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Norm Euclidean Quaternionic Orders

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Abstract

We determine the norm Euclidean orders in a positive definite quaternion algebra over $\mathbb{Q}$.

Lagrange (1770) proved the four square theorem via Euler's four square identity and a descent argument. Hurwitz [4] gave a quaternionic proof using the order $\Lambda(2)$ with $\mathbb{Z}$-basis: $1, i, j, \frac{1}{2}(1+i+j+k)$. Here $i^2 = j^2 = -1$ and $ij = -ji = k$, the standard basis of the quaternions. The key property of $\Lambda(2)$ is that it is norm Euclidean, namely, given $\alpha, \beta \in \Lambda(2)$ with $\beta \neq 0$, there exist $q, r \in \Lambda(2)$ such that $\alpha = \beta q + r$ and $N(r) < N(\beta)$. Liouville (1856) showed there are exactly seven positive definite quaternion norm forms (that is, 2-fold Pfister forms) $x^2 + ay^2 + bz^2 + abw^2$, with a, b positive integers, that represent all positive integers. These are $(1, a, b, ab) = (1, 1, 1, 1)$ and

$$\begin{array}{lll} (1,1,2,2) & (1,1,3,3) & (1,2,2,4) \\ (1,2,3,6) & (1,2,4,8) & (1,2,5,10). \end{array}$$

See volume III of Dickson [1] for more details.

Recently Deutsch constructed norm Euclidean orders to prove the universality of all but the last. Here we show there are, up to equivalence, exactly three norm Euclidean orders in a positive definite quaternion algebra over $\mathbb{Q}$. This includes one in $(\frac{-2,-5}{\mathbb{Q}})$.

1 Quaternionic orders

Q will denote a positive definite quaternion algebra over $\mathbb{Q}$. Write $Q = (\frac{a,b}{\mathbb{Q}})$. Here $a, b < 0$ are rational and Q has a basis $e_1 = 1,\ e_2, e_3, e_4$ with $e_2^2 = a$,

$e_3^2 = b$ and $e_2e_3 = -e_3e_2 = e_4$. For $\alpha = x_1e_1+x_2e_2+x_3e_3+x_4e_4$ the conjugate is $\bar{\alpha} = x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4$, the trace of α is $\mathrm{tr}(\alpha) = \alpha + \bar{\alpha}$ and the norm is $N(\alpha) = \alpha\bar{\alpha} = x_1^2 + ax_2^2 + bx_3^2 + abx_4^2$. An *order* in Q is a finitely generated $\mathbb{Z}$-module $A \subset Q$ such that A is a ring and $\mathbb{Q} \otimes_{\mathbb{Z}} A = Q$.

The following is well-known but hard to find in precisely this form. Our proof is a slight variation on that of [1] Lemma 2.

Lemma 1.1. *Let A be an order in Q.*

1. *$A \cap \mathbb{Q} \subset \mathbb{Z}$.*
2. *A is closed under conjugation.*
3. *$N(A) \subset \mathbb{Z}$.*

Proof: (1) If $p/q \in A$ for relatively prime integers p and q then all $p^n/q^n \in A$, contrary to A being finitely generated over $\mathbb{Z}$.

(2), (3) An element $\alpha \in A$ is root of $p(x) = x^2 - \mathrm{tr}(\alpha)x + N(\alpha) \in \mathbb{Q}[x]$. If $p(x)$ factors over $\mathbb{Q}$ then $\alpha \in A \cap \mathbb{Q} \subset \mathbb{Z}$ so that $\alpha = \bar{\alpha} \in A$ and $N(\alpha) \in \mathbb{Z}$. If $p(x)$ is irreducible over $\mathbb{Q}$ then, as α is integral over $\mathbb{Z}$, we have $\mathrm{tr}(\alpha)$, $N(\alpha) \in \mathbb{Z}$. And $\bar{\alpha} = \mathrm{tr}(\alpha) - \alpha \in A$. □

Let Q_1 and Q_2 be quaternion algebras over $\mathbb{Q}$ and let $A \subset Q_1, B \subset Q_2$ be orders. We call A and B *isomorphic* if they are isomorphic as rings. An isomorphism $\varphi : A \to B$ fixes $\mathbb{Z}$ so φ is a $\mathbb{Z}$-module map that preserves conjugation. In particular, $N(\varphi(\alpha)) = N(\alpha)$ so that if A is norm Euclidean then so is B. Further, if q_1 is a norm form for A and q_2 is a norm form for B then q_1 and q_2 are $\mathbb{Z}$-isometric.

Conversely, if q_1 and q_2 are $\mathbb{Z}$-isometric then the isometry extends to a $\mathbb{Q}$-isometry and so an isomorphism $Q_1 \to Q_2$. This restricts to an isomorphism $A \to B$. For details see [6] or [5].

The key to our classification is an unnamed property considered by Estes-Nipp [3]:

(*) If A is an order of Q, $\alpha \in A$ and $b \in \mathbb{Z}$ with $N(\alpha) = bc$, $(b, c) = 1$, then there exists $\beta \in A$ such that $\alpha = \beta\gamma$ for some $\gamma \in A$ and $N(\beta) = b$.

Proposition 1.2. *If A is a norm Euclidean order in Q then property (*) holds.*

Proof: Let $\alpha \in A$ with $N(\alpha) = bc$ where $(b, c) = 1$. Greatest common divisors exist in A since it is Euclidean. Set $\beta = (\alpha, b)$. Then $\alpha = \beta\gamma$ for some $\gamma \in A$. $N(\beta)$ divides $N(\alpha) = bc$ and $N(b) = b^2$. Hence $N(\beta)$ divides b. Further,

$$\begin{aligned}
\beta &= \alpha s + bt \quad \text{for some } s, t \in A \\
\bar{\beta} &= \bar{s}\bar{\alpha} + b\bar{t} \\
\beta\bar{\beta} &= N(\alpha)N(s) + [as\bar{t} + b\bar{s}\bar{\alpha} + bt\bar{t}]b \\
N(\beta) &= bcN(s) + \delta b,
\end{aligned}$$

where $\delta \in A$. Thus $N(\beta)/b = cN(s) + \delta \in \mathbb{Q} \cap A = \mathbb{Z}$ and so b divides $N(\beta)$. As Q is positive definite, $N(\beta) = b$. $\square$

Estes and Nipp show that there are, up to isomorphism, exactly 24 orders, in some positive definite quaternion algebra, with property (*). We will test the norm Euclidean property for each one to get our classification.

Lemma 1.3. *Let A be an order in Q. A is norm Euclidean iff for all $h \in Q$ there exists a $q \in A$ such that $N(h - q) < 1$.*

Proof: $(\longrightarrow)$ Let $h \in Q = \mathbb{Q} \otimes_{\mathbb{Z}} A$. Clear denominators to get $n \in \mathbb{Z}$ such that $nh \in A$. We have $nh = nq + r$ for some $q, r \in A$ with $N(r) < N(n) = n^2$. Then $N(h - q) = N(r/n) < 1$.

$(\longleftarrow)$ Let $\alpha, \beta \in A$ with $\beta \neq 0$. Set $h = \beta^{-1}\alpha \in Q$. We have by assumption $q \in A$ with $N(h - q) < 1$. Set $r_0 = h - q$. Then $\alpha = \beta q + \beta r_0$ and $N(\beta r_0) < N(\beta)$. $\square$

2 Classification

We follow the notation of Estes and Nipp. For each order we give the norm form, complete squares and deduce a $\mathbb{Z}$-basis $v_1 = 1, v_2, v_3, v_4$. We then test 1.3. We use the notation throughout of $q = \sum a_i v_i$, with each $a_i \in \mathbb{Z}$ so that $q \in A$. We also write $q = \sum q_i e_i$.

2.1 Orders in $Q = \left(\frac{-1,-1}{\mathbb{Q}}\right)$

i. $\Lambda(2)$

$$x^2+y^2+z^2+w^2+xw+yw+zw$$
$$(x+\tfrac{1}{2}w)^2+(y+\tfrac{1}{2}w)^2+(z+\tfrac{1}{2}w)^2+(\tfrac{1}{2}w)^2$$
$$v_2=e_2 \quad v_3=e_3 \quad v_4=\tfrac{1}{2}(1+e_2+e_3+e_4)$$

This is Hurwitz's algebra and so norm Euclidean.

ii. $\Lambda(4)$

$$x^2+y^2+z^2+w^2$$
$$v_2=e_2 \quad v_3=e_3 \quad v_4=e_4$$

Take $h=\frac{1}{2}(1+e_2+e_3+e_4)$. For any $q\in\Lambda(4)$, each q_i is an integer so that $N(h-q)\geq 4(\frac{1}{2})^2=1$. Hence $\Lambda(4)$ is not norm Euclidean.

iii. $\Lambda(6)'$

$$x^2+y^2+2z^2+2w^2+xy+2zw$$
$$(x+\tfrac{1}{2}y)^2+(\tfrac{1}{2}y+z)^2+(\tfrac{1}{2}y+w)^2+(-\tfrac{1}{2}y+z+w)^2$$
$$v_2=\tfrac{1}{2}(1+e_2+e_3-e_4) \quad v_3=e_2+e_4 \quad v_4=e_3+e_4$$

Take $h=\frac{1}{2}(1+e_2+e_3+e_4)$. For $q\in\Lambda(6)'$ we have:

$$q_1=a_1+\frac{1}{2}a_2 \quad q_2=a_3+\frac{1}{2}a_2 \quad q_3=a_4+\frac{1}{2}a_2 \quad q_4=a_3+a_4-\frac{1}{2}a_2.$$

If a_2 is even then the e_i-coefficients of $h-q$ have the form $(2n_i+1)/2$ with $n_i\in\mathbb{Z}$. Then $N(h-q)\geq 4(\frac{1}{2})^2=1$. If a_2 is odd then the e_i-coefficients of $h-q$ are integers. So $N(h-q)<1$ implies $h=q$. But then $a_3+\frac{1}{2}a_2=\frac{1}{2}=a_4+\frac{1}{2}a_2$ gives $a_3=a_4$ and adding $a_3+\frac{1}{2}a_2=\frac{1}{2}$ to $a_3+a_4-\frac{1}{2}a_2=\frac{1}{2}$ yields $3a_3=1$, contrary to $a_3\in\mathbb{Z}$. Hence $\Lambda(6)'$ is not norm Euclidean.

iv. $\Lambda(8)$

$$x^2+y^2+2z^2+2w^2$$
$$x^2+y^2+(z+w)^2+(z-w)^2$$
$$v_2=e_2 \quad v_3=e_3+e_4 \quad v_4=e_3-e_4$$

The same argument as for $\Lambda(4)$ shows $\Lambda(8)$ is not norm Euclidean.

v. $\Lambda(8)'$

$$x^2+y^2+3z^2+3w^2+xy+xz+xw+yw+3zw$$
$$(x+\tfrac{1}{2}y+\tfrac{1}{2}z+\tfrac{1}{2}w)^2+(\tfrac{1}{2}y-\tfrac{5}{6}z+\tfrac{5}{6}w)^2+(\tfrac{1}{2}y+\tfrac{7}{6}z+\tfrac{5}{6}w)^2+(-\tfrac{1}{2}y+\tfrac{5}{6}z+\tfrac{7}{6}w)^2$$
$$v_2=\tfrac{1}{2}(1+e_2+e_3-e_4) \quad v_3=\tfrac{1}{6}(3-5e_2+7e_3+5e_4) \quad v_4=\tfrac{1}{6}(3+5e_2+5e_3+7e_4)$$

Take $h = \frac{6}{5}(e_3 + e_4)$. For $q \in \Lambda(8)'$ we have:

$$q_1 = \frac{1}{2}(2a_1 + a_2 + a_3 + a_4) \qquad q_2 = \frac{1}{6}(3a_2 - 5a_3 + 5a_4)$$
$$q_3 = \frac{1}{6}(3a_2 + 7a_3 + 5a_4) \qquad q_4 = \frac{1}{6}(-3a_2 + 5a_3 + 7a_4)$$

To get $N(h - q) < 1$ requires each $(h_i - q_i)^2 < 1$ and so

$$q_1 = \frac{k}{2}, \quad -1 \le k \le 1 \qquad q_2 = \frac{k}{6}, \quad -5 \le k \le 5$$
$$q_3 = \frac{k}{6}, \quad 2 \le k \le 13 \qquad q_4 = \frac{k}{6}, \quad 2 \le k \le 13.$$

A computer search shows there are only seven such q arising from integral a_i:

(q_1, q_2, q_3, q_4)	$N(h - q)$
$(0, -\frac{1}{3}, \frac{5}{3}, \frac{1}{3})$	$\frac{27}{25}$
$(0, \frac{1}{3}, \frac{1}{3}, \frac{5}{3})$	$\frac{27}{25}$
$(\pm\frac{1}{2}, -\frac{5}{6}, \frac{7}{6}, \frac{5}{6})$	$\frac{27}{25}$
$(\pm\frac{1}{2}, \frac{5}{6}, \frac{5}{6}, \frac{7}{6})$	$\frac{27}{25}$
$(0, 0, 2, 2)$	$\frac{32}{25}$

Hence $\Lambda(8)'$ is not norm Euclidean.

vi. $\Lambda(10)'$

$$x^2 + y^2 + 3z^2 + 3w^2 + xz + yz + xw + yw + zw$$
$$(x + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + (y + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + (\tfrac{1}{2}z + \tfrac{3}{2}w)^2 + (\tfrac{3}{2}z - \tfrac{1}{2}w)^2$$
$$v_2 = e_2 \quad v_3 = \tfrac{1}{2}(1 + e_2 + e_3 + 3e_4) \quad v_4 = \tfrac{1}{2}(1 + e_2 + 3e_3 - e_4)$$

Take $h = \frac{1}{2}(1 + e_2 + e_3 + e_4)$. For $q \in \Lambda(10)'$ we have:

$$q_1 = a_1 + \frac{1}{2}(a_3 + a_4) \qquad q_2 = a_2 + \frac{1}{2}(a_3 + a_4)$$
$$q_3 = \frac{1}{2}(a_3 + 3a_4) \qquad q_4 = \frac{1}{2}(3a_3 - a_4).$$

If $a_3 \equiv a_4 \pmod 2$ then each $q_i \in \mathbb{Z}$ and $N(h-q) \geq 4(\frac{1}{2})^2 = 1$. If $a_3 \equiv a_4 + 1 \pmod 2$ then each q_i is of the form $(2n_i+1)/2$ for some $n_i \in \mathbb{Z}$. So $N(h-q) < 1$ implies $h = q$. But then $a_3 + 3a_4 = 1$ and $3a_3 - a_4 = 1$ and so $10a_3 = 4$, contrary to $a_3 \in \mathbb{Z}$. Hence $\Lambda(10)'$ is not norm Euclidean.

vii. $\Lambda(12)'''$

$$x^2 + 2y^2 + 2z^2 + 3w^2 + 2yz$$
$$x^2 + (y+w)^2 + (y+z-w)^2 + (z+w)^2$$
$$v_2 = e_2 + e_3 \quad v_3 = e_3 + e_4 \quad v_4 = e_2 - e_3 + e_4$$

The same argument as for $\Lambda(4)$ shows $\Lambda(12)'''$ is not norm Euclidean.

viii. $\Lambda(16)$

$$x^2 + 2y^2 + 2z^2 + 4w^2$$
$$x^2 + (y+z)^2 + (y-z)^2 + (2w)^2$$
$$v_2 = e_2 + e_3 \quad v_3 = e_2 - e_3 \quad v_4 = 2e_4$$

Again the argument for $\Lambda(4)$ shows $\Lambda(16)$ is not norm Euclidean.

ix. $\Lambda(16)''$

$$x^2 + 3y^2 + 3z^2 + 3w^2 + 2yz + 2yw - 2zw$$
$$x^2 + (y - \tfrac{1}{3}z + \tfrac{5}{3}w)^2 + (y + \tfrac{5}{3}z - \tfrac{1}{3}w)^2 + (-y + \tfrac{1}{3}z + \tfrac{1}{3}w)^2$$
$$v_2 = e_2 + e_3 - e_4 \quad v_3 = \tfrac{1}{3}(-e_2 + 5e_3 + e_4) \quad v_4 = \tfrac{1}{3}(5e_2 - e_3 + e_4)$$

Take $h = \frac{1}{6}(3 + 7e_2 + 7e_3 - e_4)$. For $q \in \Lambda(16)''$ we have $q_1 = a_1$ and

$$q_2 = a_2 + \frac{1}{3}(-a_3 + 5a_4) \quad q_3 = a_2 + \frac{1}{3}(5a_3 - a_4) \quad q_4 = -a_2 + \frac{1}{3}(a_3 + a_4).$$

To get $N(h-q) < 1$ we require each $(h_i - q_i)^2 < 1$ and so:

$$q_1 = k, \quad 0 \leq k \leq 1 \qquad q_2 = \frac{k}{3}, \quad 1 \leq k \leq 6$$
$$q_3 = \frac{k}{3}, \quad 1 \leq k \leq 6 \qquad q_4 = \frac{k}{3}, \quad -3 \leq k \leq 2.$$

A computer search shows there are only four such q arising from integral a_i: $(0,1,1,-1), (1,1,1,-1), (0,\frac{4}{3},\frac{4}{3},\frac{2}{3})$ and $(1,\frac{4}{3},\frac{4}{3},\frac{2}{3})$. In each case, $N(h-q) = 1$. Hence $\Lambda(16)''$ is not norm Euclidean.

x. $\Lambda(18)$

$$x^2 + y^2 + 5z^2 + 5w^2 + xz + yz + xw + yw + zw$$
$$(x + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + (y + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + (\tfrac{3}{2}z + \tfrac{3}{2}w)^2 + (\tfrac{3}{2}z - \tfrac{3}{2}w)^2$$
$$v_2 = e_2 \quad v_3 = \tfrac{1}{2}(1 + e_2 + 3e_3 + 3e_4) \quad v_4 = \tfrac{1}{2}(1 + e_2 + 3e_3 - 3e_4)$$

Take $h = \frac{1}{2}(1 + e_2 + e_3 + e_4)$. For $q \in \Lambda(18)$ we have:

$$q_1 = a_1 + \frac{1}{2}(a_3 + a_4) \qquad q_2 = a_2 + \frac{1}{2}(a_3 + a_4)$$
$$q_3 = \frac{3}{2}(a_3 + a_4) \qquad q_4 = \frac{3}{2}(a_3 - a_4).$$

If $a_3 \equiv a_4 \pmod 2$ then each $q_i \in \mathbb{Z}$ and $N(h-q) \geq 4(\frac{1}{2})^2 = 1$. If $a_3 \equiv 1 + a_4 \pmod 2$ then each q_i has the form $(2n_i + 1)/2$, for some $n_i \in \mathbb{Z}$. Thus $N(h - q) < 1$ implies $h = q$. In particular, $q_3 = \frac{1}{2} = q_4$ which implies $3a_3 = 1$, contrary to $a_i \in \mathbb{Z}$. So $\Lambda(18)$ is not norm Euclidean.

xi. $\Lambda(18)'$

$$x^2 + 2y^2 + 3z^2 + 5w^2 + xz + 2yz + 2yw + zw$$
$$(x + \tfrac{1}{2}z)^2 + (y + \tfrac{1}{2}z + 2w)^2 + (y + \tfrac{1}{2}z - w)^2 + (\tfrac{3}{2}z)^2$$
$$v_2 = e_2 + e_3 \quad v_3 = \tfrac{1}{2}(1 + e_2 + e_3 + 3e_4) \quad v_4 = 2e_2 - e_3$$

Take $h = \frac{1}{2}(1 + e_2 + e_3 + e_4)$. For $q \in \Lambda(18)'$ we have:

$$q_1 = a_1 + \frac{1}{2}a_3 \qquad q_2 = a_2 + \frac{1}{2}a_3 + 2a_4$$
$$q_3 = a_2 + \frac{1}{2}a_2 - a_4 \qquad q_4 = \frac{3}{2}a_3.$$

If a_3 is even then each $q_i \in \mathbb{Z}$ and $N(h - q) \geq 4(\frac{1}{2})^2 = 1$. If a_3 is odd then each q_i has the form $(2n_i + 1)/2$, for some $n_i \in \mathbb{Z}$. Thus $N(h-q) < 1$ implies $h = q$. In particular, $q_4 = \frac{1}{2}$ which implies $3a_3 = 1$, contrary to $a_i \in \mathbb{Z}$. So $\Lambda(18)'$ is not norm Euclidean.

xii. $\Lambda(22)$

$$x^2 + 2y^2 + 3z^2 + 6w^2 + xz + 2yw$$
$$(x + \tfrac{1}{2}z)^2 + (y + \tfrac{1}{2}z - w)^2 + (-y + \tfrac{1}{2}z - 2w)^2 + (\tfrac{3}{2}z + w)^2$$
$$v_2 = e_2 - e_3 \quad v_3 = \tfrac{1}{2}(1 + e_2 + e_3 + 3e_4) \quad v_4 = e_2 + 2e_3 - e_4$$

Take $h = \frac{1}{2}(1+e_2+e_3+e_4)$. For $q \in \Lambda(18)'$ we have:

$$q_1 = a_1 + \frac{1}{2}a_3 \qquad q_2 = a_2 + \frac{1}{2}a_3 + a_4$$
$$q_3 = -a_2 + \frac{1}{2}a_2 + 2a_4 \qquad q_4 = \frac{3}{2}a_3 - a_4.$$

If a_3 is even then each $q_i \in \mathbb{Z}$ and $N(h-q) \geq 4(\frac{1}{2})^2 = 1$. If a_3 is odd then each q_i has the form $(2n_i+1)/2$, for some $n_i \in \mathbb{Z}$. Thus $N(h-q) < 1$ implies $h = q$. Adding $q_2 = \frac{1}{2}$ to $q_3 = \frac{1}{2}$ gives $a_3 + 3a_4 = 1$. Combine this with $q_4 = \frac{1}{2}$ to get $11a_3 = 5$, contrary to $a_i \in \mathbb{Z}$. So $\Lambda(18)'$ is not norm Euclidean.

2.2 Orders in $Q = (\frac{-1,-3}{\mathbb{Q}})$

i. $\Lambda(3)$

$$x^2 + y^2 + z^2 + w^2 + xy + zw$$
$$(x + \tfrac{1}{2}y)^2 + (z + \tfrac{1}{2}w)^2 + 3(\tfrac{1}{2}y)^2 + 3(\tfrac{1}{2}w)^2$$
$$v_2 = \tfrac{1}{2}(1+e_3) \quad v_3 = e_2 \quad v_4 = \tfrac{1}{2}(e_2+e_4)$$

The norm form of the algebra $H_{1,3,3}$ constructed by Deutsch [1] is $x^2 + y^2 + z^2 + w^2 + xw + yz$ which is clearly isometric to above the norm form of $\Lambda(3)$. Hence $\Lambda(3)$ is isomorphic to $H_{1,3,3}$ and so norm Euclidean.

ii. $\Lambda(6)$

$$x^2 + y^2 + 2z^2 + 2w^2 + xz + yz + xw + yw + zw$$
$$(x + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + (y + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + 3(\tfrac{1}{2}z + \tfrac{1}{2}w)^2 + 3(\tfrac{1}{2}z - \tfrac{1}{2}w)^2$$
$$v_2 = e_2 \quad v_3 = \tfrac{1}{2}(1+e_2+e_3+e_4) \quad v_4 = \tfrac{1}{2}(1+e_2+e_3-e_4)$$

Take $h = 1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + e_4$. For $q \in \Lambda(6)$ we have:

$$q_1 = a_1 + \frac{1}{2}(a_3+a_4) \qquad q_2 = a_2 + \frac{1}{2}(a_3+a_4)$$
$$q_3 = \frac{1}{2}(a_3+a_4) \qquad q_4 = \frac{1}{2}(a_3-a_4).$$

Suppose $N(h-q) < 1$. Then $q_3 = 0, \frac{1}{2}$ or 1 since otherwise $N(h-q) \geq 3(\frac{1}{2}-q_3)^2 \geq 3$. If $q_3 = 0$ or 1 then $N(h-q) \geq (\frac{1}{2}-a_2)^2 + 3(\frac{1}{2})^2 \geq 1$. If $q_3 = \frac{1}{2}$ then $q_1 = a_1 + 1$ and $q_4 = \frac{1}{2} - a_4$. Then $N(h-q) \geq (\frac{1}{2}-a_1)^2 + 3(\frac{1}{2}+a_4)^2 \geq 1$. Hence $\Lambda(6)$ is not norm Euclidean.

iii. $\Lambda(12)$

$$x^2 + y^2 + 3z^2 + 3w^2$$
$$v_2 = e_2 \quad v_3 = e_3 \quad v_4 = e_4$$

Take $h = \frac{1}{2}(1 + e_2 + e_3 + e_4)$. Any $q \in \Lambda(12)$ has integral q_i so $N(h - q) \geq (\frac{1}{2})^2 + (\frac{1}{2})^2 + 3(\frac{1}{2})^2 + 3(\frac{1}{2})^2 = 2$. So $\Lambda(12)$ is not norm Euclidean.

iv. $\Lambda(12)'$

$$x^2 + y^2 + 4z^2 + 4w^2 + xy + 4zw$$
$$(x + \tfrac{1}{2}y)^2 + (2z + w)^2 + 3(\tfrac{1}{2}y)^2 + 3w^2$$
$$v_2 = \tfrac{1}{2}(1 + e_3) \quad v_3 = 2e_2 \quad v_4 = e_2 + e_4$$

Take $h = e_2$. For $q \in \Lambda(12)'$ we have:

$$q_1 = a_1 + \frac{1}{2}a_2 \quad q_2 = 2a_2 + a_4 \quad q_3 = \frac{1}{2}a_3 \quad q_4 = a_4.$$

If $a_4 \neq 0$ then $N(h - q) \geq 3a_4^2 \geq 3$. If $a_4 = 0$ then q_2 is even and $N(h - q) \geq (1 - q_2)^2 \geq 1$. So $\Lambda(12)'$ is not norm Euclidean.

v. $\Lambda(12)''$

$$x^2 + 2y^2 + 2z^2 + 4w^2 + xy + xz + xw + 2yw + 2zw$$
$$(x + \tfrac{1}{2}y + \tfrac{1}{2}z + \tfrac{1}{2}w)^2 + (y + \tfrac{5}{7}z + \tfrac{6}{7}w)^2 + 3w^2 + 3(\tfrac{1}{2}y - \tfrac{9}{14}z - \tfrac{1}{14}w)^2$$
$$v_2 = \tfrac{1}{2}(1 + 2e_2 + e_4) \quad v_3 = \tfrac{1}{14}(7 + 10e_2 - 9e_4) \quad v_4 = \tfrac{1}{14}(7 + 12e_2 + 14e_3 - e_4)$$

Take $h = \frac{1}{2}(1 + e_2 + e_3 + e_4)$. For $q \in \Lambda(12)''$ we have:

$$q_1 = a_1 + \frac{1}{2}(a_2 + a_3 + a_4) \qquad q_2 = \frac{1}{14}(7a_2 + 10a_3 + 12a_4)$$
$$q_3 = \frac{1}{14}(21a_2 - 9a_3 - a_4) \qquad q_4 = a_4.$$

Suppose $N(h - q) < 1$. Then $a_4 = 0$ or 1 else $N(h - q) \geq 3(\frac{1}{2} - a_4)^2 \geq \frac{27}{4}$. Hence $3(\frac{1}{2} - a_4)^2 = \frac{3}{4}$. Then $q_1 = \frac{1}{2}$ else $N(h - q) \geq (\frac{1}{2} - q_1)^2 + \frac{3}{4} \geq 1$. We further require $(\frac{1}{2} - q_2)^2 < \frac{1}{4}$ and $3(\frac{1}{2} - q_3)^2 < \frac{1}{4}$. Hence:

$$q_2 = \frac{k}{14}, \quad 1 \leq k \leq 13 \qquad q_3 = \frac{k}{14}, \quad 3 \leq k \leq 11.$$

A computer search shows there is no such q_i arising from integral a_i. So $\Lambda(12)''$ is not norm Euclidean.

vi. $\Lambda(24)$

$$x^2 + 3y^2 + 4z^2 + 4w^2 + 4zw$$
$$x^2 + (2z + w)^2 + 3y^2 + 3w^2$$
$$v_2 = e_3 \quad v_3 = 2e_2 \quad v_4 = e_2 + e_4$$

The same proof as for $\Lambda(6)$ shows $\Lambda(24)$ is not norm Euclidean.

2.3 Orders in $Q = (\frac{-2,-5}{\mathbb{Q}})$

i. $\Lambda(5)$

The lengthy proof that $\Lambda(5)$ is norm Euclidean is postponed to section 3.

ii. $\Lambda(10)$

$$x^2 + 2y^2 + 2z^2 + 3w^2 + xy + 2yz + yw + 2zw$$
$$(x + \tfrac{1}{2}y)^2 + 2(\tfrac{1}{2}y + z + \tfrac{1}{2}w)^2 + 5(\tfrac{1}{2}y)^2 + 10(\tfrac{1}{2}w)^2$$
$$v_2 = \tfrac{1}{2}(1 + e_2 + e_3) \quad v_3 = e_2 \quad v_4 = \tfrac{1}{2}(e_2 + e_4)$$

Take $h = \frac{1}{4}(e_2 + e_3 + e_4)$. For $q \in \Lambda(10)$ we have each $q_i \in \frac{1}{2}\mathbb{Z}$. Hence $N(h - q) \geq 2(\frac{1}{4})^2 + 5(\frac{1}{4})^2 + 10(\frac{1}{4})^2 > 1$. So $\Lambda(10)$ is not norm Euclidean.

iii. $\Lambda(20)$

$$x^2 + 2y^2 + 3z^2 + 5w^2 + 2yz$$
$$x^2 + 2(y + \tfrac{1}{2}z)^2 + 5w^2 + 10(\tfrac{1}{2}z)^2$$
$$v_2 = e_2 \quad v_3 = \tfrac{1}{2}(e_2 + e_4) \quad v_4 = e_3$$

The same proof as for the previous case shows $\Lambda(20)$ is not norm Euclidean.

2.4 Orders in $Q = (\frac{-1,-7}{\mathbb{Q}})$

i. $\Lambda(7)$

$$x^2 + y^2 + 2z^2 + 2w^2 + xz + yw$$
$$(x + \tfrac{1}{2}z)^2 + (y + \tfrac{1}{2}w)^2 + 7(\tfrac{1}{2}z)^2 + 7(\tfrac{1}{2}w)^2$$
$$v_2 = e_2 \quad v_3 = \tfrac{1}{2}(1 + e_3) \quad v_4 = \tfrac{1}{2}(e_2 + e_4)$$

Take $h = \frac{1}{4}(1 + e_2 + e_3 + e_4)$. For $q \in \Lambda(7)$ each $q_i \in \frac{1}{2}\mathbb{Z}$. Hence $N(h - q) \geq (\frac{1}{4})^2 + (\frac{1}{4})^2 + 7(\frac{1}{4})^2 + 7(\frac{1}{4})^2 = 1$. So $\Lambda(7)$ is not norm Euclidean.

ii. $\Lambda(28)$

$$x^2+3y^2+3z^2+8w^2+xy+xz-yz+2yw+2zw$$
$$(x+\tfrac{1}{2}y+\tfrac{1}{2}z)^2+(y+z+w)^2+7(\tfrac{1}{2}y-\tfrac{1}{2}z)^2+7w^2$$
$$v_2=\tfrac{1}{2}(1+2e_2+e_3) \quad v_3=\tfrac{1}{2}(1+2e_2-e_3) \quad v_4=e_2+e_4$$

The same proof as for the previous case shows $\Lambda(28)$ is not norm Euclidean.

2.5 The order in $Q=(\frac{-7,-13}{\mathbb{Q}})$

i. $\Lambda(13)$

$$x^2+2y^2+2z^2+4w^2+xy+yz+xw+yw+2zw$$
$$(x+\tfrac{1}{2}y+\tfrac{1}{2}w)^2+7(\tfrac{1}{2}y+\tfrac{1}{7}z+\tfrac{1}{14}w)^2+13(\tfrac{1}{2}w)^2+91(\tfrac{1}{7}z+\tfrac{1}{14}w)^2$$
$$v_2=\tfrac{1}{2}(1+e_2) \quad v_3=\tfrac{1}{7}(e_2+e_4) \quad v_4=\tfrac{1}{14}(7+e_2+7e_3+e_4)$$

Take $h=\frac{1}{4}(1+e_2+e_3+e_4)$. For $q\in\Lambda(13)$ we have:

$$q_1=a_1+\frac{1}{2}a_2+\frac{1}{2}a_4 \qquad q_2=\frac{1}{2}a_2+\frac{1}{7}a_3+\frac{1}{14}a_4$$
$$q_3=\frac{1}{2}a_3 \qquad q_4=\frac{1}{7}a_3+\frac{1}{14}a_4.$$

Assume $N(h-q)<1$. From $13(\frac{1}{4}-\frac{1}{2}a_3)^2<1$ we get $a_3=0$ or 1 and so $13(\frac{1}{4}-\frac{1}{2}a_3)^2=\frac{13}{16}$. Thus $(\frac{1}{4}-q_1)^2, 7(\frac{1}{4}-q_2)^2$ and $91(\frac{1}{4}-q_4)^2$ are all less than $\frac{3}{16}$. We get:

$$q_1=\frac{k}{2},\quad 0\le k\le 1 \qquad q_2=\frac{k}{14},\quad 2\le k\le 5 \qquad q_4=\frac{k}{14},\quad 3\le k\le 4.$$

A computer search shows there are only four such q_i arising from integral a_i: $(\frac{1}{2},\frac{3}{14},0,\frac{3}{14}),(\frac{1}{2},\frac{3}{14},\frac{1}{2},\frac{3}{14}),(0,\frac{2}{7},0,\frac{2}{7})$ and $(1,\frac{2}{7},\frac{1}{2},\frac{2}{7})$. In each case, $N(h-q)=1$. So $\Lambda(13)$ is not norm Euclidean.

This completes the analysis of the 24 orders with property (*). We summarize:

Theorem 2.1. *The norm Euclidean orders in a positive definite quaternion algebra over $\mathbb{Q}$ are, up to isomorphism, precisely: $\Lambda(2),\Lambda(3)$ and $\Lambda(5)$.*

Note that the three orders of (2.1) are in distinct quaternion algebras and that they are among the five maximal orders whose spinor genus and class coincide, as determined in [3] page 232.

Corollary 2.2. *Let Q be a positive definite quaternion algebra over $\mathbb{Q}$. Let A and B be orders in Q. If A and B are both norm Euclidean then they are isomorphic.*

Corollary 2.3. *If A is norm Euclidean order in a positive definite quaternion algebra over $\mathbb{Q}$ then A is maximal and $Spn(A) = Cls(A)$.*

Estes and Nipp consider a property called FNF: If A is an order of Q, $\alpha \in A$ and $b \in \mathbb{Z}$ with $N(\alpha) = bc$, then there exists $\beta \in A$ such that $\alpha = \beta\gamma$ for some $\gamma \in A$ and $N(\beta) = b$. This is property (*) without the assumption that $(b, c) = 1$.

Corollary 2.4. *Let A be an order of positive definite Q. If A is norm Euclidean then A has FNF.*

Proof: The orders of Theorem 2.1 have FNF by [3] Theorem 6. □

3 Proof for $\Lambda(5)$

The norm form and a basis of $\Lambda(5)$ are:

$$x^2 + y^2 + 2z^2 + 2w^2 + xy + xz + xw + yw + 2zw$$
$$(x + \tfrac{1}{2}(y + z + w))^2 + 2(\tfrac{1}{4}(y + 3z + 2w))^2 + 5(\tfrac{1}{2}w)^2 + 10(\tfrac{1}{4}(-y + z))^2$$
$$v_2 = \tfrac{1}{4}(2 + e_2 - e_4) \quad v_3 = \tfrac{1}{4}(2 + 3e_2 + e_4) \quad v_4 = \tfrac{1}{2}(1 + e_2 + e_3)$$

We will show that$\Lambda(5)$ is norm Euclidean. Let $h = \sum h_i e_i \in Q = (\frac{-2,-5}{\mathbb{Q}})$. We want to find a $q \in A$ with $N(h - q) < 1$.

Set $\bar{h} = (h_1, h_2, h_3, h_4)$. Note that:

$$e_2 = v_2 + v_3 - 1 \quad e_3 = 2v_4 - v_2 - v_3 \quad e_4 = v_3 - 3v_2 + 1,$$

are all in A. Hence we may assume each $h_i \in [0, 1]$. Suppose $h_1 \in [\frac{1}{2}, 1]$ and that for $h^* = (1 - h_1) + \sum_{i \geq 2} h_i e_i$ we have $q^* = \sum q_i e_i \in A$ with $N(h^* - q^*) < 1$. Then, as $q_1 \in \frac{1}{2}\mathbb{Z}$, we can write $q_1 = k/2$ for some integer k. So $q = (1 - q_1) + \sum_{i \geq 2} q_i e_i = q^* - (k - 1) \in A$ and $N(h - q) = N(h^* - q^*) < 1$. Thus we may assume $h_1 \in [0, \frac{1}{2}]$. As $q_3 \in \frac{1}{2}\mathbb{Z}$ also, the same reasoning shows we may assume $h_3 \in [0, \frac{1}{2}]$. Lastly, $\frac{1}{2}(\pm e_2 + e_4) \in \Lambda(5)$ implies that we may assume $e_4 \in [0, \frac{1}{2}]$ also. In summary, we may assume

$$\bar{h} \in [0, \frac{1}{2}] \times [0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}].$$

Set $\bar{q} = (q_1, q_2, q_3, q_4)$. The following $\bar{q}$ represent elements of A:

$$-v_2 + v_4 \to (0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}) \qquad -v_2 + v_3 \to (0, \frac{1}{2}, 0, \frac{1}{2})$$
$$v_1 - v_2 \to (\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}) \qquad 1 - 2v_2 + v_4 \to (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$$
$$v_3 \to (\frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}) \qquad v_4 \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0).$$

Set $n(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2^2 + 5x_3^2 + 10x_4^2$ so that $N(\sum x_i e_i) = n(\bar{x})$.

In the table below, for each block of $\bar{h}$'s we give a $\bar{q} \in A$ and compute $n(\bar{x})$ where x_i is the maximum, over the block, of $|h_i - q_i|$.Each $n(\bar{x})$ is less than one, proving that $\Lambda(5)$ is norm Euclidean.

	h_1	h_2	h_3	h_4	q	maximal norm
1.	$[0, \frac{1}{4}]$	$[0, \frac{1}{4}]$	$[0, \frac{1}{4}]$	$[0, \frac{3}{16}]$	$0, 0, 0, 0$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}) = \frac{109}{128}$
			$[0, \frac{1}{8}]$	$[\frac{3}{16}, \frac{1}{4}]$	$0, 0, 0, 0$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}) = \frac{57}{64}$
			$[\frac{1}{8}, \frac{1}{4}]$		$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{16}) = \frac{119}{128}$
2.	$[0, \frac{1}{4}]$	$[0, \frac{1}{4}]$	$[0, \frac{1}{8}]$	$[\frac{1}{4}, \frac{5}{16}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{16}) = \frac{111}{128}$
				$[\frac{5}{16}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{3}{16}) = \frac{127}{128}$
		$[0, \frac{1}{8}]$	$[\frac{1}{8}, \frac{1}{6}]$	$[\frac{1}{4}, \frac{1}{3}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{12}) = \frac{553}{576}$
				$[\frac{1}{3}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}) = \frac{47}{48}$
			$[\frac{1}{6}, \frac{1}{4}]$	$[\frac{1}{4}, \frac{2}{5}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{3}{20}) = \frac{697}{720}$
				$[\frac{2}{5}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}) = \frac{39}{40}$
		$[\frac{1}{8}, \frac{1}{4}]$	$[\frac{1}{8}, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{3}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{8}, \frac{3}{8}, \frac{1}{12}) = \frac{499}{576}$
				$[\frac{1}{3}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{6}) = \frac{269}{288}$
3.	$[0, \frac{1}{4}]$	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{3}]$	$[0, \frac{1}{8}]$	$0, 0, 0, 0$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{8}) = \frac{259}{288}$
			$[\frac{1}{3}, \frac{1}{2}]$		$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}) = \frac{137}{144}$
			$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{8}, \frac{1}{4}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{21}{32}$
4.	$[0, \frac{1}{4}]$	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{4}, \frac{3}{8}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{21}{32}$
				$[\frac{3}{8}, \frac{1}{2}]$	$\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}$	$n(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{27}{32}$

	h_1	h_2	h_3	h_4	q	maximal norm
5.	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{12}{25}]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$0,0,0,0$	$n(\frac{1}{4},\frac{12}{25},\frac{1}{4},\frac{1}{8})=\frac{19841}{20000}$
		$[\frac{12}{25},\frac{1}{2}]$	$[0,\frac{1}{5}]$		$0,0,0,0$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{5},\frac{1}{8})=\frac{147}{160}$
			$[\frac{1}{5},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{3}{10},\frac{1}{8})=\frac{157}{160}$
	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$
	$[0,\frac{1}{8})$		$[\frac{1}{8},\frac{1}{4}]$		$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{8})=1$
	$[\frac{1}{8},\frac{1}{4}]$	$[\frac{1}{4},\frac{3}{8}]$			$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{8},\frac{3}{8},\frac{1}{8})=\frac{61}{64}$
		$[\frac{3}{8},\frac{1}{2}]$			$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{3}{8},\frac{3}{8},\frac{1}{4},\frac{1}{8})=\frac{57}{64}$
6.	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$[\frac{1}{4},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{4})=\frac{57}{64}$
			$[\frac{1}{8},\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{12})=\frac{553}{576}$
				$[\frac{1}{3},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{6})=\frac{7}{9}$
7.	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
8.	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{3}{8}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
			$[\frac{1}{4},\frac{5}{16}]$	$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{5}{16},\frac{1}{8})=\frac{213}{256}$
			$[\frac{5}{16},\frac{1}{2}]$		$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{16},\frac{1}{4})=\frac{253}{256}$
9.	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{4}]$	$[0,\frac{7}{40}]$	$0,0,0,0$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{7}{40})=\frac{159}{160}$
				$[\frac{7}{40},\frac{1}{4}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{3}{40})=\frac{149}{160}$
10.	$[\frac{1}{4},\frac{5}{16}]$	$[0,\frac{3}{20}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{21}{50}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{2}{5},\frac{1}{4},\frac{17}{100})=\frac{123}{125}$
				$[\frac{21}{50},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{5}{16},\frac{1}{2},\frac{1}{4},\frac{2}{25})=\frac{31173}{32000}$
		$[\frac{3}{20},\frac{1}{4}]$		$[\frac{1}{4},\frac{1}{3}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{12})=\frac{17}{18}$
				$[\frac{1}{3},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{5}{16},\frac{7}{20},\frac{1}{4},\frac{1}{6})=\frac{53737}{57600}$
	$[\frac{5}{16},\frac{3}{8}]$	$[0,\frac{3}{20}]$		$[\frac{1}{4},\frac{54}{125}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{3}{16},\frac{2}{5},\frac{1}{4},\frac{91}{500})=\frac{799117}{800000}$
				$[\frac{54}{125},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{3}{8},\frac{1}{2},\frac{1}{4},\frac{17}{250})=\frac{199873}{200000}$
		$[\frac{3}{20},\frac{1}{4}]$		$[\frac{1}{4},\frac{1}{3}]$	$\frac{1}{2},-\frac{1}{4},0,\frac{1}{4}$	$n(\frac{3}{16},\frac{1}{2},\frac{1}{4},\frac{1}{12})=\frac{2113}{2304}$

	h_1	h_2	h_3	h_4	q	maximal norm
				$[\frac{1}{3}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{3}{8}, \frac{7}{20}, \frac{1}{4}, \frac{1}{6}) = \frac{14053}{14400}$
	$[\frac{3}{8}, \frac{2}{5}]$	$[0, \frac{1}{10}]$		$[\frac{1}{4}, \frac{9}{20}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{8}, \frac{7}{20}, \frac{1}{4}, \frac{1}{5}) = \frac{1557}{1600}$
				$[\frac{9}{20}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{2}{5}, \frac{1}{2}, \frac{1}{4}, \frac{1}{20}) = \frac{399}{400}$
		$[\frac{1}{10}, \frac{1}{4}]$		$[\frac{1}{4}, \frac{3}{8}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{8}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}) = \frac{63}{64}$
				$[\frac{3}{8}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{2}{5}, \frac{2}{5}, \frac{1}{4}, \frac{1}{8}) = \frac{759}{800}$
	$[\frac{2}{5}, \frac{1}{2}]$	$[0, \frac{1}{10}]$	$[0, \frac{1}{8}]$	$[\frac{1}{4}, \frac{1}{2}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{10}, \frac{7}{20}, \frac{1}{8}, \frac{1}{4}) = \frac{1533}{1600}$
			$[\frac{1}{8}, \frac{1}{4}]$	$[\frac{1}{4}, \frac{3}{8}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{10}, \frac{7}{20}, \frac{1}{4}, \frac{1}{8}) = \frac{579}{800}$
				$[\frac{3}{8}, \frac{1}{2}]$	$\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}$	$n(\frac{1}{10}, \frac{1}{10}, \frac{3}{8}, \frac{1}{8}) = \frac{1423}{1600}$
		$[\frac{1}{10}, \frac{3}{20}]$	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{2}{5}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{10}, \frac{2}{5}, \frac{1}{4}, \frac{3}{20}) = \frac{347}{400}$
				$[\frac{2}{5}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{1}{10}) = \frac{393}{400}$
		$[\frac{3}{20}, \frac{1}{4}]$		$[\frac{1}{4}, \frac{3}{8}]$	$\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$	$n(\frac{1}{10}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}) = \frac{783}{800}$
				$[\frac{3}{8}, \frac{1}{2}]$	$0, \frac{1}{2}, 0, \frac{1}{2}$	$n(\frac{1}{2}, \frac{7}{20}, \frac{1}{4}, \frac{1}{8}) = \frac{771}{800}$
11.	$[\frac{1}{4}, \frac{1}{2}]$	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{5}{16}]$	$[0, \frac{1}{16}]$	$0, 0, 0, 0$	$n(\frac{1}{2}, \frac{1}{4}, \frac{5}{16}, \frac{1}{16}) = \frac{231}{256}$
			$[\frac{5}{16}, \frac{1}{2}]$		$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0$	$n(\frac{1}{4}, \frac{1}{2}, \frac{3}{16}, \frac{1}{16}) = \frac{199}{256}$
	$[\frac{1}{4}, \frac{3}{8}]$		$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{16}, \frac{1}{4}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}) = \frac{119}{128}$
	$[\frac{3}{8}, \frac{1}{2}]$				$\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}$	$n(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}) = \frac{103}{128}$
12.	$[\frac{1}{4}, \frac{1}{2}]$	$[0, \frac{1}{4}]$	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{4}, \frac{3}{8}]$	$0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{27}{32}$
				$[\frac{3}{8}, \frac{1}{2}]$	$\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}) = \frac{21}{32}$
13.	$[\frac{1}{4}, \frac{3}{8}]$	$[\frac{1}{4}, \frac{2}{5}]$	$[0, \frac{1}{4}]$	$[0, \frac{3}{20}]$	$0, 0, 0, 0$	$n(\frac{3}{8}, \frac{2}{5}, \frac{1}{4}, \frac{3}{20}) = \frac{1597}{1600}$
				$[\frac{3}{20}, \frac{1}{4}]$	$\frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}) = \frac{39}{40}$
		$[\frac{2}{5}, \frac{1}{2}]$		$[0, \frac{3}{50}]$	$0, 0, 0, 0$	$n(\frac{3}{8}, \frac{1}{2}, \frac{1}{4}, \frac{3}{50}) = \frac{7913}{8000}$
				$[\frac{3}{50}, \frac{1}{4}]$	$\frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{7}{20}, \frac{1}{4}, \frac{19}{100}) = \frac{981}{1000}$
	$[\frac{3}{8}, \frac{2}{5}]$	$[\frac{1}{4}, \frac{2}{5}]$		$[0, \frac{3}{25}]$	$0, 0, 0, 0$	$n(\frac{2}{5}, \frac{2}{5}, \frac{1}{4}, \frac{3}{25}) = \frac{1873}{2000}$
				$[\frac{3}{25}, \frac{1}{4}]$	$\frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}$	$n(\frac{1}{8}, \frac{1}{2}, \frac{1}{4}, \frac{13}{100}) = \frac{7977}{8000}$

	h_1	h_2	h_3	h_4	q	maximal norm
		$[\frac{2}{5},\frac{1}{2}]$		$[0,\frac{1}{20}]$	$0,0,0,0$	$n(\frac{2}{5},\frac{1}{2},\frac{1}{4},\frac{1}{20})=\frac{399}{400}$
				$[\frac{1}{20},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{8},\frac{7}{20},\frac{1}{4},\frac{1}{5})=\frac{1557}{1600}$
	$[\frac{2}{5},\frac{1}{2}]$	$[\frac{1}{4},\frac{3}{8}]$	$[0,\frac{1}{8}]$	$[0,\frac{1}{10}]$	$0,0,0,0$	$n(\frac{1}{2},\frac{3}{8},\frac{1}{8},\frac{1}{10})=\frac{227}{320}$
				$[\frac{1}{10},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{8},\frac{3}{20})=\frac{1301}{1600}$
			$[\frac{1}{8},\frac{1}{4}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{10},\frac{1}{4},\frac{3}{8},\frac{1}{8})=\frac{1591}{1600}$
				$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{4},\frac{1}{8})=\frac{783}{800}$
		$[\frac{3}{8},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$[0,\frac{1}{8}]$	$0,0,0,0$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$
			$[\frac{1}{8},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{10},\frac{1}{8},\frac{3}{8},\frac{1}{8})=\frac{1441}{1600}$
			$[0,\frac{1}{4}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{10},\frac{3}{8},\frac{1}{4},\frac{1}{8})=\frac{19}{25}$
14.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{12})=\frac{17}{18}$
				$[\frac{1}{3},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{6})=\frac{139}{144}$
15.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
16.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{2}{5}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{20})=\frac{73}{80}$
				$[\frac{2}{5},\frac{1}{2}]$	$\frac{1}{2},0,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
17.	$[0,\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{5}]$	$[0,\frac{1}{8}]$	$0,1,0,0$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{5},\frac{1}{8})=\frac{147}{160}$
			$[\frac{1}{5},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{3}{10},\frac{1}{8})=\frac{157}{160}$
			$[0,\frac{1}{4}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
18.	$[0,\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{3}{8}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
19.	$[0,\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{7}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{7})=\frac{699}{784}$
				$[\frac{1}{7},\frac{1}{4}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{3}{28})=\frac{97}{98}$
20.	$[0,\frac{1}{10}]$	$[\frac{1}{2},\frac{11}{20}]$	$[\frac{1}{4},\frac{3}{8}]$	$[\frac{1}{4},\frac{3}{8}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{3}{40},\frac{3}{10},\frac{1}{4},\frac{1}{8})=\frac{1047}{1600}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{3}{40},\frac{1}{20},\frac{3}{8},\frac{1}{8})=\frac{87}{100}$

	h_1	h_2	h_3	h_4	q	maximal norm
			$[\frac{3}{8},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{2}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{3}{10},\frac{1}{8},\frac{1}{4})=\frac{1429}{1600}$
		$[\frac{11}{20},\frac{3}{5}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{9}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{7}{20},\frac{1}{4},\frac{1}{5})=\frac{387}{400}$
				$[\frac{9}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{9}{20},\frac{1}{4},\frac{1}{20})=\frac{397}{400}$
		$[\frac{3}{5},\frac{7}{10}]$		$[\frac{1}{4},\frac{2}{5}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{9}{20},\frac{1}{4},\frac{3}{20})=\frac{381}{400}$
				$[\frac{2}{5},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{2}{5},\frac{1}{4},\frac{1}{10})=\frac{393}{400}$
		$[\frac{7}{10},\frac{3}{4}]$		$[\frac{1}{4},\frac{7}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{10},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{369}{400}$
				$[\frac{7}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{3}{10},\frac{1}{4},\frac{3}{20})=\frac{387}{400}$
	$[\frac{1}{10},\frac{1}{5}]$	$[\frac{1}{2},\frac{3}{5}]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{9}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{5},\frac{7}{20},\frac{1}{4},\frac{1}{5})=\frac{399}{400}$
				$[\frac{9}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{2}{5},\frac{1}{2},\frac{1}{4},\frac{1}{20})=\frac{399}{400}$
		$[\frac{3}{5},\frac{3}{4}]$		$[\frac{1}{4},\frac{9}{25}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{5},\frac{1}{2},\frac{1}{4},\frac{11}{100})=\frac{1947}{2000}$
				$[\frac{9}{25},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{2}{5},\frac{2}{5},\frac{1}{4},\frac{7}{50})=\frac{1977}{2000}$
	$[\frac{1}{5},\frac{1}{4}]$	$[\frac{1}{2},\frac{3}{5}]$		$[\frac{1}{4},\frac{11}{25}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{7}{20},\frac{1}{4},\frac{19}{100})=\frac{981}{1000}$
				$[\frac{11}{25},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{3}{10},\frac{1}{2},\frac{1}{4},\frac{3}{50})=\frac{1877}{2000}$
		$[\frac{3}{5},\frac{3}{4}]$		$[\frac{1}{4},\frac{7}{20}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
				$[\frac{7}{20},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{3}{10},\frac{2}{5},\frac{1}{4},\frac{3}{20})=\frac{379}{400}$
21.	$[0,\frac{1}{4}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$0,1,0,0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
22.	$[0,\frac{1}{4}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{2}{5}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{20})=\frac{73}{80}$
				$[\frac{2}{5},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
23.	$[0,\frac{1}{4}]$	$[\frac{3}{4},\frac{7}{8})$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{1}{8}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{3}{8},\frac{1}{4},\frac{1}{8})=1$
		$[\frac{7}{8},1]$	$[\frac{1}{4},\frac{3}{8}]$		$0,1,0,0$	$n(\frac{1}{4},\frac{1}{8},\frac{3}{8},\frac{1}{8})=\frac{61}{64}$
			$[\frac{3}{8},\frac{1}{2}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8})=\frac{63}{64}$
		$[\frac{3}{4},\frac{7}{8}]$	$[\frac{1}{4},\frac{1}{3}]$	$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{8},\frac{1}{3},\frac{1}{8})=\frac{143}{144}$

	h_1	h_2	h_3	h_4	q	maximal norm
			$[\frac{1}{3},\frac{1}{2}]$		$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{6},\frac{1}{8})=\frac{247}{288}$
		$[\frac{7}{8},1]$	$[\frac{1}{4},\frac{1}{2}]$		$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{3}{8},\frac{1}{4},\frac{1}{8})=\frac{13}{16}$
24.	$[0,\frac{1}{4}]$	$[\frac{3}{4},1]$	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{5}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{12})=\frac{17}{18}$
				$[\frac{1}{3},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{6})=\frac{139}{144}$
25.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{8}]$	$[0,\frac{1}{16}]$	$0,1,0,0$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{16})=\frac{111}{128}$
			$[\frac{1}{8},\frac{1}{4}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
			$[0,\frac{1}{4}]$	$[\frac{1}{16},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$
26.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{3}{8}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
				$[\frac{3}{8},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
27.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[\frac{1}{4},\frac{1}{2}]$	$[0,\frac{3}{16}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$
			$[\frac{1}{4},\frac{3}{8}]$	$[\frac{3}{16},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
			$[\frac{3}{8},\frac{1}{2}]$		$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{4})=\frac{57}{64}$
28.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{1}{2},\frac{3}{4}]$	$[\frac{1}{4},\frac{1}{3}]$	$[\frac{1}{4},\frac{2}{5}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{3},\frac{3}{20})=\frac{697}{720}$
				$[\frac{2}{5},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{10})=\frac{39}{40}$
			$[\frac{1}{3},\frac{1}{2}]$	$[\frac{1}{4},\frac{1}{3}]$	$0,\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{1}{12})=\frac{23}{24}$
				$[\frac{1}{3},\frac{1}{2}]$	$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{2},\frac{1}{6},\frac{1}{6})=\frac{47}{48}$
29.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[0,\frac{1}{8}]$	$0,1,0,0$	$n(\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{27}{32}$
				$[\frac{1}{8},\frac{1}{4}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8})=\frac{21}{32}$
30.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[0,\frac{1}{4}]$	$[\frac{1}{4},\frac{7}{16}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16})=\frac{109}{128}$
			$[0,\frac{1}{8}]$	$[\frac{7}{16},\frac{1}{2}]$	$0,\frac{1}{2},0,\frac{1}{2}$	$n(\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{16})=\frac{111}{128}$
			$[\frac{1}{8},\frac{1}{4}]$		$\frac{1}{2},1,\frac{1}{2},\frac{1}{2}$	$n(\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{16})=\frac{119}{128}$
31.	$[\frac{1}{4},\frac{1}{2}]$	$[\frac{3}{4},1]$	$[\frac{1}{4},\frac{5}{16}]$	$[0,\frac{1}{10}]$	$0,1,0,0$	$n(\frac{1}{2},\frac{1}{4},\frac{5}{16},\frac{1}{10})=\frac{1233}{1280}$
				$[\frac{1}{10},\frac{3}{20}]$	$\frac{1}{2},\frac{3}{4},0,\frac{1}{4}$	$n(\frac{1}{4},\frac{1}{4},\frac{5}{16},\frac{3}{20})=\frac{1153}{1280}$
			$[\frac{5}{16},\frac{3}{8}]$	$[0,\frac{3}{20}]$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},0$	$n(\frac{1}{4},\frac{1}{2},\frac{3}{16},\frac{3}{20})=\frac{1233}{1280}$

	h_1	h_2	h_3	h_4	q	maximal norm
			$[\frac{1}{4}, \frac{3}{8}]$	$[\frac{3}{20}, \frac{1}{4}]$	$\frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{10}) = \frac{317}{320}$
			$[\frac{3}{8}, \frac{1}{2}]$	$[0, \frac{1}{8}]$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0$	$n(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}) = \frac{51}{64}$
				$[\frac{1}{8}, \frac{1}{4}]$	$0, \frac{5}{4}, \frac{1}{2}, \frac{1}{4}$	$n(\frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}) = \frac{63}{64}$
32.	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{3}{4}, 1]$	$[\frac{1}{4}, \frac{3}{8}]$	$[\frac{1}{4}, \frac{5}{16}]$	$\frac{1}{2}, \frac{3}{4}, 0, \frac{1}{4}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{16}) = \frac{119}{128}$
			$[\frac{3}{8}, \frac{1}{2}]$		$\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}) = \frac{57}{64}$
			$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{5}{16}, \frac{1}{2}]$	$\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}$	$n(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}) = \frac{109}{128}$

4 Extending Hurwitz's proof

We return to the notation of the introduction: $(1, a, b, ab)$ denotes the form $x^2 + ay^2 + bz^2 + abw^2$. There are exactly seven such forms, with a, b positive integers, that represent all positive integers: $(1,1,1,1), (1,1,2,2), (1,1,3,3)$, $(1,2,2,4), (1,2,3,6), (1,2,4,8)$ and $(1,2,5,10)$. The first form, $(1,1,1,1)$, gives the four square theorem, proven by Hurwitz via the norm Euclidean order $\Lambda(2)$.

Deutsch [1] constructed three norm Euclidean orders: $H_{1,2,2}$ used to prove the universality of $(1,1,2,2), (1,2,2,4)$ and $(1,2,4,8)$, $H_{1,1,3}$ for $(1,1,3,3)$ and $H_{2,3,6}$ for $(1,2,3,6)$. He was unable to find a norm Euclidean order in $(\frac{-2,-5}{\mathbb{Q}})$ to prove $(1,2,5,10)$ is universal.

Now the forms $(1,1,1,1)$ and $(1,1,2,2)$ are isometric over $\mathbb{Q}$. Hence $H_{1,2,2}$ is isomorphic to the Hurwitz algebra $\Lambda(2)$. Indeed, $H_{1,2,2}$ can be found as the image of $\Lambda(2)$ under the isomorphism $(\frac{-1,-1}{\mathbb{Q}}) \to (\frac{-1,-2}{\mathbb{Q}})$ given by $1 \mapsto 1, i \mapsto e_2, j \mapsto \frac{1}{2}(e_4 + e_3), k \mapsto \frac{1}{2}(e_4 - e_3)$. Similarly, $(\frac{-1,-1}{\mathbb{Q}})$ is isomorphic to $(\frac{-2,-3}{\mathbb{Q}})$ and so $H_{2,3,6}$ is also isomorphic to the Hurwitz algebra. And $H_{1,3,3}$ is isomorphic to $\Lambda(3)$ by Corollary 2.2.

Lastly, we have shown that $\Lambda(5)$ is a norm Euclidean order in $(\frac{-2,-5}{\mathbb{Q}})$. Following Hurwitz's proof with this order yields: given a positive integer n, there exists $\alpha \in \Lambda(5)$ such that $N(\alpha) = n$. To prove $(1,2,5,10)$ is universal in the style of Hurwitz requires showing that for every $\alpha \in \Lambda(5)$ there exist units $u_i \in \Lambda(5)$ such that $u_1 \alpha u_2 \in L(5) \equiv \mathbb{Z} - \text{span}\{1, e_2, e_3, e_4\}$. But the units of $\Lambda(5)$ are $\pm 1, \pm v_2, \pm(1 - v_2)$ and a check of the cases shows that there not units u_1, u_2 such that $u_1(1+v_2)u_2 \in L(5)$.

Now, $4\Lambda(5) \subset L(5)$ which implies that $(1,2,5,10)$ represents $16n$ for any positive integer n. But we have been unable to find an extension of Euler's trick (which shows that if $2n$ is a sum of four squares then so is n). In short, we have the required norm Euclidean order of $(\frac{-2,-5}{\mathbb{Q}})$ but not an extension of Hurwitz's proof to $(1,2,5,10)$.

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