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Controllability and Local Accessibility—A Normal Form Approach

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Controllability and Local Accessibility—A Normal Form Approach

Wei Kang, Mingqing Xiao, and Issa Amadou Tall

***Abstract*—Given a system with an uncontrollable linearization at the origin, we study the controllability of the system at equilibria around the origin. If the uncontrollable mode is nonzero, we prove that the system always has other equilibria around the origin. We also prove that these equilibria are linearly controllable provided a coefficient in the normal form is nonzero. Thus, the system is qualitatively changed from being linearly uncontrollable to linearly controllable when the equilibrium point is moved from the origin to a different one. This is called a bifurcation of controllability. As an application of the bifurcation, systems with a positive uncontrollable mode can be stabilized at a nearby equilibrium point. In the last part of this paper, simple sufficient conditions are proved for local accessibility of systems with an uncontrollable mode. Necessary conditions of controllability and local accessibility are also proved for systems with a convergent normal form.**

***Index Terms*—Linearly controllable, nonlinear systems, normal forms, stabilizable.**

I. Introduction

It is well known that a system with an uncontrollable mode in the right-half plane is not stabilizable using smooth feedback. Stabilization by nonsmooth or time-dependent feedbacks was studied by many researchers. A large number of publications and elegant results can be found in the literature (see, for instance, [4]–[6], [9], [16], [17], and [19]–[22]).

In this paper, we study the controllability, stabilizability and local accessibility of systems with a single uncontrollable mode. The term *controllability* is used in this paper to represent the controllability of the linearized system. The viewpoint and approach adopted in this paper are fundamentally different from existing publications on nonsmooth or time-dependent feedback stabilization. Instead of focusing on the stability of a single equilibrium point, we study the existence and the controllability of all equilibria in a neighborhood of the point of interest. The theoretical approach in this paper is based on the normal form of nonlinear control systems, a relatively new theoretical tool that has been actively developed during the last ten years. Different from the results in [16], [17], [19], and [20], we do not restrict our attention on homogeneous or generalized triangular systems. The original system is not required to have any triangular structure. On the other hand, the stability achieved by the feedback in this paper is local, while the results from [16], [17], [19], and [20] are global. In addition, we do assume that the system has a single uncontrollable mode and a single input. Given the rapid development in the normal form theory of systems with multiple uncontrollable modes and multiple inputs, we hope that this restriction will be removed in the future.

Given a system with a nonzero uncontrollable mode and a linearly controllable part, we prove that there always exists an infinite number of equilibria around this point. More importantly, it is proved that the controllability of a system could change when the equilibrium point is varied. This qualitative change in controllability is called a *bifurcation* of the control system [12]. As a by-product of the bifurcation in controllability, a system with a positive uncontrollable mode at an equilibrium point may have infinitely many, arbitrarily close equilibrium points at which the system becomes linearly controllable. Thus, the system can be stabilized at a point sufficiently close to the desired equilibrium. In addition to the stabilization, we also proved a simple relationship between the normal form of a system and its local accessibility. Even if a system has uncontrollable mode, its local accessibility can be easily determined based on its normal form. Necessary conditions for controllability and local accessibility are also addressed.

This paper is organized as follows. Sections II and III focus on the bifurcation of controllability. Theorems are introduced in Section II. They are proved in Section III. In Section IV, examples are introduced for the problem of feedback stabilization. The feedback is designed based on the bifurcation of controllability. In Section V, a partial result on the necessary condition for linearly controllable systems is proved. In Section VI, the local accessibility of nonlinear systems is addressed. Results on both sufficient and necessary conditions for local accessibility are introduced and proved.

II. Equilibrium Set and Controllability

Before the introduction of the theory, we use the following simple example to illustrate some basic concepts and ideas. Consider the following system:

$$\dot{x} = x - u^2.$$

The linearization of the system has a positive uncontrollable eigenvalue at the origin $x = 0$. The system cannot be stabilized at $x = 0$ by any state feedback because $x(t)$ approaches $-\infty$ if the initial condition satisfies $x_0 < 0$. On the other hand, the origin is not the only equilibrium point of the system. In fact, given any initial state $x_0 > 0$, we define the control input as

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$u = \sqrt{x_0}$. Then, the system has a constant solution $x(t) = x_0$, i.e. x_0 is an equilibrium point of the system. Furthermore, the linearization of the system at $(x, u) = (x_0, \sqrt{x_0})$ is

$$(A, B) = (1, -2\sqrt{x_0})$$

which is controllable. Therefore, the system can be practically stabilized at an equilibrium point close to the origin provided the initial state satisfies $x_0 > 0$. In this example, the set of equilibrium points helps to practically stabilize the system when it is impossible to asymptotically stabilize the solution $x = 0$ by any state feedback. This is not an isolated case. In this paper, we prove that a large family of uncontrollable systems has similar properties. We use the invariants of nonlinear control systems to characterize the equilibrium set and the controllability of systems with an uncontrollable mode.

Consider a nonlinear system with a single input in the following form:

$$\dot{x} = f(x, u) \tag{2.1}$$

where $x \in \Re^n$ is the state variable, and $u \in \Re$ is the control input. $f\colon \Re^{n+1} \rightarrow \Re^n$ is assumed to be C^k for sufficiently large k.

Definition 1: A point $x_e \in \Re^n$ is an equilibrium or equilibrium point of (2.1) if and only if $\exists u_e \in \Re$ so that

$$f(x_e, u_e) = 0. \tag{2.2}$$

System (2.1) is said to be linearly controllable at (x_e, u_e) if its linearization

$$\dot{\bar{x}} = F\bar{x} + Gu \tag{2.3}$$

is controllable where

$$F = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} \quad G = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)}.$$

Without loss of generality, we assume $f(0, 0) = 0$.

Assumption 1: We assume that $f(x, u)$ is C^k for sufficiently large k. We also assume that the linearization (F, G) at the origin $x_e = 0$ has one uncontrollable mode with eigenvalue $\lambda \neq 0$.

From Assumption 1, we adopt the following normal form for the linearization at $x_e = 0$:

$$F = \begin{bmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.4}$$

If the original linearization is not in this form, it is well known that it can be transformed to (2.4) by a linear change of coordinates and linear feedback. With the linearization (2.4), the nonlinear control system is in the following form:

$$\begin{aligned} \dot{z} &= \lambda z + f_1^{[2+]}(z, x, u) \\ \dot{x}_1 &= x_2 + f_{2,1}^{[2+]}(z, x, u) \\ \dot{x}_2 &= x_3 + f_{2,2}^{[2+]}(z, x, u) \\ &\vdots \\ \dot{x}_{n-1} &= u + f_{2,n-1}^{[2+]}(z, x, u) \end{aligned} \tag{2.5}$$

where the superscript in $f_1^{[2+]}(z, x, u)$ implies that the Taylor expansion of the function f_1 starts with quadratic or higher degree terms. Equivalently

$$f_1^{[2+]}(0, 0, 0) = 0 \quad \frac{\partial f_1^{[2+]}}{\partial (z, x, u)}(0, 0, 0) = 0. \tag{2.6}$$

The definition of the superscripts in $f_{2,j}^{[2+]}$ is similar. It is proved in this paper that the set of equilibria around the origin is a smooth curve. At two different equilibria, the controllability of the system may change. Following [12], if a *qualitative* property such as controllability is changed, we say that the control system has a bifurcation. In this paper, we study the bifurcation of the controllability around an uncontrollable point.

The analysis and proofs in this paper are based on the normal form theory of control systems. In the case $u \equiv 0$, it is well known from the Poincaré–Dulac theorem [1] that (2.1) can be reduced to the canonical form

$$\dot{y} = Fy + w(y) + O(y)^{d+1} \tag{2.7}$$

by means of a formal change of variables $x = y + \theta(y)$, where all monomials in w are resonant. Kang and Krener [14] initiated the extension of the Poincaré normal form to control systems. More general results on normal forms were obtained by several authors (see [2], [7], [11], and [25]). The normal form theory covered a large family of systems including controllable, uncontrollable, continuous and discrete-time systems. In this paper, we study the controllability around a linearly uncontrollable equilibrium of nonlinear systems by making use of the normal form from [2], [12], and [25]. We prove that the homogeneous terms in the normal form characterizes the controllability of a system at the points in the equilibrium set.

Theorem 2.1: Consider a system (2.5). Suppose $\lambda \neq 0$. Then, in a neighborhood of the origin, its set of equilibrium constitutes a smooth curve in $\Re^n$ which passes through the origin. Furthermore, the curve of equilibrium points can be parameterized as a function of x_1 satisfying

$$\begin{aligned} z &= z_e(x_1) \quad z_e(0) = 0 \quad \text{and} \quad \frac{dz_e}{dx_1}(0) = 0 \\ x_2 &= x_{e,2}(x_1) \quad x_{e,2}(0) = 0 \quad \text{and} \quad \frac{dx_{e,2}}{dx_1}(0) = 0 \\ &\vdots \\ x_{n-1} &= x_{e,n-1}(x_1) \quad x_{e,n-1}(0) = 0 \\ &\text{and} \quad \frac{dx_{e,n-1}}{dx_1}(0) = 0 \end{aligned} \tag{2.8}$$

i.e., x_1 is a parameter, and x_1-axis is the tangent line of the curve at $(z, x) = (0, 0)$.

The proof of the theorem can be found in Section III. According to Theorem 2.1, the existence of equilibrium points around an uncontrollable point is guaranteed. The geometry of the set of equilibrium points is illustrated by the parameterization. The following theorems are about the controllability of these equilibrium points. In the next theorem, f_1 represents the right-hand side of the $\dot{z}$ equation in (2.5), i.e.,

$$f_1 = \lambda z + f_1^{[2+]}(z, x, u).$$

In (2.9), the variable x_n represents the control input u to simplify the notation.

Theorem 2.2: Consider a control system (2.5) with a nonzero uncontrollable mode λ. Suppose

$$\left.\frac{\partial^2 f_1(z,x,x_n)}{\partial x_1^2}\right|_{(0,0,0)} + \lambda \left.\frac{\partial^2 f_1(z,x,x_n)}{\partial x_1 \partial x_2}\right|_{(0,0,0)} + \cdots + \lambda^{n-1} \left.\frac{\partial^2 f_1(z,x,x_n)}{\partial x_1 \partial x_n}\right|_{(0,0,0)} \neq 0. \quad (2.9)$$

Then, there exists a neighborhood of the origin in which (2.5) is linearly controllable at any equilibrium point except for $(z, x) = (0, 0)$.

The condition (2.9) is a sufficient condition for controllability. It does not require the system to be in normal form. However, if (2.9) is not satisfied, it is still possible for (2.5) to be controllable. In such a case, the conclusion of the theorem is based on the normal form. In the following, the normal form of control systems is introduced without proof (see [15] and [25]). They play a key role in the proof of all the main theorems. A system is equivalent to its normal form under change of coordinates and feedback. Therefore, to prove the main theorems about general control systems, it is enough to prove the result for systems in normal form, which significantly simplifies the proof of the theorems. According to [15] and [25], a transformation consists of the following change of coordinates and feedback:

$$\begin{bmatrix} \tilde{z} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} z \\ x \end{bmatrix} - \sum_{k=2}^{d} \begin{bmatrix} \phi_1^{[k]}(z,x) \\ \phi_2^{[k]}(z,x) \end{bmatrix}$$
$$\tilde{u} = u - \sum_{k=2}^{d} \alpha^{[k]}(z,x,u) \quad (2.10)$$

where $\phi_1^{[k]}$ and $\alpha^{[k]}$ are homogeneous polynomials of degree k in its arguments, $\phi_2^{[k]}$ is a $k-1$-dimensional vector whose entries are homogeneous polynomials of degree k. The highest degree d is selected to be large enough so that adequate information about the local performance of a system can be extracted from the Taylor expansion. It was proved in [15] and [25] that there exists a transformation under which (2.5) can be transformed to

$$\begin{aligned}
\dot{z} &= \lambda z + \sum_{j=1}^{n} x_j^2 Q_j(z, \bar{x}_j) + x_1 S(z) + O(z,x,u)^{d+1} \\
\dot{x}_1 &= x_2 + \sum_{j=3}^{n} x_j^2 P_{1j}(z, \bar{x}_j) + O(z,x,u)^{d+1} \\
\dot{x}_2 &= x_3 + \sum_{j=4}^{n} x_j^2 P_{2j}(z, \bar{x}_j) + O(z,x,u)^{d+1} \\
&\vdots \\
\dot{x}_i &= x_{i+1} + \sum_{j=i+2}^{n} x_j^2 P_{ij}(z, \bar{x}_j) + O(z,x,u)^{d+1} \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} + x_n^2 P_{n-2,n}(z, \bar{x}_n) + O(z,x,u)^{d+1} \\
\dot{x}_{n-1} &= u + O(z,x,u)^{d+1}
\end{aligned} \quad (2.11)$$

where $\bar{x}_j = (x_1, x_2, \ldots, x_j)^T$ (we also denote $\bar{x}_{n-1} = x$ and $\bar{x}_1 = x_1$), and

$$\begin{aligned}
Q_j(z, \bar{x}_j) &= Q_j^{[0]} + Q_j^{[1]}(z, \bar{x}_j) + \cdots + Q_j^{[d-2]}(z, \bar{x}_j) \\
S(z) &= S^{[1]}(z) + S^{[2]}(z) + \cdots + S^{[d-1]}(z) \\
P_{ij}(z, \bar{x}_j) &= P_{ij}^{[0]} + P_{ij}^{[1]}(z, \bar{x}_j) + \cdots + P_{ij}^{[d-2]}(z, \bar{x}_j).
\end{aligned} \quad (2.12)$$

Once again, the variable x_n in (2.11) represents the control input u. To simplify the notation, we use (z, x) and u instead of ($\tilde{z}$, $\tilde{x}$) and $\tilde{u}$ as the state variable and the input in the normal form. According to [11], the computation of the normal form for a given system is equivalent to solving systems of linear algebraic equations. So, there is no fundamental obstacle toward the computation of the normal form. Therefore, the computation of the normal form can be carried out using the software equipped with linear algebraic equation solvers such as MAPLE, Mathematica, and Matlab.

Theorem 2.3: Consider a control system (2.5) with a nonzero λ. Suppose the normal form of (2.5) is (2.11). Suppose there exists an integer $k \geq 2$ so that

$$Q_1^{[k-2]}(0, x_1) \not\equiv 0. \quad (2.13)$$

Then, there exists a neighborhood of the origin in which (2.5) is linearly controllable at all equilibrium points except for $(z, x) = (0, 0)$.

The theorems state that under assumption (2.9) or (2.13), the system (2.5) is controllable at all equilibrium points in a neighborhood of the origin, although the system is not linearly controllable at $(z, x) = (0, 0)$. Thus the origin is an isolated uncontrollable equilibrium point. It suggests that if a system is not linearly controllable at this equilibrium, we can control the system by stabilizing it at a nearby equilibrium. The proofs of the theorems are given in the next section. In the proof, it is shown that $Q_1^{[0]} \neq 0$ is equivalent to condition (2.9). So, (2.9) is more restrictive than (2.13). However, it is easy to check (2.9) because the derivatives in (2.9) can be computed without transforming the system into its normal form. Condition (2.13) requires the system to be transformed to normal form, which could be cumbersome for high degree terms. Nevertheless, Theorem 2.3 reveals an interesting fact: around an uncontrollable equilibrium of a nonlinear system, it is highly possible for the system to be controllable at other equilibrium points in a neighborhood. Theorem 2.3 is equivalent to the fact that a system not controllable in a neighborhood must satisfy the following infinite number of algebraic equations:

$$Q_1^{[k-2]}(0, x_1) = 0$$

for all $k \geq 2$.

From Theorem 2.3, it is proved that (2.13) is a sufficient condition for a system to be linearly controllable. In Section V, it is proved that (2.13) is also a necessary condition, provided that the normal form is convergent.

III. Proof of the Theorems

The proof of theorem 2.1: Denote the right-hand side of (2.5) by $f(z, x, u)$. It is easy to check that

$$\left.\frac{\partial f(z,x,u)}{\partial (z\ x_2 \cdots x_{n-1}\ u)}\right|_{(z,x_2,\cdots,x_{n-1},u)=(0,\cdots,0)} = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (3.1)$$

Since $\lambda \neq 0$, the matrix has full rank. According to the implicit function theorem, there exists a neighborhood of $(z, x, u) = (0, 0, 0)$ such that $f(z, x, u) = 0$ has a unique solution

$$\begin{aligned} z &= z_e(x_1) \\ x_2 &= x_{e,2}(x_1) \\ &\cdots \\ x_{n-1} &= x_{e,n-1}(x_1) \\ u &= u_e(x_1) \end{aligned} \quad (3.2)$$

satisfying

$$z_e(0) = x_{e,2}(0) = \cdots = x_{e,n-1}(0) = u_e(0) = 0. \quad (3.3)$$

Now we prove that the function $z_e(x_1)$ and $x_{e,i}(x_1)$ consists of quadratic and higher degree terms only. From (2.5) and the definition of equilibrium point

$$x_{e,i}(x_1) = -f_{2,i-1}^{[2+]}(z_e(x_1), x_1, x_{e,2}(x_1), \cdots, x_{e,n-1}(x_1), u_e(x_1)). \quad (3.4)$$

The right-hand side are terms of $O(x_1)^2$. So

$$\frac{dx_{e,i}}{dx_1}(0) = 0$$

for $i = 1, \cdots, n-1$. Similarly, it can be proved that the derivatives of $z_e(x_1)$ and $u_e(x_1)$ at $x_1 = 0$ equal zero. □

In the following, we prove Theorem 2.3 first. Then, it will be shown that Theorem 2.2 is a corollary of Theorem 2.3. The following Lemma is used in the proof of Theorem 2.3.

Lemma 3.1: Consider a system in the normal form (2.11) of degree d. Let k_0, $2 \leq k_0 \leq d$, be the smallest positive integer so that

$$Q^{[k_0-2]}(0, x_1) \not\equiv 0. \quad (3.5)$$

Then the equilibrium set of (2.11) satisfies

$$z_e = O(x_1)^{k_0}, x_{e,2} = O(x_1)^{k_0}, \ldots \quad x_{e,n-1} = O(x_1)^{k_0}, u_e = O(x_1)^{k_0}. \quad (3.6)$$

Proof: Since the normal form of higher degree terms does not appear in the proof, we consider the normal form to the degree k_0 only, i.e., consider the case $d = k_0$. The equation

$$u_e + O(z_e, x_1, x_{e,2}, \cdots, x_{e,n-1}, u_e)^{k_0} = 0$$

implies that

$$u_e = O(x_1)^{k_0}. \quad (3.7)$$

From the equation of $\dot{x}_{n-2}$ in (2.11), the nonlinear term in the normal form contains x_n^2. From (3.7), we have $x_{e,n-1}(x_1) = O(x_1)^{k_0}$. Repeating this procedure, it can be proved that

$$x_i = O(x_1)^{k_0} \quad (3.8)$$

for $i = 2, \ldots, n-1$. Now we consider the first equation of (2.11). Suppose the lowest nonzero term in the Taylor expansion of $z_e(x_1)$ has degree k_1, i.e.

$$z_e(x_1) = z_e^{[k_1]}(x_1) + O(x_1)^{k_1+1}$$

where $z_e^{[k_1]}(x_1) \neq 0$. We must prove $k_0 \leq k_1$. Consider the equilibrium equation

$$\lambda z_e + \sum_{i=1}^{n} x_{e,i}^2 Q(z_e, \bar{x}_{e,i}) + x_{e,1} S(z_e) + O(z_e, x_e, u_e)^{k_0+1} = 0 \quad (3.9)$$

where $x_{e,1} = x_1$. Equation (3.8) implies

$$\begin{aligned} &x_{e,2}^2 Q_2(z_e, \bar{x}_{e,i}) + \cdots + x_{e,n-1}^2 Q_{n-1}(z_e, \bar{x}_{e,i}) = O(x_1)^{k_0+1}; \\ &x_{e,1} S(z_e) = O(x_1)^{k_1+1} \\ &\lambda z_e = \lambda z^{[k_1]}(x_1) + O(x_1)^{k_1+1} \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} x_1^2 Q(z_e, x_1) &= x_1^2 Q(0, x_1) + O(x_1)^{k_1+2} \\ &= x_1^2 Q^{[k_0-2]}(0, x_1) + O(x_1)^{k_0+1} \\ &\quad + O(x_1)^{k_1+2}. \end{aligned} \quad (3.11)$$

If $k_1 < k_0$, then (3.10) and (3.11) imply that the only degree k_1 term in (3.9) is in λz. Thus, (3.9) does not hold. So, $k_1 \geq k_0$. □

Remark: Actually, z_e satisfies

$$z_e^{[k_0]}(x_1) = -\frac{1}{\lambda} Q_1^{[k_0-2]}(0, x_1) x_1^2. \quad (3.12)$$

◃

Now, we are ready to prove Theorem 2.3.

The proof of theorem 2.3: Consider a system (2.11) in normal form. Suppose that $k = k_0$ is the smallest positive integer so that (2.13) holds. We assume $d = k_0$ in the normal form because all terms of degree greater than k_0 are not involved in the proof. Denote the right-hand side of (2.11) by $f(z, x, u)$. Notice that (3.13) and (3.14), shown at the bottom of the page, hold, The equilibrium points satisfy $x_{e,i} = O(x_1)^{k_0}$, for $i = 2, \ldots, n$, and $z_e = O(x_1)^{k_0}$ (Lemma 3.1). Evaluating (3.13) at an equilibrium point yields

$$\left.\frac{\partial f}{\partial(z, x_1, \cdots, x_{n-1})}\right|_{(z=z_e, x_2=x_{e,2}, \cdots, x_n=x_{e,n})} \times \begin{bmatrix} c_{11} & \frac{\partial}{\partial x_1}\left(x_1^2 Q_1^{[k_0-2]}(0, x_1)\right) & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(x_1)^{k_0}. \quad (3.15)$$

At an equilibrium point

$$\left.\frac{\partial f}{\partial u}\right|_{(z_e, x_1, x_{e,2}, \cdots, x_{e,n})} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + O(x_1)^{k_0}. \quad (3.16)$$

Thus, the controllability matrix is

$$\begin{bmatrix} 0 & \cdots & 0 & \frac{\partial}{\partial x_1}\left(x_1^2 Q_1^{[k_0-2]}(0, x_1)\right) \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} + O(x_1)^{k_0}. \quad (3.17)$$

The determinant of the matrix is

$$D(x_1) = -\frac{\partial}{\partial x_1}\left(x_1^2 Q_1^{[k_0-2]}(0, x_1)\right) + O(x_1)^{k_0}. \quad (3.18)$$

Therefore, $D(x_1)$ is not zero if x_1 is small and $x_1 \neq 0$, provided (3.5) holds. This implies that the system is controllable at all equilibria around the origin. □

Remark: If the eigenvalue corresponding to the uncontrollable mode is zero, more bifurcation analysis is needed. It is proved in [12] that the set of equilibrium points may be either a single point set, or a smooth curve, depending on the nonlinear normal form of the system. The topology of the equilibrium set may change, which is addressed in [12] as a bifurcation of equilibrium sets. ◁

In the following, Theorem 2.2 is proved as a corollary of Theorem 2.3. In the proof, we need the following lemma.

Lemma 3.2: Given a system (2.5), the value defined by the left-hand side of (2.9) is invariant under any transformation of the form (2.10).

Proof: In the following, $\mathcal{L}$ is the operator defined as follows:

$$\mathcal{L}(f_1) := \left.\frac{\partial^2 f_1(z, x, x_n)}{\partial x_1^2}\right|_{(0,0,0)} + \lambda \left.\frac{\partial^2 f_1(z, x, x_n)}{\partial x_1 \partial x_2}\right|_{(0,0,0)} + \cdots + \lambda^{n-1} \left.\frac{\partial^2 f_1(z, x, x_n)}{\partial x_1 \partial x_n}\right|_{(0,0,0)}. \quad (3.19)$$

Its value is completely determined by the quadratic terms in $f_1^{[2]}(z, x)$. It is known that $\phi_i^{[k]}$, $i = 1, 2$, in (2.10) with $k > 2$ does not change the quadratic part of a system. Therefore, we only consider quadratic transformations in the following form:

$$\begin{aligned} \tilde{z} &= z - \phi_1^{[2]}(z, x) \\ \tilde{x} &= x - \phi_2^{[2]}(z, x) \\ \tilde{u} &= u - \alpha^{[2]}(z, x, u). \end{aligned} \quad (3.20)$$

$$\frac{\partial f}{\partial(z, x_1, \cdots, x_{n-1})} = \begin{bmatrix} c_{11} & c_{12} & \sum_{i=2}^{n} \frac{\partial x_i^2 Q_i}{\partial x_2} & \cdots & \sum_{i=n-1}^{n} \frac{\partial x_{n-1}^2 Q_{n-1}}{\partial x_{n-1}} \\ \sum_{j=3}^{n} x_i^2 \frac{\partial P_{1,j}}{\partial z} & \sum_{j=3}^{n} \frac{\partial x_j^2 P_{1,j}}{\partial x_1} & 1 + \sum_{j=3}^{n} \frac{\partial x_j^2 P_{1,j}}{\partial x_2} & \cdots & \sum_{j=3}^{n} \frac{\partial\left(x_j^2 P_{1,j}\right)}{\partial x_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=n-1}^{n} x_i^2 \frac{\partial P_{n-3,j}}{\partial z} & \sum_{j=n-1}^{n} \frac{\partial x_j^2 P_{n-3,j}}{\partial x_1} & \sum_{j=n-1}^{n} \frac{\partial x_j^2 P_{n-3,j}}{\partial x_2} & \cdots & \sum_{j=n-1}^{n-1} \frac{\partial\left(x_j^2 P_{n-3,j}\right)}{\partial x_{n-1}} \\ x_n^2 \frac{\partial P_{n-2,n}}{\partial z} & x_n^2 \frac{\partial P_{n-2,n}}{\partial x_1} & x_n^2 \frac{\partial P_{n-2,n}}{\partial x_2} & \cdots & 1 + x_n^2 \frac{\partial P_{n-2,n}}{\partial x_{n-1}} \\ 0 & 0 & 0 & & 0 \end{bmatrix} + O(z, y)^{k_0} \quad (3.13)$$

where

$$\begin{aligned} c_{11} &= \lambda + \sum_{i=1}^{n} x_i^2 \frac{\partial Q_i}{\partial z} + x_1 \frac{\partial S}{\partial z} \\ c_{12} &= \frac{\partial x_1^2 Q_1}{\partial x_1} + \sum_{i=2}^{n} x_i^2 \frac{\partial Q_i}{\partial x_1} + S(z). \end{aligned} \quad (3.14)$$

Applying (3.20) to (2.5) yields a new system. Suppose that the uncontrollable part in the new system is

$$\dot{\tilde{z}} = \lambda\tilde{z} + \tilde{f}_1^{[2]}(\tilde{z},\tilde{x},\tilde{u}) + \cdots$$

where $+\cdots$ represents terms of degree three and higher in ($\tilde{z}$, $\tilde{x}$, and $\tilde{u}$). Then

$$\begin{aligned}\tilde{f}_1^{[2]}(\tilde{z},\tilde{x},\tilde{u}) &= f_1^{[2]}(\tilde{z},\tilde{x},\tilde{u}) + \lambda\phi_1^{[2]}(\tilde{z},\tilde{x})\\ &\quad -\frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{z}}\lambda\tilde{z} - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{x}}(A_2\tilde{x}+B_2\tilde{u}) \end{aligned} \quad (3.21)$$

where

$$A_2=\begin{bmatrix}0&1&0&\cdots 0&\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix}_{(n-1)\times(n-1)} \qquad B_2=\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}_{n-1\times 1}.$$

To prove the value of the left-hand side of (2.9) is invariant under the transformation, it is enough to prove

$$\mathcal{L}\left(\tilde{f}_1^{[2]}\right) = \mathcal{L}\left(f_1^{[2]}\right).$$

First, let us assume $\phi_1^{[2]}(z,x) = a_{1j}x_1x_j$, $1 \le j \le n-1$. Then

$$\begin{aligned}\lambda\phi_1^{[2]}(\tilde{z},\tilde{x}) - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{z}}\lambda\tilde{z} - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{x}}A_2\tilde{x}\\ = \lambda a_{1j}\tilde{x}_1\tilde{x}_j - a_{1j}\tilde{x}_2\tilde{x}_j - a_{1j}\tilde{x}_1\tilde{x}_{j+1}.\end{aligned}$$

Therefore

$$\mathcal{L}\left(\lambda\phi_1^{[2]}(\tilde{z},\tilde{x}) - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{z}}\lambda\tilde{z} - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{x}}A_2\tilde{x}\right) = 0.$$

Next, if $\phi_1^{[2]}$ is independent of x_1, then the expression

$$\lambda\phi_1^{[2]}(\tilde{z},\tilde{x}) - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{z}}\lambda\tilde{z} - \frac{\partial\phi_1^{[2]}(\tilde{z},\tilde{x})}{\partial\tilde{x}}A_2\tilde{x} \quad (3.22)$$

does not generate any term of the form $\tilde{x}_1\tilde{x}_j$. Obviously, applying $\mathcal{L}$ to (3.22) yields zero. So, for any choice of $\phi_1^{[2]}$, (3.21) implies

$$\mathcal{L}\left(\tilde{f}_1^{[2]}\right) = \mathcal{L}\left(f_1^{[2]}\right).$$

□

Proof of Theorem 2.2: Consider a system (2.5). From Lemma 3.2, $\mathcal{L}(f_1)$ is invariant, where $\mathcal{L}$ is defined by (3.19). So, its value equals the value computed using its normal form (2.11). However, it is easy to check that the invariant number computed from the normal form is $2Q_1^{[0]}$. Therefore

$$2Q_1^{[0]}(0,0) = \mathcal{L}(f_1).$$

Condition (2.9) implies $Q_1^{[0]} \neq 0$. Then, Theorem 2.2 follows the result of Theorem 2.3. □

IV. Examples of Feedback Stabilization

We consider a nonlinear system

$$\begin{aligned}\dot{z} &= z + x_1^p + x_2^q\\ \dot{x}_1 &= x_2\\ \dot{x}_2 &= u\end{aligned} \quad (4.1)$$

where p, q are positive integers greater than one. Note that in this case $k_0 = p$ and

$$Q^{[p-2]}(0,x_1) = x_1^{p-2}. \quad (4.2)$$

The linearization of the system is uncontrollable at the origin. Because the uncontrollable mode is positive, it is not stabilizable at (0, 0, 0) by any smooth feedback. However, the system has infinite number of equilibrium points around the origin. The equilibrium points are

$$\begin{aligned}z_e &= -x_1^p\\ x_{e,2} &= 0\\ u_e &= 0.\end{aligned} \quad (4.3)$$

The controllability matrix is

$$\begin{bmatrix}0&0&px_1^{p-1}\\0&1&0\\1&0&0\end{bmatrix}. \quad (4.4)$$

According to Theorem 2.3, if we choose $x_1 \neq 0$ but close to zero, then the system is controllable and thus is stabilizable. To practically stabilize the system, we pick up an equilibrium point that is different from the origin. Both linear and nonlinear feedbacks are designed to test the stability.

In the simulations, we define $p = q = 4$. An even number for p and q makes the stabilization extremely difficult. In fact, existing results on stabilization of uncontrollable systems require the dominating nonlinear terms to have odd number degree, if the uncontrollable mode is positive. In this case, (4.1) cannot be stabilized by any kind of feedback, no matter the feedback is smooth, continuous, or time dependent. However, all other equilibria of (4.1) are locally stabilizable because they are linearly controllable. Therefore, it is possible to stabilize the system at a point close to the origin. Furthermore, we will show that the domain of attraction for (4.1) is unbounded. For example, consider an equilibrium point around the origin

$$(z_e,x_{1e},x_{2e},u_e) = \left(-\sqrt[3]{\left(\frac{1}{4}\right)^4}, \sqrt[3]{\frac{1}{4}}, 0, 0\right). \quad (4.5)$$

The first feedback that we use is a simple linear control stabilizing the system at the equilibrium point. Let

$$u = -24\left(z+\sqrt[3]{\left(\frac{1}{4}\right)^4}\right) - 18\left(x_1 - \sqrt[3]{\frac{1}{4}}\right) - 7x_2.$$

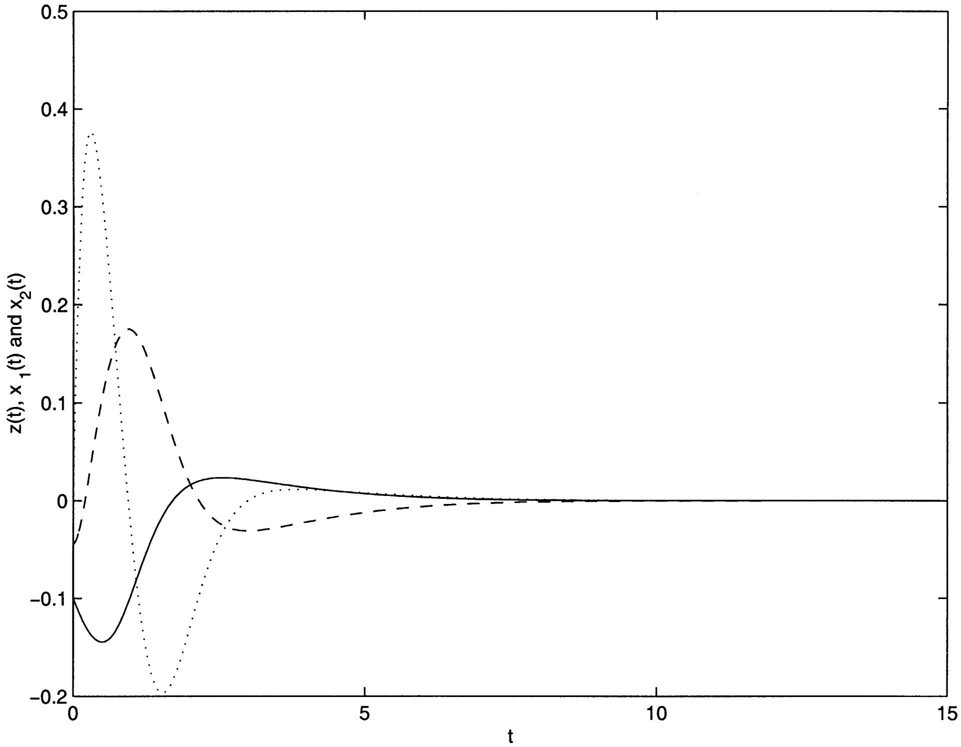

Fig. 1. Solid: $z - z_e$. Dashed: $x_1 - x_{1e}$. Dotted: $x_2 - x_{2e}$.

Then, the eigenvalues of the linearized system at (z_e, x_{1e}, x_{2e}, u) are $\{-1, -2, -3\}$. We set the initial condition to be $(z(0), x_1(0), x_2(0)) = (-0.2575, 0.5857, 0)$. Fig. 1 is a plot of the trajectories $z(t) - z_e$, $x_1(t) - x_{1e}$ and $x_2(t) - x_{2e}$. It shows that the system is stabilized at the equilibrium point. In Fig. 1, we shift the equilibrium point (4.5) to the origin.

While the linear feedback is easy to design and easy to implement, the domain of attraction is small. Furthermore, as the equilibrium point gets closer to the origin, the gain becomes higher. In the following, a nonlinear feedback is designed to significantly increase the size of the domain of attraction. Let us again consider our previous example in the case of $p = q = 4$, i.e.,

$$\begin{aligned} \dot{z} &= z + x_1^4 + x_2^4 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u. \end{aligned} \tag{4.6}$$

It can be shown that $z(t) \geq 0$ for all $t \geq 0$ if $z(0) > 0$. Because all equilibrium points satisfy $z \leq 0$, we only consider the case of $z(0) \leq 0$. Define $\bar{z} = \sqrt[4]{-z}$. Then

$$\begin{aligned} \dot{\bar{z}} &= \frac{1}{4}(-z)^{\frac{-3}{4}} \cdot (-1) \cdot \dot{z} \\ &= \frac{1}{4} \cdot \frac{(-1)}{\bar{z}^3}\left(-\bar{z}^4 + x_1^4 + x_2^4\right) \\ &= \frac{1}{4}\left(\bar{z} - \frac{x_1^4 + x_2^4}{\bar{z}^3}\right). \end{aligned} \tag{4.7}$$

Hence, the set of equilibrium points of the new system is defined by the equations $|\bar{z}_e| = |x_{1e}|$, $x_{2e} = 0$, $u_e = 0$, where x_1 is selected as the parameter. The linearization of the system at an equilibrium is

$$A = \begin{bmatrix} \frac{1}{4}\left(1 + \frac{3\left(x_{1e}^4 + x_{2e}^4\right)}{\bar{z}^4}\right) & -\frac{x_{1e}^3}{\bar{z}^3} & -\frac{x_{2e}^3}{\bar{z}^3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.8}$$

Suppose $[k_1, k_2, k_3]$ is the feedback gain, then

$$u = -k_1(\bar{z} - \bar{z}_e) - k_2(x_1 - x_{1e}) - k_3 x_2. \tag{4.9}$$

In the original coordinates

$$u = -k_1\left((-z)^{\frac{1}{4}} - \bar{z}_e\right) - k_2(x_1 - x_{1e}) - k_3 x_2. \tag{4.10}$$

The feedback is continuous whenever $z \leq 0$. It is smooth at all equilibrium points except for the origin. In this simulation, the control gain is $k_1 = -50.6092$, $k_2 = 36.4671$, and $k_3 = 13.1504$, which are obtained by applying LQR to (4.8) with

$$R = 1 \quad Q = 100I$$

where I is the 3 × 3 identity matrix. In the following simulation, the equilibrium is (4.5) and the initial condition is (−1.157 490 1, 0.629 960 5, 0). The trajectories $z(t)$, $x_1(t)$ and $x_2(t)$ is shown in Fig. 2, in which the equilibrium point is transformed to the origin. In comparison with the linear controller, the simulation using the nonlinear feedback has an initial error of 1.0, while the initial error in Fig. 1 is less than 0.11. For this particular example, numerical experiments using

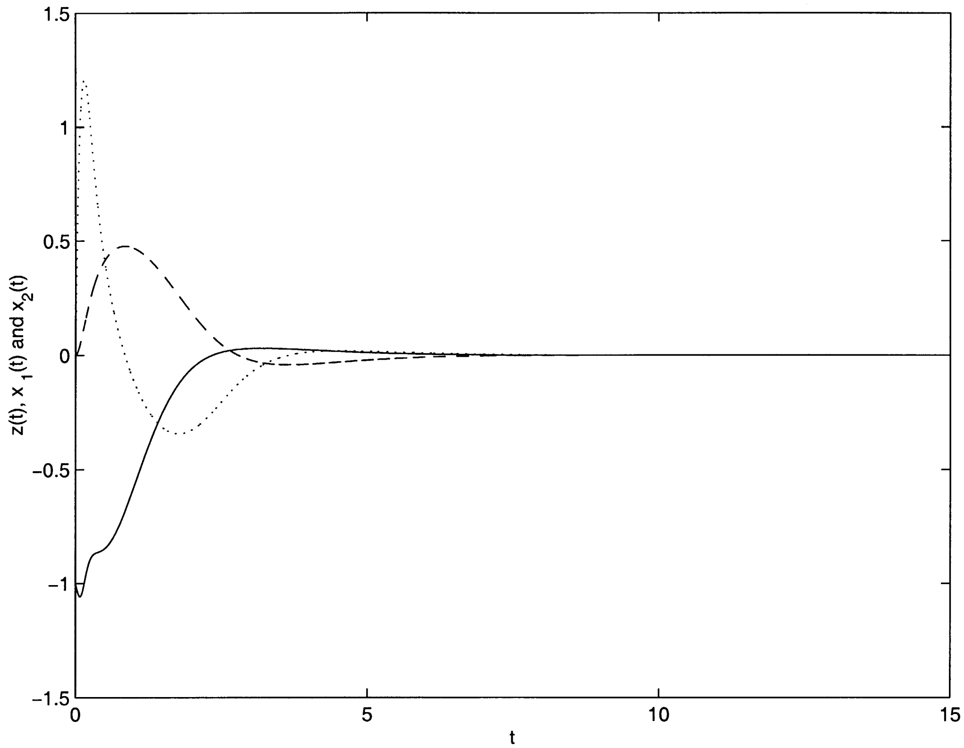

Fig. 2. Solid: $z - z_e$. Dashed: $x_1 - x_{1e}$. Dotted: $x_2 - x_{2e}$.

different initial conditions show that the nonlinear feedback has a larger domain of attraction than the linear feedback.

Once the system is stabilized at an equilibrium point, the state of the system can be driven along the set of equilibrium. For instance, suppose the system is to be stabilized at

$$(10^{-8}, 10^{-2}, 0) \tag{4.11}$$

which is very close to the origin. Under a feedback control, the domain of attraction around the equilibrium point (4.11) is very small. If the initial condition is not close to (4.11), we can stabilize the system at another equilibrium. Then, slowly move the equilibrium point along the curve (4.3) in the direction of (4.11). In the following simulation, we first stabilize the system at the equilibrium (4.5), which has a much larger domain of attraction. Then, the equilibrium point is moved slowly from the initial equilibrium point (4.5) toward (4.11). In the simulation shown in Fig. 3, the initial condition of the system is (4.5), and the feedback is (4.10) with $k_1 = -50.6092$, $k_2 = 36.4671$, and $k_3 = 13.1504$. Every ten seconds, the value of x_{1e} in the controller (4.10) is reduced by Δx_{1e}. The value of z_e is changed accordingly. At the end of every iteration, the state of the system is driven closer to the origin than before. Several step sizes

$$\Delta x_{1e} = 0.1, 0.05, 0.01, 0.005$$

are used in the simulation. The step size 0.1 is used for the first few iterations. As the state gets closer and closer to the origin, the smaller and smaller step sizes are used. At $t = 160$, i.e., after 15 iterations, the state of the system is stabilized at the destination equilibrium (4.11). Notice that the equilibrium set (4.3) is unbounded. Therefore, the domain in which the system can be driven to a neighborhood of the origin is unbounded.

Although (2.13) is based on the normal form, it is not necessary to transform a system to its normal form to design a feedback. Based on Theorem 2.3, if a system is not linearly controllable at $(z, x) = (0, 0)$, it is worthy to check the controllability of the system at other equilibrium points in a neighborhood of the origin because they are likely to be controllable. If the computation of the normal form for the given system is complicated, the computation of the controllability matrix is relatively straightforward, which does not require the normal form. For example, consider the following system that is not in the normal form (2.11):

$$\begin{aligned} \dot{z} &= z - x_1^2 \\ \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1^2 + x_1 x_2 + u. \end{aligned} \tag{4.12}$$

Clearly, the set of equilibrium points is $z = x_1^2$, $x_2 = 0$, $u = -x_1^2$. The system is not controllable at the origin. The controllability matrix at these equilibria is

$$\begin{bmatrix} 0 & 0 & 2x_1 \\ 0 & -1 & -x_1 \\ 1 & x_1 & -2x_1 + x_1^2 \end{bmatrix}. \tag{4.13}$$

Thus, the system is controllable at the equilibrium points around the origin with $x_1 \neq 0$.

V. Necessary Condition for Controllability

Theorem 2.3 is a sufficient condition for systems to be linearly controllable in a vicinity of the origin. Is the condition also necessary? The answer depends on the convergence of the normal form. As we know that the Poincaré normal form may not converge as the degree approaches infinity. The same is true

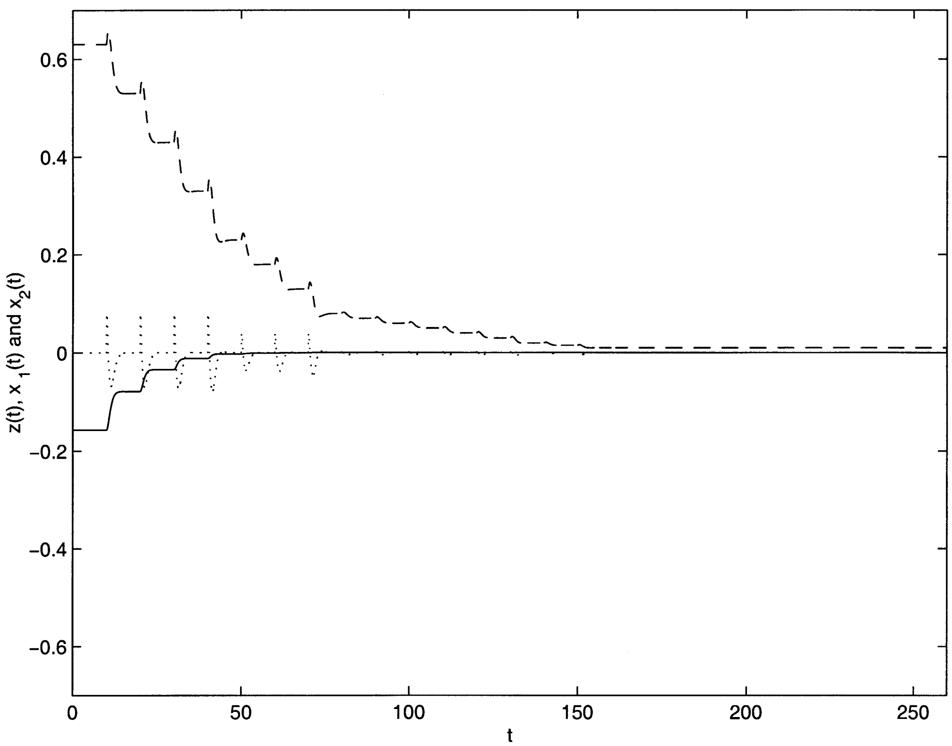

Fig. 3. Solid: z. Dashed: x_1. Dotted: x_2.

for control systems. However, if the formal series of normal form converges for a control system, then (2.13) in Theorem 2.3 is also a necessary condition for the controllability of nonzero equilibrium points in a neighborhood of the origin.

Theorem 5.1: Consider a system in normal form (2.11). Consider the case in which $d = \infty$, i.e., the homogeneous parts of all degrees are known. Suppose the normal form converges as a power series in a neighborhood of the origin. If

$$Q_1^{[k-2]}(0, x_1) = 0 \tag{5.1}$$

for all $k \geq 2$, then there exists a neighborhood, V, of the origin so that the system is not linearly controllable at all equilibrium points in V.

Proof: If the normal form (2.11) contains all the homogenous parts of arbitrary degree, then $O(z, x)^{d+1}$ is not in the normal form. Because $Q_1(0, x_1) = 0$, the equilibrium set is simple

$$\{(z, x) | z = x_2 = \cdots = x_{n-1} = 0, x_1 = \epsilon, \epsilon \in \Re\}. \tag{5.2}$$

Now, consider the linearization matrix (3.13). Substituting any equilibrium point (5.2) into the matrix yields

$$\begin{bmatrix} c_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The vector B in the linearization is $[0 \ \ 0 \ \ \ldots \ \ 1]^T$. It is easy to check that the controllability matrix has rank $n - 1$. So, the system is not linearly controllable at any equilibrium point around the origin. □

Theorem 5.1 has two major limitations. It assumes a convergent normal form as a power series. However, it is not easy to prove the convergence of a normal form. It is known that Poincaré normal form does not always converge. The same is true for the normal form of control systems. In fact, the problem of convergence of a normal form for control systems is still largely an open problem, although a lot of examples do have convergent normal forms. Another limitation of the necessary condition is due to the fact that systems may not be analytic. If the normal form is not analytic, then (2.13) is not a necessary condition. This is proved in the following example.

Example: Consider the system

$$\begin{aligned} \dot{z} &= z + p(x_1) \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{5.3}$$

where

$$p(x_1) = \begin{cases} e^{-\frac{1}{x_1^2}}, & x_1 \neq 0 \\ 0, & x_1 = 0 \end{cases}. \tag{5.4}$$

The system has an equilibrium set defined by

$$\{(z, x) | z = -p(x_1), x_2 = 0, x_1 = \epsilon, \epsilon \in \Re\}.$$

The function $p(x_1)$ is C^∞, but not analytic. Its Taylor expansion is zero. Therefore, (5.3) satisfies (5.1). However, the system is linearly controllable at all the equilibrium points except for the origin. This is true because of the following controllability matrix derived from (5.3) at any equilibrium point $(z, x_1, x_2) = (0, \epsilon, 0)$ with $\epsilon \neq 0$,

$$\begin{bmatrix} 0 & 0 & \frac{3}{x_1^3} e^{-\frac{1}{x_1^2}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

△

VI. Local Accessibility

Controllability and accessibility are fundamental properties of nonlinear control systems. It is proved that the accessibility of a control system is closely related to the dimension of the accessibility distribution. However, for systems with uncontrollable linearization, the computation of the dimension of the accessibility distribution is not straightforward, if it is possible. In this section, we prove a simple relationship between the normal form and its local accessibility for systems with a single nonzero uncontrollable mode. Based on this result, it is easy to check the local accessibility for systems in normal form.

In this section, we consider affine systems of the following form:

$$\Sigma : \dot{x} = f(x) + g(x)u \quad x(\cdot) \in \Re^n \quad u(\cdot) \in \mathcal{U} \tag{6.1}$$

where $\mathcal{U}$ is the space of piecewise continuous functions also called *admissible inputs*. The vector fields f and g are either smooth or analytic or of class C^k for sufficiently large k. Given a state x_0. Let V be a neighborhood of x_0. From [18], we denote $R^V(x_0, T)$ the reachable set from x_0 at time $T > 0$, following trajectories which remain for $t \leq T$ in V, and denote

$$R_T^V(x_0) = \cup_{\tau \leq T} R^V(x_0, \tau).$$

Definition 6.1 [18]: The system Σ is locally accessible from x_0 if $R_T^V(x_0)$ contains a nonempty open set of $\Re^n$ for all neighborhood V of x_0 and all $T > 0$.

Denote by $\mathcal{C}$ the smallest Lie algebra of vector fields on $\Re^n$ containing f and g. Let Δ be the involutive distribution generated by $\mathcal{C}$, that is

$$\Delta(x) = \text{span}\,\{X(x),\ X \in \mathcal{C}\}, \qquad \text{for any } x \in \mathcal{M}.$$

It is well-known that Σ is locally accessible from x_0 if $\dim \Delta(x_0) = n$.

In the following, we assume that (6.1) is in the normal form defined by (2.11). Because (6.1) is affine in control, x_n do not appear in the nonlinear part of (2.11) (see [12]). In this section, the nonlinear normal form of degree less than or equal to m_0 is used, where m_0 is an integer to be specified later. Thus, (6.1) has the following form:

$$\begin{aligned} \dot{z} &= f_1(z,x) + g_1(z,x)u \\ &= \lambda z + \sum_{m=2}^{m_0} f_1^{[m]}(z,x) + O(z,x,u)^{m_0+1} \\ \dot{x} &= f_2(z,x) + g_2(z,x)u \\ &= Ax + Bu + \sum_{m=2}^{m_0} f_2^{[m]}(z,x) + O(z,x,u)^{m_0+1} \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} f_1^{[m]}(z,x) &= \sum_{j=1}^{n-1} x_j^2 Q_j^{[m-2]}(z, \bar{x}_j) + x_1 S^{[m-1]}(z) \\ f_{2,i}^{[m]}(z,x) &= \sum_{j=i+2}^{n-1} x_j^2 P_{i,j}^{[m-2]}(z, \bar{x}_j) \end{aligned}$$

for $m \geq 2$ and $1 \leq i \leq n-3$.

Theorem 6.1: Consider (6.2). Suppose

$$f_1^{[m]}(0,x) \neq 0 \tag{6.3}$$

for some positive integer $m > 1$. Then, the distribution Δ has full rank at $(z,x) = (0,0)$. Thus, (6.2) is locally accessible at the origin.

From the normal form (6.2), the condition (6.3) is equivalent to the existence of nonnegative integers $i_1, \ldots, i_n$, with $i_1 + \cdots + i_n \geq 2$, such that

$$\frac{\partial^{i_1+\cdots+i_n} f_1}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(0,0) \neq 0. \tag{6.4}$$

Let $m_0 = i_1 + i_2 + \cdots + i_n$ be the smallest positive integer that satisfies the condition (6.4). Following differential geometry, the vector fields f and g are also denoted by

$$\begin{aligned} f(z,x) &= f_1(z,x)\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot f_2(z,x) \\ g(z,x) &= g_1(z,x)\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \cdot g_2(z,x) \end{aligned}$$

where

$$\frac{\partial}{\partial x} = \left[\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2}, \ldots, \qquad \frac{\partial}{\partial x_{n-1}}\right].$$

In $f_1^{[m_0]}(z,x)$, the terms independent of z form a new function, denoted by $f_1^{[m_0]}(x)$, i.e.,

$$f_1^{[m_0]}(x) = f_1^{[m_0]}(z,x)|_{z=0}.$$

Let $Q_x(z)$ be any function of (z,x) satisfying $O_x(0) = 0$. Then, the function $f_1(z,x)$ has the following form:

$$f_1(z,x) = f_1^{[m_0]}(x) + O_x(z) + O(z,x)^{m_0+1}. \tag{6.5}$$

Because

$$f^{[m_0]}(x) = \sum_{j=1}^{n-1} x_j^2 Q_j^{[m_0-2]}(0, \bar{x}_j) \neq 0$$

let s be the largest positive integer so that $Q_s^{[m_0-2]}(0, \bar{x}_s) \neq 0$. Therefore

$$\frac{\partial f_1^{[m_0]}(x)}{\partial x_j} = 0, \qquad \text{if } j > s. \tag{6.6}$$

To prove Theorem 6.1, we must derive the formulas for the vectors $ad_f^k(g)$. Given

$$\begin{aligned} f &= \left(f_1^{[m_0]}(x) + O_x(z) + O(z,x)^{m_0+1}\right)\frac{\partial}{\partial z} \\ &\quad + \frac{\partial}{\partial x} \cdot \left(Ax + O(z,x)^2\right) \\ g &= \frac{\partial}{\partial x_{n-1}} + O(z,x)^{[m_0+1]}\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \\ &\quad \cdot O(z,x)^{[m_0+1]} \end{aligned} \tag{6.7}$$

and the (6.6), it is straightforward to prove the following equation:

$$\begin{aligned}(-1)ad_f(g) &= [g,f] \\ &= \left(O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z}+\frac{\partial}{\partial x_{n-2}} \\ &\quad+\frac{\partial}{\partial x}\cdot O(z,x)\end{aligned} \tag{6.8}$$

if $s < n-1$. Once again, based on (6.8) and (6.6), it can be proved that

$$\begin{aligned}ad_f^2(g) &= [-ad_f(g),f] \\ &= \left(O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z}+\frac{\partial}{\partial x_{n-3}}+\frac{\partial}{\partial x}\cdot O(z,x)\end{aligned}$$

provided $s < n-2$. In general, if $k < n-s$, we have

$$\begin{aligned}(-1)^k ad_f^k(g) &= \left(O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z}+\frac{\partial}{\partial x_{n-k-1}} \\ &\quad+\frac{\partial}{\partial x}\cdot O(z,x), \qquad \text{for } k<n-s.\end{aligned} \tag{6.9}$$

When $k = n-s-1$, the vector field ad_f^{n-s-1} has a term $\partial/\partial x_s$. By the definition of s

$$\frac{\partial f_1^{[m_0]}(x)}{\partial x_s} \neq 0.$$

Therefore, if $k = n-s$, we have

$$\begin{aligned}(-1)^{n-s}ad_f^{n-s}(g) &= \left(\frac{\partial f_1^{[m_0]}(x)}{\partial x_s}+O_x(z)+O(z,x)^{m_0}\right) \\ &\quad\times\frac{\partial}{\partial z}+\frac{\partial}{\partial x_{s-1}}+\frac{\partial}{\partial x}\cdot O(z,x).\end{aligned} \tag{6.10}$$

In general, for any positive integer k satisfying $n-2 > k > n-s$

$$\begin{aligned}(-1)^k ad_f^k(g) &= \left(O(x)^{[m_0-1]}+O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z} \\ &\quad+\frac{\partial}{\partial x_{n-k-1}}+\frac{\partial}{\partial x}\cdot O(z,x).\end{aligned} \tag{6.11}$$

The formula of the vector field $ad_f^k(g)$ is summarized in the following lemma.

Lemma 6.1: Define the vector fields $h_k = (-1)^k ad_f^k(g)$, for $k = 1,2,\ldots,n-2$, then

$$\begin{aligned}h_k &= (-1)^k ad_f^k(g) \\ &= \left(O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z}+\frac{\partial}{\partial x_{n-k-1}} \\ &\quad+\frac{\partial}{\partial x}\cdot O(z,x), \qquad k<n-s \\ h_{n-s} &= (-1)^{n-s}ad_f^{n-s}(g) \\ &= \left(\frac{\partial f_1^{[m_0]}(x)}{\partial x_s}+O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z}+\frac{\partial}{\partial x_{s-1}} \\ &\quad+\frac{\partial}{\partial x}\cdot O(z,x) \\ h_k &= (-1)^k ad_f^k(g) \\ &= \left(O(x)^{[m_0-1]}+O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z} \\ &\quad+\frac{\partial}{\partial x_{n-k-1}}+\frac{\partial}{\partial x}\cdot O(z,x), \qquad k>n-s.\end{aligned} \tag{6.12}$$

Proof of Theorem 6.1: Define the vector field $\hat{f}$ by the following equation:

$$\begin{aligned}\hat{f} &= [h_{n-s-1},h_{n-s}] \\ &= \Big[(-1)^{n-s-1}ad_f^{n-s-1}(g), \\ &\qquad (-1)^{n-s}ad_f^{n-s}(g)\Big].\end{aligned} \tag{6.13}$$

From Lemma 6.1, we have

$$\begin{aligned}\hat{f} &= \left(\frac{\partial^2 f_1^{[m_0]}(x)}{\partial x_s^2}+O_x(z)+O(z,x)^{m_0-1}\right)\frac{\partial}{\partial z} \\ &\quad+\frac{\partial}{\partial x}\cdot V(z,x)\end{aligned} \tag{6.14}$$

where $V(z,x)$ is any vector field, which is not important for the derivation that follows. Suppose $i_1, i_2, \cdots, i_s$ is a sequence of nonnegative integers so that $i_1+i_2+\cdots+i_s = m_0$ and

$$\frac{\partial^{m_0} f_1^{[m_0]}(x)}{\partial x_1^{i_1}\partial x_2^{i_2}\ldots\partial x_s^{i_s}} \neq 0. \tag{6.15}$$

From the definition of m_0 and the structure of normal form, we know that the sequence $\{i_1,i_2,\ldots,i_s\}$ exists and $i_s \geq 2$. From Lemma 6.1 and the (6.14), it is straightforward to derive the following equation:

$$\begin{aligned}ad_{h_{n-j-1}}(\hat{f}) &= \left[(-1)^{n-j-1}ad_f^{n-j-1}(g),\hat{f}\right] \\ &= \left(\frac{\partial^3 f_1^{[m_0]}(x)}{\partial x_j\partial x_s^2}+O_x(z)+O(z,x)^{m_0-2}\right) \\ &\quad\times\frac{\partial}{\partial z}+\frac{\partial}{\partial x}\cdot V(z,x)\end{aligned} \tag{6.16}$$

where $V(z,x)$ is any vector field. It is a different function from the vector V in (6.14). However, because its value is not important for the derivation, we keep the same notation, V, for the reason of simplicity.

In general, we have

$$\begin{aligned}&ad_{h_{n-2}}^{i_1}\left(ad_{h_{n-3}}^{i_2}\left(\cdots\left(ad_{h_{n-s-1}}^{i_s-2}(\hat{f})\right)\ldots\right)\right) \\ &\quad= \left(\frac{\partial^{m_0} f_1^{[m_0]}(x)}{\partial x_1^{i_1}\partial x_2^{i_2}\ldots\partial x_s^{i_s}}+O_x(z)+O(z,x)^{m_0}\right)\frac{\partial}{\partial z} \\ &\quad\quad+\frac{\partial}{\partial x}\cdot V(z,x).\end{aligned} \tag{6.17}$$

Denote this vector by h_{n-1}. At the origin $(z,x) = (0,0)$, the vectors $h_{n-1}, h_{n-2}, \ldots, h_1$, and g are

$$\frac{\partial^{m_0} f_1^{[m_0]}(x)}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_s^{i_s}}\frac{\partial}{\partial z}+\frac{\partial}{\partial x}\cdot V(0,0), \qquad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \ldots, \quad \frac{\partial}{\partial x_{n-1}}$$

respectively. From (6.15) and the definition of h_k, the dimension of the distribution Δ is n at the origin, therefore, the system is locally accessible from $(z,x) = (0,0)$. □

The following remark is a partially converse result of Theorem 6.1

Remark: Given a system in the form of

$$\begin{aligned}\dot{z} &= f_1(z,x)\\ \dot{x} &= Ax + Bu + O(z,x)^2. \end{aligned} \tag{6.18}$$

Suppose

$$f_1(z,x)|_{z=0} = 0.$$

Then, (6.18) is not locally accessible from the origin. ◁

The proof of the remark is straightforward. If the initial condition satisfies $z = 0$, then the trajectory, $(z(t),\ x(t))$, of the system satisfies $z(t) = 0$ for all $t \geq 0$. Therefore, $R_T^V(0,0)$ is a subset of the subspace $\{(z,x)|z=0\}$, which does not contain an open set of $\Re^n$. Therefore, the system is not locally accessible from the origin.

As we know that the normal form, (6.2) is in the form of (6.18) if we let $m \to \infty$. In this case, the normal form is a series with infinite number of terms. The result in the remark can be considered as a converse result of Theorem 6.1 for analytic systems whose normal form is a convergent series. In this case, System (6.1) is accessible from the origin if and only if the condition (6.3) holds.

As an illustrative example, let us consider the following system:

$$\begin{aligned}\dot{z} &= z + a_{01}x_1 z + (a_{12}x_1 + a_{22}x_2)x_2^2\\ \dot{x}_1 &= x_2 + x_3^2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u.\end{aligned}$$

In this example $m_0 = 3$ and the integer $s = 2$. If $a_{12}a_{22} \neq 0$, then (6.3) is satisfied for $m = 3$. Therefore, the system must be locally accessible from the origin, although the system is not linearly controllable at the origin.

In this section, we focus on the accessibility from the equilibrium point. According to [8] and [23], the rank condition and the accessability of analytic systems in an open set are equivalent. The relationship between the rank condition and the invariants in an open neighborhood is a promising topic for future research.

VII. Conclusion

Three related topics are addressed in this paper, namely the bifurcation of controllability, the stabilization of systems with a positive uncontrollable mode, and the local accessibility of nonlinear systems. Both sufficient and necessary conditions for systems to be linearly controllable at other equilibrium points around an uncontrollable equilibrium are proved. The qualitative change in terms of linear controllability is used to design feedbacks for practical system stabilization in the presence of positive uncontrollable mode. Sufficient conditions are proved for systems to be locally accessible. The theoretical approach in all the proofs is based on the normal forms of nonlinear control systems. A nonlinear normal form consists of the core nonlinearity in a system that cannot be canceled by change of coordinates and state feedback. It is proved in this paper that the normal form of a control system characterizes the linear controllability and the local accessibility of the system.

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