Southern Illinois University Carbondale

OpenSIUC

Articles

Department of Electrical and Computer Engineering

1-1992

Asymptotic Performance of a Distributed Detection System in Correlated Gaussian Noise

V. Aalo
Florida Atlantic University

R. Viswanathan
Southern Illinois University Carbondale, viswa@engr.siu.edu

Recommended Citation

Correspondence

Asymptotic Performance of a Distributed Detection System in Correlated Gaussian Noise

V. Aalo and R. Viswanathan

Abstract—In this correspondence we consider the detection of a constant signal in noise with a large set of geographically dispersed sensors. The noise at the sensors are correlated Gaussian. Two correlation models are considered: one where the correlation coefficient between any two sensors decreases geometrically as the sensor separation increases, and the other where the correlation coefficient between any two sensors is a constant. For both correlation models, the asymptotic (as the number of sensors becomes large) performances of a distributed detection system and a central system are examined.

I. INTRODUCTION

Consider a distributed detection problem in which a large number N of geographically dispersed identical detectors make decisions $\{u_i, i = 1, 2, \cdots, N\}$ for the underlying binary hypothesis testing problem based on their local observations $\{X_i\}$. Each local detector transmits its decision to the fusion center where a final decision u_0 is obtained. The distributed detection problem has been studied extensively for the case where the local observations are conditionally independent given either hypothesis (see [1] for a review). However, the asumption of conditional independence may not be valid in some cases of practical interest [2].

Tsitsiklis [3] shows that for the binary hypothesis case, under mild regularity conditions, it is asymptotically optimal to operate all the local sensors with identical tests if the conditional independence assumption is valid. In [4] it is shown that if the fusion center performs a counting (k out of N) rule, the probability of miss for finite k (or finite $N - k$)) does not go to zero asymptotically unless the probability distributions under the hypotheses satisfy certain conditions.

Here we consider two correlation models for the observations in the distributed detection of a known constant signal in correlated Gaussian noise. In the first model, the correlation coefficient between the observation at a given sensor and that at any other sensor decreases geometrically as the separation between the two sensors increases. With large, but finite N, this model could approximate some real situations. In the second model, any pair of sensors receive equicorrelated observations. In both cases we investigate the asymptotic performances of the distributed detection system employing a counting rule and that of the central system which derives

its decision based on the set of observations $\{X_i\}$. In Section II the detection problem is stated and in Section III the asymptotic performances of the central and the distributed systems are discussed.

II. PROBLEM STATEMENT

Consider the problem of detecting a constant signal in additive Gaussian noise, as described by the following hypotheses testing:

$$H_0: X_i = n_i$$

$$H_i: X_i = n_i + m \tag{1}$$

$i = 1, 2, \cdots, N$ and $\{n_i\}$ are dependent zero-mean Gaussian noise with unit variance and m (>0) is a known constant. Each local sensor performs an identical test

$$X_i \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{2}$$

The binary decisions are therefore

$$u_i = \begin{cases} 1 & \text{if the } i\text{th sensor decides } H_1 \\ 0 & \text{if the } i\text{th sensor decides } H_0. \end{cases}$$

In (2) every sensor uses the same threshold t. Optimizing the thresholds with correlated observations in general is a difficult problem [1], [2]. In a centralized detection scheme, the sensors send all their observations to the fusion center where an optimum test can be performed. The optimum (likelihood ratio) test in such a case is given by [5]:

$$l(X) = M^T \Lambda^{-1} X \underset{H_0}{\overset{H_1}{\gtrless}} \lambda^* \tag{3}$$

where $X = \{X_1, X_2, \cdots, X_N\}^T$, $M = m(1, 1, \cdots, 1)^T$, Λ is the covariance matrix and λ^* is the threshold at the fusion center determined by the required false alarm probability. In the distributed scheme, a counting rule is considered at the fusion center. That is,

$$l(u) = \sum_{i=1}^{N} u_i \underset{H_0}{\overset{H_1}{\gtrless}} \beta \tag{4}$$

where $u = (u_1, u_2, \cdots, u_N)^T$ and β is the fusion center threshold.

III. CORRELATION MODELS AND PERFORMANCES OF CENTRAL AND DISTRIBUTED SYSTEMS

Denote the correlation coefficient between X_i and X_j as ρ_{ij}, $i, j = 1, 2, \cdots, N$.

a) Let $\rho_{ij} = \rho^{|i-j|}$ where $0 \leq \rho < 1$. $\tag{5}$

In this case the (optimum) centralized test in (3) becomes

$$l = \frac{1}{1 + \rho} (X_1 + X_N) + \frac{1 - \rho}{1 + \rho} \sum_{i=2}^{N-1} X_i \underset{H_0}{\overset{H_1}{\gtrless}} \lambda. \tag{6}$$

For every N, l is distributed as Gaussian. Upon computing the mean and the variance of l, we obtain the following as $N \to \infty$:

$$\lambda = \sqrt{N \frac{1-\rho}{1+\rho}} \, Q^{-1}(P_F)$$

$$P_M = 1 - P_D = Q\left(m \sqrt{\frac{N(1-\rho)}{(1+\rho)}}\right) \tag{7}$$

where P_F, P_D, P_M denote the probabilities of detection, false alarm, and miss, respectively, and $Q(y) = 1 - F(y)$, $F(y)$ is the standard normal CDF. For a given P_F, the probability of a miss for the test (6) goes to zero exponentially with N at a rate $(m^2/2)(1 - \rho/1 + \rho)$.

Next, consider the correlation model in (5) when the local sensors send only their decisions to the fusion center. A stationary Gaussian sequence $\{X_i\}$ is ergodic iff its spectral distribution function is continuous [6]. For the assumed correlation model, it can be shown that the spectral distribution is continuous. It follows that $\{u_i\}$ is also ergodic and stationary. Therefore, $1/N \sum_{i=1}^{N} u_i$ tends to $Q(t)$ as $N \to \infty$ under H_0 and to $Q(t-m)$ under H_1. A test based on $\sum_{i=1}^{N} u_i$ therefore achieves zero probability of error, asymptotically. Alternatively, we establish a similar result, using a central limit theorem. For a given $P_F > 0$, it is shown that $1 - P_D \to 0$ and $N \to \infty$. In the process of arriving at this result, we derive an inequality relating the correlation coefficients between X_i and X_j, and u_i and u_j.

We first obtain a bound on the bivariate normal integral. Let

$$Q_2(t \mid \rho) = \int_t^{\infty} \int_t^{\infty} f(x, y; \rho) \, dx \, dy \tag{8}$$

$$F_2(t \mid \rho) = \int_{-\infty}^{t} \int_{-\infty}^{t} f(x, y; \rho) \, dx \, dy \tag{9}$$

where $f(x, y; \rho)$ is the standard bivariate normal density with correlation coefficient ρ. Let $f(x)$ denote the standard normal density. Several equivalent expressions for $F_2(t \mid \rho)$ exist [7], [13]. One of them is given by

$$F_2(t \mid \rho) = \int_{-\infty}^{\infty} F^2\left(\frac{t - \sqrt{\rho} y}{\sqrt{1-\rho}}\right) f(y) \, dy. \tag{10}$$

Lemma: For any $0 \le \rho \le 1$, and all t we have

$$F_2(t \mid \rho) \le \rho F(t) + (1-\rho) F^2(t) \tag{11}$$

and

$$Q_2(t \mid \rho) \le \rho Q(t) + (1-\rho) Q^2(t). \tag{12}$$

Proof: Consider (11). The result is seen true for $\rho = 0$. For $\rho = 1$, $\lim_{\rho \to 1} F_2(t \mid \rho) = F(t)$ [12], [13].

For $0 < \rho < 1$, we show that $F_2(t \mid \rho)$ is convex in ρ. That is $(d^2/d\rho^2) F_2(t \mid \rho) > 0$ for all t. From [8] we have

$$\frac{d^2}{d\rho^2} F_2(t \mid \rho) = \frac{d^2}{d\rho^2}\left[\int_{-\infty}^{t} \int_{-\infty}^{t} f(x, y; \rho) \, dx \, dy\right]$$

$$= \frac{d}{d\rho}\left[F^2(t) + \int_0^{\rho} f(t, t; z) \, dz\right] > 0.$$

With an appropriate change of variable, the second inequality (12) follows.

A consequence of the lemma is that the correlation coefficient between two sensor decisions (ρ_u) cannot exceed that between the corresponding sensor observation (ρ_x). In order to see this, consider under H_0,

$$\rho_u = \frac{E(u_1 u_2) - E(u_1) E(u_2)}{E(u_1) - E(u_1)^2} = \frac{Q_2(t \mid \rho_x) - Q^2(t)}{Q(t) - Q^2(t)} \le \rho_x. \tag{13}$$

Since t in (8) and (9) is arbitrary, the above bound is valid for the hypothesis H_1 also.

Note that $\rho_u \le \rho_x$ has been established by Kedem in [9] for the special case of $t = 0$.

Next, we present the definition of maximal correlation coefficient of a sequence $\{X_i\}$ and a related central limit theorem [10], [11]. The maximal correlation coefficient of a stationary sequence $\{X_i\}_{i=1}^{\infty}$ between the past $\{X_i\}_{i=1}^{k}$, and the future $\{X_i\}_{i=k+n}^{\infty}$, is defined by

$$\rho(n) \triangleq \sup \left\{ \frac{|E((y_1 - E(y_1))(y_2 - E(y_2))|}{[|E((y_1 - E(y_1)))^2 E(y_2 - E(y_2))^2|^{1/2}} \right\} \tag{14}$$

where the supremum is taken over all second order random variables y_1 and y_2 such that for any arbitrary positive integer k, $y_1 \in \{X_1, X_2, \cdots, X_k\}$ and $y_2 \in \{X_{k+n}, X_{k+n+1}, \cdots\}$. A central limit theorem for sequences of stationary random variables in which the past and distant future are asymptotically independent (i.e., $\rho(n) \to 0$) is as follows:

If $\rho(n) \to 0$ as $n \to \infty$, $E\{|X_1|^{2+\delta}\} < \infty$ for some $\delta > 0$ and

$\sigma_n^2 = E\{(\sum_{j=1}^{n} X_j - E(X_j)))^2\} \to \infty$ as $n \to \infty$, then $\sum_{j=1}^{n}(X_j - E(X_j))/\sigma_n \to N(0, 1)$.

For the sequence $\{X_i\}_{i=1}^{n}$, and the correlation model (5), the maximal correlation coefficient is given by ρ^n · $\{u_i\}_{i=1}^{n}$ is a bounded sequence of random variables. Using (13), the maximal correlation coefficient of this sequence is given by

$$\rho_u(n) \le \rho^n \to 0 \text{ as } n \to \infty. \tag{15}$$

The Gaussian observations $\{X_1\}_{i=1}^{n}$ and hence the decisions $\{u_i\}_{i=1}^{n}$ are stationary. Using the lemma, it can be seen that

$$\sigma_N^2 \ge \sum_{i=1}^{N} [E(u_i^2) - E(u_i)^2] = N[Q(d) - Q^2(d)] \tag{16}$$

where d is an appropriate constant, depending on H_0 or H_1 and N is the number of sensors. Therefore, $\sigma_N^2 \to \infty$ as $N \to \infty$ for finite d. Using this fact and (15), we can apply the above central limit theorem to the sequence of decisions. We are unable to obtain an exact value of the variance σ_N^2 because of the bivariate integrals and will therefore derive a bound on the performance of the distributed detection system. It can be shown that when N is large, the following bounds are true [12]:

$$\sigma_{N0}^2 \le N\left(\frac{1+\rho}{1-\rho}\right)[P_f - P_f^2] \text{ under } H_0$$

$$\sigma_{N1}^2 \le N\left(\frac{1+\rho}{1-\rho}\right)[P_d - P_d^2] \text{ under } H_1 \tag{17}$$

where $P_f = Q(t)$ and $P_d = Q(t-m)$.

To obtain the lower bound on the probability of detection (for a fixed probability of false alaram at the fusion center), we use (17). The probability of false alarm is given by (using the CLT men-

tioned before)

$$P_F = P\left(\sum_{i=1}^{N} u_i > \beta \mid H_0\right)$$

$$= P\left(\frac{\sum_{i=1}^{N} u_i - NP_f}{\sigma_{NO}} > \frac{\beta - NP_f}{\sigma_{NO}}\right) = Q(h) \qquad (18)$$

where β is a constant chosen so that $\beta = NP_f + \sqrt{N}h\psi$ and $h = Q^{-1}(P_F)$, $\psi\sqrt{N} = \sigma_{NO}$. The probability of detection is given by

$$P_D = P\left(\sum_{i=1}^{N} u_i > \beta \mid H_1\right)$$

$$\geq Q\left(h\left(\frac{\sigma_{NO}}{\sigma_{N1}}\right) - \frac{\sqrt{N}(P_d - P_f)}{\sqrt{\left(\frac{1+\rho}{1-\rho}\right)(P_d - P_d^2)}}\right). \qquad (19)$$

As the number of sensors becomes very large, we have

$$P_M \leq Q\left(\sqrt{N}\frac{(P_d - P_f)}{\sqrt{\left(\frac{1+\rho}{1-\rho}\right)(P_d - P_d^2)}}\right) \to 0 \text{ as } N \to \infty.$$

b) Let $\rho_{ij} = \rho$ \hfill (20)

where $i, j = 1, 2, \cdots, N$, and $-1/N - 1 < \rho < 1$. When all the observations are available at the fusion center, the test in (3) becomes

$$l(X) = \frac{1-\rho}{(N-1)\rho^2 - (N-2)\rho - 1} \sum_{i=1}^{N} X_i \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \qquad (21)$$

or

$$\sum_{i=1}^{N} X_i \underset{H_0}{\overset{H_1}{\gtrless}} \bar{\lambda} \qquad (22)$$

$$P_D = Q\left(\frac{\bar{\lambda} - Nm}{\sqrt{N(1-\rho) + N^2\rho}}\right) = Q\left(C - \frac{Nm}{\sqrt{N(1-\rho) + N^2\rho}}\right) \qquad (23)$$

where $C = Q^{-1}(P_F) = \bar{\lambda}/(\sqrt{\rho}N)$ for large N. As $N \to \infty$, $P_D \to Q(C - m/\sqrt{\rho})$, which is a constant not equal to one. Hence the probability of a miss does not go to zero as $N \to \infty$. In this correlation model, an infinite set of such sensors is just equivalent to a single sensor receiving the constant signal $m/\sqrt{\rho}$.

Since the performance of a distributed detection system is bounded from above by that of the central system, the probability of a miss for any distributed detection system will not go to zero as well.

IV. CONCLUSION

We have studied the distributed detection of a constant known signal in correlated Gaussian noise for the case of two correlation models. The asymptotic performances of the central system and the distributed system for the cases of these correlation models, are summarized in Table I.

TABLE I
MISS PROBABILITY (AT A GIVEN FALSE ALARM PROBABILITY > 0 FOR $N \to \infty$)

Detection System \ Correlation Model	Geometric Decrease with Sensor Separation	Equal Correlation
Distributed	0	Fixed, >0
Central	Approaches 0 exponentially with $N(m^2/2)(1 - \rho/1 + \rho)$	Fixed, >0

REFERENCES

[1] J. N. Tsitsiklis, "Decentralized detection," in Advances in Statistical Signal Processing, vol. 2, Signal Detection, H. V. Poor and J. B. Thomas, Eds. Greenwich, CT: JAI Press, 1990.

[2] G. S. Lauer and N. R. Sandell, Jr., "Distributed detection of known signal in correlated noise," ALPHATECH, Burlington, MA, Rep. Mar. 1982.

[3] J. N. Tsitsiklis, "Decentralized detection by a large number of sensors," Math Contr. Signal Syst., vol. 1, pp. 167–182, 1988.

[4] R. Viswanathan and V. Aalo, "On counting rules in distributed detection," IEEE Trans. Acoust., Speech, Signal Processing, vol. 37, pp. 772–775, May 1989.

[5] H. L. VanTrees, Detection, Estimation, and Modulation Theory, vol. 1. New York: Wiley, 1968.

[6] J. L. Doob, Stochastic Processes. New York: Wiley, 1967.

[7] S. Gupta, "Probability integrals of multivariate normal and multivariate t," Ann. Math. Stat., vol. 34, pp. 792–828, Sept.–Dec., 1963.

[8] Y. L. Tong, Probability Inequalities in Multivariate Distributions. New York: Academic, 1980.

[9] B. Kedem, Binary Time Series, vol. 52, Lecture Notes on Pure and Applied Mathematics. New York: Marcel Dekker, 1980, ch. 4.

[10] D. R. Halverson and G. L. Wise, "Asymptotic memoryless detection of random signals in dependent noise," J. Franklin Inst., vol. 312, pp. 13–29, 1981.

[11] I. A. Ibragimov, "A note on the central limit theorem for dependent random variables," Theory Prob. Appl., vol. 20, pp. 135–140, 1975.

[12] V. Aalo, "Performance study of some distributed detection rules," Ph.D. dissertation, Dep. Elec. Eng., Southern Illinois University, Carbondale, IL (in preparation).

[13] Abromowitz and Stegun, Handbook of Mathematical Functions. New York: Dover, 1972.

A Unified Approach to Three Eigendecomposition Methods for Frequency Estimation

Zoran Banjanin, J. R. Cruz, and Dusan S. Zrnić

Abstract—We present a unified approach to three eigendecomposition-based methods for frequency estimation in the presence of noise. These are the Tufts-Kumaresan (TK) method, the minimum-norm (MN) method, and the total least squares (TLS) method. It is shown that: 1) the MN method is a modified version of the TK method; 2) the TLS method is a generalization of the MN method; 3) the TLS solution vector can be expressed in matrix form, and an alternate way of computing it is presented; 4) the MN and the TLS methods exhibit some improvement over the TK method.