epsilon_1(m+1)} & \epsilon_2 t_2^{\epsilon_2(m+1)} & \epsilon_3 t_3^{\epsilon_3(m+1)} \end{vmatrix}. \tag{7.3.2}$$

Now expand each of these determinants by minors on the first row, we see that this is equal to

$$\sum_{\epsilon\in\{\pm1\}^3}\sum_{i\in\mathbb{Z}/3\mathbb{Z}}\epsilon_i t_i^{\epsilon_i\cdot(n+3)}\cdot\begin{vmatrix} \epsilon_{i+1}t_{i+1}^{\epsilon_{i+1}(m+2)} & \epsilon_{i+2}t_{i+2}^{\epsilon_{i+2}(m+2)} \\ \epsilon_{i+1}t_{i+1}^{\epsilon_{i+1}(m+1)} & \epsilon_{i+2}t_{i+2}^{\epsilon_{i+2}(m+1)} \end{vmatrix}$$

$$= \sum_{\epsilon\in\{\pm1\}^3}\sum_{i\in\mathbb{Z}/3\mathbb{Z}}\epsilon_i t_i^{\epsilon_i\cdot(n+3)}\cdot\begin{vmatrix} \epsilon_{i+1}t_{i+1}^{2\epsilon_{i+1}} & \epsilon_{i+2}t_{i+2}^{2\epsilon_{i+2}} \\ \epsilon_{i+1}t_{i+1}^{\epsilon_{i+1}} & \epsilon_{i+2}t_{i+2}^{\epsilon_{i+2}} \end{vmatrix}\cdot t_{i+1}^{\epsilon_{i+1}m}t_{i+2}^{\epsilon_{i+2}m}$$

Now summing over m, we see that

$$\sum_{m=0}^{n}(-1)^{n-m}\cdot\begin{vmatrix} s(t_1^{n+3}) & s(t_2^{n+3}) & s(t_3^{n+3}) \\ s(t_1^{m+2}) & s(t_2^{m+2}) & s(t_3^{m+2}) \\ s(t_1^{m+1}) & s(t_2^{m+1}) & s(t_3^{m+1}) \end{vmatrix}$$

$$= \sum_{\epsilon\in\{\pm1\}^3}\sum_{i\in\mathbb{Z}/3\mathbb{Z}}(-1)^n\cdot\epsilon_1\epsilon_2\epsilon_3\cdot t_i^{(n+3)\epsilon_i}t_{i+1}^{\epsilon_{i+1}}t_{i+2}^{\epsilon_{i+2}}\cdot(t_{i+1}^{\epsilon_{i+1}}-t_{i+2}^{\epsilon_{i+2}})\cdot\frac{1-(-t_{i+1}^{\epsilon_{i+1}}t_{i+2}^{\epsilon_{i+2}})^{n+1}}{1+t_{i+1}^{\epsilon_{i+1}}t_{i+2}^{\epsilon_{i+2}}}. \tag{7.3.3}$$

If we let $T_i = t_i^n$, then this last expression is a linear combination $T_i^{\epsilon_i}$ and $T_1^{\epsilon_1}T_2^{\epsilon_2}T_3^{\epsilon_3}$ with coefficients given by rational functions of t_i's essentially independent of n (the only dependence on n being the factor $(-1)^n$).

Similarly, we expand the RHS of the desired identity in a manner analogous to (7.3.2). It is equal to:

$$\sum_{\epsilon\in\{\pm 1\}^3} \epsilon_1\epsilon_2\epsilon_3 T_1^{\epsilon_1}T_2^{\epsilon_2}T_3^{\epsilon_3}\cdot \begin{vmatrix} t_1^{5\epsilon_1} & t_2^{5\epsilon_2} & t_3^{5\epsilon_3} \\ t_1^{3\epsilon_1} & t_2^{3\epsilon_2} & t_3^{3\epsilon_3} \\ t_1^{\epsilon_1} & t_2^{\epsilon_2} & t_3^{\epsilon_3} \end{vmatrix}.$$

Treat both sides of the desired identity as polynomials in $T_1, T_2, T_3, T_1^{-1}, T_2^{-1}, T_3^{-1}$, with coefficients in $\mathbb{C}(t_1, t_2, t_3)$. Then we need to show:

- the coefficient of $T_i^{\epsilon_i}$ on the LHS vanishes. By symmetry, it suffices to show

$$\sum_{\epsilon\in\{\pm 1\}^2} \frac{\epsilon_1\epsilon_2 t_1^{\epsilon_1}t_2^{\epsilon_2}\cdot(t_1^{\epsilon_1}-t_2^{\epsilon_2})}{1+t_1^{\epsilon_1}t_2^{\epsilon_2}} = 0.$$

  This is immediately verifiable by hand.
- the coefficient of $T_1^{\epsilon_1}T_2^{\epsilon_2}T_3^{\epsilon_3}$ on both sides agree. By symmetry again, it suffices to verify:

$$\left(\frac{t_1^3t_2^2t_3^2(t_2-t_3)}{1+t_2t_3} + \frac{t_2^3t_3^2t_1^2(t_3-t_1)}{1+t_3t_1} + \frac{t_3^3t_1^2t_2^2(t_1-t_2)}{1+t_1t_2}\right)\cdot\Delta(t) = \begin{vmatrix} t_1^5 & t_2^5 & t_3^5 \\ t_1^3 & t_2^3 & t_3^3 \\ t_1^1 & t_2^1 & t_3^1 \end{vmatrix}.$$

  Again, this is easily verified by hand.

The proposition is proved.

□

7.4. **Proof of Lemma 7.1.1.** Finally, we need to give the proof of Lemma 7.1.1.

For this, we use a formula which goes back to Murnaghan and Littlewood, and has been given a nice interpretation in terms of "universal characters" in [Ko] and [K-T]. This is discussed in greater detail in the appendix. We state it in the notation of the appendix. We associate the highest weight of $V^{Sp_6}_{k_1,k_2,k_3}$ with the partition $(k_1+k_2+k_3, k_2+k_3, k_3)$, and for λ a partition with at most 3 parts, let $\chi_{Sp_6}(\lambda)$ denote the character of the representation with the corresponding highest weight. We consider also finite formal sums of partitions, and extend χ_{Sp_6} to these by $\mathbb{Z}$-linearity.

Thus, the right hand side of the identity is

$$(1-xy)\left(\sum_{k=0}^{\infty}(-x)^k\chi_{Sp_6}(k)\right)\left(\sum_{0\le\mu_2\le\mu_1} y^{\mu_1}(-1)^{\mu_1-\mu_2}\chi_{Sp_6}(\mu_1\mu_2^2)\right). \tag{7.4.1}$$

On the set of all partitions, we define operations $\cdot$ and $/$ by

$$\lambda\cdot\mu = \sum_\nu LR^{\nu}_{\lambda,\mu}\nu$$

and

$$\lambda/\mu = \sum_\nu LR^{\lambda}_{\mu,\nu}\nu,$$

where the Littlewood-Richardson coefficient $LR^{\lambda}_{\mu,\nu}$ counts the number of ways to add the boxes that make up the Young diagram of μ to the diagram of ν in order to obtain that of

λ subject to certain constraints (see [F-H, p.455-56]), and so each of these sums is actually finite.

The formula may be stated as

$$\chi_{Sp_6}(\lambda)\chi_{Sp_6}(\mu) = \pi_{Sp_6}\left(\sum_{\zeta} \chi_{Sp}((\lambda/\zeta)\cdot(\mu/\zeta))\right), \tag{7.4.2}$$

where $\chi_{Sp}(\nu)$ is a certain "universal character," which is defined for all partitions ν (while χ_{Sp_6} is defined only for those with ≤ 3 parts) and extended to finite formal sums by $\mathbb{Z}$-linearity, and π_{Sp_6} is a projection from the ring of universal characters to that of characters of Sp_6. In our case, by the interpretation of the Littlewood-Richardson coefficients in terms of adding boxes to the Young diagram, we get

$$\chi_{Sp_6}(k)\chi_{Sp_6}(\mu_1\mu_2^2) = \sum_{\nu,\kappa} \pi_{Sp_6}(\chi_{Sp}(\nu)),$$

where the sum is over partitions ν and κ satisfying

$$0 = \kappa_4 \leq \nu_4 \leq \kappa_3 \leq \nu_3 \leq \mu_2 = \kappa_2 \leq \nu_2 \leq \kappa_1 \leq \nu_1, \mu_1$$

and

$$\mu_1 + \mu_2 + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 - 2\kappa_1 - \kappa_3 = k.$$

For ν having 3 or fewer parts, we have

$$\pi_{Sp_6}(\chi_{Sp}(\nu)) = \chi_{Sp_6}(\nu).$$

For ν having exactly 4 parts, it is zero. So we find that (7.4.1) is equal to

(7.4.3)

$$(1-xy)\sum_{0\leq\nu_3\leq\nu_2\leq\nu_1} \chi_{Sp_6}(\nu) \sum_{\kappa_3=0}^{\nu_3}\sum_{\mu_2=\nu_3}^{\nu_2}\sum_{\kappa_1=\nu_2}^{\nu_1}\sum_{\mu_1=\kappa_1}^{\infty} x^{\mu_1+\nu_1+\nu_2+\nu_3-2\kappa_1-2\kappa_3}(-1)^{\mu_2+\nu_1+\nu_2+\nu_3}y^{\mu_1}.$$

Now,

$$\sum_{\mu_1=\kappa_1}^{\infty}(xy)^{\mu_1} = (xy)^{\kappa_1}(1-xy)^{-1}.$$

and

$$\sum_{\mu_2=\nu_3}^{\nu_2}(-1)^{\mu_2} = (-1)^{\nu_3}\cdot\epsilon_2(\nu_2-\nu_3+1).$$

Plugging these in, we obtain

$$\sum_{0\leq\nu_3\leq\nu_2\leq\nu_1} \chi_{Sp_6}(\nu)\epsilon_2(\nu_2-\nu_3+1)\sum_{\kappa_3=0}^{\nu_3}\sum_{\kappa_1=\nu_2}^{\nu_1} x^{\nu_1+\nu_2+\nu_3-\kappa_1-2\kappa_3}y^{\kappa_1}(-1)^{\nu_1+\nu_2}. \tag{7.4.4}$$

Summing over κ_1 yields

$$(x^{-1}y)^{\nu_2}\frac{1-(x^{-1}y)^{\nu_1-\nu_2+1}}{1-x^{-1}y}.$$

We then write $\nu_3 - 2\kappa_3 = -\nu_3 + 2r_3$, where $r_3 = \nu_3 - \kappa_3$, and sum r_3 from 0 to ν_3 obtaining

$$\frac{1-x^{2(\nu_3+1)}}{1-x^2}.$$

We now have

$$\sum_{0\leq\nu_3\leq\nu_2\leq\nu_1} \chi_{Sp_6}(\nu)\epsilon_2(\nu_2-\nu_3+1)x^{\nu_1-\nu_3}y^{\nu_2}(-1)^{\nu_1-\nu_2}\frac{1-x^{2(\nu_3+1)}}{1-x^2}\frac{1-(x^{-1}y)^{\nu_1-\nu_2+1}}{1-x^{-1}y}.$$

Translating this back to the other notation via the correspondence

$$\chi_{Sp_6}(\nu_1,\nu_2,\nu_3) = \mathrm{Tr}(\cdot|V^{Sp_6}_{\nu_1-\nu_2,\nu_2-\nu_3,\nu_3}),$$

and plugging in $\overline{s_\pi}$, we obtain the desired result. Lemma 7.1.1 is proved.

8. **Unramified Computation:** $E = F^3$

Finally, we come to the split case. This case is the most complicated of the three, because it involves decomposing the tensor product of 3 representations of $Spin_8(\mathbb{C})$.

8.1. **The identity.** Let us set

$$x_i = q^{-s_i} \quad \text{for } i = 1, 2 \text{ or } 3.$$

Then the LHS of the identity in Theorem 5.5.3 is:

$$(1 - x_1x_2x_3)^{-1} \cdot \sum_{k_i=0}^{\infty} (k_0+1) \cdot V_{k_0,k_1,k_2,k_3} x_1^{k_0+k_1} x_2^{k_0+k_2} x_3^{k_0+k_3} \times$$

$$\frac{1-(x_1^{-1}x_2x_3)^{k_1+1}}{1-(x_1^{-1}x_2x_3)} \frac{1-(x_2^{-1}x_1x_3)^{k_2+1}}{1-(x_2^{-1}x_1x_3)} \frac{1-(x_3^{-1}x_1x_2)^{k_3+1}}{1-(x_3^{-1}x_1x_2)}.$$

On the other hand, the RHS of the identity is:

$$\prod_{i=1}^{3} \frac{1-x_i^2}{\det(1-x_i s_\pi|V_{\mu_i})}.$$

Using the fact that (cf. [Br, Pg. 13])

$$Sym^n(V_{\mu_i}) = Sym^{n-2}(V_{\mu_i}) \oplus V_{n\mu_i},$$

this is equal to

$$\sum_{\ell_i=0}^{\infty} V_{0,\ell_1,0,0} \otimes V_{0,0,\ell_2,0} \otimes V_{0,0,0,\ell_3} x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3}.$$

Thus the identity we wish to prove is:

$$\sum_{\ell_i=0}^{\infty} V_{0,\ell_1,0,0} \otimes V_{0,0,\ell_2,0} \otimes V_{0,0,0,\ell_3} x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3}$$

$$= \frac{1}{1-x_1x_2x_3} \sum_{k_i=0}^{\infty} (k_0+1) V_{k_0,k_1,k_2,k_3} x_1^{k_0+k_1} x_2^{k_0+k_2} x_3^{k_0+k_3}$$

$$\frac{1-(x_1^{-1}x_2x_3)^{k_1+1}}{1-(x_1^{-1}x_2x_3)} \frac{1-(x_2^{-1}x_1x_3)^{k_2+1}}{1-(x_2^{-1}x_1x_3)} \frac{1-(x_3^{-1}x_1x_2)^{k_3+1}}{1-(x_3^{-1}x_1x_2)}.$$

8.2. **Decomposition of tensor products.** We make use of the fact (cf [GH, Lemma 3.3]) that

$$V_{0,0,\ell_2,0} \otimes V_{0,0,0,\ell_3} = \bigoplus_{k=0}^{\min(\ell_2,\ell_3)} V_{0,k,\ell_2-k,\ell_3-k},$$

to rewrite the identity as

$$\sum_{\ell,m_i=0}^{\infty} V_{0,\ell,0,0} \otimes V_{0,m_1,m_2,m_3} x_1^{\ell} x_2^{m_1+m_2} x_3^{m_1+m_3}$$

$$= \frac{1}{1-x_1x_2x_3} \sum_{k_i=0}^{\infty} (k_0+1) V_{k_0,k_1,k_2,k_3} x_1^{k_0+k_1} x_2^{k_0+k_2} x_3^{k_0+k_3}$$

$$\frac{1-(x_1^{-1}x_2x_3)^{k_1+1}}{1-(x_1^{-1}x_2x_3)} \frac{1-(x_2^{-1}x_1x_3)^{k_2+1}}{1-(x_2^{-1}x_1x_3)} \frac{1-(x_3^{-1}x_1x_2)^{k_3+1}}{1-(x_3^{-1}x_1x_2)}. \tag{8.2.1}$$

We will be using a method of evaluating tensor products which is due to Black, King, and Wybourne [B-K-W]. They make use of an identification of certain weights of $Spin_8$ with partitions. Specifically, we identify the highest weight of V_{k_0,k_1,k_2,k_3} with the quadruple $(k_0+k_1+\frac{k_2+k_3}{2}, k_0+\frac{k_2+k_3}{2}, \frac{k_2+k_3}{2}, \frac{k_3-k_2}{2})$ which, if k_3-k_2 is even and non-negative, is then identified with a partition by dropping any terminal zeros. We could, of course, have made such an identification in five other ways, and picking this one in particular privileges the eight dimensional representation $V_{0,1,0,0}$ over the other two, and fixes one particular identification of the representation ring of SO_8 with a subring of that of $Spin_8$. For μ a dominant weight, let $\chi_{Spin_8}(\mu)$ denote the character of the representation with highest weight μ. If μ consists of integers, then this representation factors through $V_{0,1,0,0}$ and we may also write χ_{SO_8}.

Let ι denote the involution of $Spin_8$ which reverses the last two fundamental weights. Then the set of dominant weights is the union of the sets $\{\mu, \mu+(\frac{1}{2})^4, \mu^{\iota}, (\mu+(\frac{1}{2})^4)^{\iota}\}$, where μ ranges over partitions. The union is disjoint, but if μ has fewer than 4 parts, then $\mu=\mu^{\iota}$. Hence the character of the LHS of (8.2.1) is given by

$$\begin{aligned}\sum_{\ell,\mu} \chi_{SO_8}(\ell) x_1^{\ell} \Big(&\chi_{SO_8}(\mu) x_2^{\mu_1-\mu_4} x_3^{\mu_1+\mu_4} + \chi_{SO_8}(\mu^{\iota}) x_2^{\mu_1+\mu_4} x_3^{\mu_1-\mu_4} + \\ &\chi_{Spin_8}(\mu+(\frac{1}{2})^4) x_2^{\mu_1-\mu_4} x_3^{\mu_1+\mu_4+1} + \chi_{Spin_8}((\mu+(\frac{1}{2})^4)^{\iota}) x_2^{\mu_1+\mu_4+1} x_3^{\mu_1-\mu_4} \Big) \\ &- \sum_{\ell,\mu:\mu_4=0} \chi_{SO_8}(\ell)\chi_{SO_8}(\mu) x_1^{\ell} (x_2x_3)^{\mu_1},\end{aligned} \tag{8.2.2}$$

where ℓ is summed over $\mathbb{Z}_{\geq 0}$ and μ is summed over partitions satisfying $\mu_2=\mu_3$.

We describe the method of computing the necessary products of characters in brief here, and more completely in the appendix. First, the computation of the product of characters of $Spin_8$ is reduced to one of characters of GL_4, which are indexed by pairs of partitions having at most four total parts. Such a pair of partitions also gives a character of GL_n for each $n>4$. Each character is expressed as a determinant involving elementary symmetric polynomials and an identity in these determinantal expressions gives the product for all n sufficiently large or, as we prefer, in a suitably defined projective limit. We thus obtain a finite sum of "universal characters" $\chi_{GL}(\bar{\tau};\nu)$. (We follow [B-K-W] in writing $(\bar{\tau};\nu)$, rather than (τ, ν)). We then must analyze the contribution of each of these universal characters to the original product under projection.

We now write down a formula for the sum of universal characters that must be considered (cf. formula (11.4.4) in the appendix). Since one of our highest weights is the one-part partition ℓ, the formula is a bit simpler than in the general case. It is:

$$\sum_{\eta,\zeta} \chi_{GL}(\bar{\eta}; (\ell/(\eta \cdot \zeta)) \cdot \mu)). \tag{8.2.3}$$

Here ζ and η are summed over all partitions, but we only get contributions when each is a single integer and their sum does not exceed ℓ. Making further use of the interpretation of the Littlewood-Richardson coefficients in terms of adding boxes to the Young diagram, we obtain

$$\sum_{\nu,\kappa} \chi_{GL}(\overline{\ell'(\mu,\nu,\kappa,\ell)}; \nu),$$

where the sum is over ν and κ partitions satisfying

$$\nu_5 \le \kappa_4 \le \mu_4, \nu_4 \le \kappa_3 \le \nu_3 \le \mu_3 = \kappa_2 = \mu_2 \le \nu_2 \le \kappa_1 \le \mu_1, \nu_1,$$

and subject to the condition that

$$\ell'(\mu,\nu,\kappa,\ell) := \ell + 2\kappa_1 + 2\kappa_3 + 2\kappa_4 - \mu_1 - \mu_4 - \nu_1 - \nu_2 - \nu_3 - \nu_4 - \nu_5$$

is non-negative. The precise manner in which a term $\chi_{GL}(\ell'; \nu)$ contributes to the product depends on the precise relationship between the highest weight of our second representation and the partition μ. This is the topic of the next subsection. For now, we prove a formula for

$$\sum_{\mu,\ell} x^{\ell} y^{\mu_1-\mu_4} z^{\mu_1+\mu_4} \sum_{\nu,\kappa} \chi_{GL}(\overline{\ell'(\mu,\nu,\kappa,\ell)}; \nu), \tag{8.2.4}$$

where the sum in ν and κ is as above. Let

$$D(n_3, n_4; x, z) = (1 - x^{2n_3})(1 - (xz^{-1})^{n_4}) - (xz)^{n_3}(1 - x^{2n_4})(1 - (xz^{-1})^{n_3}).$$

Then we prove

Proposition 8.2.5. *The sum* (8.2.4) *is equal to*

$$F(x,y,z)^{-1} \sum_{\ell',\nu} \chi_{GL}(\bar{\ell}'; \nu) x^{\ell'} c(\nu; x, y, z)$$

with

$$F(x,y,z) = (1 - xy)(1 - x^{-1}y)(1 - xz)(1 - xz^{-1})(1 - x^2)$$

and

$$c(\nu; x, y, z) = x^{\nu_1-\nu_3+\nu_5} y^{\nu_2} z^{\nu_4} (\nu_2 - \nu_3 + 1)(1 - (x^{-1}y)^{\nu_1-\nu_2+1}) D(\nu_3 - \nu_4 + 1, \nu_4 - \nu_5 + 1; x, z).$$

Proof. We first bring the sum over ℓ inside, and get

$$\sum_{\nu,\ell'} \chi_{GL}(\bar{\ell}'; \nu) x^{\ell'} \sum_{\mu} y^{\mu_1} z^{\mu_4} P(\mu, \nu; x_1),$$

where

$$P(\mu, \nu; x) = \sum_{\kappa} x^{\mu_1+\mu_4+\nu_1+\nu_2+\nu_3+\nu_4+\nu_5-2\kappa_1-2\kappa_3-2\kappa_4}.$$

Observe that

$$P(\mu, \nu; x) = P_1(\mu_1, \nu_1, \nu_2; x) P_2(\mu_4, \nu_3, \nu_4, \nu_5; x)$$

where

$$P_1(\mu_1,\nu_1,\nu_2;x) = \sum_{\kappa_1=\nu_2}^{\min(\mu_1,\nu_1)} x^{\mu_1+\nu_1+\nu_2-2\kappa_1}$$

and

$$P_2(\mu_4,\nu_3,\nu_4,\nu_5;x) = \sum_{\kappa_4=\nu_5}^{\min(\mu_4,\nu_4)} \sum_{\kappa_3=\max(\mu_4,\nu_4)}^{\nu_3} x^{\mu_4+\nu_3+\nu_4+\nu_5-2\kappa_4-2\kappa_3}.$$

Now, let

$$Q_1(\nu_1,\nu_2;x,y) = \sum_{\mu_1=\nu_2}^{\infty} P_1(\mu_1,\nu_1,\nu_2;x)y^{\mu_1} = \sum_{\kappa_1=\nu_2}^{\nu_1}\sum_{\mu_1=\kappa_1}^{\infty} x^{\nu_1+\nu_2+\mu_1-2\kappa_1}y^{\mu_1},$$

and let

$$Q_2(\nu_3,\nu_4,\nu_5;x,z) = \sum_{\mu_4=\nu_5}^{\nu_3} P_2(\mu_4,\nu_3,\nu_4,\nu_5;x)z^{\mu_4} = \sum_{\kappa_4=\nu_5}^{\nu_4}\sum_{\kappa_3=\nu_4}^{\nu_3}\sum_{\mu_4=\kappa_4}^{\kappa_3} x^{\mu_4+\nu_3+\nu_4+\nu_5-2\kappa_4-2\kappa_3}z^{\mu_4}.$$

Then we have

$$c(\nu;x,y,z) = (\nu_2-\nu_3+1)Q_1(\nu_1,\nu_2;x,y)Q_2(\nu_3,\nu_4,\nu_5;x,z).$$

The $(\nu_2-\nu_3+1)$ comes because in original sum we had a free variable (say, μ_3) ranging from ν_3 to ν_2. The Q_i can be evaluated explicitly:

$$Q_1(\nu_1,\nu_2;x,y) = (1-xy)^{-1}(1-x^{-1}y)^{-1}x^{\nu_1}y^{\nu_2}(1-(x^{-1}y)^{\nu_1-\nu_2+1}).$$

and

$$Q_2(\nu_3,\nu_4,\nu_5;x,z) = (1-xz)^{-1}(1-xz^{-1})^{-1}(1-x^2)^{-1}z^{\nu_4}x^{-\nu_3+\nu_5}$$
$$\left((1-x^{2(\nu_3-\nu_4+1)})(1-(xz^{-1})^{\nu_4-\nu_5+1})-(xz)^{\nu_3-\nu_4+1}(1-x^{2(\nu_4-\nu_5+1)})(1-(xz^{-1})^{\nu_3-\nu_4+1})\right).$$

The result follows. □

In the next subsection, we describe how each $\chi_{GL}(\bar{\ell}';\nu)$ gives a contribution to the product

$$\chi_{SO_8}(\ell)\chi_{Spin_8}(\xi), \quad \text{with } \xi = (\mu+\varepsilon^4) \text{ or } (\mu+\varepsilon^4)^\iota \text{ and } \varepsilon = 0 \text{ or } \frac{1}{2}.$$

Just as the precise contribution depends on ε and the presence or absence of ι, so too does the polynomial which will accompany it in (8.2.2). As may be seen from (8.2.2), and the fact that $F(x,y,z) = F(x,y,z^{-1})$, this will be

$$F(x_1,x_2x_3,x_2x_3^{-1})^{-1}x_1^{\ell'} \times \begin{cases} c(\nu;x_1,x_2x_3,x_2^{-1}x_3) \text{ for } \mu; \\ c(\nu;x_1,x_2x_3,x_2x_3^{-1}) \text{ for } \mu^\iota; \\ x_3c(\nu;x_1,x_2x_3,x_2^{-1}x_3) \text{ for } \mu+(\frac{1}{2}); \\ x_2c(\nu;x_1,x_2x_3,x_2x_3^{-1}) \text{ for } (\mu+(\frac{1}{2}))^\iota. \end{cases}.$$

We also record for later use the analogous statement for the smaller sum being subtracted at the end of (8.2.2). The proof is easily obtained by adapting that of the previous proposition:

Lemma 8.2.6. *We have*

$$\sum_{\ell,\mu:\mu_4=0} x_1^{\ell}(x_2x_3)^{\mu_1}\chi_{GL}(\overline{\ell'(\mu,\nu,\kappa,\ell)};\nu) = \sum_{\ell',\nu}\chi_{GL}(\bar{\ell}';\nu)x_1^{\ell'}c'(\nu;x_1,x_2x_3)$$

where ν *can have at most* four *parts, and*

$$c'(\nu; x_1, x_2x_3) = \frac{(\nu_2 - \nu_3 + 1)x_1^{\nu_1-\nu_3+\nu_4}(x_2x_3)^{\nu_2}(1 - (x_1^{-1}x_2x_3)^{\nu_1-\nu_2+1})(1 - x_1^{2(\nu_3-\nu_4+1)})}{(1 - x_1x_2x_3)(1 - x_1^{-1}x_2x_3)(1 - x_1^2)}.$$

8.3. **From χ_{GL}'s to χ_{Spin_8}'s.** Now, we briefly describe the contribution of $\chi_{GL}(\bar{\ell}'; \nu)$ to $\chi_{SO_8}(\ell)\chi_{Spin_8}(\mu+\varepsilon^4)$. The answer is always zero or $\pm\chi_{Spin_8}(\xi)$ for some weight ξ. The corresponding contribution to $\chi_{SO_8}(\ell)\chi_{Spin_8}((\mu+\varepsilon^4)^\iota)$ would then be zero or $\pm\chi_{Spin_8}(\xi^\iota)$. First, there is a projection from universal characters to characters of GL_4 which sends $\chi_{GL}(\bar{\tau}; \nu)$ either to 0 or to $\pm\chi_{GL_4}(\bar{\tau}^\circ; \nu^\circ)$ for a pair of partitions $(\bar{\tau}^\circ; \nu^\circ)$ with at most four total parts. In our case, $\tau = \ell'$, and ν can have at most five parts. Among such pairs, the only ones which have more than four total parts, (recall that $\ell' = 0$ gives the empty partition having no parts) and are not killed are those of the form $(\bar{1}; \sigma 1)$, where $\sigma 1$ is the partition obtained by appending a 1 to the partition σ having exactly four parts. The projection of $\chi_{GL}(\bar{1}; \sigma 1)$ is $-\chi_{GL_4}(0; \sigma)$. If the total number of parts is already less than four, then $\chi_{GL}(\bar{\ell}'; \nu)$ projects to $\chi_{GL_4}(\bar{\ell}'; \nu)$.

The parametrization of representations by pairs of partitions is such that $\chi_{GL_4}(\bar{\ell}'; \nu)$ is the character of the representation whose highest weight ω is ν if $\ell' = 0$ and $(\nu_1, \nu_2, \nu_3, -\ell')$ otherwise. (Here we append terminal zeros to ν if it has fewer than 3 parts.) Now, we must add ε^4 to ω. The contribution to $\chi_{SO_8}(\ell)\chi_{Spin_8}(\mu+\varepsilon^4)$ is as follows:

$$\text{(8.3.1)} \qquad \begin{cases} \chi_{Spin_8}(\nu+\varepsilon^4), \text{ if } \ell' = 0 \\ 0, \text{ if } \omega+\varepsilon^4+\rho \text{ has a nontrivial stabilizer in the Weyl group of } Spin_8, \\ sgn(w)\chi_{Spin_8}(\xi), \text{ if } \omega+\varepsilon^4+\rho = w(\xi+\rho) \text{ for } \xi \text{ dominant.} \end{cases}$$

Here ρ is half the sum of the positive roots of $Spin_8$, which corresponds to the quadruple $(3, 2, 1, 0)$. The last case will produce a weight of the form $(\sigma+\varepsilon^4)^\iota$, with σ a partition. Indeed, $(\sigma+\varepsilon^4)^\iota$ arises directly as the weight corresponding to $(\overline{\sigma_4+2\varepsilon}; \sigma_1, \sigma_2, \sigma_3)$ and the weights corresponding to the three other pairs:

$$(\overline{\sigma_3+1+2\varepsilon}; \sigma_1, \sigma_2, \sigma_4-1), (\overline{\sigma_2+2+2\varepsilon}; \sigma_1, \sigma_3-1, \sigma_4-1), (\overline{\sigma_1+3+2\varepsilon}; \sigma_2-1, \sigma_3-1, \sigma_4-1)$$

are related to it by Weyl elements w with signs $-, +$ and $-$ respectively, provided σ has four parts. Hence these terms in the sum for $(\mu+\varepsilon^4)^\iota$ contribute to $(\sigma+\varepsilon^4)$.

8.4. **Completion of proof.** We now fix a partition σ and compute the coefficient of $\chi_{Spin_8}(\sigma+\varepsilon^4)$ in (8.2.2). Since $\chi_{Spin_8}(\sigma+\varepsilon^4)$ is the character of $V_{\sigma_2-\sigma_3,\sigma_1-\sigma_2,\sigma_3-\sigma_4,\sigma_3+\sigma_4+2\varepsilon}$, we should get

$$F(x_1, x_2x_3, x_2x_3^{-1})^{-1}(1-x_1^2)(\sigma_2-\sigma_3+1)x_1^{\sigma_1-\sigma_3}\sigma_2^{\sigma_2-\sigma_4}x_3^{\sigma_2+\sigma_4+2\varepsilon}$$
$$(1-(x_1^{-1}x_2x_3)^{\sigma_1-\sigma_2+1})(1-(x_1x_2^{-1}x_3)^{\sigma_3-\sigma_4+1})(1-(x_1x_2x_3^{-1})^{\sigma_3+\sigma_4+2\varepsilon+1}).$$

Furthermore, once we check this, we are done, since both sides of (8.2.1) are symmetric if we replace the subscript 2 everywhere by 3 and vice versa.

Clearing denominators, we just need to check some identities of polynomials. As will be evident from the discussion above, there are fewer terms if σ has fewer than four parts. However, if we append zeros to σ, the case $\sigma_4 = 0$ is recovered from the general case. For example, the term corresponding to $(\overline{\sigma_3+1+2\varepsilon}; \sigma_1, \sigma_2, \sigma_4-1)$ should not be there, but there is no harm in including it, because $D(n_3, 0; x, z)$ is equal to zero anyway. Thus, there are only two identities to check.

First, we perform a simplification that is relevant to both; namely one shows that

$$x_1^{\sigma_1-\sigma_3+\sigma_4} x_2^{\sigma_2} x_3^{\sigma_2} (\sigma_2 - \sigma_3 + 1)(1 - (x_1^{-1} x_2 x_3)^{\sigma_1-\sigma_2+1}) D(\sigma_3 + 1, 1; x_1, x_2 x_3^{-1})$$

$$-x_1^{\sigma_1+\sigma_3-\sigma_4+2} x_2^{\sigma_2} x_3^{\sigma_2} (\sigma_2 - \sigma_4 + 2)(1 - (x_1^{-1} x_2 x_3)^{\sigma_1-\sigma_2+1}) D(\sigma_4, 1; x_1, x_2 x_3^{-1})$$

$$+x_1^{\sigma_1+\sigma_2-\sigma_4+3} x_2^{\sigma_3-1} x_3^{\sigma_3-1} (\sigma_3 - \sigma_4 + 1)(1 - (x_1^{-1} x_2 x_3)^{\sigma_1-\sigma_3+2}) D(\sigma_4, 1; x_1, x_2 x_3^{-1})$$

$$-x_1^{\sigma_1+\sigma_2-\sigma_4+3} x_2^{\sigma_3-1} x_3^{\sigma_3-1} (\sigma_3 - \sigma_4 + 1)(1 - (x_1^{-1} x_2 x_3)^{\sigma_2-\sigma_3+1}) D(\sigma_4, 1; x_1, x_2 x_3^{-1})$$

is equal to

$$(\sigma_2 - \sigma_3 + 1) x_1^{\sigma_1-\sigma_3+\sigma_4} (x_2 x_3)^{\sigma_2} (1 - (x_1^{-1} x_2 x_3)^{\sigma_1-\sigma_2+1})$$

$$(D(\sigma_3 + 1, 1; x_1, x_2 x_3^{-1}) - x_1^{2(\sigma_3-\sigma_4+1)} D(\sigma_4, 1; x_1, x_2 x_3^{-1})).$$

Canceling from both sides of (8.2.1) the expression

$$(\sigma_2 - \sigma_3 + 1) x_1^{\sigma_1-\sigma_3} x_2^{\sigma_2-\sigma_4} x_3^{\sigma_2+\sigma_4+2\varepsilon} (1 - (x_1^{-1} x_2 x_3)^{\sigma_1-\sigma_2+1}),$$

which appears in $c(\sigma; x_1, x_2 x_3, x_2^{-1} x_3)$, $c(\sigma 1; x_1, x_2 x_3, x_2^{-1} x_3)$ and $c'(\sigma; x_1, x_2 x_3, x_2 x_3^{-1})$ (when $\varepsilon = 0$), and making use of (8.3.1), the identities to be checked boil down to:

(i) corresponding to $\varepsilon = 0$,

$$D(\sigma_3 - \sigma_4 + 1, \sigma_4 + 1; x_1, x_2^{-1} x_3) - x_1^2 D(\sigma_3 - \sigma_4 + 1, \sigma_4; x_1, x_2^{-1} x_3) + (x_1 x_2 x_3^{-1})^{\sigma_4}$$

$$(D(\sigma_3+1, 1; x_1, x_2 x_3^{-1}) - x_1^{2(\sigma_3-\sigma_4+1)} D(\sigma_4, 1; x_1, x_2 x_3^{-1}) - (1 - x_1 x_2^{-1} x_3)(1 - x_1 x_2 x_3^{-1})(1 - x_1^{2(\sigma_3-\sigma_4+1)}))$$

$$= (1 - (x_1 x_2^{-1} x_3)^{\sigma_3-\sigma_4+1})(1 - (x_1 x_2^{-1} x_3^{-1})^{\sigma_3+\sigma_4+1})(1 - x_1^2)$$

(ii) corresponding to $\varepsilon = \frac{1}{2}$,

$$D(\sigma_3 - \sigma_4 + 1, \sigma_4 + 1; x_1, x_2^{-1} x_3) - x_1^2 D(\sigma_3 - \sigma_4 + 1, \sigma_4; x_1, x_2^{-1} x_3)$$

$$+(x_1 x_2 x_3^{-1})^{\sigma_4+1} (D(\sigma_3 + 1, 1; x_1, x_2 x_3^{-1}) - x_1^{2(\sigma_3-\sigma_4+1)} D(\sigma_4, 1; x_1, x_2 x_3^{-1}))$$

$$= (1 - (x_1 x_2^{-1} x_3)^{\sigma_3-\sigma_4+1})(1 - (x_1 x_2^{-1} x_3)^{\sigma_3+\sigma_4+2})(1 - x_1^2).$$

Each of these may be checked by hand.

This completes the proof of Theorem 5.5.3 in the split case.

9. **Ramified Factors**

Finally, we shall address the analytic properties of the local zeta factor at a ramified place $v \in S$. Thus, we continue to work locally and suppress v from the notations.

9.1. **The local zeta integral.** Recall that:

$$Z(\varphi,\Phi,f,\underline{s}) = \int_{V'_E(F)\backslash G_E(F)} W_{V'_E,\psi}(g\cdot\varphi)\cdot\overline{L_{\chi_0}(g\cdot f)}\cdot\widehat{\Phi_s}(g)\,dg$$

$$(9.1.1)\qquad = \int_{V'_E(F)\backslash G_E(F)}\int_{U_{-\alpha}(F)} W_{V'_E,\psi}(g\cdot\varphi)\cdot\overline{L_{\chi_0}(g\cdot f)}\cdot\psi(u)\cdot\Phi_s(ug)\,du\,dg$$

$$(9.1.2)\qquad = \int_{S_E}\int_K\int_{U_{-\alpha}} W_{V'_E,\psi}(tk\cdot\varphi)\cdot\overline{L_{\chi_0}(tk\cdot f)}\cdot\psi(u)\cdot\Phi_s(utk)\cdot\delta_{B'_E}(t)^{-1}\,du\,dk\,dt.$$

In this section, by the local zeta integral $Z(\varphi,\Phi,f,\underline{s})$, we mean this last integral (9.1.2).

Assuming absolute convergence of (9.1.2), the manipulations above would be justified and we can collapse the two integrals in (9.1.1) to get:

$$\begin{aligned}
&Z(\varphi,\Phi,f,\underline{s})\\
&=\int_{N_E(F)\backslash G_E(F)} W_{V'_E,\psi}(g\cdot\varphi)\cdot\overline{L_{\chi_0}(g\cdot f)}\cdot\Phi_s(g)\,dg\\
&=\int_{V_E(F)\backslash G_E(F)} \Phi_{\underline{s}}(g)\cdot\left(\int_{U_\alpha} W_{V'_E,\psi}(ug\cdot\varphi)\cdot\overline{L_{\chi_0}(ug\cdot f)}\,du\right)\,dg\\
&=\int_K \Phi(k)\left(\int_{S_E}\chi_{\underline{s}}(t)\cdot\delta_{B_E}(t)^{-1}\cdot\int_{U_\alpha} W_{V'_E,\psi}(utk\cdot\varphi)\cdot\overline{L_{\chi_0}(utk\cdot f)}du\,dt\right)\,dk.
\end{aligned}$$

Setting $Y(\varphi,f,\underline{s},k)$ to be the inner integral, we would have

$$Z(\varphi,\Phi,f,\underline{s}) = \int_K \Phi(k)\cdot Y(\varphi,f,\underline{s},k)\,dk$$

.

The above formal manipulation is justified by the following proposition, which is the main result of this section:

Proposition 9.1.3. *(i) The local zeta integral $Z(\varphi,\Phi,f,\underline{s})$ (i.e. the integral (9.1.2)) converges absolutely for $\underline{s}\in\Omega_R$ (cf. (2.6)) when R is sufficiently large. In particular, $Y(\varphi,f,\underline{s},k)$ converges absolutely in Ω_R as well, and we have*

$$Z(\varphi,\Phi,f,\underline{s}) = \int_K \Phi(k)\cdot Y(\varphi,f,\underline{s},k)\,dk$$

for $\underline{s}\in\Omega_R$.

(ii) Both $Y(\varphi,f,\underline{s},-)$ and $Z(\varphi,\Phi,f,\underline{s})$ admit meromorphic continuation to the whole of $\mathbb{C}^{r-1}$ where $r=rank_F(G_E)$.

(iii) For a fixed $\underline{s_0}\in\mathbb{C}^{r-1}$, there is a choice of φ, Φ and f such that $Z(\varphi,\Phi,f,\underline{s})$ is holomorphic at $\underline{s_0}$ and non-zero there.

As a consequence, we have:

Corollary 9.1.4. *The partial Spin L-function $L^S(\underline{s},\pi,V_{\mathrm{Spin},E})$ admits meromorphic continuation to $\mathbb{C}^{r-1}$.*

The rest of this section is devoted to the proof of the proposition.

9.2. **Asymptotics of functions.** With $r = rank_F(G_E)$, we identify $S_E(F)$ with $(F^\times)^r$ using the fundamental coweights $\{\mu_i'\}$ as described in (5.1). We first note the following lemmas:

Lemma 9.2.1. *There are finitely many finite functions ξ_i on $(F^\times)^r$ such that for any $\varphi \in \pi$, one can find Schwarz-Bruhat functions φ_i (depending on φ) on $F^r \times K$ satisfying*

$$W_{V_E',\psi}(tk \cdot \varphi) = \sum_i \varphi_i(t,k) \cdot \xi_i(t).$$

This lemma is well-known. See [JS, §4, Props. 1 and 3] and [S, §4, Thm. 1]. We have a similar lemma for the asymptotics of the linear functional L_{χ_0} on the minimal representation.

Lemma 9.2.2. *There are finitely many finite functions η_i on $(F^\times)^r$ such that given any $f \in \Pi_E$, one can find Schwarz-Bruhat functions f_i on $F \times K$ satisfying*

$$L_{\chi_0}(tk \cdot f) = \sum_i f_i(t_0,k) \cdot \eta_i(t).$$

Proof. This follows from the construction of the linear functional L_{χ_0} described in the proof of Prop. 3.4.1. Indeed, if $L_E'U_E'$ is the 3-step parabolic containing B_E', then L_{χ_0} was described as the composition of an U_E'-invariant map to a representation σ of L_E' followed by the Whittaker functional on σ. Thus the lemma follows by the analog of the previous lemma for Whittaker functions on L_E. □

9.3. **Proof of Prop. 9.1.3(i).** We are now ready to prove Prop. 9.1.3(i), i.e. that the integral

$$\int_K \int_{S_E} \int_{U_{-\alpha}} |W_{V_E',\psi}(tk \cdot \varphi)| \cdot |\overline{L_{\chi_0}(tk \cdot f)}| \cdot |\Phi_s(utk)| \cdot \delta_{B_E'}(t)^{-1} \, du \, dt \, dk$$

converges when $\underline{s} \in \Omega_R$ with $R \gg 0$.

If we commute t across u in the above integral and change variables in u, as we did in the calculation before Prop. 5.5.2, we see that we need to prove the convergence of

$$\int_K \int_{S_E} |W_{V_E',\psi}(tk \cdot \varphi)| \cdot |\overline{L_{\chi_0}(tk \cdot f)}| \cdot \delta_{\underline{s}}(t)^{-1} \cdot \left(\int_{U_{-\alpha}} |\Phi_{\underline{s}}(uk)| \, du \right) \, dt \, dk$$

where $\delta_{\underline{s}}$ is an explicit character of t which can be read off from the computation before Prop. 5.5.2.

Now the integral over $U_{-\alpha}$ is a standard intertwining operator which converges in Ω_R for $R \gg 0$ and defines a smooth function of k. On the other hand, the convergence of the integrals over S_E and K follows from Lemmas 9.2.1 and 9.2.2. This proves Prop. 9.1.3(i).

9.4. **Proof of Prop. 9.1.3(ii).** To prove meromorphic continuation, we first note:

Lemma 9.4.1. *(i) When $\underline{s} \in \Omega_R$ with $R \gg 0$, the integral defining $\widehat{\Phi_{\underline{s}}}(g)$ converges absolutely.*

(ii) It admits a holomorphic continuation to $\mathbb{C}^{r-1}$.

(iii) Moreover, there are finitely many finite functions $\chi_{\underline{s},i}$ on $(F^\times)^r$ depending holomorphically on $\underline{s}$ such that for any flat section $\Phi_{\underline{s}}$, one can find Schwartz-Bruhat functions $\phi_{\underline{s},i}$ on $(F^\times)^{r-1} \times K$ depending holomorphically on $\underline{s}$ satisfying

$$\widehat{\Phi}_{\underline{s}}(t) = \sum_i \phi_{\underline{s},i}(t_1, ..., t_{r-1}, k) \cdot \chi_{\underline{s},i}(t).$$

The finite functions $\chi_{\underline{s},i}$ are precisely those which appear in Proposition 5.5.2.

Proof. (i) The absolute convergence of the integral defining $\widehat{\Phi}_{\underline{s}}$ has been addressed in the proof of Prop. 9.1.3(i) above.

(ii) Observe that the value $\widehat{\Phi_{\underline{s}}}(g)$ is obtained by restricting $g \cdot \Phi_{\underline{s}}$ to M_E, thus obtaining a section in a family of principal series of M_E, followed by applying the Jacquet integral for this family of principal series (which is a Whittaker functional). The desired holomorphic continuation was shown in Jacquet's thesis [J1].

(iii) This follows by an analog of Lemma 9.2.1 with parameters. Such a result is proved in [S, §4, Thm. 4], as well as in the recent article [J2]. □

As a consequence of Lemmas 9.2.1, 9.2.2 and 9.4.1, the local zeta integral is equal (for $\underline{s} \in \Omega_R$) to a sum of various integrals over $S_E \times K$ of finite functions ξ_i on S_E against Schwarz-Bruhat functions ϕ_i on $F^r \times K$. If the Schwarz functions ϕ_i are independent of $\underline{s}$, the meromorphic continuation of this type of integrals is well-known (cf. [JS, §3]). Indeed, in the p-adic case, the integral is easily seen to be equal to a rational function in q^{-s_i}. For a general local field, it was shown in [JS, §3] that the resulting meromorphic function is the product of various abelian L-functions (which are independent of the ϕ_i's) and an entire function depending on the ϕ_i's.

Now in our case, the Schwarz functions ϕ_i do depend holomorphically on $\underline{s}$ (via Lemma 9.4.1). The meromorphic continuation of our integrals then follows from the case discussed in the previous paragraph and [J2, Lemma 1, Pg. 377]. This proves the meromorphic continuation of $Z(\varphi, \Phi, f, \underline{s})$. The same argument gives the meromorphic continuation of the integral Y; one simply omits the integration over K. This proves Prop. 9.1.3(ii).

9.5. **Proof of Prop. 9.1.3(iii).** Recall that

$$Z(\varphi, \Phi, f, \underline{s}) = \int_K \Phi(k) \cdot Y(\varphi, f, \underline{s}, k)\, dk.$$

The function Φ is an arbitrary smooth function on K subject to the condition that

$$\Phi(lk) = \Phi(k) \quad \text{for all } l \in K \cap Q_E(F).$$

The function $Y(\varphi, f, \underline{s}, k)$ on K is easily seen to be left invariant under $K \cap U_E(F)$. Thus to show that data can be chosen to ensure the non-vanishing of $Z(\varphi, \Phi, f, \underline{s})$ at $\underline{s} = \underline{s_0}$, we need to show that the integral

$$Y'(\varphi, f, \underline{s}) = \int_{K \cap L_E(F)} Y(\varphi, f, \underline{s}, l)\, dl$$

is non-zero for some choices of f and φ. Note that since Y has meromorphic continuation to $\mathbb{C}^{r-1}$, so does the integral Y'.

Now we are going to massage the expression for Y' as follows:

$$
\begin{aligned}
&Y'(\varphi, f, \underline{s}) \\
&= \int_{K\cap L_E} \left(\int_{S_E} \int_{U_\alpha} \chi_{\underline{s}}(t) \cdot \delta_{B_E}(t)^{-1} \cdot W_{V'_E,\psi}(utk \cdot \varphi) \cdot \overline{L_{\chi_0}(utk \cdot f)}\, du\, dt \right) dk \\
&= \int_{U_\beta \backslash L_E} \int_{U_\alpha} \chi_{\underline{s}}(l) \cdot \delta_{Q_E}(l)^{-1} \cdot W_{V'_E,\psi}(ul \cdot \varphi) \cdot \overline{L_{\chi_0}(ul \cdot f)}\, du\, dl \\
&= \int_{T_E} \int_{U_{-\beta}} \int_{U_\alpha} \chi_{\underline{s}}(t) \cdot \delta_{Q_E}(t)^{-1} \cdot W_{V'_E,\psi}(u_\alpha u_{-\beta} t \cdot \varphi) \cdot \overline{L_{\chi_0}(u_\alpha u_{-\beta} t \cdot f)}\, du_\alpha\, du_{-\beta}\, dt.
\end{aligned}
$$

In the above, to obtain the second equality, we have used the Iwasawa decomposition for L_E, and to obtain the last equality, we have used the Bruhat decomposition to replace the integral over $U_\beta \backslash L_E$ by the integral over the open dense subset $U_{-\beta} \cdot T_E$.

Now the above manipulations are initially valid for $\underline{s} \in \Omega_R$ (with $R \gg 0$), where the integral on the RHS of the last equality converges absolutely. However, we shall presently show that for suitable choices of f, this integral admits meromorphic continuation to $\mathbb{C}^{r-1}$.

Let us write the torus T_E as $T_0 \times T_1$, where $T_0 \cong F^\times$ via the coweight μ'_0, and $T_1 \cong E^\times$ via the coweights $\{\mu'_1, \mu'_2, \mu'_3\}$. Correspondingly, we shall write the torus element t as $t_0 t_1$. If we conjugate t_1 to the left across $u_\alpha u_{-\beta}$, and change variables, we see that

$$
Y'(\varphi, f, \underline{s}) = \int_{T_0} \int_{T_1} \int_{U_\alpha \times U_{-\beta}} \mu_{\underline{s}}(t_0 t_1) \cdot W_{V'_E,\psi}(t_1 u_\alpha u_{-\beta} t_0 \cdot \varphi) \cdot \overline{L_{\chi_0}(t_1 u_\alpha u_{-\beta} t_0 \cdot f)}\, du_\alpha\, du_{-\beta}\, dt_0\, dt_1,
$$

where $\mu_{\underline{s}}$ is the resulting character of T_E after these manipulations.

At this point, we need the following lemma which gives an alternative construction of the functional L_{χ_0} on the minimal representation Π_E.

Lemma 9.5.1. *Let $P'' = M''N''$ be the Heisenberg parabolic subgroup of G_E so that the center of N'' is $Z'' = U_{\alpha'_0} = U_{3\alpha+\beta}$. Let $P''_{ss} = M''_{ss}N''$ be the derived group of P''. The unique irreducible representation of the Heisenberg group N'' with central character ψ can be realized on $\mathcal{S}(U_\alpha \times U_{-\beta}) = \mathcal{S}(E \times F)$ and this representation extends uniquely to the Weil representation ω_ψ of P''_{ss}.*

(i) There is a P''-equivariant injective map

$$
\iota : \Pi_E \longrightarrow Ind_{P''_{ss}}^{P''} \omega_\psi.
$$

(ii) We may realize the latter induced representation on the space of smooth functions on $T_0 \cong F^\times$ taking values in $\mathcal{S}(U_\alpha \times U_{-\beta})$. Thus a function in this space may be denoted by $\phi(t_0; u_\alpha, u_{-\beta})$. Moreover, the image of ι contains $\mathcal{S}(T_0) \otimes \mathcal{S}(U_\alpha \times U_{-\beta})$.

(iii) The linear functional L_{χ_0} is given by

$$
L_{\chi_0}(f) = \iota(f)(1; 0, 0)
$$

for $f \in \Pi_E$. Thus, we have:

$$
L_{\chi_0}(t_1 u_\alpha u_{-\beta} t_0 \cdot f) = \chi(t_1) \cdot \iota(f)(t_0; u_\alpha, u_{-\beta}),
$$

where χ is a character of T_1 which we will not make explicit here.

Proof. We shall give a sketch of the proof. The statements (i) and (ii) follow from the very construction of the minimal representation (cf. [K] and [GS]). To deduce (iii), we observe that via Frobenius reciprocity, ι gives a P''_{ss}-equivariant map (the local Fourier-Jacobi map)

$$FJ_\psi : \Pi_E \longrightarrow \mathcal{S}(U_\alpha \times U_{-\beta}).$$

This is explicitly described by

$$FJ_\psi(f)(u_\alpha, u_{-\beta}) = \iota(f)(1; u_\alpha, u_{-\beta}).$$

In the p-adic case, this is in fact the unique such map, since $(\Pi_E)_{Z'',\psi} \cong \omega_\psi$. If we let

$$R = \langle Z'', U_{3\alpha+2\beta}, U_{2\alpha+\beta}, U_{\alpha+\beta}\rangle \subset N_E,$$

then the composition of FJ_ψ with evaluation at $(0,0)$ gives a linear functional l on Π_E satisfying

$$l(r \cdot f) = \chi_0(r) \cdot l(f).$$

In the p-adic case, we have $(\omega_\psi)_{R,\chi_0} \cong \mathbb{C}$, so that

$$(\Pi_E)_{R,\chi_0} \cong \mathbb{C}.$$

Thus, $l = L_{\chi_0}$ up to scaling. The archimedean case can be finessed from this by a global argument using weak approximation; we omit the details. $\square$

In view of the lemma, we have

$$(9.5.2)\quad Y'(\varphi, f, \underline{s}) = \int_{T_0}\int_{U_\alpha\times U_{-\beta}} \overline{\iota(f)(t_0; u_\alpha, u_{-\beta})} \cdot \mu_{\underline{s}}(t_0) \cdot Y''(u_\alpha u_{-\beta} t_0 \cdot \varphi, \underline{s})\, du_\alpha\, du_{-\beta}\, dt_0,$$

where

$$Y''(\varphi, \underline{s}) = \int_{T_1} \mu_{\underline{s}}(t_1) \cdot \chi(t_1) \cdot W_{V'_E,\psi}(t_1 \cdot \varphi)\, dt_1.$$

Now as we noted in the proof of Prop. 9.1.3(ii) (after the proof of Lemma 9.4.1), an integral of the type defining $Y''(\varphi, \underline{s})$ (i.e. the Mellin transform of a Whittaker function) admits meromorphic continuation to $\mathbb{C}^{r-1}$ (cf. [JS, §3]). Further, at a point $\underline{s_0}$ of holomorphy, one can show that the linear functional $\varphi \mapsto Y''(\varphi, \underline{s_0})$ is continuous. Thus the map $(t_0, u_\alpha, u_{-\beta}) \mapsto Y''(t_0 u_\alpha u_{-\beta} \cdot \varphi, \underline{s})$ defines a smooth function.

Now by Lemma 9.5.1(ii), we may take f so that

$$\overline{\iota(f)(t_0; u_\alpha, u_{-\beta})} = f_0(t_0) \cdot f_1(u_\alpha, u_{-\beta})$$

for arbitrary $f_0 \in C_c^\infty(T_0)$ and $f_1 \in C_c^\infty(U_\alpha \times U_{-\beta})$. Thus, the integrals over T_0 and $U_\alpha \times U_{-\beta}$ converges absolutely and the equation (9.5.2) gives the meromorphic continuation of Y'.

By the above discussion, we see that if $Y'(\varphi, f, \underline{s})$ vanishes for all choices of f, then $Y''(\varphi, \underline{s})$ vanishes for all choices of φ. We are thus reduced to showing that there exists φ such that

$$\int_{T_1} W_{V'_E,\psi}(t_1 \cdot \varphi) \cdot \mu_{\underline{s}}(t_1) \cdot \chi(t_1)\, dt_1$$

is absolutely convergent for all $\underline{s}$ and non-zero for a particular $\underline{s}$. For this, we take $\phi \in \mathcal{S}(U_{-\alpha}) = \mathcal{S}(E)$ and replace φ by

$$\phi * \varphi = \int_{U_{-\alpha}} \phi(u) \cdot \pi(u)\varphi\, du.$$

Then

$$W_{V'_E,\psi}(t_1 \cdot (\phi * \varphi)) = \widehat{\phi}(t_1) \cdot W_{V'_E,\psi}(t_1 \cdot \varphi)$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. We may choose ϕ so that $\widehat{\phi}$ is an arbitrary compactly supported function. Then, for a suitable choice of such a ϕ, the integral over T_1 converges absolutely for all $\underline{s}$ and can be arranged to be non-zero for any given $\underline{s_0}$.

This completes the proof of Prop. 9.1.3(iii).

10. **Polynomial Invariants**

In the case of split groups, Ginzburg has observed that all L-functions $L(s, \pi, \rho)$ which are known to be representable by some Rankin-Selberg integrals have the property that the representation (ρ, V) of ${}^LG$ has the property that $\mathbb{C}[V]^{{}^LG}$ is a polynomial ring. This observation appears, for example, in [GH2] (where the word "split" was mistakenly omitted). Ginzburg has confirmed in private communication that his observation does not address the non-split cases.

In any case, the referee of this paper suggested that we investigate if Ginzburg's observation remains valid in our case. Thus, we shall describe briefly the algebras of ${}^LG^0$- and ${}^LG$-invariant polynomials on

$$V_{\mathrm{Spin}} = V_{\mu_1} \oplus V_{\mu_2} \oplus V_{\mu_3}.$$

For each i, there is a (unique up to scaling) nondegenerate quadratic form Q_i on V_{μ_i} which is fixed by ${}^LG^0$. We may normalize these so that the action of S_3 permutes them. It is well-known that

$$\mathbb{C}[V_{\mu_i}]^{{}^LG^0} = \mathbb{C}[Q_i].$$

In addition, there is an ${}^LG^0$-invariant trilinear linear form R (unique up to scaling) on $V_{\mu_1} \times V_{\mu_2} \times V_{\mu_3}$. In fact, one can show that $\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG^0}$ is the polynomial algebra generated by these four elements, as may be deduced from the local computations when $E = F \times F \times F$.

In fact, using the geometric description of V_{Spin} in (2.8), one can write down the above invariants easily. If $(x_1, x_2, x_3) \in V_{\mathrm{Spin}} = \mathbb{O}^3$, then for $i = 1, 2, 3$, we have:

$$Q_i(x_1, x_2, x_3) = N(x_i) \quad \text{and} \quad R(x_1, x_2, x_3) = Tr(x_1 x_2 x_3).$$

From this description, it is easy to verify that the Q_i's are permuted by S_3 whereas R is fixed by S_3. For example, when σ is the transposition such that

$$\sigma : (x_1, x_2, x_3) \mapsto (\overline{x_2}, \overline{x_1}, \overline{x_3}),$$

then

$$R(\sigma(x_1, x_2, x_3)) = Tr(\overline{x_2 x_1 x_3}) = Tr(x_3 x_1 x_2) = R(x_1, x_2, x_3).$$

So for example,

$$\mathbb{C}[V_{\mathrm{Spin}}]^{\mathrm{Spin}_8 \rtimes S_3} = \mathbb{C}[\Sigma_1, \Sigma_2, \Sigma_3, R]$$

where the Σ_i's are the elementary symmetric functions in the Q_i's.

There are thus four possibilities for $\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG}$ corresponding to the four possibilities for the image of the Galois group in S_3:

- when $E = F \times F \times F$,
$$\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG} = \mathbb{C}[Q_1, Q_2, Q_3, R].$$
- when $E = F \times K$,
$$\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG} = \mathbb{C}[Q_1, Q_2 + Q_3, Q_2 Q_3, R].$$

- when E is Galois cubic,

$$\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG} = \mathbb{C}[Q_1, Q_2, Q_3]^{A_3}[R],$$

  where $\mathbb{C}[Q_i]^{A_3}$ is the direct sum of the symmetric and skew-symmetric polynomials in the Q_i's.

- when E is non-Galois cubic,

$$\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG} = \mathbb{C}[\Sigma_1, \Sigma_2, \Sigma_3, R].$$

Observe that $\mathbb{C}[V_{\mathrm{Spin}}]^{{}^LG}$ is a polynomial ring except when E is Galois cubic. This suggests that, if Ginzburg's observation were to extend to the non-split case, it is the ${}^LG^0$-invariants, rather than the ${}^LG$-invariants, that should play a role.

11. **Appendix: Tensor Products and Universal Characters**

In this section, all groups are over $\mathbb{C}$, so the "$\mathbb{C}$"'s are suppressed throughout. Thus GL_n means $GL_n(\mathbb{C})$, etc. For G a connected reductive Lie group, let $R(G)$ denote the representation ring of G. Our primary goal here is to give an exposition of some of the results in [B-K-W] from a point of view akin to that of [K-T].

It is perhaps worthwhile to comment on a potential source of confusion: the relationship between highest weights and partitions (i.e., nonincreasing sequences of integers of finite length). For classical groups there is a natural identification between a subset of the set of partitions (those with a number of parts bounded suitably in terms of the group) and a subset of the set of weights (for example, for $Spin_{2n+1}$, the weights which factor through the projection to SO_{2n+1}). Greek letters, λ, μ, ν, etc., are hence used for both partitions and weights. One must therefore be a bit careful about what "λ" is in any given sentence: is it a partition, which may have too many parts, and hence not be a weight, or is it a weight, which may not lie in the subset parametrized by partitions?

For $G = GL_n$, we consider the subring $R_+(G)$ of "polynomial representations," i.e., representations whose characters are polynomials in the coordinates of an element of the torus. (The character of a general representation being the product of one of these times a power of the inverse of the determinant.) The highest weight of an element of $R_+(G)$ is a partition with at most n parts, λ, the value at $diag(t_1, \ldots, t_n)$ being $\prod_{i=1}^n t_i^{\lambda_i}$. Here, if λ has fewer than n parts, we add terminal zeros.

Since the map that sends a representation to its character is injective, we shall generally speak in terms of characters, rather than representations. We introduce a little more general notation. For G as above, and λ a dominant weight of G we let $\Gamma_G(\lambda)$ denote the irreducible finite dimensional representation of G with highest weight λ, and $\chi_G(\lambda)$ its character.

11.1. **Review and Summary.** We review the method of universal characters as discussed in [K-T]. For each n, let $\Lambda_n = R_+(GL_n) = \mathbb{Z}[t_1, \ldots, t_n]^{\mathfrak{S}_n}$. For $m > n$, let $\tilde{\rho}_{m,n} : \Lambda_m \to \Lambda_n$ be the ring homomorphism that sends t_i to t_i if $i \leq n$ and 0 if $i > n$. Then $(\Lambda_n, \tilde{\rho}_{m,n})$ is a projective system. The projective limit (in the category of $\mathbb{Z}_{\geq 0}$-graded algebras), Λ, with maps $\tilde{\rho}_n$ to each Λ_n, was defined by MacDonald [M] and is called the universal character ring.

For each n, we define the elementary symmetric polynomials, $e_k^n(t)$ by

$$\prod_{i=1}^{n}(1+t_i x) = \sum_{k=-\infty}^{\infty} e_k^n(t)x^k.$$

In particular, $e_k^n(t) = 0$ if $k < 0$ or $k > n$. Then $\mathbb{Z}[t_1, \ldots, t_n]^{\mathfrak{S}_n} = \mathbb{Z}[e_1^n, \ldots, e_n^n]$. Clearly $\tilde{\rho}_{m,n}(e_k^m) = e_k^n$, so we get elements e_k of Λ. Similarly, we consider p_i^n defined by

$$\prod_{i=1}^{n}(1-xt_i)^{-1} = \sum_{i=-\infty}^{\infty} p_i^n(t)x^i,$$

so that $p_i^n = 0$ for $i < 0$ but not for i large. Then $\tilde{\rho}_{m,n}(p_i^m) = p_i^n$, and so once again we get elements of Λ.

We have the classical determinantal expressions

$$\chi_{GL_n}(\lambda) = |p^n_{\lambda_i+i-j}| = |e^n_{\lambda'_i+i-j}|. \tag{11.1.1}$$

Here and throughout $|f(i,j)|$ denotes the determinant of the square matrix whose i,j entry is $f(i,j)$, and λ' is the transpose of the partition λ. Note that in each expression, the only dependence of n is the superscripts on the p_i's (resp. e_i's). In particular, the size of the determinant is equal to the number of nonzero parts of λ (resp. λ') and does not depend on n. Making use of these, together with the elements $p_i, e_i \in \Lambda$ above, we define $\chi_{GL}(\lambda) \in \Lambda$. For each n we have the natural projection $\pi_n : \Lambda \to \Lambda_n$. Then clearly $\pi_n(\chi_{GL}(\lambda)) = \chi_{GL_n}(\lambda)$ if λ has at most n parts. What is more, if λ has more than n parts, then the top row of the second determinantal expression in (11.1.1) is all zeros, so that $\pi_n(\chi_{GL}(\lambda)) = 0$.

The universal characters $\chi_{GL}(\lambda)$ are a $\mathbb{Z}$-basis for Λ, and the corresponding structure constants are the Littlewood-Richardson coefficients $LR^{\lambda}_{\mu,\nu}$. See [F-H, p. 455-56]. Thus,

$$\chi_{GL}(\mu)\chi_{GL}(\nu) = \sum_{\lambda} LR^{\lambda}_{\mu,\nu}\chi_{GL}(\lambda),$$

where the sum is over all partitions λ, whereas

$$\chi_{GL_n}(\mu)\chi_{GL_n}(\nu) = \sum_{\lambda} LR^{\lambda}_{\mu,\nu}\chi_{GL_n}(\lambda),$$

where now the sum is only over those partitions with at most n parts.

We turn now to Sp_{2n}. Highest weights are once again naturally identified with partitions of length $\leq n$. We have

$$\chi_{Sp_{2n}}(\lambda) = |p^{2n}_{\lambda_i+i-j} + \delta_{j,1}p^{2n}_{\lambda_i-i-j+2}|,$$

where $\delta_{i,j}$ is a Kronecker δ and p_i^{2n} has been restricted from the torus of GL_{2n} to that of Sp_{2n}. Once again this is nearly independent of n and allows us to define $\chi_{Sp}(\lambda) \in \Lambda$. However, it is no longer true that the determinant simply vanishes when λ has more than n parts. The universal characters $\chi_{Sp}(\lambda)$ give a second $\mathbb{Z}$-basis for Λ. The corresponding structure constants may be given in terms of the Littlewood-Richardson coefficients, as in [Ko], Theorem 7.5. This has a nice expression if we adopt one of the notational innovations in [B-K-W], namely the definition

$$\lambda/\zeta := \sum_{\xi} LR^{\lambda}_{\zeta,\xi}\xi. \tag{11.1.2}$$

Let z, x and ℓ denote the sums of the parts of ζ, ξ and λ respectively. The coefficient $LR^{\lambda}_{\zeta,\xi}$ counts the number of ways to add the z boxes to the Young diagram of ξ (or x to that of ζ) so as to obtain the Young diagram of λ, subject to certain conditions. In particular, it is zero for $x + z \neq \ell$, so the above sum is effectively finite, and zero for almost all ζ. With this notation, we have

$$\chi_{Sp}(\lambda)\chi_{Sp}(\mu) = \sum_{\zeta} \chi_{Sp}((\lambda/\zeta) \cdot (\mu/\zeta)), \tag{11.1.3}$$

where the interior product is given by the Littlewood-Richardson rule. Here we extend the definition of χ_{Sp} to finite sums of partitions by $\mathbb{Z}$-linearity. Hence

$$\chi_{Sp_{2n}}(\lambda)\chi_{Sp_{2n}}(\mu) = \pi_{Sp_{2n}}\left(\sum_{\zeta} \chi_{Sp}((\lambda/\zeta) \cdot (\mu/\zeta))\right). \tag{11.1.4}$$

Here $\pi_{Sp_{2n}}$ is the projection from Λ to $R(Sp_{2n})$ obtained by projecting to $\Lambda_{2n} \subset R(GL_{2n})$ and then restricting to Sp_{2n}. This formula essentially goes back to Littlewood and Newell. Only the interpretation in terms of universal characters is new.

As noted above, it is not the case that $\pi_{Sp_{2n}}$ gives the naive projection onto the set of partitions with at most n parts. A nice description of the projection and its kernel is given in [Ko, Section 9]. For purposes of our application to Sp_6, we only need to know that

$$\pi_{Sp_6}(\chi_{Sp}(\lambda)) = \begin{cases} \chi_{Sp_6}(\lambda) \text{ if } \lambda \text{ has at most 3 parts;} \\ 0, \text{ if } \lambda \text{ has exactly 4 parts.} \end{cases}$$

11.2. **New Material.** We turn now to the formulae in section 7 of [B-K-W]. These are derived by similar means, but in contrast to the above, they relate characters of representations of classical groups to those of general linear groups not by restricting, say, from GL_{2n} to SO_{2n}, but rather by restricting from SO_{2n} to GL_n embedded as the Levi factor of the Siegel parabolic. The symplectic and odd orthogonal cases are also considered in section 7 of [B-K-W], but it is only the even orthogonal case where the method there really offers substantial improvement over the method of section 5 of that paper and the previous section of these notes.

It is convenient to think of SO_{2n} as a subset of GL_{2n}, with the maximal torus of diagonal elements $t = diag(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1)$, and identify the partition λ with the map $t \mapsto \prod_i t_i^{\lambda_i}$. Indeed, we have already done so in referring to "the Siegel parabolic." Of course, there is another identification which works just as well (using the other parabolic) and in the case $n = 4$, which is our primary interest here, there are still more possibilities arising from the additional symmetry of the Dynkin diagram. We fix once and for all one of these identifications. When $n \neq 4$, the highest weight of the unique projection to SO_{2n} is identified with the partition 1 having one part. When $n = 4$, we will also refer to the representation whose highest weight has been identified with this partition as "the" projection. Because of our choice of identification, it is no longer on equal footing with the other two.

There are three matters which must be addressed.

(i) The first is that the restriction of a representation of SO_{2n} to GL_n will not be contained in $\Lambda_n = R_+(GL_n)$. Hence we can anticipate the need for some larger "universal" ring which covers all of $R(GL_n)$.

(ii) The second is that we are not satisfied with $R(SO_{2n})$, but want all of $R(Spin_{2n})$. It will turn out that for the cases we need here this second issue may to a certain extent be ducked. We must replace GL_n by its inverse image in Spin_{2n}, which is a double cover $\widetilde{GL_n}$, but as we shall see later, the double cover enters the computation only in a trivial way. Alternatively, everything could be phrased in terms of the Lie algebras.

(iii) The third issue is that, while the product of two elements of $R(\widetilde{GL_n})$, each of which is a restriction from $R(Spin_{2n})$ is certainly the restriction of something, we need a way to recognize what element of $R(Spin_{2n})$ it is a restriction of. This is handled by the following Lemma, akin to Brauer's method, which is very pretty in its own right. (It will be applied with $G = Spin_{2n}$ and $H = \widetilde{GL_n}$ the Levi subgroup of the Siegel parabolic.) It is due to King [Ki].

To state the lemma we need some notation, as before, let G be a connected reductive Lie group. Fix maximal torus T and a choice of positive roots, and let H be a standard Levi (there is a more general formulation, but restricting to standard Levi subgroups is sufficient for our purposes here and simplifies certain things a bit). Then we obtain a set of positive roots for H with respect to T, and the notions of G-dominance and H-dominance for elements of the lattice L of weights. Let L_G^+ denote the set of G-dominant weights and L_H^+ the set of H-dominant ones. Then $L_G^+ \subset L_H^+$ since G-dominance is more restrictive than H-dominance.

For $\lambda \in L$ and $t \in T$, let

$$A_\lambda^G(t) = \sum_{w \in W_G} sgn(w) t^{w\lambda}$$

and

$$\chi_G(\lambda) = A_{\lambda+\rho_G}/A_{\rho_G}.$$

For λ G-dominant, this agrees with the previous definition as the character of $\Gamma_G(\lambda)$. For λ not G-dominant, it is equal to $sgn(w)\chi_\tau$ if $\tau = w(\lambda + \rho_G) - \rho_G$ is G-dominant, and is zero otherwise.

Lemma 11.2.1. *With this notation, we have*

$$\chi_G(\lambda)\chi_G(\mu) = \sum_{\tau \in L_H^+} \left(\sum_\eta n_\eta^\lambda c(H)_{\eta,\mu}^\tau \right) \chi_G(\tau),$$

where n_η^λ is the multiplicity of $\Gamma_H(\eta)$ in the restriction $\Gamma_G(\lambda)$ to H, and $c(H)_{\lambda,\mu}^\tau$ is the multiplicity of $\Gamma_H(\tau)$ in $\Gamma_H(\lambda) \otimes \Gamma_H(\mu)$.

Note that this decomposition is in terms of χ_G's, but the τ's are only H-dominant. Before we proceed to the proof, let us note that

$$\chi_G(\lambda)\chi_H(\mu) = \sum_{\tau \in L_H^+} \left(\sum_\eta n_\eta^\lambda c(H)_{\eta,\mu}^\tau \right) \chi_H(\tau),$$

as is immediate from the definitions. So the lemma may be interpreted as follows. Suppose we have a formula for the restriction from G to H and a method of computing tensor products of representations of H. Then we may compute the product of two irreducible representations of G as follows: restrict one of them to H, take the product of this restriction with the irreducible representation of H that has the same highest weight as the other, and then simply replace each of the Γ_H's in the answer by the corresponding Γ_G. This is precisely the method employed in [B-K-W] section 7, the requisite branching rules having been obtained in earlier work.

We now remark why it is possible to essentially duck the issue of the difference between $R(SO_{2n})$ and $R(Spin_{2n})$ in the cases we are dealing with. This is because we never need to compute a product where *both* of the characters involved are in $R(Spin_{2n}) - R(SO_{2n})$. When one of them is, we make the corresponding weight "μ" rather than "λ." Suppose λ is a weight of $Spin_{2n}$, but not of SO_{2n}. Then either it is of the form $(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}) + p$ with p a partition, or it is associated to a weight of this form by the automorphism that reverses the last two fundamental weights. It is enough to consider the case when $\lambda = (\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}) + p$. It is not a weight of GL_n, but of the double cover $\widetilde{GL_n}$. But since

$$\chi_{\widetilde{GL_n}}((\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}) + p) = \det{}^{\frac{1}{2}} \chi_{GL_n}(p),$$

the double cover comes in only in a trivial way.

Proof of Lemma 11.2.1: Let ρ_G denote half the sum of the positive roots of G and ρ_H half the sum of the positive roots in H. Let W_G denote the Weyl group of G with respect to T. Note that the Weyl group W_H of H is naturally identified with a subgroup of the Weyl group W_G of G, and that $\rho_G - \rho_H$ is W_H-stable.

Let $m^G_\lambda(\nu)$ denote the multiplicity of the weight ν in $\Gamma_G(\lambda)$, and define $m^H_\lambda(\nu)$ similarly. Then

$$\chi_G(\lambda)\chi_G(\mu) = \sum_\nu m^G_\lambda(\nu)\chi_G(\nu + \mu) = \sum_\eta \sum_\nu n^\lambda_\eta m^H_\eta(\nu)\chi_G(\nu + \mu). \tag{11.2.2}$$

For ν a weight we define $|\nu|_G$ and $\frac{\nu}{|\nu|_G}$ as follows. If the stabilizer of $\nu + \rho_G$ in W_G is trivial, then $|\nu|_G$ is the unique dominant weight τ such that $\tau + \rho_G = w(\nu + \rho_G)$ for some $w \in W_G$ and $\frac{\nu}{|\nu|_G}$ is the sign of this w. If the stabilizer of $\nu + \rho_G$ in W_G is not trivial, then $\frac{\nu}{|\nu|_G} = 0$, and it does not matter what $|\nu|_G$ is (since it is multiplied by zero).

Then from the first equality above, we obtain

$$c(G)^\tau_{\lambda,\mu} = \sum_{|\nu+\mu|_G = \tau} m^G_\lambda(\nu)\frac{\nu + \mu}{|\nu + \mu|_G} = \sum_{w \in W_G} sgn(w) m^G_\lambda(w(\tau + \rho_G) - \mu - \rho_G).$$

Similarly,

$$c(H)^\tau_{\mu,\eta} = \left(\sum_{\nu : |\mu+\nu|_H = \tau} m^H_\eta(\nu)\frac{\mu + \nu}{|\mu + \nu|_H} \right).$$

Now $|\mu + \nu|_H = \tau$ means that $w(\mu + \nu + \rho_H) = \tau + \rho_H$ for some $w \in W_H$ in which case $w(\mu + \nu + \rho_G) = \tau + \rho_G$, since $\rho_G - \rho_H$ is W_H -stable. Thus (11.2.2) equals

$$\sum_{\tau \in L^+_H} \sum_{\eta \in L} n^\lambda_\eta \sum_{\nu : |\mu+\nu|_H = \tau} m^H_\eta(\nu)\frac{\mu + \nu}{|\mu + \nu|_H}\chi_G(\tau),$$

and the result follows. ■

Remark: One can, and King does, formulate and prove a more general result applicable to the case $G = G_2, H = SL_3$. In this case $\rho_G - \rho_H$ is no longer W_H-stable, so the expression is more complicated:

$$\chi_G(\lambda)\chi_G(\mu) = \sum_{\tau \in L_H^+} \sum_{\eta \in L} n_\eta^\lambda c(H)^\tau_{\mu+\rho_G-\rho_H,\eta} \chi_G(\tau - \rho_G + \rho_H).$$

11.3. **An analog of Λ.** As noted above, in order to place the branching rule method of King and his collaborators within a framework analogous to that of [K-T], we need an analog of Λ that surjects onto all of $R(GL_n)$ for each n, not only $R_+(GL_n)$.

Let $R_n = \mathbb{Z}[t_1, \ldots, t_n, u_1, \ldots, u_n]^{\mathfrak{S}_n \times \mathfrak{S}_n}$, where $\mathfrak{S}_n$ is the symmetric group, the first acting by permuting the t-indices and the second permutes the u-indices. Let π_n be the map $R_n \to R(GL_n)$, defined by $u_i \mapsto t_i^{-1}$. Also, for $m > n$ there is a surjective map $\rho_{m,n} : R_m \to R_n$ defined by $t_i \mapsto t_i, u_i \mapsto u_i,\ i \leq n$ and $t_i \mapsto 0, u_i \mapsto 0,\ i > n$. Clearly, $\pi_n \circ \rho_{m,n}$ does not factor through π_m. However, $(R_n, \rho_{m,n})$ forms a projective system. We let R denote the projective limit in the category of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ graded algebras, and let ρ_n denote the natural map $R \to R_n$.

We denote the n-tuples $(t_1, \ldots, t_n), (u_1, \ldots, u_n), (t_1^{-1}, \ldots, t_n^{-1})$ etc., more briefly by t, u, t^{-1}, etc.

Lemma 11.3.1. *For each n, we have $R_n = \mathbb{Z}[e_1^n(t), \ldots, e_n^n(t), e_1^n(u), \ldots, e_n^n(u)]$.*

From this it is clear that the map π_n is surjective, since $R(GL_n)$ is generated by $e_i^n(t), i = 1, \ldots, n$ and $\det^{-1} = \pi_n(e_n^n(u))$.

Proof: This is clear: $\mathbb{Z}[t_1, \ldots, t_n]^{\mathfrak{S}_n} = \mathbb{Z}[e_1^n(t), \ldots, e_n^n(t)]$ (likewise with u's) and

$$R_n = \mathbb{Z}[t_1, \ldots, t_n]^{\mathfrak{S}_n} \otimes_{\mathbb{Z}} \mathbb{Z}[u_1, \ldots, u_n]^{\mathfrak{S}_n}.$$

■

Since $\rho_{m,n}(e_k^m(t)) = e_k^n(t)$ (likewise with u), we obtain elements of R, which we denote by $e_{k,+}$ and $e_{k,-}$, such that $\rho_n(e_{k,+}) = e_k^n(t)$, and $\rho_n(e_{k,-}) = e_k^n(u)$ for each n. The notation $e_{k,-}$ is, of course, motivated by the fact that $\pi_n \circ \rho_n(e_{k,-}) = e_k^n(t^{-1})$.

It follows from Lemma 11.3.1 that $R = \mathbb{Z}[\{e_{k,\pm}\}]$.

Lemma 11.3.2. *For every element χ of R, there is an N such that if $n > N$, then χ is not in the kernel of $\pi_n \circ \rho_n$.*

Proof: Clearly, for each n, the map ρ_n is injective on $\mathbb{Z}[\{e_{k,\pm} : k \leq n\}]$. The kernel of π_n is generated by $\{e_n^n(t)e_i^n(u) - e_{n-i}^n(t) : i = 0, \ldots, n\}$. Given $\chi \in R$, we define K_1 to be the largest k such that $e_{k,+}$ appears in the expression of χ in terms of the $e_{i,\pm}$, and K_2 to be the largest k such that $e_{k,-}$ appears. Then $N = K_1 + K_2 + 1$ is sufficient. ■

Next, we wish to define an element of R associated to any pair of partitions λ and μ. This is given in terms of the transpose partitions λ' and μ', and it will be convenient to denote λ_1, which is equal to the number of parts in λ', more briefly by r. Similarly, we let $\mu_1 = s$.

Then, let

$$\chi_{GL}(\bar{\mu};\lambda) = \begin{vmatrix} e_{\mu'_s,-} & \cdots & e_{\mu'_s-r-s+1,-} \\ \vdots & \ddots & \vdots \\ e_{\mu'_1+s-1,-} & \cdots & e_{\mu'_1-r,-} \\ e_{\lambda'_1-s,+} & \cdots & e_{\lambda'_1+r-1,+} \\ \vdots & \ddots & \vdots \\ e_{\lambda'_r-r-s+1,+} & \cdots & e_{\lambda'_r,+} \end{vmatrix}.$$

Let n be at least $\lambda'_1 + \mu'_1$. Then, making use of the identity $\det^{-1}(t)e^n_k(t) = e_{n-k}(t^{-1})$, it is easy to check that

$$\pi_n \circ \rho_n \left(\chi_{GL}(\bar{\mu};\lambda)\right) = \det{}^{-s}\chi_{GL_n}(\tau(\lambda,\mu)),$$

where τ is the partition such that $\tau(\lambda,\mu)' = (n-\mu'_s,\dots,n-\mu'_1,\lambda'_1,\dots,\lambda'_r)$. (This $\tau(\lambda,\mu)$ may be visualized as follows: start with the Young diagram of λ. Add μ_1 columns of length n at the left side. Then take the Young diagram of μ, rotate it 180 degrees, and *subtract* it from the $n \times \mu_1$ rectangle.) So, $\pi_n \circ \rho_n$ maps $\{\chi_{GL}(\bar{\mu};\lambda) | \lambda'_1 + \mu'_1 \le n\}$ bijectively onto the set of characters of irreducible representations of GL_n.

Observe that Λ is naturally identified with a subring of R, Λ_n with a subring of R_n, and the restriction of $\rho_{m,n}$ is $\tilde{\rho}_{m,n}$ for each m,n. Also, when μ is the empty partition, $\chi_{GL}(\bar{\mu};\lambda)$ is precisely the universal character $\chi_{GL}(\lambda) \in \Lambda$ defined above.

We may now offer a remark about why we do not simply generate $R(GL_n)$ by $\{e^n_i : i = 1,\dots,n\}$ and $\det^{-1}$. Let λ be a partition. The expression for $\chi_{SO_{2n}}(\lambda)$ as a sum of characters of GL_n is stable (in the sense that it is independent of n sufficiently large) when these characters are expressed as $\pi_n \circ \rho_n \chi_{GL}(\bar{\mu};\nu)$. On the other hand, the expression in terms of $\det{}^{-s}\chi_{GL_n}(\tau(\nu,\mu))$ does not stabilize as $n \to \infty$. Indeed, the partition $\tau(\nu,\mu)$ is different for each value of n.

11.4. **A review of our application.** With the background established, let us go back over the application to the local computations when $E = F \times F \times F$. We must compute the product of two characters. One of them is the character of a representation whose highest weight not only is a partition, but is one with strictly less than n parts. Let us call this partition λ.

We first need a formula for the restriction of $\chi_{SO_{2n}}(\lambda)$ from SO_{2n} to GL_n. Once again the "/" notation of (11.1.2) allows for a nice formulation. The relevant formula, which appears in [B-K-W] as (A4) on p. 1586 is, in our notation:

$$\chi_{SO_{2n}}(\lambda) = \sum_{\xi,\beta} \pi_n \circ \rho_n \left(\chi_{GL_n}(\bar{\xi};\lambda/(\xi\cdot\beta))\right), \tag{11.4.1}$$

where λ has at most n parts, ξ is summed over all partitions, and β is summed only over partitions such that each part appears with even multiplicity. This is a finite sum, since there are only finitely many pairs (ξ,β) such that $\lambda/(\xi\cdot\beta)$ is nonzero. But note that this holds only for those λ which (a) are partitions, as not all weights of SO_{2n}, much less those of $Spin_{2n}$ are, and (b) as partitions, have strictly fewer than n parts,

Next, we take the product of (11.4.1) with the character of $\widetilde{GL}_n$ which has the same highest weight as our other representation of $Spin_{2n}$. Let us assume that this weight is $\mu+(\varepsilon,\dots,\varepsilon,\varepsilon)$, where μ is a partition, and $\varepsilon = 0$ or $\frac{1}{2}$. Then this character is $\chi_{GL_n}(\mu)\det^{\varepsilon}$. So, we obtain

$$\chi_{SO_{2n}}(\lambda)\chi_{\widetilde{GL}_n}(\mu+\varepsilon^n) = \det{}^{\varepsilon}\pi_n \circ \rho_n \left(\sum_{\xi,\beta} \chi_{GL}(\bar{\xi};\lambda/(\xi\cdot\beta))\chi_{GL}(\mu) \right). \tag{11.4.2}$$

We now need to evaluate these products of characters of GL_n, which we wish to do in R and then project. The product in R is described by formula (5.2) p. 1571 of [B-K-W]. In our notation:

$$\chi_{GL}(\bar{\mu};\lambda)\chi_{GL}(\bar{\kappa};\nu) = \sum_{\sigma,\tau} \chi_{GL}(\overline{(\mu/\sigma)\cdot(\kappa/\tau)};(\lambda/\tau)\cdot(\nu/\sigma)).$$

Plugging this into (11.4.2) we get

$$\chi_{SO_{2n}}(\lambda)\chi_{\widetilde{GL}_n}(\mu+\varepsilon^n) = \det{}^{\varepsilon} \sum_{\eta,\zeta,\beta} \pi_n \circ \rho_n(\chi_{GL}(\bar{\eta};(\lambda/(\zeta\eta\beta))\cdot(\mu/\zeta))),$$

where we have extended χ_{GL} to pairs of finite formal sums of partitions by $\mathbb{Z}$-linearity. Let us now define, for each n a map $T_{\widetilde{GL}_n\to Spin_{2n}}$ from $R(\widetilde{GL}_n)$ to $R(Spin_{2n})$ which is $\mathbb{Z}$-linear and sends $\chi_{\widetilde{GL}_n}(\lambda)$ to $\chi_{Spin_{2n}}(\lambda)$ for each weight λ. Then we get

$$\chi_{SO_{2n}}(\lambda)\chi_{Spin_{2n}}(\mu+\varepsilon^n) = \sum_{\xi,\beta} T_{\widetilde{GL}_n\to Spin_{2n}} \left(\det{}^{\varepsilon}\chi_{GL_n}(\bar{\xi};\lambda/(\xi\cdot\beta))\chi_{GL_n}(\mu)\right) \tag{11.4.3}$$

$$= \sum_{\eta,\zeta,\beta} T_{\widetilde{GL}_n\to Spin_{2n}} \left(\det{}^{\varepsilon}\pi_n \circ \rho_n(\chi_{GL}(\bar{\eta};(\lambda/(\zeta\eta\beta))\cdot(\mu/\zeta)))\right), \tag{11.4.4}$$

All that is needed now to compute $\chi_{SO_{2n}}(\lambda)\chi_{GL_n}(\mu)$ is an efficient algorithm for reducing χ_{GL} modulo the kernel of $\pi_n \circ \rho_n$. This is given in the next subsection. The answer is always either 0 or $\pm\chi_{GL_n}(\bar{\tau}^\circ;\nu^\circ)$ for a single pair of partitions τ°,ν° with at most n total parts.

11.5. **Reduction Modulo $Ker(\pi_n \circ \rho_n)$.** In this section we show how, for any pair of partitions $(\bar{\mu};\lambda)$ with any numbers of parts, to reduce $\chi_{GL}(\bar{\mu};\lambda)$ modulo the kernel of $\pi_n \circ \rho_n$, obtaining either zero, or a pair $(\bar{\mu}^\circ;\lambda^\circ)$ with at most n total parts, such that $\pi_n \circ \rho_n(\chi_{GL}(\bar{\mu};\lambda)) = \pm\chi_{GL_n}(\bar{\mu}^\circ;\lambda^\circ)$, with an explicit description of the sign. This corresponds to the U_n, SU_n modification rule in Table 3 of [B-K-W], although it may be necessary to apply that rule more than once to obtain the answer we describe below. By contrast, the modification rules in Tables 4 and 5 correspond to finding the dominant weight $|\nu|_G$ associated to ν by the action of the Weyl group, shifted by ρ_G.

For any infinite sequence $a = a_1, a_2, \ldots, a_m, \ldots$ of integers with the property that $a_m = 1-m$ for $m > N$, we define $E(a)$ to be the $N\times N$ determinant whose (i,j) entry is e_{a_i+j-1}, and define $E^n(a)$ analogously with e^n's. Observe that this definition is independent of the choice of specific N having the requisite property. If ℓ is an infinite sequence of integers such that $\ell_m = 0$ for all m sufficiently large, we follow [Ko] in defining the associated β-sequence $\beta(\ell)$ by $\beta(\ell)_m = \ell_m + 1 - m$. Observe that if λ is a partition, then $\chi_{GL}(\lambda) = E(\beta(\lambda'))$, where we have identified the partition λ' with the infinite sequence obtained by appending zeros at the end.

We now consider $\pi_n \circ \rho_n(\chi_{GL_n}(\bar{\mu};\lambda))$. Making use of the identity $e_n^i(t^{-1}) = e_n^n(t)^{-1}e_{n-i}^n(t)$, we find that

$$\pi_n \circ \rho_n(\chi_{GL_n}(\bar{\mu};\lambda)) = (e_n^n)^{-s}E^n(\beta(a(\lambda,\mu))),$$

where

$$a(\lambda, \mu) = (n - \mu'_s, \dots, n - \mu'_1, \lambda'_1, \dots, \lambda'_r).$$

Here, we have once again denoted λ_1 and μ_1 by r and s, as is convenient to do when they show up as the number of parts of the transpose partitions λ' and μ'. When $n - \mu'_1 \geq \lambda'_1$ this expresses $\pi_n \circ \rho_n(\chi_{GL_n}(\bar{\mu}; \lambda)$ as the character of a polynomial representation of GL_n times a power of $\det^{-1}$.

Now suppose that $a(\lambda, \mu)$ is not a nonincreasing sequence (i.e., that $\mu'_1 + \lambda'_1 > n$.) Denote $\beta(a(\lambda, \mu))$ more briefly by β. If there exists $i \neq j$ such that $\beta_i = \beta_j$, then $E(\beta) = 0$. Otherwise, we may rearrange the terms to obtain a sequence $\hat{\beta}$ which is strictly decreasing such that $E(\hat{\beta}) = \pm E(\beta)$, with the sign depending on the number of order reversals. We observe that if $\hat{\beta}_1 > n$, then $E(\hat{\beta}) = 0$. Furthermore, if $\hat{\beta}_1 = n$, then the initial e_n^n is the only nonzero entry in the first row of the determinant defining $E(\hat{\beta})$ and so we may express it as e_n^n times the lower right minor. In this way, we see that

$$E(\hat{\beta}) = (e_n^n)^k E(\beta^\circ)$$

for some $k \in \mathbb{Z}_{\geq 0}$ and β° a strictly decreasing sequence of integers, such that $\beta^\circ_m = 1 - m$ for m sufficiently large and $\beta^\circ_1 < n$. We may then recover two partitions λ° and μ° such that

$$(\lambda^\circ)'_1 + (\mu^\circ)'_1 \leq n, \quad \mu^\circ_1 = s - k$$

and

$$\beta(a(\lambda^\circ, \mu^\circ)) = \beta^\circ.$$

Thus we have

$$\chi_{GL_n}(\bar{\mu}; \lambda) = \pm \chi_{GL_n}(\bar{\mu}^\circ; \lambda^\circ)$$

with the sign being the one obtained from the order reversals to get $\hat{\beta}$ from β.

11.6. **Dictionary.** Now, for the reader familiar with [B-K-W] or interested in consulting it now, we offer a partial dictionary to translate from our notation to theirs.

The translation is not precise since the notion of a projective limit does not appear in [B-K-W], only various identities which "stabilize" once n is sufficiently large and hence are ripe for interpretation in terms of this notion. In the text of this appendix, we have used λ and μ both for partitions (which might not be weights) and for weights (which might not be partitions); throughout this table, λ and μ are *partitions,* while other Greek letters are *weights.* Some of the notations depend on the number of parts of the partitions. These will be denoted by p and q respectively. We subdivide into three parts: notations common in the majority of [B-K-W], notations appearing mainly in section 6, and then a few logical equivalencies. We remark that $[\lambda]_\pm$ and $[\Delta; \lambda]_\pm$ are also defined for $p > n$. For a "universal character" interpretation, see [Ko]. Note that $[\bar{\emptyset}; \lambda]_\pm$ and $[\Delta; \bar{\emptyset}; \lambda]_\pm$ are *not* equal to $[\lambda]_\pm$ and $[\Delta; \lambda]_\pm$ for $p > n$.

Our notation	Their notation
det	ε
$LR^{\nu}_{\lambda,\mu}$	$m^{\nu}_{\lambda\mu}$
$\pi_n \circ \rho_n(\chi_{GL}(\bar{\mu};\lambda))$	$\{\bar{\mu};\lambda\}$
$\tilde{\rho}_n(\chi_{GL}(\lambda)) = \pi_n \circ \rho_n(\chi_{GL}(\bar{\emptyset};\lambda))$	$\{\lambda\}$
$\pi_n \circ \rho_n(\chi_{GL}(\bar{\mu};\emptyset))$	$\{\bar{\mu}\}$
$\pi_{Sp_{2n}}(\chi_{Sp}(\lambda))$	$\langle\lambda\rangle$
ι	$\dagger$ or $\cdots]_+ \to \cdots]_-$
$\chi_{SO_{2n}}(\lambda)$	$[\lambda]_+$ if $p \leq n$
$\chi_{SO_{2n}}(\lambda) + \chi_{SO_{2n}}(\lambda^{\iota})$	$[\lambda]$ if $p = n$
$\chi_{SO_{2n}}(\lambda) = \chi_{SO_{2n}}(\lambda^{\iota})$	$[\lambda] = [\lambda]_+ = [\lambda]_-$ if $p < n$
$\chi_{Spin_{2n}}(\lambda + (\frac{1}{2})^n)$	$[\Delta;\lambda]_+$ if $p \leq n$
$\chi_{Spin_{2n}}(\lambda + (\frac{1}{2})^n) + \chi_{SO_{2n}}((\lambda + (\frac{1}{2})^n)^{\iota})$	$[\Delta;\lambda]$ if $p \leq n$
$\chi_{Spin_{2n}}(\lambda + 1^n)$	$[\Box;\lambda]_+$ if $p \leq n$
$\chi_{Spin_{2n}}(\lambda + 1^n) + \chi_{Spin_{2n}}((\lambda + 1^n)^{\iota})$	$[\Box;\lambda]$ if $p \leq n$
$T_{\widetilde{GL_n} \to Spin_{2n}} \circ \pi_n \circ \rho_n(\chi_{GL}(\bar{\mu};\lambda))$	$[\bar{\mu};\lambda]_+$
$T_{\widetilde{GL_n} \to Spin_{2n}}(\det^{\frac{1}{2}} \pi_n \circ \rho_n(\chi_{GL}(\bar{\mu};\lambda)))$	$[\Delta;\bar{\mu};\lambda]_+$
$\chi_G(\nu)$	$(\nu)_G$ (alternate notation of section 6.)
ρ_G	δ_G
$c(H)^{\tau}_{\sigma,\nu+\rho_G-\rho_H}$	$K^{\tau_H}_{\sigma_H,\nu_G+\delta_G-\delta_H}$
n^{ν}_{η}	$B^{\eta_H}_{\nu_G}$
$m^G_{\nu}(\sigma)$	$M^{\sigma_T}_{\nu_G}$
Equation (11.4.3)	Equations (7.1a) & (7.1b)
Equation (11.4.4)	Equations (7.5a) & (7.5b)
$(\bar{\mu};\lambda) \mapsto \pm(\bar{\mu}^{\circ};\lambda^{\circ})$	U_n, SU_n Modification rule of Table 3 (possibly applied more than once)
$\chi_G(\nu) \mapsto \frac{\nu}{\lvert nu\rvert_G}\chi_G(\lvert\nu\rvert_G)$	Modification rule for G in Tables 4 &5 (possibly applied more than once)

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