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Correction to "Packetized Predictive Control of Stochastic Systems over Bit-Rate Limited Channels with Packet Loss"

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***Abstract*— We correct the results in Section V of the above mentioned manuscript.**

In [1], we showed that a particular class of networked control system (NCS) with quantization, i.i.d. dropouts and disturbances can be described as a Markov jump linear system of the form

$$\theta_{k+1} = \bar{A}(d_k)\theta_k + \bar{B}(d_k)\nu_k, \tag{1}$$

where

$$\theta_k \triangleq \begin{bmatrix} x_k \\ b_{k-1} \end{bmatrix} \in \mathbb{R}^{n+N}, \quad \nu_k \triangleq \begin{bmatrix} w_k \\ n_k \end{bmatrix} \in \mathbb{R}^{m+N}$$

and $\{d_k\}_{k\in\mathbb{N}_0}$ is a Bernoulli dropout process, with

$$\text{Prob}(d_k = 1) = p \in (0, 1).$$

Throughout [1] we showed that properties of the NCS can be conveniently stated in terms of the expected system matrices

$$\begin{aligned} \mathcal{A}(p) &= \mathbb{E}\{\bar{A}(d_k)\} \\ \mathcal{B}(p) &= \mathbb{E}\{\bar{B}(d_k)\} = \begin{bmatrix} \mathcal{B}_w & \mathcal{B}_n(p) \end{bmatrix} \end{aligned}$$

and the matrix $\widetilde{\mathcal{A}} = \bar{A}(1) - \bar{A}(0)$. Unfortunately, Theorem 4 in Section V-A of [1] is incorrect. For white disturbances $\{w_k\}_{k\in\mathbb{N}_0}$, the statement should be as given below. Non-white $\{w_k\}_{k\in\mathbb{N}_0}$ can be accommodated by using standard state augmentation techniques; see, e.g., [2].

Theorem 4: Suppose that (1) is MSS and AWSS and that $\{w_k\}_{k\in\mathbb{N}_0}$ is white with $\sigma_w^2 = \text{tr}R_w(0)$. Define

$$\begin{aligned} \mathcal{F}(z) &\triangleq \big(zI - \mathcal{A}(p)\big)^{-1} \\ \mathcal{C}(p) &\triangleq (\sigma_w^2/m)\mathcal{B}_w\mathcal{B}_w^T + (\sigma_n^2/N)(1-p)\mathcal{E} \in \mathbb{R}^{(n+N)\times(n+N)}, \end{aligned} \tag{2}$$

where (see [1, Sec.2] for definitions)

$$\mathcal{E} \triangleq \frac{\mathcal{B}_n(p)\mathcal{B}_n(p)^T}{(1-p)^2} = \begin{bmatrix} B_1 e_1^T(\Psi^T\Psi)^{-1}e_1 B_1^T & B_1 e_1^T(\Psi^T\Psi)^{-1} \\ (\Psi^T\Psi)^{-1}e_1 B_1^T & (\Psi^T\Psi)^{-1} \end{bmatrix}. \tag{3}$$

Then, the spectral density of $\{\theta_k\}_{k\in\mathbb{N}_0}$ is given by

$$S_\theta(e^{j\omega}) = \mathcal{F}(e^{j\omega})\big(p(1-p)\widetilde{\mathcal{A}}R_\theta(0)\widetilde{\mathcal{A}}^T + \mathcal{C}(p)\big)\mathcal{F}^T(e^{-j\omega}), \tag{4}$$

where $R_\theta(0)$ solves the following linear matrix equation:

$$R_\theta(0) = \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T + p(1-p)\widetilde{\mathcal{A}}R_\theta(0)\widetilde{\mathcal{A}}^T + \mathcal{C}(p). \tag{5}$$

Proof: See the appendix. ∎

To further elucidate the situation, we note that (5) is linear and that its solution can be stated as the linear combination

$$R_\theta(0) = (\sigma_w^2/m)R_\theta^w(0) + (\sigma_n^2/N)R_\theta^n(0), \tag{6}$$

where $R_\theta^w(0)$ and $R_\theta^n(0)$ satisfy

$$\begin{aligned} R_\theta^w(0) &= \mathcal{A}(p)R_\theta^w(0)\mathcal{A}(p)^T + p(1-p)\widetilde{\mathcal{A}}R_\theta^w(0)\widetilde{\mathcal{A}}^T + \mathcal{B}_w\mathcal{B}_w^T \\ R_\theta^n(0) &= \mathcal{A}(p)R_\theta^n(0)\mathcal{A}(p)^T + p(1-p)\widetilde{\mathcal{A}}R_\theta^n(0)\widetilde{\mathcal{A}}^T + (1-p)\mathcal{E}. \end{aligned}$$

Therefore, the distortion D defined by (52) in [1] is given by

$$D \triangleq \text{tr}(\tilde{Q}R_\theta(0)) + \lambda[0 \quad e_1^T]R_\theta(0)[0 \quad e_1^T]^T,$$

where $\tilde{Q}$ is given in terms of the Kronecker product

$$\tilde{Q} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes Q.$$

Thus, $D = \alpha\sigma_n^2 + \beta$, with

$$\begin{aligned} \alpha &= (1/N)\text{tr}(\tilde{Q}R_\theta^n(0)) + (\lambda/N)[0 \quad e_1^T]R_\theta^n(0)[0 \quad e_1^T]^T \\ \beta &= (\sigma_w^2/m)\text{tr}(\tilde{Q}R_\theta^w(0)) + (\lambda\sigma_w^2/m)[0 \quad e_1^T]R_\theta^w(0)[0 \quad e_1^T]^T. \end{aligned}$$

The above expressions replace Lemma 11 of [1].

To derive a noise-shaping model, (6) can be substituted into into (4) to provide

$$S_\theta(e^{j\omega}) = \mathcal{F}(e^{j\omega})\big((\sigma_w^2/m)\mathcal{K}_w\mathcal{K}_w^T + (\sigma_n^2/N)\mathcal{K}_n\mathcal{K}_n^T\big)\mathcal{F}^T(e^{-j\omega}),$$

where $\mathcal{K}_w$ and $\mathcal{K}_n$ are obtained from the factorizations

$$\begin{aligned} \mathcal{K}_w\mathcal{K}_w^T &= \mathcal{B}_w\mathcal{B}_w^T + p(1-p)\widetilde{\mathcal{A}}R_\theta^w(0)\widetilde{\mathcal{A}}^T \\ \mathcal{K}_n\mathcal{K}_n^T &= (1-p)\big(\mathcal{E} + p\widetilde{\mathcal{A}}R_\theta^n(0)\widetilde{\mathcal{A}}^T\big). \end{aligned}$$

If we define

$$\mathcal{H}(z) \triangleq \begin{bmatrix} I & 0 \end{bmatrix}\mathcal{F}(z),$$

then the above provides the noise-shaping model depicted in Fig. 2. The latter replaces Fig. 2 and Corollary 1 of [1].

Remark 1: We would like to emphasize that Theorem 4 can also be proven by adapting results in [3]–[5]. However, the noise shaping interpretation in Fig. 2 does not explicitly need an additional noise term to quantify second-order dropout effects, as opposed to what is done in [3]–[5]. □

The upper bound on the coding rate provided by Theorem 5 in [1] is also no longer correct, since it relied upon $R_\theta(0)$. The new Theorem 5 is provided below:

Theorem 5: For any $1 \leq N \in \mathbb{N}$, the minimum bit-rate R of $\vec{u}_k$ satisfies:

$$R(D) \leq \frac{1}{2}\log_2\big(\det(I + (N/\sigma_n^2)R_\xi(0))\big) + \frac{N}{2}\log_2\left(\frac{\pi e}{6}\right) + 1, \tag{7}$$

where

$$R_\xi(0) = \begin{bmatrix} \Gamma & 0 \end{bmatrix} R_\theta(0) \begin{bmatrix} \Gamma & 0 \end{bmatrix}^T.$$

Proof: Follows immediately from (73) in [1] by omitting the last step where $R_\xi(0)$ was written in terms of $R_x(0)$ and (50) was used. ∎

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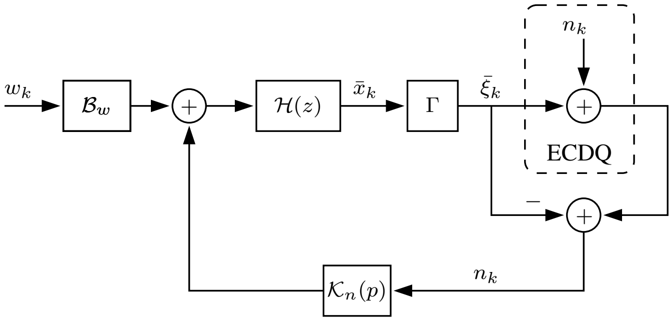

Fig. 2. Noise-Shaping Model of the NCS

Note that, in view of (6), the bound in (7) provides

$$\lim_{\sigma_n^2\to\infty} R(D) \leq \frac{1}{2}\log_2\left(\det(I+[\Gamma \quad 0]R_\theta^n(0)[\Gamma \quad 0]^T)\right) + \frac{N}{2}\log_2\left(\frac{\pi e}{6}\right)+1,$$

expression, which is positively bounded away from zero and replaces (58) in [1].

Remark 2: By using results in [6, Sec.5], the covariance matrix $R_\theta(0)$ can be expressed explicitly in terms of Kronecker products and matrix inversions. Specifically, let

$$G \triangleq \mathcal{A}(p)\otimes\mathcal{A}(p)^T + p(1-p)\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{A}}^T$$

and let $c\in\mathbb{R}^{(n+N)^2}$ be the vectorized version of the matrix $\mathcal{C}(p)$ given in (2). Then, the vectorized version of $R_\theta(0)$ is simply given by $r=(I-G)^{-1}c$. Using this approach, it is straight-forward to numerically evaluate the rate and distortion in (7). □

We finalize this note by revisiting the NCS considered in Section V-C of [1]. Fig. 3 illustrates the rate and distortion trade-off for different horizon lengths and a fixed packet loss probability $p=0.0085$. It may be noticed that the distortion can be reduced by using a longer horizon length in addition to increasing the bit-rate. Fig. 4 shows that when the packet-loss probability increases, it is necessary to use a larger horizon length to guarantee stability and thereby reduce the distortion.

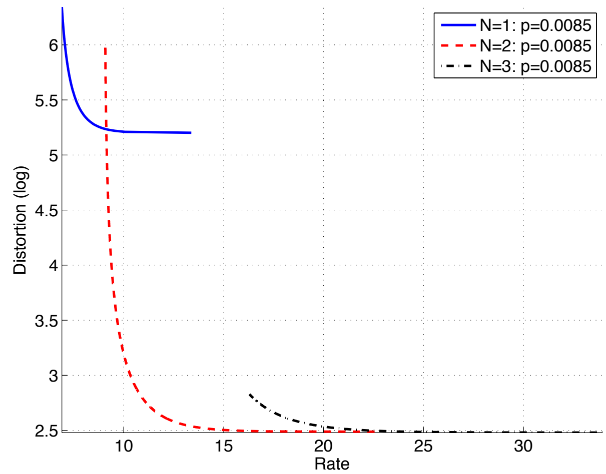

Fig. 3. Bound on $D(R)$ obtained from (7) for a fixed $p=0.0085$ and different horizon lengths $N=1,2,3$. The distortion is here expressed in the log-domain.

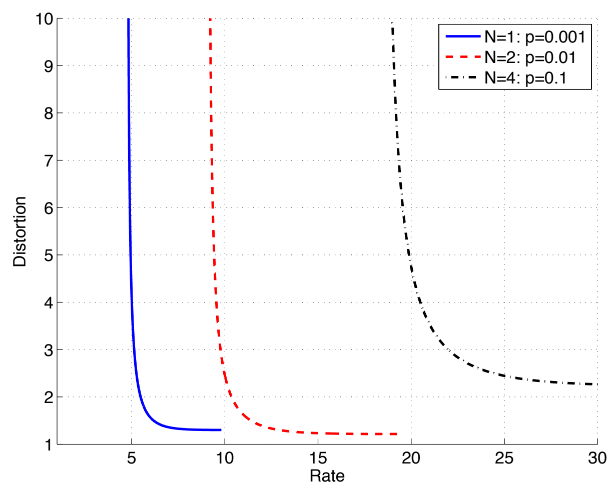

Fig. 4. Bound on $D(R)$ obtained from (7) for different packet loss probabilities and different horizon lengths.

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Appendix
Proof of Theorem 4

Since $\{\nu_k\}_{k\in\mathbb{N}_0}$ is white and thus $\mathbb{E}\{\theta_k\nu_k^T\}=0$, the system recursion (1) provides

$$\mathbb{E}\{\theta_{k+1}\theta_{k+1}^T\}=\mathbb{E}\{\bar{A}(d_k)\theta_k\theta_k^T\bar{A}(d_k)^T\}+\mathbb{E}\{\bar{B}(d_k)\nu_k\nu_k^T\bar{B}(d_k)^T\}.$$

Therefore, by conditioning on d_k and using the law of total expectation, we obtain:

$$\begin{aligned}\mathbb{E}\{\theta_{k+1}\theta_{k+1}^T\} &= p\mathbb{E}\{\bar{A}(d_k)\theta_k\theta_k^T\bar{A}(d_k)^T \mid d_k=1\}\\ &\quad+(1-p)\mathbb{E}\{\bar{A}(d_k)\theta_k\theta_k^T\bar{A}(d_k)^T \mid d_k=0\}\\ &\quad+p\mathbb{E}\{\bar{B}(d_k)\nu_k\nu_k^T\bar{B}(d_k)^T \mid d_k=1\}\\ &\quad+(1-p)\mathbb{E}\{\bar{B}(d_k)\nu_k\nu_k^T\bar{B}(d_k)^T \mid d_k=0\}\\ &= p\bar{A}(1)\mathbb{E}\{\theta_k\theta_k^T\}\bar{A}(1)^T+(1-p)\bar{A}(0)\mathbb{E}\{\theta_k\theta_k^T\}\bar{A}(0)^T\\ &\quad+p\bar{B}(1)R_\nu(0)\bar{B}(1)^T+(1-p)\bar{B}(0)R_\nu(0)\bar{B}(0)^T,\end{aligned}$$

where we have used the fact that $\{d_k\}_{k\in\mathbb{N}_0}$ is Bernoulli and ν_k and θ_k are independent of d_k. Direct algebraic manipulations allow us to rewrite the above as

$$\begin{aligned}\mathbb{E}\{\theta_{k+1}\theta_{k+1}^T\} = \mathcal{A}(p)\mathbb{E}\{\theta_k\theta_k^T\}\mathcal{A}(p)^T \\ + p(1-p)\widetilde{\mathcal{A}}\mathbb{E}\{\theta_k\theta_k^T\}\widetilde{\mathcal{A}}^T + \mathcal{C}(p).\end{aligned} \tag{8}$$

In a similar way, one can derive that

$$\begin{aligned}\mathbb{E}\big\{\theta_{k+\ell+1}\theta_k^T\big\} &= \mathbb{E}\big\{(\bar{A}(d_{k+\ell})\theta_{k+\ell} + \bar{B}(d_{k+\ell})\nu_{k+\ell})\theta_k^T\big\} \\ &= \mathbb{E}\big\{\bar{A}(d_{k+\ell})\theta_{k+\ell}\theta_k^T\big\} + \mathbb{E}\big\{\bar{B}(d_{k+\ell})\nu_{k+\ell}\theta_k^T\big\} \\ &= \mathcal{A}(p)\mathbb{E}\big\{\theta_{k+\ell}\theta_k^T\big\} + \mathcal{B}(p)\mathbb{E}\big\{\nu_{k+\ell}\theta_k^T\big\} \\ &= \mathcal{A}(p)\mathbb{E}\big\{\theta_{k+\ell}\theta_k^T\big\}, \qquad \forall \ell \in \mathbb{N}_0,\end{aligned} \tag{9}$$

since $\{\nu_k\}_{k\in\mathbb{N}_0}$ is white and θ_k and $\theta_{k+\ell}$ are independent of $d_{k+\ell}$ for non-negative values of ℓ. Equation (9) gives the explicit expression

$$\mathbb{E}\big\{\theta_{k+\ell}\theta_k^T\big\} = \mathcal{A}(p)^\ell\mathbb{E}\big\{\theta_k\theta_k^T\big\}, \quad \forall \ell \in \mathbb{N}_0. \tag{10}$$

Since the system is AWSS, we have $\lim_{k\to\infty}\mathbb{E}\{\theta_{k+1}\theta_{k+1}^T\} = R_\theta(0)$, the stationary covariance matrix of $\{\theta_k\}_{k\in\mathbb{N}_0}$. By (8) and results in [7], [8], the latter is given by the solution to (5).

On the other hand, in steady state, (10) gives that the covariance function

$$R_\theta(\ell) = \mathcal{A}(p)^\ell R_\theta(0), \quad \forall \ell \in \mathbb{N}_0. \tag{11}$$

Consequently, the positive real part of the spectrum of $\{\theta_k\}_{k\in\mathbb{N}_0}$ is given by

$$\begin{aligned}S_\theta^+(z) &= \frac{1}{2}R_\theta(0) + \sum_{\ell=1}^{\infty} R_\theta(\ell)z^{-\ell} \\ &= \big((1/2)I + \mathcal{A}(p)\big(zI - \mathcal{A}(p)\big)^{-1}\big)R_\theta(0),\end{aligned}$$

where we have used the fact that, by assumption, (1) is MSS and AWSS, thus $\mathcal{A}(p)$ is Schur (see Lemma 4 in [1]) and the geometric series

$$\sum_{n=0}^{\infty}\big(\mathcal{A}(p)z^{-1}\big)^n = \big(I - \mathcal{A}(p)z^{-1}\big)^{-1}.$$

Since $\{\theta_k\}_{k\in\mathbb{N}_0}$ is AWSS, its spectrum satisfies [9]

$$\begin{aligned}S_\theta(z) &= S_\theta^+(z) + \big(S_\theta^+(z^{-1})\big)^T \\ &= R_\theta(0) + \mathcal{A}(p)\big(zI - \mathcal{A}(p)\big)^{-1}R_\theta(0) \\ &\quad + R_\theta(0)\big(z^{-1}I - \mathcal{A}(p)\big)^{-T}\mathcal{A}(p)^T.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\big(zI - \mathcal{A}(p)\big)S_\theta(z)\big(z^{-1}I - \mathcal{A}(p)\big)^T \\ &= \big(zI - \mathcal{A}(p)\big)R_\theta(0)\big(z^{-1}I - \mathcal{A}(p)\big)^T \\ &\quad + \big(zI - \mathcal{A}(p)\big)\mathcal{A}(p)\big(zI - \mathcal{A}(p)\big)^{-1}R_\theta(0)\big(z^{-1}I - \mathcal{A}(p)\big)^T \\ &\quad + \big(zI - \mathcal{A}(p)\big)R_\theta(0)\big(z^{-1}I - \mathcal{A}(p)\big)^{-T}\mathcal{A}(p)^T\big(z^{-1}I - \mathcal{A}(p)\big)^T \\ &= \big(zI - \mathcal{A}(p)\big)R_\theta(0)\big(z^{-1}I - \mathcal{A}(p)\big)^T \\ &\quad + \mathcal{A}(p)R_\theta(0)\big(z^{-1}I - \mathcal{A}(p)\big)^T + \big(zI - \mathcal{A}(p)\big)R_\theta(0)\mathcal{A}(p)^T,\end{aligned}$$

since $(zI - \mathcal{A}(p))\mathcal{A}(p)(zI - \mathcal{A}(p))^{-1} = \mathcal{A}(p)$. Thus,

$$\begin{aligned}&\mathcal{F}^{-1}(z)S_\theta(z)\mathcal{F}^{-T}(z^{-1}) \\ &= \big(zR_\theta(0) - \mathcal{A}(p)R_\theta(0)\big)\big(z^{-1}I - \mathcal{A}(p)\big)^T + z^{-1}\mathcal{A}(p)R_\theta(0) \\ &\quad - \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T + zR_\theta(0)\mathcal{A}(p)^T - \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T \\ &= R_\theta(0) - z^{-1}\mathcal{A}(p)R_\theta(0) - zR_\theta(0)\mathcal{A}(p)^T \\ &\quad + \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T + z^{-1}\mathcal{A}(p)R_\theta(0) - \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T \\ &\quad + zR_\theta(0)\mathcal{A}(p)^T - \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T \\ &= R_\theta(0) - \mathcal{A}(p)R_\theta(0)\mathcal{A}(p)^T,\end{aligned}$$

and (5) establishes (4). $\square$