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2006

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Preprint of a paper published in *Topology Proceedings*, 30(1), 403-416.

Submitted to
Topology Proceedings

FACTORING POSITIVE BRAIDS VIA BRANCHED MANIFOLDS

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Abstract. We show that a positive braid is composite if and only if the factorization is "visually obvious" by placing the braid k in a specially constructed smooth branched 2-manifold $B(k)$ and studying how a would-be cutting sphere meets $B(k)$. This gives an elementary proof of a theorem due to Peter Cromwell.

Peter Cromwell has shown that positive braids can only be factored in the visually obvious ways [3]. This proved a conjecture by Bob Williams that positive braids with a full twist are prime [8]. The present author independently proved Williams' conjecture in [7] using techniques different from Cromwell's. Here we prove Cromwell's stronger result using the techniques developed in [7] and [8]. The essential difference is that Cromwell studies the ways cutting spheres intersect a spanning surface of a knot while we will place knots in a smooth branched 2-manifold, an annulus with "tabs," and study the intersection with cutting spheres.

Makoto Ozawa [6] has also proven Cromwell's theorem, and extended to positive knots. His work uses the machinery of incompressible surfaces. The present proof, although tedious, could in principle be followed be an undergraduate.

In [5] Bill Menasco proved that alternating knots only factor in visually obvious ways. Cromwell modeled his theorem, though not

Date: October 13, 2005.
2000 *Mathematics Subject Classification.* 57M25.
Key words and phrases. Branched manifolds, factoring knots, knot theory, positive braids.

his proof, on Menasco's. Interestingly, Menasco's proof is based on placing the knot in a branched 2-manifold, a sphere with "bubbles", though he does not exploit smoothness. This leads us to wonder if there is a unified approach to the two results or even a general algorithm for knot factoring based on placing a knot in a suitable branched manifold.

Knot and braid theory is described very briefly in the Appendix.

Definition 0.1. A knot is a *positive braid* if it can be represented as a braid, all of whose crossing are of the same type.

Definition 0.2. A braid diagram is *irreducible* if it has no cut points.

Definition 0.3. A braid diagram D is *decomposable* if there is a simple closed curve C meeting D just twice, transversely, such that neither $D \cap D_1$ nor $D \cap D_2$ is an arc, where D_1 and D_2 are the two disks bounding C. If there is no such closed curve the braid diagram is *indecomposable.*

Theorem 0.4 (Cromwell). *Let D be a positive irreducible braid diagram for a knot k. Then k is prime if and only if D is indecomposable.*

Proof. If an irreducible positive braid diagram $D = D(k)$ is decomposable then it is obvious how to construct a nontrivial factorization of k.

Let k be a knot with a braid diagram $D(k)$ that is indecomposable, positive, and irreducible. We associate to k a branched surface, $B = B(D(k))$. The surface is an annulus A, with half disks or *tabs* attached along their diameters. There is one tab for each crossing of $D(k)$. The attachings are smooth and all tabs come into the A from the same direction. The smoothness is needed so that B will have a well defined normal bundle.

We can find in B a knot of knot type k using one tab for each crossing. Figure 1 gives an example. The embedding of k into B is piecewise smooth with one cusp point on each branch line.

Suppose that k is a composite knot. Then there is a 2-sphere S that factors k. We can take S to be transverse to B. Further, we can require them to be perpendicular. Let $I = B \cap S$. Let β be the union of the branch lines of B. Divide I into segments whose end points are in $\partial B \cup \beta$.

Assume that the number of segments in I is minimal over all the transverse cutting spheres for k. We will give an algorithm that traces out a path in I with infinitely many segments, contradicting transversality, and thus proving the theorem.

Minimality tells us that in I there are no trivial loops that miss β or arcs that connect a segment of ∂B to itself. (We regard the trivial loops as segments for counting proposes.) Also, segments of I that connect one segment of ∂A to another without meeting β would be pierced by the knot more than once from the same side.

Here are the remaining types of segments that could be in I:

- Same branch segments:
  - $\cup$-joints connect two points on the same branch line from below.

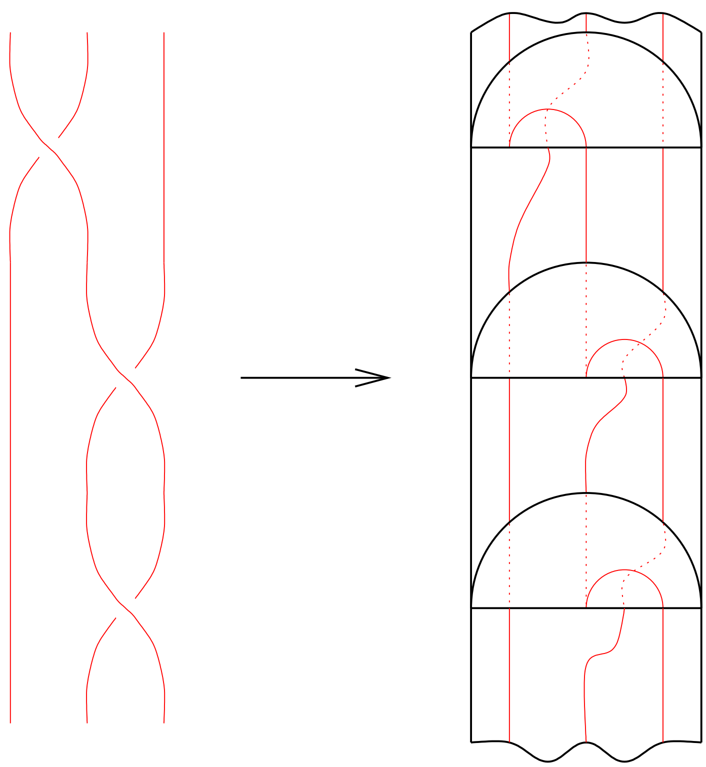

Figure 1. A braid in a branch surface

  - $\cap$-joints connect two points on the same branch line from above.
- Branch to edge segments:
  - $\lfloor$-joints connect a branch point to a point on ∂A below and to the right.
  - $\rfloor$-joints connect a branch point to a point on ∂A below and to the left.
  - $\lceil$-joints connect a branch point to a point on ∂A above and to the right.
  - $\rceil$-joints connect a branch point to a point on ∂A above and to the left.
  - $|$-joints connect a branch point to a point on $\partial B - A$.
- *bb*-segments, or *branch-to-branch* segments connect one branch line to another.

Lemma 0.5. *If $p \in I \cap \beta$ then either p is the lower end point of a branch-to-branch segment or it is the right end point of a $\cap$-joint.*

Proof. Consider a $\cap$-joint. Label its left and right end points p and q respectively. We may take it to be innermost. This implies that I does not meet the arc in β bounded by p and q. Call this arc $\overline{pq}$.

Since k does not meet $\overline{pq}$ then we can deform S so as to push the $\cap$-joint through the branch line and reduce the number of segments without gaining or losing intersection points between the knot and the sphere. That is we have a new transverse cutting sphere with fewer segments in I; see Figure 2.

Moreover, if the only points in $k \cap \overline{pq}$ come from arcs of the knot that hit the $\cap$-joint, then we can do the same move as before; the single intersection point is dragged along with the deformation of S. The same is true if the only point in $k \cap \overline{pq}$ is a cusp point; see Figure 3. We summarize this discussion by saying: *A $\cap$-joint must be guarded by an arc of k from the opposite branch.* Similarly, $\lceil$-joints must be guarded by an arc of k on the corresponding tab.

We will now show that there are no $\rceil$-joints in I. Suppose that there is one and let its end point on β be p and the left end point of this component of β be e. Call the open arc from e to p, $\overline{ep}$. Assume the $\rceil$-joint to be left-most. If I meets $\overline{ep}$ there would have to be a $\cap$-joint on the annulus A that cannot be guarded. This is because the only way to guard the $\cap$-joint would force k to cross the $\rceil$-joint twice from the same side. But then k would pierce S twice from the same side; see Figure 4.

Since k can meet our $\rceil$-joint at most once, there are only three possibilities to consider: k misses the joint and hence misses $\overline{ep}$, k meets $\overline{ep}$ at a cusp point, or k meets $\overline{ep}$ transversely. In each case, we can deform S so as to reduce the number of segments in I; see Figure 5. (We might say that it is impossible to guard a $\rceil$-joint.)

Let $p \in I \cap \beta$, and assume the lemma is false. Then one of the three cases below must occur, yet each leads to a contradiction.

CASE 1: Suppose p is the common left end point of two $\cap$-joints. We assume the structure is innermost. Then I must miss the interior of the arc of the branch line connecting the end points of the shorter $\cap$-joint.

If the right end points are equal, it is impossible to guard both $\cap$-joints without puncturing the sphere twice from the same side.

Suppose the that the front $\cap$-joint's right end point falls short of the back $\cap$-joint's right end point. In order to guard the front $\cap$-joint a single arc of k must pass under it and hence pierce the other $\cap$-joint. We see in Figure 6 a that we can still deform S so as to reduce the number of segments even though the $\cap$-joint was guarded. The joint wasn't *guarded well enough.*

If the back $\cap$-joint is the shorter than we must have the over-crossing arc of k to guard it, but again we just can't guard it well enough. This is shown in Figure 6b.

CASE 2: Suppose a $\lceil$-joint meets the left end point of a $\cap$-joint at p. Then the only way to guard the $\cap$-joint forces k to pierce the $\lceil$-joint. Figure 7 shows how to deform S.

CASE 3: Suppose a $|$-joint and a $\lceil$-joint meet at p. Assume this is the right-most such case. If I meets the open interval in β from p to the right end point, then one of the combinations of cases 1 or 2 must occur. Thus, I misses this interval.

The only way to guard the $\lceil$-joint is by the arc of k that comes up from the cusp point onto the tab. However, the cusp point is to the right of p, since otherwise the $\lceil$-joint is pierced twice by k from the same side. Just draw the picture. But now k pierces the $|$-joint. The reader can check that as before, the $\lceil$-joint isn't guarded well enough; see Figure 8. $\square$

The algorithm. Pick a point p in $I \cap \beta$. Such a point exists since segments that miss β where ruled out. Choose one of the two segments meeting p from above according to the following rules: If possible choose a $\cap$-joint to the left and repeat the algorithm from its other end point. If not, then at least one of the two segments is a bb-segment. Choose it and repeat from the point where it meets the branch line above.

Lemma 0.5 guarantees that the algorithm is valid and gives a path P that misses ∂B. It only remains to show that P has no loops, for this will imply that P has infinitely many distinct segments.

Lemma 0.6. *There are no loops in P.*

Proof. Suppose L is a closed loop in P. Then L is made up of bb-segments and $\cap$-joints. At times it will be useful abuse terminology and regard L as a braid with the same braid axis as k. Let n be the number of strands in L when it is so regarded.

Suppose that there are no $\cap$-joints in L. Then $n = 1$, since L cannot cross itself. We consider three cases. If L never passes under a crossing of k, then k is a separable link. If L passes under exactly one crossing then $D(k)$ has a cut point. Suppose L passes under two or more crossings of k. Then for each time L passes under k, there must be a point where L crosses k from right to left. But if such a crossing occurs more than once k will have entered (or exited) the sphere twice. This is impossible. Thus, we conclude that there are $\cap$-joints in our would-be loop L.

Let f be the number of $\cap$-joints of L in the tabs and let b be the number of $\cap$-joints of L in the annulus A. Call these *front* and *back* $\cap$-joints, respectively. Any back $\cap$-joint in P must be followed by a front $\cap$-joint. Thus, $f \geq b$.

Consider a unit normal bundle to L. Think of it as a ribbon R. Since the sphere is normal to B, we can place R in the sphere. Hence R must be untwisted. When we travel up along L and meet a $\cap$-joint we are going from right to left. When we get to the other end point and proceed upward from there, a half twist is created in R. If we go from front to back we get a $+\frac{1}{2}$ twist, while if we go from back to front we get a $-\frac{1}{2}$ twist; see Figure 9. (Figure 9 actually shows tangent bundles. These are easier to draw but have the same twist as the normal bundles.)

Suppose $n > 1$. Then L crosses itself at least $n-1$ times as it is an unknotted braid on n strands. There are three ways L may cross itself. These are illustrated in Figure 10. The first involves a $\cap$-joint and a bb-segment and is positive. The next two involve two $\cap$-joints. The center one is positive while the last one is negative. However, in the cases involving two $\cap$-joints the crossing must be quickly "undone" by another crossing of the opposite sign. Thus, along any branch line the contribution to the singed crossing number of L is nonnegative. Since "essential" crossings must be positive we see that L has $n-1$ essential crossings. Thus, the net twisting in the ribbon R is given by the formula

$$T = n - 1 + (f - b)/2.$$

But this number is positive, a contradiction. We conclude that $n = 1$. It also follows that $f = b$.

The linking number of L and k is not defined in general since they may intersect. However, we can deform L in a small neighborhood of each intersection point, staying within the cutting sphere S, and produce a new loop $\bar{L} \subset S$. If L crosses k from right to left (going down) we will push L below k. If L crosses k from left to right we will push L over k. The linking number of $\bar{L}$ and k is well defined and determined by L. We call this linking number the *adjusted linking number* of L and k, and point out that it must be 0 or ± 1. We claim this forces $f + b = 2$, that is that there is a single front-back pair of $\cap$-joints in L.

Consider a front-back $\cap$-joint pair. We will show that it either contributes positively to the adjusted linking number. But, since intersection can only give nonnegative contribution to the adjusted linking number and L can only cross over k at front-back $\cap$-joint pairs, there can only be one such pair.

Label the three points where a front-back $\cap$-joint pair meet the branch line a, b and c, from left to right as in Figure 12. Label the cusp point of k on this branch line p. If p is to the right of b then we cannot guard the back $\cap$-joint. If p is in between a and b the adjusted linking number is too large; see Figure 12 and notice that L and k must meet in this case. Thus p is to the left of a as in Figure 12. This contributes a plus 1 to the adjusted linking number between L and k. Since there is no way to reduce the adjusted linking number this can happen at most once.

Assume L is a loop in P that has a single pair of $\cap$-joints. Note that L cannot cross k going down from left to right since this would force the adjusted linking number above 1. Also, L cannot cross k from right to left more than once, since to do so would force L to cross k from left to right. Likewise L can cross under k at most once. Further, if L does cross under k it cannot intersect k and vice verse. An obvious projection into the plane shows that the diagram is decomposable. □

Appendix A. Knots and Braids

A *knot* k is an embedding of the oriented circle S^1 into R^3; $k : S^1 \to R^3$. Although technically a knot is a function, it is common practice to confound k with its its image $k(S^1)$. A *knot diagram* is a projection D of $k(S^1)$ into some plane such that $D(k(S^1))$ is in *general position*, that is, it has no triple points and any double points are transverse. The double points are called *crossings* and come in two types, *positive* or *negative*.

Knots are classified up to ambient isotopy of R^3. It was shown by Reidemeister that ambient isotopies are generated by certain moves on knot diagrams. These are now called *Reidemeister moves*.

Let D be a knot diagram in a plane P. If there is a point $a \in P \backslash D$ and a positive integer n such that every ray starting at a meets D n times, counting double points twice, then D is a *braid on n strands*. A key fact, proved by Alexander, is that any knot diagram can be deformed via Reidemeister moves into a braid.

There are many textbooks on the theory of knots and braids: [1, 2, 4].

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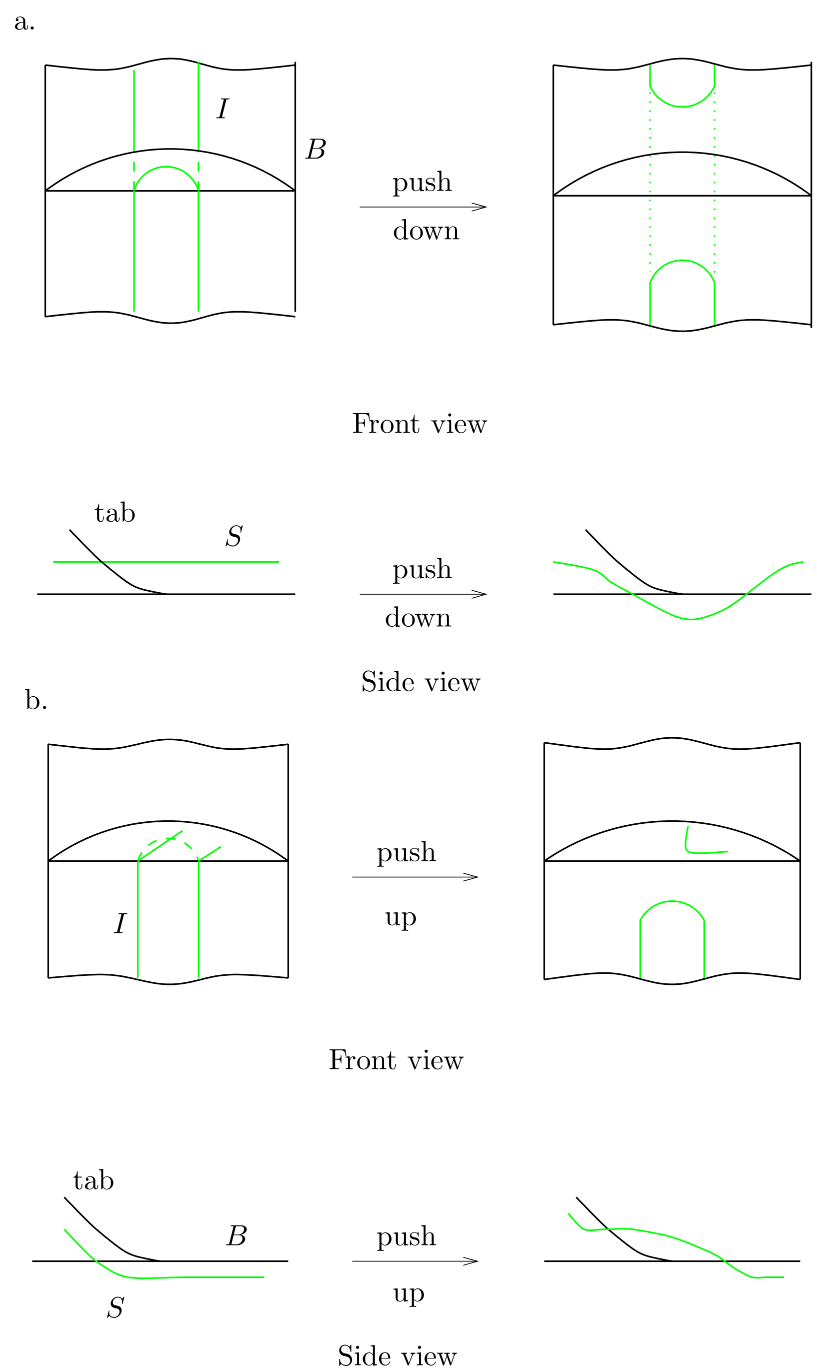

FIGURE 2. a) Removing an unguarded front ∩-joint. b) Removing an unguarded back ∩-joint. c) Guarded ∩-joints. (Next page.)

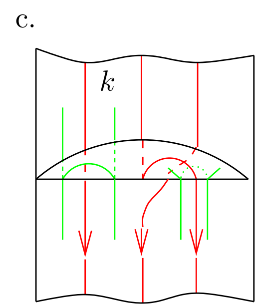

FIGURE 2. Continued from previous page

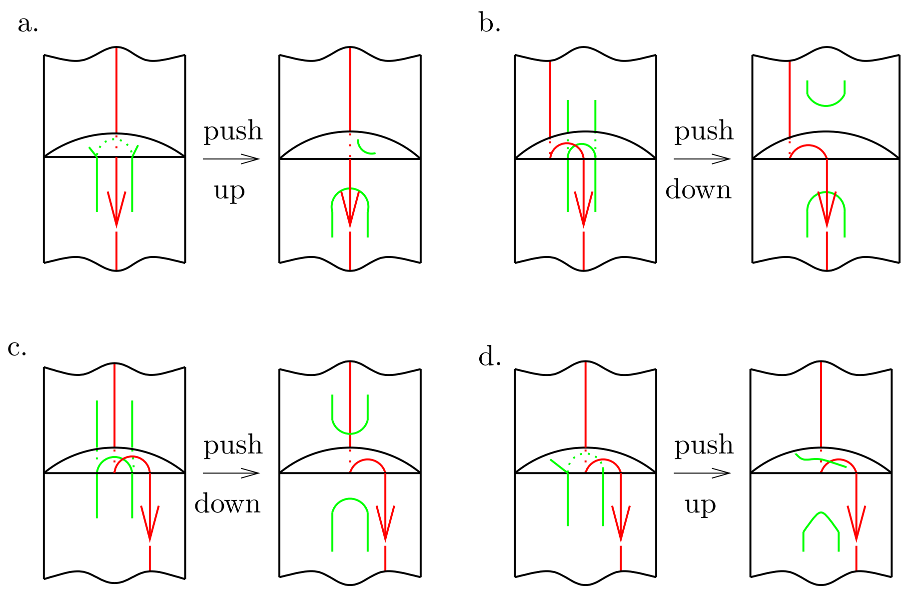

FIGURE 3. Removing unguarded ∩-joints that meet the knot k

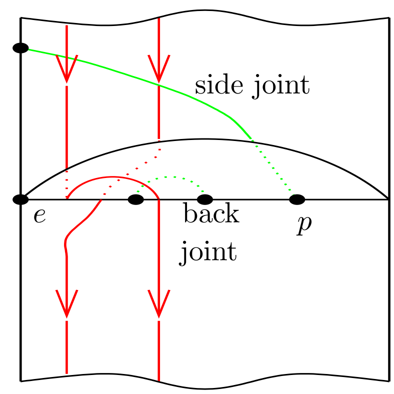

FIGURE 4. Double piercing of S from same side

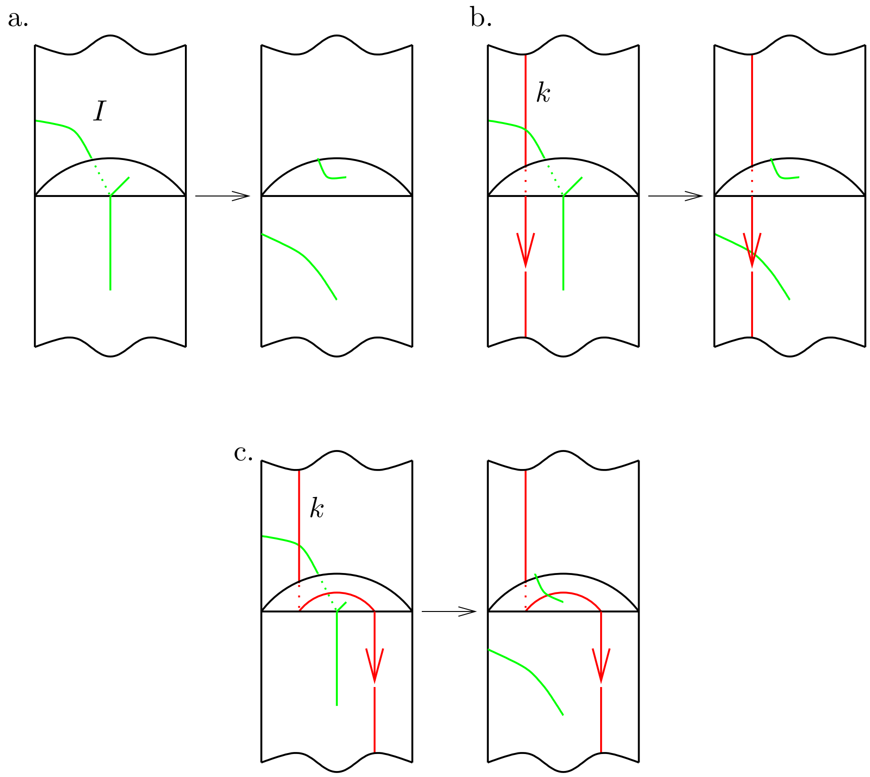

FIGURE 5. Removing edge-joints

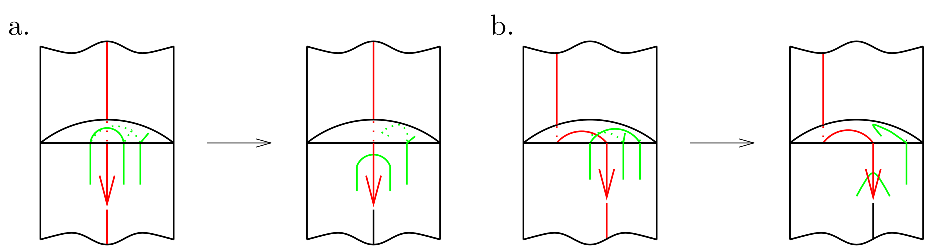

FIGURE 6. Removing ∩-joints that double back

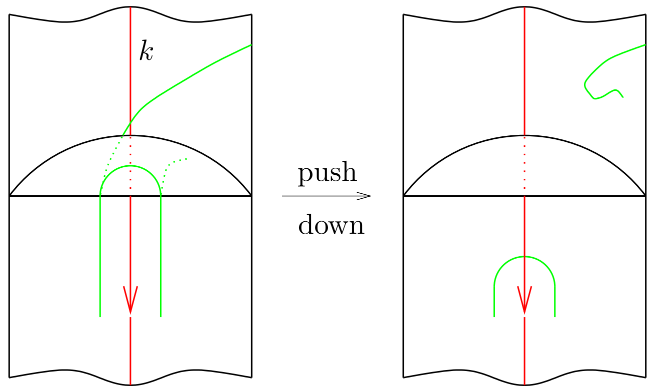

FIGURE 7. Case 2

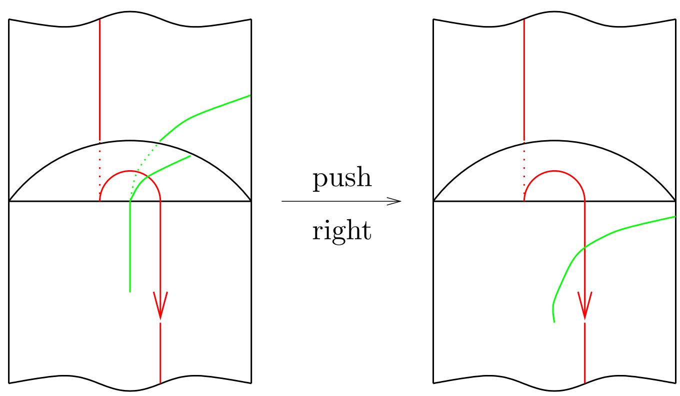

FIGURE 8. Case 3

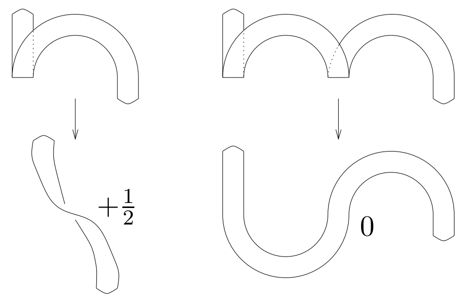

FIGURE 9. Tangent bundles

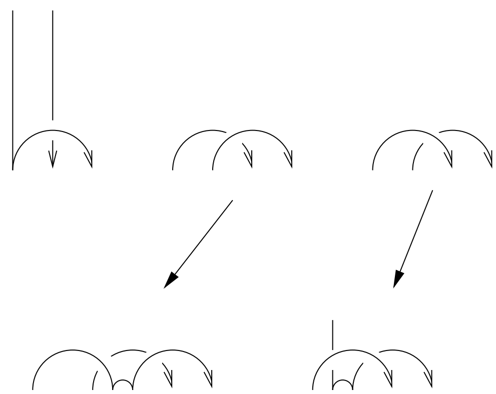

FIGURE 10. Three ways L can cross itself

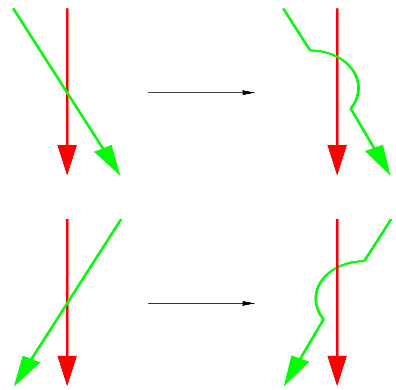

FIGURE 11. Computing adjusted linking numbers

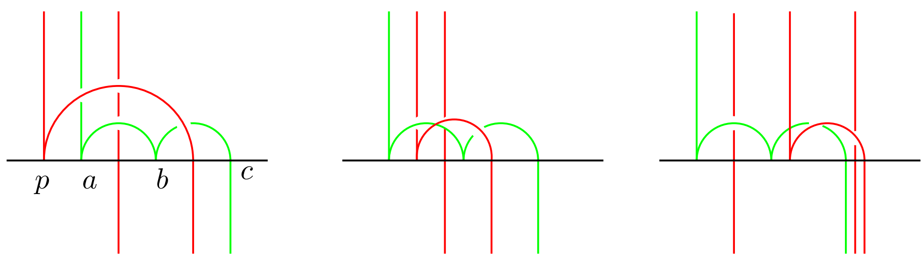

FIGURE 12. ∩-joint pairs