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Optimal Filtering Scheme for Bilinear Discrete-Time Systems: a Linear Matrix Inequality Approach

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Abstract: The filtering problem is among the fundamental issues in control and signal processing. Several approaches such as H_2 optimal filtering and H_∞ optimal filtering have been developed to address this issue. While the optimal H_2 filtering problem has been extensively studied in the past for linear systems, to the best of our knowledge, it has not been studied for bilinear systems. This is indeed surprising, since bilinear systems are important class of nonlinear systems with well-established theories and applications in the variety of fields. The problem of H_2 optimal filtering for discrete-time bilinear systems is addressed in this paper. The filter design problem is formulated in the convex optimization framework using linear matrix inequalities. The results are used for the optimal filtering of a bilinear model of an electro-hydraulic drive.

1. INTRODUCTION

The filtering problem is among the fundamental problems in control theory and signal processing and therefore over the past it has received a lot of attention. Several approaches such as H_2 optimal filtering and H_∞ optimal filtering have been developed to address this issue (Anderson and Moore [1979]). In general, there are two approaches to solve the filtering problem: the Riccati-like approaches and the linear matrix inequality (LMI) approach. Generally,the algorithms for optimal filtering of linear discrete-time systems which has been proposed in the past is based the optimization of an H_2 norm (Anderson and Moore [1979]). This makes sense, because the statistical knowledge of the input signal, particularly a white-noise process, corrupting the measurement output is described as the sum of the output variances which leads to the H_2 norm. The necessary and sufficient conditions based on a Riccati filtering equation for the existence of an estimator structure associated with the first category were derived (Basar and Bernhard [1995], Petersen and Mcfarlane [1994]).

The focus of this paper is on the second category of the filtering techniques. An example of such filtering methods is the one proposed in Palhares and Peres [1998]. In Palhares and Peres [1998], the filtering problem has been casted in terms of linear matrix inequalities (LMI's). In this framework, the global optimal solutions are attained through convex optimization procedures, which can be efficiently solved. The H_2 optimal filtering design was derived from the state-space definition of the H_2 norm of the transfer function which relates the noise signal to the estimation error.

Both families of the methods have been extended for robust filtering of linear systems (See e. g. Petersen and McFarlane [1996],Tuan et al. [2001] ,Gao et al. [2008]) . While the optimal H_2 filtering problem has been extensively studied in the past for linear systems, to the best of our knowledge, it has not been studied for bilinear systems. This is indeed surprising, since bilinear systems are important class of nonlinear systems with well established theories and applications. These systems are used in the variety of fields to describe the processes ranging from electrical networks, hydraulic systems to heat transfer, and chemical processes. Moreover, many highly nonlinear systems may be modeled as bilinear systems with appropriate state feedback or can be approximated as bilinear systems in the so-called bilinearization process. See e.g. Svoronos et al. [1980]. The problem of H_2 optimal filtering for discrete-time bilinear systems is addressed in this paper. The filter design problem is formulated in a the convex optimization framework using linear matrix inequalities. This is an extension of the optimal filtering scheme in Palhares and Peres [1998], to support bilinear systems. The results are used for the optimal filtering of a bilinear model of an electro-hydraulic drive system.

The notation used in this paper is as follows: M^* denotes transpose of matrix if $M \in R^{n\times m}$ and complex conjugate transpose if $M \in C^{n\times m}$. The $Tr(M)$ denotes the trace of the matrix M. The $\otimes$ stands for the Kronecker Product. The standard notation $>$, $\geq$ ($<$, $\leq$) is used to denote the positive (negative) definite and semidefinite ordering of matrices.

2. BILINEAR SYSTEMS, GRAMIANS AND H_2 NORM

Let Σ be a bilinear dynamical system which is described by:

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + \displaystyle\sum_{j=1}^{m} N_j x(k) u_j(k) + Bu(k), \\ y(k) = Cx(k). \end{cases} \quad (1)$$

where $x(t) \in R^n,\ u(t) \in R^m, y(t) \in R^m$.
The controllability gramian for this system is defined as

(Dorissen [1989],D'Alessandro et al. [1974],Zhang et al. [2003] and Zhang and Lam [2002]):

$$P := \sum_{i=1}^{\infty} \sum_{k_i=0}^{\infty} ... \sum_{k_1=0}^{\infty} P_i P_i^*, \qquad (2)$$

where:

$P_1(k_1) = A^{k_1} B$,

$P_i(k_1, ..., k_i) = A^{k_i} [\, N_1 P_{i-1} \ N_2 P_{i-1} \ \cdots \ N_m P_{i-1} \,]$,

and the observability gramian is defined as:

$$Q := \sum_{i=1}^{\infty} \sum_{k_i=0}^{\infty} ... \sum_{k_1=0}^{\infty} Q^*{}_i Q_i, \qquad (3)$$

where:

$$Q_1(k_1) = CA^{k_1} \ ,$$

$$Q_i(k_1, ..., k_i) = \begin{bmatrix} Q_{i-1}N_1 \\ Q_{i-1}N_2 \\ \vdots \\ Q_{i-1}N_m \end{bmatrix} A^{k_i}.$$

If A is stable, the gramians are given by the solutions of the generalized Lyapunov equations (Zhang et al. [2003] and Zhang and Lam [2002]):

$$APA^* - P + \sum_{j=1}^{m} N_j P N_j^* + BB^* = 0, \qquad (4)$$

$$A^*QA - Q + \sum_{j=1}^{m} N_j^* Q N_j + C^*C = 0. \qquad (5)$$

These generalized Lyapunov equation (4) has a unique solution if and only if:

$$W = (A \otimes A - I + \sum_{j=1}^{m} N_j \otimes N_j) \qquad (6)$$

is nonsingular. The dual condition can be found for (5). See (Zhang and Lam [2002]) for more details.

The generalized Lyapunov equations can be solved iteratively.
The controllability gramian P is obtained by (Zhang et al. [2003] and Zhang and Lam [2002]):

$$P = \lim_{i\to\infty} \hat{P}_i \qquad (7)$$

where:

$$A\hat{P}_1 A^* - \hat{P}_1 + BB^* = 0,$$
$$A\hat{P}_i A^* - \hat{P}_i + \sum_{j=1}^{m} N_j \hat{P}_{i-1} N_j^* + BB^* = 0, \quad i = 2, 3, ... \qquad (8)$$

The observability gramian is dually obtained by :

$$Q = \lim_{i\to\infty} \hat{Q}_i \qquad (9)$$

where:

$$A^*\hat{Q}_1 A - \hat{Q}_1 + C^*C = 0,$$
$$A^*\hat{Q}_i A - \hat{Q}_i + \sum_{j=1}^{m} N_j{}^* \hat{Q}_{i-1} N_j + C^*C = 0, \qquad (10)$$
$$i = 2, 3, ...$$

The controllability and observability gramians show how difficult a system is to control and to observe. The gramians for bilinear systems have an important property which will be used for H_2 filtering in the next section. For bilinear system Σ , if A is stable and the reachability gramian P (or observability gramian Q) exists; then its H_2 norm can be computed from (Zhang and Lam [2002]):

$$\|\Sigma\|_2 = \sqrt{Tr(CPC^*)} = \sqrt{Tr(B^*QB)} \qquad (11)$$

3. H_2 OPTIMAL FILTERING

Consider the following bilinear time-invariant discrete-time system given by:

$$S : \begin{cases} x(k+1) = Ax(k) + \sum_{j=1}^{m} N_j x(k) u_j(k) + Bw(k), \\ y(k) = Cx(k) + Dw(k), \\ z(k) = Lx(k). \end{cases} \qquad (12)$$

where $x(k) \in R^n$ is the state vector, $y(k) \in R^r$ is the measurements output vector, $w(k) \in R^m$ is the noise signal vector (including process and measurement noises) and $z(k) \in R^p$ is the signal to be estimated. It is assumed that (A, C) is detectable. This guarantees that there exists an observer constant gain such that the filter is asymptotically stable. The goal is to design an asymptotically stable linear filter described by:

$$F : \begin{cases} \hat{x}(k+1) = A\hat{x}(k) + \sum_{j=1}^{m} N_j \hat{x}(k) w_j(k) \\ \qquad\qquad + K(y(k) - C\hat{x}(k)), \\ \hat{z}(k) = L\hat{x}(k) \end{cases} \qquad (13)$$

where $K \in R^{n\times r}$ is the filter constant gain to be determined.
The state error is defined as $e(k) := x(k) - \hat{x}(k)$, then the dynamics of the estimation error is described by:

$$E : \begin{cases} e(k+1) = A_\psi e(k) + \sum_{j=1}^{m} N_j e(k) w_j(k) \\ \qquad\qquad + B_\psi w(k), \\ \tilde{z}(k) = Le(k), \end{cases} \qquad (14)$$

where $z(k) := z(k) - \hat{z}(k)$ is the estimation error, and:

$$A_\psi := A - KC,$$
$$B_\psi := B - KD. \qquad (15)$$

In the optimal H_2 filtering scheme, a bilinear filter F needs to be determined such that the estimation error variance is minimized. The problem therefore will be:

$$\min_K \|E\|_2^2 \qquad (16)$$

The H_2 norm of estimation error dynamics according to (11) can be computed as:

$$\|E\|_2 = \sqrt{Tr(B_\psi^* Q_\psi B_\psi)} \qquad (17)$$

where Q_ψ is the observability gramian of the bilinear system (14) and is the solution to the generalized Lyapunov equation:

$$A_\psi^* Q_\psi A_\psi - Q_\psi + \sum_{j=1}^{m} N_j^* Q_\psi N_j + L^*L = 0 \qquad (18)$$

In the following some results are stated and proved which will be used later to reformulate our problem in LMI framework:

Lemma 1. Let A be stable and W which is defined as:

$$\mathrm{W} = (\mathrm{A}^* \otimes A^* - I + \sum_{j=1}^{m} N_j^* \otimes N_j^*) \tag{19}$$

be nonsingular. If X satisfies:

$$A^*XA - X + \sum_{j=1}^{m} N_j^* X N_j \ \leq 0, \tag{20}$$

then: $X \geq 0$.

Proof:

Let X satisfies (20), there exist $M \geq 0$ for which:

$$A^*XA - X + \sum_{j=1}^{m} N_j^* X N_j \ + M = 0, \tag{21}$$

On the other hand, A is stable and W which is obtained by duality from (6) as:

$$W = (A^* \otimes A^* - I + \sum_{j=1}^{m} N_j^* \otimes N_j^*)$$

is nonsingular, The generalized Lyapunov equation (21) has a unique solution which is obtained as:

$$X = \lim_{i \to \infty} \hat{X}_i \tag{22}$$

where:

$$\begin{aligned} &A^*\hat{X}_1 A - \hat{X}_1 + M = 0, \\ &A^*\hat{X}_i A - \hat{X}_i + \sum_{j=1}^{m} N_j{}^* \hat{X}_{i-1} N_j + M = 0, \\ &\qquad\qquad i = 2, 3, ... \end{aligned} \tag{23}$$

Since $M \geq 0$, then $\hat{X}_1 = \sum_{k=0}^{\infty} (A^k) M A^k \geq 0$. For $i = 2, 3, ...$, we have:

$$\sum_{j=1}^{m} N_j{}^* \hat{X}_{i-1} N_j + M \geq 0,$$

consequently: $\hat{X}_i \geq 0$. Therefore: $X = \lim_{i\to\infty} \hat{X}_i \geq 0$.

□

Proposition 2. Let for bilinear system (1), A be stable and W which is defined as:

$$\mathrm{W} = (\mathrm{A}^* \otimes A^* - I + \sum_{j=1}^{m} N_j^* \otimes N_j^*) \tag{24}$$

be nonsingular. Assume that Q is the observability gramian of (1). If $\bar{Q}$ satisfies:

$$A^*\bar{Q}A - \bar{Q} + \sum_{j=1}^{m} N_j^* \bar{Q} N_j + C^*C \leq 0 \tag{25}$$

then: $\bar{Q} \geq Q$.

Proof:
Subtract (25) from (5) and apply *Lemma 1* with $X = \bar{Q} - Q$.

□

This proposition has an interesting consequence:

Corollary 3. Let A_ψ be stable and W_ψ which is defined as:

$$W_\psi = ({A_\psi}^* \otimes {A_\psi}^* - I + \sum_{j=1}^{m} N_j^* \otimes N_j^*) \tag{26}$$

is nonsingular. Let Q_ψ be the observability gramian for the bilinear system (14) and assume that there exist $\bar{Q_\psi}$ which satisfies:

$$A_\psi^* \bar{Q_\psi} A_\psi - \bar{Q_\psi} + \sum_{j=1}^{m} N_j^* \bar{Q_\psi} N_j + L^*L \leq 0, \tag{27}$$

then:

$$Tr(B_\psi^* \bar{Q}_\psi B_\psi) \geq Tr(B_\psi^* Q_\psi B_\psi). \tag{28}$$

In the following, the H_2 optimal filtering problem for bilinear system is cast as a convex optimization problem:

Theorem 4. Consider the following optimization problem:

$$\min_{J,Y,V} \ Tr(J) \tag{29}$$

Subject To:

$$\begin{bmatrix} J & B^*Y - D^*V^* \\ YB - VD & Y \end{bmatrix} \geq 0, \tag{30}$$

$$\begin{bmatrix} Y & \sum_{j=1}^{m} YN_j + YA - VC & 0 \\ \sum_{j=1}^{m} N_j^*Y + A^*Y - C^*V^* & Y & L^* \\ 0 & L & I \end{bmatrix} \geq 0, \tag{31}$$

$$Y \geq 0 \tag{32}$$

where $Y = Y^* \in R^{n\times n}$, $V \in R^{n\times r}$ and $J = J^* \in R^{m\times m}$. The optimal solution is such that:

$$Tr(J) = \min \|E\|_2^2 \tag{33}$$

and the optimal H_2 filtering gain is given by:

$$K = Y^{-1}V. \tag{34}$$

Proof:

Suppose that there exist $Y \geq 0$ and V, satisfying (31). Then, from Schur's complement we have:

$$\begin{gathered}(\sum_{j=1}^{m} YN_j + YA - VC)^* Y^{-1} (\sum_{j=1}^{m} YN_j + YA - VC) \\ \text{- Y + L}^*L \leq 0\end{gathered}$$

Equivalently, we have:

$$\begin{gathered}(\sum_{j=1}^{m} N_j + A - Y^{-1}VC)^* Y (\sum_{j=1}^{m} N_j + A - Y^{-1}VC) \\ \text{- Y + L}^*L \leq 0\end{gathered}$$

For $K = Y^{-1}V$ and $Y = \bar{Q}_\psi$:

$$A_{\psi}^{*}\bar{Q_{\psi}}A_{\psi}-\bar{Q_{\psi}}+\sum_{j=1}^{m}N_j^{*}\bar{Q_{\psi}}N_j+L^{*}L\leq 0,$$

Corollary 3 applies and therefore we have:

$$Tr(B_{\psi}^{*}YB_{\psi})\geq Tr(B_{\psi}^{*}Q_{\psi}B_{\psi}). \tag{35}$$

On the other hand, from (30) using Schur complement we get:

$$J-(YB-VD)^{*}Y^{-1}(YB-VD)\geq 0$$

Therefore:

$$\begin{aligned} Tr(J)&\geq Tr((YB-VD)^{*}Y^{-1}(YB-VD))\\ &=\mathrm{Tr}((\mathrm{B}-\mathrm{Y}^{-1}VD)^{*}Y(B-Y^{-1}VD))\\ &=\mathrm{Tr}(\mathrm{B}_{\psi}^{*}YB_{\psi})\end{aligned}$$

From this and (35), we have:

$$Tr(J)\geq Tr(B_{\psi}^{*}Q_{\psi}B_{\psi})=\|E\|_2^2\,. \tag{36}$$

Since no other constraint is imposed on the J, the minimization of the linear cost ensures that:

$$Tr(J)=min\|E\|_2^2 \tag{37}$$

□

This theorem is used for H_2 optimal filtering of a bilinear hydraulic derive system in the next section.

4. H_2 OPTIMAL FILTERING OF A BILINEAR HYDRAULIC DERIVE SYSTEM

In general, hydraulic systems are highly nonlinear dynamical systems. The linear models are not sufficiently accurate to describe them and consequently the controllers which are designed based on the linear models of the hydraulic systems quite often do not end up with satisfying results. On the other hand, due to the complexity of the highly nonlinear hydraulic models, methods for analyzing them , filtering and synthesizing their controllers are not well developed and often they are difficult to apply in practice. In between the spectrum of different models to describe a hydraulic system from linear model to highly nonlinear model, the bilinear model often offers an adequately accurate model with a well-developed theory for the analysis and control. In the sequel, the H_2 optimal filter is designed for a bilinear model of an electro-hydraulic drive system. This hydraulic drive has been studied in Schwartz and Ingenbleek [1994] and Shaker and Tahavori [2011]. The model is discretized by Euler's forward discretization method with the sampling time 0.1. The resulting discrete-time model is in the form:

$$\begin{cases} x(k+1)=Ax(k)+Nx(k)u(k)+Bw(k),\\ y(k)=Cx(k),\\ z(k)=Lx(k).\end{cases} \tag{38}$$

where:

$$A=\begin{bmatrix}1&0&0&-0.00002\\0.1&1&0&-0.00225\\0&0.1&1&-0.06600\\0&0&0.1&0.41370\end{bmatrix},B=\begin{bmatrix}0.00014\\0\\0\\0\end{bmatrix},$$

$$C=[\,0\;0\;0\;1\,],\quad L=[\,0\;0\;0\;1\,],$$

$$N=\begin{bmatrix}0&0&0&0.00003\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Applying Theorem 1, the solution to the optimization problem is:

$$Y=\begin{bmatrix}0.00036&-0.00432&0.05733&-0.77669\\-0.00432&0.08441&-1.40225&21.68466\\0.05733&-1.40225&27.68576&-487.88060\\-0.77669&21.68466&-487.88060&9585.27364\end{bmatrix},$$

$$V=\begin{bmatrix}-0.58546327196\\19.36260729131\\-514.30992621789\\11598.15417677910\end{bmatrix},\; J=7.55316291347\mathrm{e}-012.$$

The optimal H2 filtering constant gain is given by:

$$K=\begin{bmatrix}946.6941734529101\\423.9133743225211\\65.0927812342745\\3.6408492106137\end{bmatrix}.$$

The H_2 estimation error is:

$$\|E\|_2^2=7.55316291347\mathrm{e}-012.$$

5. CONCLUSION

Over the last few decades, several approaches such as H_2 optimal filtering and H_∞ optimal filtering have been developed for filtering. While the H_2 optimal filtering problem has been extensively studied in the past for linear systems, to the best of our knowledge, it has not been studied for bilinear systems. Due to the importance of this class of nonlinear systems , the problem of H_2 optimal filtering for discrete-time bilinear systems has been addressed in this paper. The filter design problem has been formulated in the convex optimization framework using linear matrix inequalities. The results have been successfully used for the optimal filtering of a bilinear model of an electro-hydraulic drive.

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