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FRAMES FOR DECOMPOSITION SPACES GENERATED BY A SINGLE FUNCTION

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ABSTRACT. In this paper we present a construction of frames generated by a single band-limited function for decomposition smoothness spaces on $\mathbb{R}^d$ of modulation and Triebel-Lizorkin type. A perturbation argument is then used to construct compactly supported frame generators.

1. INTRODUCTION

Sparse representations of signals relative to a fixed time-frequency system play an important role in approximation theory, numerical analysis, and for practical applications. The most prominent example being sparse wavelet representation of signals, which is an approach with many applications to modern technology. However, wavelets are not a universal fix for every sparse representation problem, and depending on the "nature" of the signals to be analysed, it might be more advantageous to choose a representation system with significantly different time-frequency properties such as Gabor systems or perhaps curvelet-type frames.

Two aspects of sparse representations that will be in focus in this paper are the flexibility to adapt the representation system to the signal and the fact that a discrete sparse representation of a function (signal) is often linked to some notion of smoothness of the function. The fact that sparseness in general is linked to smoothness is an observation that that goes all the way back to the study of Fourier series for C^1-functions, but the link is perhaps even more transparant in the modern theory of function spaces based on Littewood-Paley theory.

The path we follow here is to study the link between discrete sparse representations and smoothness within the framework of so-called decomposition smoothness spaces. We also insist that smoothness should be linked to sparseness using a simple representation system generated by a single function similar to case for wavelet and Gabor systems.

Decomposition spaces were introduced by Feichtinger and Gröbner [10] and Feichtinger [7], and are based on structured coverings $\mathbb{R}^d$ considered either as the direct (time-variable) space or the frequency space. One major advantage of this set up is the flexibility. Very general decompositions of the frequency space fit in this framework yielding flexibility that also allows an anisotropic setup to be considered without much added complexity. For example, classical Triebel-Lizorkin and Besov spaces fit nicely into the decomposition space

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model and they correspond to dyadic coverings of the frequency space, see [25]. Modulation spaces also fit into the model and they correspond to uniform coverings of the frequency place, see [8]. In recent years, many authors have found the decomposition approach useful, see e.g. [5, 6, 12, 15, 16, 21, 22].

One of the advantages of the abstract decomposition space approach is the fact that tight frames can easily be constructed for such smoothness spaces of both Besov and Triebel-Lizorkin type in both an isotropic and an anisotropic setting, see [4]. In fact, in [4] an explicit construction is proposed yielding frames of the form

$$\Big\{\eta_{k,n}(x) = t_k^{\frac{\nu}{2}}\mu_k(\delta_{t_k}^\top x - \frac{\pi}{a}n)e^{ix\cdot\xi_k}\Big\}_{k,n\in\mathbb{Z}^d}, \tag{1.1}$$

where $\{\mu_k\}$ is a sequence of smooth localized atoms (functions), $\{t_k\}$ is a sequence of dilation parameters that depend on the particular covering of the frequency space, and $\{\delta_t\}_{t>0}$ is a one-parameter group of dilations that incorporate the possible anisotropic properties of the setting. Frames for particular types of decomposition spaces have been considered earlier, see e.g. [1, 5, 9, 11].

However, it was pointed out to us recently by Hans G. Feichtinger[1] that frames of the type (1.1) are somewhat problematic from a computational point of view due to the fact that we have an infinite number of generators $\{\mu_k\}$ unlike e.g. wavelet or Gabor systems that are generated by a small number of functions facilitating fast associated algorithms.

The main contribution of the present paper is to present a construction of frames for decomposition spaces generated by a single function. Moreover, we show that the single generator can in fact be chosen with compact support. We recall that compact support is very desirable from an application point-of-view since it allows for calculation of frame coefficient based only on local information about the signal.

The construction is through a two-stage process. First we construct a frame based on a single band-limited generator. Any band-limited generator clearly lacks compact support, but the second step is to use an approximation procedure proposed for decomposition spaces in [18] to get our hands on a compactly supported generator.

Let us finally mention that we carry through the construction for Triebel-Lizorkin type decomposition spaces. The construction can also be carried out for modulation type decomposition space, and the proofs in the modulation case are in fact significantly simpler. However, our agenda is also to try to increase the awareness of the option to study Triebel-Lizorkin type spaces in the decomposition setup, and we are hopeful that some of the tools that will be discussed in the present paper can also turn out to be useful for other purposes.

2. Decomposition type smoothness spaces on $\mathbb{R}^d$

We begin by introducing some machinery needed for the definition of decomposition spaces. First we study a homogeneous structure on $\mathbb{R}^d$ that will be used to generated general coverings of the the frequency space, and moreover, it also offers the flexibility to incorporate anisotropy into the general construction.

[1]Private communication.

2.1. **A homoneneous structure on $\mathbb{R}^d$.** Here we define homogeneous type spaces on $\mathbb{R}^d$ which will be used later to construct a suitable covering of the frequency space. These spaces are created with a quasi-norm induced by a one-parameter group of dilations.
Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$ induced by the inner product $\langle\cdot,\cdot\rangle$. We assume that A is a real $d\times d$ matrix with eigenvalues having positive real parts. For $t>0$ define the group of dilations $\delta_t:\mathbb{R}^d\to\mathbb{R}^d$ by $\delta_t := \exp(A\ln t)$ and let $\nu := \text{trace}(A)$. The matrix A will be kept fixed throughout the paper. Some well-known properties of δ_t are (see [24]),

- $\delta_{ts}=\delta_t\delta_s$.
- $\delta_1 = Id$ (identity on $\mathbb{R}^d$).
- $\delta_t\xi$ is jointly continuous in t and ξ, and $\delta_t\xi\to 0$ as $t\to 0^+$.
- $|\delta_t| := \det(\delta_t) = t^\nu$.

According to [24, Proposition 1.7] there exists a strictly positive symmetric matrix P such that for all $\xi\in\mathbb{R}^d$,

$$[\delta_t\xi]_P := \langle P\delta_t\xi,\delta_t\xi\rangle^{\frac{1}{2}}$$

is a strictly increasing function of t. This helps use introduce a quasi-norm $|\cdot|_A$ associated with A.

Definition 2.1. We define the function $|\cdot|_A:\mathbb{R}^d\to\mathbb{R}_+$ by $|0|_A := 0$ and for $\xi\in\mathbb{R}^d\setminus\{0\}$ by letting $|\xi|_A$ be the unique solution t to the equation $[\delta_{1/t}\xi]_P = 1$.

It can be shown that:

- There exists a constant $C_A>0$ such that
$$|\xi+\zeta|_A \le C_A(|\xi|_A+|\zeta|_A),\ \xi,\zeta\in\mathbb{R}^d.$$
- $|\delta_t\xi|_A = t|\xi|_A$.
- There exists constants $C_1, C_2, \alpha_1, \alpha_2 > 0$ such that

$$C_1\min(|\xi|_A^{\alpha_1},|\xi|_A^{\alpha_2}) \le |\xi| \le C_2\max(|\xi|_A^{\alpha_1},|\xi|_A^{\alpha_2}),\qquad \xi\in\mathbb{R}^d. \tag{2.1}$$

Example 2.2. For $A=\text{diag}(\beta_1,\beta_2,\dots,\beta_d)$, $\beta_i>0$, we have $\delta_t = \text{diag}(t^{\beta_1},t^{\beta_2},\dots,t^{\beta_d})$, and one can verify that

$$|\xi|_A \asymp \sum_{j=1}^d |\xi_j|^{\frac{1}{\beta_j}}, \xi\in\mathbb{R}^d.$$

We need an extension of the quantity $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ to the non-isotropic setting. Let $\tilde{A}$ be the $(d+1)\times(d+1)$ matrix given by

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A\end{bmatrix},$$

and define $D_t = \exp(\tilde{A}\ln t)$. For $(\zeta,\xi)\in\mathbb{R}\times\mathbb{R}^d$, we have $D_t(\zeta,\xi) = (t\zeta,\delta_t\xi)$. We let $|(\zeta,\xi)|_{\tilde{A}}$ be the unique solution t to $[[D_{1/t}(\zeta,\xi)]]_P = 1$, where $[[(\zeta,\xi)]]_P := (\zeta^2+[\xi]_P^2)^{1/2}$. Notice that $|(1,0)|_{\tilde{A}} = 1$ and $|(0,\xi)|_{\tilde{A}} = |\xi|_A$. For $\xi\in\mathbb{R}^d$, we define the bracket $\langle\xi\rangle_A := |(1,\xi)|_{\tilde{A}}$.

Finally, we define the balls $\mathscr{B}_A(\xi,r) := \{\zeta\in\mathbb{R}^d : |\xi-\zeta|_A<r\}$. It can be verified that $|\mathscr{B}_A(\xi,r)| = r^\nu\omega_d^A$, where $\omega_d^A := |\mathscr{B}_A(0,1)|$, so $(\mathbb{R}^d,|\cdot|_A,d\xi)$ is a space of homogeneous type with homogeneous dimension ν.

The transpose of A with respect to $\langle\cdot,\cdot\rangle$, $B := A^\top$, will be useful for generating coverings of the direct space $\mathbb{R}^d$. Since the eigenvalues of B have positive real parts we can repeat the above construction for the group $\delta_t^\top := \exp(B \ln t)$, $t > 0$. We let $|\cdot|_B$ denote the quasi-norm induced by $\delta_t^\top$, $\mathscr{B}_B(x, r)$, and $\langle \cdot \rangle_B$ denote the quasi-distance and bracket corresponding to the group $\delta_t^\top$. The balls associated with $|\cdot|_B$, and C_B the equivalent of C_A in (2.1). Furthermore, we have that the constants α_1 and α_2 in (2.1) also hold with B and trace$(B) = \nu$. Notice that if $g_m(x) := m^\nu g(\delta_m^\top x)$, $g \in L_2(\mathbb{R}^d)$, then $\hat{g}_m(\xi) = \hat{g}(\delta_{\frac{1}{m}} \xi)$. We use the convention that δ_t acts on the frequency space while $\delta_t^\top$ acts on the direct space.

The following adaption of the Fefferman-Stein maximal inequality to the quasi-norm $|\cdot|_B$ will be essential for showing the boundedness of almost diagonal matrices. For $0 < r < \infty$, the parabolic maximal function of Hardy-Littlewood type is defined by

$$M_r^B u(x) := \sup_{t>0} \Big(\frac{1}{\omega_d^B \cdot t^\nu} \int_{\mathscr{B}_B(x,t)} |u(y)|^r \, dy \Big)^{\frac{1}{r}}, u \in L_{r,\text{loc}}(\mathbb{R}^d), \tag{2.2}$$

where $\omega_d^B := |\mathscr{B}_B(0,1)|$. We also need the vector-valued Fefferman-Stein maximal inequality. For $0 < p, q \le \infty$, and a sequence $f = \{f_j\}_{j\in\mathbb{N}}$ of $L_p(\mathbb{R}^d)$ functions, we define the norm

$$\|f\|_{L_p(\ell_q)} := \Big\| \Big(\sum_{j\in\mathbb{N}} |f_j|^q \Big)^{1/q} \Big\|_{L_p(\mathbb{R}^d)}.$$

With this notation, there exists $C > 0$ so that the following vector-valued maximal inequality holds for $r < q \le \infty$ and $r < p < \infty$ (see [23, Chapters I&II]),

$$\big\| \{M_r^B f_k\}_k \big\|_{L_p(\ell_q)} \le C \big\| \{f_k\}_k \big\|_{L_p(\ell_q)}. \tag{2.3}$$

If $q = \infty$, then the inner l_q-norm is replaced by the l_∞-norm.

We introduce the following Petree type maximal function. Let $u(x)$ be a continuous function on $\mathbb{R}^d$. We define

$$u^*(a, R; x) := \sup_{y\in\mathbb{R}^d} \langle y \rangle_B^{-a} |u(x - \delta_R^{-\top} y)|, \qquad a, R > 0, x \in \mathbb{R}^d,$$

where we use the notation $\delta_R^{-\top} := (\delta_R^{-1})^\top$.

It is clear that $u^*(a, R; x)$ is finite whenever u is bounded. However, for band-limited functions we can obtain a much more interesting estimate of $u^*(a, R; x)$ in terms of the parabolic maximal function. It can be shown (see [25, Theorem 1.3.1]) that for $r, R > 0$, there exist a constant $C := C(R, r)$ such that for any function $u(x)$ on $\mathbb{R}^d$ with $\text{supp}(\hat{u}) \subset \mathscr{B}_A(0, R)$, we have

$$u^*(\nu/r, R; x) \le C M_r^B u(x), \qquad \forall x \in \mathbb{R}^d. \tag{2.4}$$

This result can be considered in the vector-valued setting. Let $\Omega = \{\Omega_n\}$ be a sequence of compact subsets of $\mathbb{R}^d$, and let

$$L_p^\Omega(\ell_q) := \{\{f_n\}_{n\in\mathbb{N}} \in L_p(\ell_q) \mid \text{supp}(\hat{f}_n) \subseteq \Omega_n, \forall n\}.$$

Then the following proposition holds.

Proposition 2.3. *Suppose $0 < p < \infty$ and $0 < q \le \infty$, and let $\Omega = \{T_k \mathscr{C}\}_{k\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{R}^d$ generated by a family $\{T_k = \delta_{t_k} \cdot + \xi_k\}_{k\in\mathbb{N}}$ of invertible affine transformations on $\mathbb{R}^d$, with $\mathscr{C}$ a fixed compact subset of $\mathbb{R}^d$. If $0 < r < \min(p, q)$, then there exists a* *constant K such that*

$$\Big\| \Big\{ \sup_{z\in\mathbb{R}^d} \langle \delta_{t_k}^\top z\rangle_B^{-\nu/r} |f_k(\cdot - z)| \Big\} \Big\|_{L_p(\ell_q)} \le K \|\{f_k\}\|_{L_p(\ell_q)}, \tag{2.5}$$

for all $f \in L_p^\Omega(\ell_q)$, *where* $f = \{f_k\}_{k\in\mathbb{N}}$.

Finally we consider a result on vector-valued Fourier multipliers that will be needed to study stability of smoothness spaces in the following section. For $s \in \mathbb{R}$ we let

$$\|f\|_{H_2^s} := \Big(\int |\mathcal{F}^{-1} f(x)|^2 \langle x\rangle_B^{2s}\, dx \Big)^{1/2} \tag{2.6}$$

denote the (anisotropic) Sobolev space norm. Notice that for $f \in H_2^s$, $0 < \varepsilon_0 \le t < \infty$, and $\xi_k \in \mathbb{R}^d$, we have $\|f(\delta_t \cdot + \xi_k)\|_{H_2^s} \le C t^{s-\nu/2} \|f\|_{H_2^s} < \infty$ with C depending only on ε_0. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $\Psi \in \mathcal{S}'(\mathbb{R}^d)$, we define $\Psi(D)f := \mathcal{F}^{-1}(\Psi \hat{f})$.

Theorem 2.4. *Suppose* $0 < p < \infty$ *and* $0 < q \le \infty$. *Let* $\Omega = \{T_k \mathcal{C}\}_{k\in\mathbb{N}}$ *be a sequence of compact subsets of* $\mathbb{R}^d$ *generated by a family* $\{T_k = \delta_{t_k} \cdot + \xi_k\}_{k\in\mathbb{N}}$ *of invertible affine transformations, with* $\mathcal{C}$ *a fixed compact subset of* $\mathbb{R}^d$. *Assume* $\{\psi_j\}_{j\in\mathbb{N}}$ *is a sequence of functions satisfying* $\psi_j \in H_2^s$ *for some* $s > \frac{\nu}{2} + \frac{\nu}{\min(p,q)}$. *Then there exists a constant* $C < \infty$ *such that*

$$\|\{\psi_k(D) f_k\}\|_{L_p(\ell_q)} \le C \sup_j \|\psi_j(T_j \cdot)\|_{H_2^s} \cdot \|\{f_k\}\|_{L_p(\ell_q)}$$

for all $\{f_k\}_{k\in\mathbb{N}} \in L_p^\Omega(\ell_q)$.

2.2. **Decomposition spaces and adapted frames.** We now introduce so-called admissible coverings and show how to generate them. These coverings are then used to construct a suitable resolution of unity and next define Triebel-Lizorkin type smoothness spaces and associated modulation spaces. Finally, we construct a frame which will be used in the following sections to generate compactly supported frame expansions.

Definition 2.5. A set $\mathcal{Q} := \{Q_k\}_{k\in\mathbb{Z}^d}$ of measurable subsets $Q_k \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} Q_k$ and there exists $n_0 < \infty$ such that $\#\{j \in \mathbb{Z}^d : Q_k \cap Q_j \ne \emptyset\} \le n_0$ for all $k \in \mathbb{Z}^d$.

To generate an admissible covering we will use a suitable collection of $|\cdot|_A$-balls, where the radius of a given ball is a so-called moderate function of its center.

Definition 2.6. A function $h : \mathbb{R}^d \to [\varepsilon_0, \infty)$ for $\varepsilon_0 > 0$ is called ($|\cdot|_A$-)moderate if there exists constants $\rho_0, R_0 > 0$ such that $|\xi - \zeta|_A \le \rho_0 h(\xi)$ implies $R_0^{-1} \le h(\zeta)/h(\xi) \le R_0$.

Remark 2.7. Since we consider the quasi-distance $|\cdot|_A$ fixed, we will slightly abuse notation and refer to a $|\cdot|_A$–moderate function simply as a moderate function.

Example 2.8. Let $0 \le \alpha \le 1$. Then

$$h_\alpha(\xi) := (1 + |\xi|_A)^\alpha \tag{2.7}$$

is moderate.

With a moderate function h, it is then possible to construct an admissible covering by using balls (see [7, Lemma 4.7] and [4, Lemma 5]):

Lemma 2.9. *Given a moderate function h with constants $\rho_0, R_0 > 0$, there exists a countable admissible covering $\mathscr{C} := \{\mathscr{B}_A(\xi_k, \rho h(\xi_k))\}_{k\in\mathbb{Z}^d}$ for $\rho < \rho_0/2$, and there exists a constant $0 < \rho' < \rho$ such that the sets in $\mathscr{C}$ are pairwise disjoint.*

□

By using that $\{\mathscr{B}_A(\xi_k, \rho' h(\xi_k))\}$ are disjoint it can be shown that $\{\mathscr{B}_A(\xi_k, 2\rho h(\xi_k))\}$ also give an admissible covering. Notice that the covering $\mathscr{C}$ from Lemma 2.9 is generated by a family of invertible affine transformations applied to $\mathscr{B}_A(0,\rho)$ in the sense that

$$\mathscr{B}_A(\xi_k, \rho h(\xi_k)) = T_k\mathscr{B}_A(0,\rho),\ T_k := \delta_{\rho h(\xi_k)}\cdot + \xi_k.$$

An important property of the covering we will need is that whenever $\mathscr{B}_A\big(\xi_j, 2\rho h(\xi_j)\big) \cap \mathscr{B}_A\big(\xi_k, 2\rho h(\xi_k)\big) \neq \emptyset$ then $h(\xi_j) \asymp h(\xi_k)$ uniformly in j and k which follows from the fact that h is moderate and $2\rho < \rho_0$. We deduce that there exists a uniform constant K such that

$$\|\delta^{-1}_{h(\xi_k)}\delta_{h(\xi_j)}\|_{\ell_2(\mathbb{R}^{d^2})} \le K \qquad \text{whenever} \quad \mathscr{B}_A\big(\xi_j, 2\rho h(\xi_j)\big) \cap \mathscr{B}_A\big(\xi_k, 2\rho h(\xi_k)\big) \neq \emptyset. \tag{2.8}$$

We are now in a position to generate a suitable resolution of unity which additionally (due to technical reasons) has to satisfy the following conditions.

Definition 2.10. Let $\mathscr{C} := \{T_k\mathscr{B}_A(0,\rho)\}_{k\in\mathbb{Z}^d}$ be an admissible covering of $\mathbb{R}^d$ from Lemma 2.9. A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\{\Psi_k\}_{k\in\mathbb{Z}^d} \subset \mathscr{S}$ satisfying:

- $\text{supp}(\Psi_k) \subseteq T_k\mathscr{B}_A(0,2\rho)$, $k \in \mathbb{Z}^d$.
- $\sum_{k\in\mathbb{Z}^d} \Psi_k(\xi) = 1$, $\xi \in \mathbb{R}^d$.
- $\sup_{k\in\mathbb{Z}^d} \|\Psi_k(T_k\cdot)\|_{H_2^s} < \infty$, $s > 0$,

where the H_2^s-norm is defined in (2.6).

A standard trick for generating a BAPU for $\mathscr{C}$ is to pick $\phi \in C^\infty(\mathbb{R}^d)$ non-negative with $\text{supp}(\phi) \subseteq \mathscr{B}_A(0,2\rho)$ such that there is $\delta > 0$ satisfying $\phi(\xi) \ge \delta$ for $\xi \in \mathscr{B}_A(0,\rho)$. We define

$$\phi_k(\xi) := \phi(T_k^{-1}). \tag{2.9}$$

One can show that

$$\Psi_k(\xi) := \frac{\phi_k(\xi)}{\sum_{j\in\mathbb{Z}^d}\phi_k(\xi)}$$

defines a BAPU for $\mathscr{C}$. For later use, we introduce

$$\Phi_k(\xi) := \frac{\phi_k(\xi)}{G(\xi)}, \tag{2.10}$$

with

$$G(\xi) := \sqrt{\sum_{j\in\mathbb{Z}^d}\phi_k^2(\xi)}. \tag{2.11}$$

Notice that $\{\Phi_k\}$ in a certain sense defines a square root of the BAPU. We also notice that there exists a constant $0 < L < \infty$ such that

$$L^{-1} \le \sum_k \phi(T_k^{-1}\cdot) \le L, \quad \text{and} \quad L^{-1} \le G(\cdot) \le L,$$

since $\mathscr{C}$ is a covering and $\{T_k\mathscr{B}_A(0,2\rho)\}_k$ is admissible.

Remark 2.11. For a BAPU $\{\Psi_k\}_{k\in\mathbb{N}}$ associated with the admissible covering $\{T_k\mathscr{B}_A(0,2\rho)\}_{k\in\mathbb{N}}$ we define

$$\Psi_k^* := \sum_{j\in\tilde{k}} \Psi_j, \tag{2.12}$$

where $\tilde{k} := \{j \in \mathbb{Z}^d : \text{supp}(\Psi_j) \cap T_k\mathscr{B}_A(0,2\rho) \neq \emptyset\}$. We will use extensively that $\Psi_k\Psi_k^* = \Psi_k$ and that theere exists M independent of k such that $\#\tilde{k} \leq M$, see [7, 10].

With a BAPU in hand we can now define the T-L type spaces and the associated modulation spaces. For $f \in \mathscr{S}'(\mathbb{R}^d)$ and $\Psi \in \mathscr{S}(\mathbb{R}^d)$, we define $\Psi(D)f := \mathscr{F}^{-1}(\Psi\hat{f})$.

Definition 2.12. Let h be a moderate function satisfying

$$C_1(1+|\xi|_A)^{\gamma_1} \leq h(\xi) \leq C_2(1+|\xi|_A)^{\gamma_2},\ \xi \in \mathbb{R}^d, \tag{2.13}$$

for some $0 < \gamma_1 \leq \gamma_2 < \infty$. Let $\mathscr{Q}$ be an admissible covering generated by h of the type considered in Lemma 2.9, and suppose $\mathscr{T} = \{T_k\}_{k\in\mathbb{N}}$, $T_k = \delta_{\rho h(\xi_k)} \cdot + \xi_k$, is the induced family of invertible affine transformations, and let $\{\Psi_k\}_{k\in\mathbb{N}}$ be a corresponding BAPU. Put $t_k := \rho h(l\xi_k)$.

- For $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$ we let $F^s_{p,q}(h)$ denote the set of functions $f \in \mathscr{S}'(\mathbb{R}^d)$ satisfying

$$\|f\|_{F^s_{p,q}(h)} := \Big\| \Big(\sum_{k\in\mathbb{Z}^d} t_k^{sq} [\Psi_k(D)f]^q \Big)^{1/q} \Big\|_{L_p(\mathbb{R}^d)} < \infty.$$

- For $s \in \mathbb{R}$, and $0 < p, q \leq \infty$ we let $M^s_{p,q}(h)$ denote the set of functions $f \in \mathscr{S}'(\mathbb{R}^d)$ satisfying

$$\|f\|_{M^s_{p,q}(h)} := \Big(\sum_{k\in\mathbb{Z}^d} \|t_k^{sq} \Psi_k(D)f\|^q_{L_p(\mathbb{R}^d)} \Big)^{1/q} < \infty.$$

Remark 2.13. It can be shown that $F^s_{p,q}(h)$ depends only on h up to equivalence of the norms (see [4, Proposition 5.3]), so the T-L type spaces are well-defined. Similar for the modulation spaces. Furthermore, they both constitute quasi-Banach spaces, and for $p, q < \infty$, $\mathscr{S}$ is dense in both (see [4, Proposition 5.2]).

Next, we construct a frame for the T-L type spaces and the associated modulation spaces. Consider the system $\{\Phi_k\}_{k\in\mathbb{Z}^d}$ from (2.10) which in a sense is a square root of a BAPU. Let K_a be a cube in $\mathbb{R}^d$ which is aligned with the coordinate axes and has side-length $2a$ satisfying $\mathscr{B}_A(0,2\rho) \subseteq K_a$. For the sake of convenience, put

$$t_k := \rho h(\xi_k), \tag{2.14}$$

with ρ given by Lemma 2.9. We first push forward the Fourier basis on K_a using the transformations $\{T_k\}_k$ to obtain the functions

$$e_{k,n}(\xi) := (2a)^{-\frac{d}{2}} t_k^{-\frac{\nu}{2}} \chi_{K_a}(T_k^{-1}\xi) e^{-i\frac{\pi}{a} n \cdot T_k^{-1}\xi},\ n, k \in \mathbb{Z}^d,$$

and

$$\hat{\eta}_{k,n} := \Phi_k e_{k,n},\ n, k \in \mathbb{Z}^d. \tag{2.15}$$

One can verify that $\{\eta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a tight frame for $L_2(\mathbb{R}^d)$, see [4]. By defining $\hat{\mu}_k(\xi) := \Phi_k(T_k\xi)$, we get an explicit representation of $\eta_{k,n}$ in direct space

$$\eta_{k,n}(x) = (2a)^{-\frac{d}{2}} t_k^{\frac{\nu}{2}} \mu_k(\delta_{t_k}^\top x - \frac{\pi}{a} n) e^{ix\cdot\xi_k}. \tag{2.16}$$

From a computational point-of-view it is not very satisfactory to have an infinite number of generators for the system given by (2.16). To address this problem, we define an associated unnormalized system by

$$\hat{\psi}_{k,n} := \phi_k e_{k,n}. \tag{2.17}$$

Notice that $\{\psi_{k,n}\}$ is generated by a single function. In fact, let $\psi := \check{\phi}$. Then

$$\psi_{k,n}(x) = (2a)^{-\frac{d}{2}} t_k^{\frac{\nu}{2}} \psi(\delta_{t_k}^\top x - \frac{\pi}{a} n) e^{ix\cdot\xi_k}, \tag{2.18}$$

which is considerably simpler than the system given by (2.16). As a candidate for dual system to $\{\psi_{k,n}\}$, we define

$$\hat{\tilde{\psi}}_{k,n} = \frac{\phi_k e_{k,n}}{\sum_k \phi_k^2(\cdot)}. \tag{2.19}$$

By defining

$$\hat{\gamma}_k(\xi) := \frac{\phi_k(T_k\xi)}{\sum_j \phi_j^2(T_j\xi)},$$

we get an explicit representation of $\tilde{\psi}_{k,n}$ in direct space analog to (2.18),

$$\tilde{\psi}_{k,n}(x) = (2a)^{-\frac{d}{2}} t_k^{\frac{\nu}{2}} \gamma_k(\delta_{t_k}^\top x - \frac{\pi}{a} n) e^{ix\cdot\xi_k}. \tag{2.20}$$

The generators $\{\gamma_k\}$ are in fact uniformly well-localized, which will be verified in the proof of Lemma 2.17.

2.3. **Stability of the frame system generated by a single function.** We now turn to the task of showing that the system

$$\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$$

is stable in the smoothness spaces $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$. To show that $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ constitutes a frame for $F^s_{p,q}(h)$ and $M^s_{p,q}(h)$, we need to introduce associated sequence spaces. The following point sets will be useful for that,

$$Q(k,n) = \Big\{y \in \mathbb{R}^d : \delta_{t_k}^\top y - \frac{\pi}{a} n \in \mathscr{B}_B(0,1)\Big\}. \tag{2.21}$$

Notice that $Q(k,n)$ can be considered the "effective" support of $\psi_{k,n}$. It can easily be verified that there exists $n_0 < \infty$ such that uniformly in x and k, $\sum_{n\in\mathbb{Z}^d} \chi_{Q(k,n)}(x) \le n_0$. With this property in hand, we can define the associated sequence spaces.

Definition 2.14. Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$. We then define the sequence space $f^s_{p,q}(h)$ as the set of sequences $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$ satisfying

$$\|s_{k,n}\|_{f^s_{p,q}(h)} := \Big\| \Big(\sum_{k,n\in\mathbb{Z}^d} \big(t_k^{s+\frac{\nu}{2}} |s_{k,n}|\big)^q \chi_{Q(k,n)} \Big)^{1/q} \Big\|_{L_p} < \infty.$$

Let $s \in \mathbb{R}$, $0 < p \le \infty$, and $0 < q < \infty$. We then define the sequence space $m^s_{p,q}(h)$ as the set of sequences $\{s_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset \mathbb{C}$ satisfying

$$\|s_{k,n}\|_{m^s_{p,q}(h)} := \Big\| t_k^{s+\frac{\nu}{2}-\frac{\nu}{p}} \Big(\sum_{n\in\mathbb{Z}^d} |s_{k,n}|^p \Big)^{1/p} \Big\|_{l_q} < \infty.$$

If $p = \infty$ or $q = \infty$, then the l_p-norm or l_q-norm, respectively, is replaced by the l_∞-norm.

We now show that the system $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ and $\{\tilde{\psi}_{k,n}\}_{k,n\in\mathbb{Z}^d}$ given by (2.17) and (2.19) generate stable frame decomposition of $F^s_{p,q}(h)$.

Remark 2.15. We mention that Theorem 2.16 below also holds for the modulation spaces $M^s_{p,q}(h)$ and the associated sequence space $m^s_{p,q}(h)$. The proof in the modulation case runs parallel with the proof presented below, but it is in fact significantly simpler due to the fact that one does not have to call on results for vector valued maximal functions. We leave the details to the reader (see also the approach in [3]).

Theorem 2.16. *Let $h : \mathbb{R}^d \to (0,\infty)$ be a moderate function satisfying the conditions in Definition 2.12, and let $\mathscr{C} := \{T_k\mathscr{B}_A(0,\rho)\}_{k\in\mathbb{Z}^d}$ be an associated admissible covering of $\mathbb{R}^d$ of the type considered in Lemma 2.9. Let $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ and $\{\tilde{\psi}_{k,n}\}_{k,n\in\mathbb{Z}^d}$ be the function systems given by* (2.17) *and* (2.19)*, respectively. Then*

i. *The coefficient operators defined formally by*

$$Cf = \{\langle f, \psi_{k,n}\rangle\}_{k,n} \qquad \text{and} \qquad \tilde{C}f = \{\langle f, \tilde{\psi}_{k,n}\rangle\}_{k,n}$$

extend to bounded operators

$$C : F^s_{p,q}(h) \to f^s_{p,q}(h) \qquad \text{and} \qquad \tilde{C} : F^s_{p,q}(h) \to f^s_{p,q}(h).$$

ii. *The reconstruction operators defined formally by*

$$R : \{c_{k,n}\}_{k,n} \to \sum_{k,n} c_{k,n}\tilde{\psi}_{k,n}, \qquad \text{and} \qquad \tilde{R} : \{c_{k,n}\}_{k,n} \to \sum_{k,n} c_{k,n}\psi_{k,n}$$

extend to bounded operators

$$R : f^s_{p,q}(h) \to F^s_{p,q}(h), \qquad \tilde{R} : f^s_{p,q}(h) \to F^s_{p,q}(h).$$

Moreover, it holds true that

$$RC = \mathrm{Id}_{F^s_{p,q}(h)} \qquad \text{and} \qquad \tilde{R}\tilde{C} = \mathrm{Id}_{F^s_{p,q}(h)}\,.$$

All of the above results also hold for the spaces $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$ in place of $F^s_{p,q}(h)$ and $f^s_{p,q}(h)$.

The proof of Theorem 2.16 is completely parallel for the two cases: C and R and $\tilde{C}$ and $\tilde{R}$, so we only give the details for C and R. However, before we get to the details of the proof, we need to prove three lemmas of independent interest. Moreover, the proof in the modulation space case for the pair $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$ is somewhat simpler that for $F^s_{p,q}(h)$ and $f^s_{p,q}(h)$, so we only give the argument in the Triebel-Lizorkin case.

Lemma 2.17. *Let $\{T_k = \delta_{t_k}\cdot + \xi_k\}_{k\in\mathbb{Z}^d}$ be a family of invertible affine transformations on $\mathbb{R}^d$ generated by Lemma 2.9 based on the moderate function h. Suppose $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$. Then*

$$\|S^s_q(f)\|_{L_p} \le C\|f\|_{F^s_{p,q}(h)}, \qquad f \in F^s_{p,q}(h),$$

where

$$S_q^s(f) := \Big(\sum_k \sum_{n\in\mathbb{Z}^d} t_k^{s+\nu/2} |\langle f, \tilde{\psi}_{k,n}\rangle| \chi_{Q(k,n)})^q\Big)^{1/p},$$

where $Q(k,n)$ *is defined by* (2.21)*, and* t_k *is given by* (2.14)*.*

Proof. Take $f \in F_{p,q}^s(h)$. According to Eq. (2.16),

$$t_k^{\nu/2} |\langle f, \tilde{\psi}_{k,n}\rangle| = (2a)^{-d/2} \big|(\gamma_k * f)(\frac{\pi}{a}\delta_{t_k}^{-\top} n)\big|,$$

where we let $t_k = h(\xi_k)$. Moreover, if $Q(k,n)\cap Q(k,n') \neq \emptyset$ and $u \in Q(k,n)$, $v \in Q(k,n')$ then $|u-v|_B \le K t_k^{-1}$ for some uniform constant K. Hence,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} (|\langle f, \tilde{\psi}_{k,n}\rangle| t_k^{\nu/2} \chi_{Q(k,n)}(x))^q &\le C \sum_{n\in\mathbb{Z}^d} (\sup_{y\in Q(k,n)} |(\gamma_k(D)f)(y)| \chi_{Q(k,n)}(x))^q \\ &\le C' \sup_{z\in \mathscr{B}_B(0,Kt_k^{-1})} \big(\langle \delta_{t_k}^\top z\rangle_B^{-\nu/r} |(\gamma_k(D)f)(x-z)|\big)^q \cdot \langle \delta_{t_k}^\top z\rangle_B^{\nu q/r} \\ &\le C'' \big(\sup_{z\in\mathbb{R}^d} \langle \delta_{t_k}^\top z\rangle_B^{-\nu/r} |(\gamma_k(D)f)(x-z)|\big)^q. \end{aligned} \tag{2.22}$$

We would like to apply Proposition 2.3 and then Theorem 2.4 to (2.22), so we have to verify that $\{\gamma_k\}_k$ are uniformly localized. To see this, we first notice that,

$$\sum_k \phi^2(T_k^{-1}\xi) = \sum_{k\in F_\xi} \phi^2(T_k^{-1}\xi),$$

with $F_\xi := \{k\in\mathbb{N} : \xi \in T_k \mathscr{B}_A(0,2\rho)\}$, where the cardinality if F_ξ is uniformly bounded in ξ. We have

$$\hat{\gamma}_j(\xi) = \frac{\phi(\xi)}{\sum_k \phi^2(T_k^{-1}T_j\xi)} = \frac{\phi(\xi)}{\sum_k \phi^2(\delta_{t_k^{-1}}\delta_{t_j}\xi + \delta_{t_k^{-1}}\xi_j - \delta_{t_k^{-1}}\xi_k)}.$$

Now consider $\partial^\alpha \hat{\gamma}_j$. For $f(\xi) := \phi^2(\delta_{t_k^{-1}}\delta_{t_j}\xi + \delta_{t_k^{-1}}\xi_j - \delta_{t_k^{-1}}\xi_k)$, the chain rule shows that $\partial^\eta f = \sum_{\beta:|\beta|=|\eta|} p_\beta \partial^\beta[\phi^2]$, where p_β are monomials of degree $|\eta|$ in the entries of $\delta_{t_k^{-1}}\delta_{t_j}$. It follows from the estimate (2.8) that

$$|\partial^\beta \hat{\gamma}_j(\xi)| \le C_\beta K^{|\beta|} \chi_{\mathscr{B}_A(0,2\rho)}(\xi), \qquad \beta\in\mathbb{N}_0^d, \tag{2.23}$$

with C_β a constant that does not depend on j. Thus, we have for any $N\in\mathbb{N}$,

$$\begin{aligned}|\mathscr{F}^{-1}\hat{\gamma}_k(x)| &\le C(1+|x|)^{-N} \Big|\sum_{|\beta|\le N} x^\beta \mathscr{F}^{-1}\hat{\gamma}_k(x)\Big| \le C(1+|x|)^{-N} \sum_{|\beta|\le N} \|\partial_\xi^\beta \hat{\gamma}_k\|_{L_1} \\ &\le C_N(1+|x|)^{-N} \le C_N' \langle x\rangle_B^{-N\alpha_1}. \end{aligned} \tag{2.24}$$

So, in particular, Theorem 2.4 applies to the multipliers induced by $\{\gamma_k\}_k$ since $\mathrm{supp}(\hat{\gamma}_k) \subset T_k\mathscr{B}_A(0,2\rho)$. We let $\{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a BAPU associated with $F_{p,q}^s(h)$, and recall the system $\{\Psi_k^*\}$ given by (2.12). Hence, by Proposition 2.3, Theorem 2.4, and the estimate above,

$$\begin{aligned}\|S_q^s(f)\|_{L_p} &\le C\|h(\xi_k)^s \gamma_k(D) f\|_{L_p(\ell_q)} \\ &= C\|h(\xi_k)^s \gamma_k(D)\Psi_k^*(D) f\|_{L_p(\ell_q)} \\ &\le C'\|h(\xi_k)^s \Psi_k^*(D) f\|_{L_p(\ell_q)} \\ &\le C''\|f\|_{F_{p,q}^s(h)}. \end{aligned}$$

□

The following technical lemma estimates the absolute value of an expansion on a fixed scale relative to a well-localized system in terms of the maximal function. The lemma is a slight variation on a well-known result, see [4, 17].

Lemma 2.18. *Let* $0 < r \le 1$. *Suppose there is a constant* C *such that the system* $\{\tilde{\psi}_{k,n}\}_{k,n\in\mathbb{Z}^d}$ *defined on* $\mathbb{R}^d$ *satisfies*

$$|\tilde{\psi}_{k,n}(x)| \le C t_k^{\nu/2}(1+|\tfrac{\pi}{a}n+\delta_{t_k}^{\top}x|_B)^{-N}, \qquad \forall k,n\in\mathbb{Z}^d,$$

for some $N > \nu/r$. *There exists a constant* C' *such that for any sequence* $\{s_{k,n}\}_{k,n}\subset\mathbb{C}$, *we have*

$$\sum_n |s_{k,n}||\tilde{\psi}_{k,n}| \le C' t_k^{\nu/2} M_r^B\Big(\sum_n |s_{k,n}|\chi_{Q(k,n)}\Big).$$

Proof. We may without loss of generality suppose $x\in Q(k,0)$. Let $A_0=\{n\in\mathbb{Z}^d : |\frac{\pi}{a}n|_B\le 1\}$, and for $j\in\mathbb{N}$, let $A_j=\{n\in\mathbb{Z}^d : 2^{j-1}<|\frac{\pi}{a}n|_B\le 2^j\}$. Notice that $\cup_{n\in A_j}Q(k,n)$ is a bounded set contained in the ball $\mathscr{B}_B(0,c2^{j+1}t_k^{-1})$. Now,

$$\begin{aligned}
\sum_{n\in A_j}|s_{k,n}|(1+|\tfrac{\pi}{a}n+\delta_{t_k}^{\top}x|_B)^{-N} &\le C2^{-jN}\sum_{n\in A_j}|s_{k,n}| \le C2^{-jN}\Big(\sum_{n\in A_j}|s_{k,n}|^r\Big)^{1/r}\\
&\le C2^{-jN}t_k^{\nu/r}\Big(\int\sum_{n\in A_j}|s_{k,n}|^r\chi_{Q(k,n)}(y)\,dy\Big)^{1/r}\\
&\le Cn_0^{1-r}2^{-jN}t_k^{\nu/r}\Big(\int_{\mathscr{B}_B(0,c2^{j+1}t_k^{-1})}\Big(\sum_{n\in A_j}|s_{k,n}|\chi_{Q(k,n)}(y)\Big)^r\,dy\Big)^{1/r}\\
&\le C'2^{-j(N-\nu/r)}M_r^B\Big(\sum_{n\in\mathbb{Z}^d}|s_{k,n}|\chi_{Q(k,n)}\Big)(x).
\end{aligned}$$

The result now follows by summing over $j\in\mathbb{N}_0$. □

Next we use Lemma 2.18 to study the reconstruction operator $\tilde{R}$.

Lemma 2.19. *Suppose* $s\in\mathbb{R}$, $0<p<\infty$, *and* $0<q\le\infty$. *Then for any finite sequence* $\{s_{k,n}\}_{k,n}$, *we have*

$$\Big\|\sum_{k,n}s_{k,n}\tilde{\psi}_{k,n}\Big\|_{F^s_{p,q}(h)} \le C\|\{s_{k,n}\}\|_{f^s_{p,q}}.$$

Proof. Let $\{\Psi_k\}_{k\in\mathbb{N}}$ be a BAPU associated with $F^s_{p,q}(h)$. By Remark 2.11 and Theorem 2.4 we get

$$\begin{aligned}
\Big\|\sum_{k,n}s_{k,n}\tilde{\psi}_{k,n}\Big\|_{F^s_{p,q}} &= \Big\|\Big\{t_k^s\cdot\Psi_k(D)\Big(\sum_{\ell,n}s_{\ell,n}\tilde{\psi}_{\ell,n}\Big)\Big\}_k\Big\|_{L_p(\ell_q)}\\
&\le C\Big\|\Big\{t_k^s\sum_{\ell\in N(k)}\sum_n s_{\ell,n}\tilde{\psi}_{\ell,n}\Big\}_k\Big\|_{L_p(\ell_q)},
\end{aligned}$$

where $N(k)=\{\ell\in\mathbb{N}:\ \mathrm{supp}(\Psi_k)\cap\mathrm{supp}(\tilde{\psi}_\ell)\ne\emptyset\}$. According to [10] on equivalence for admissible coverings, it holds true that $\#N(k)$ is uniformly bounded. Moreover, h is a moderate weight, so it follows that we obtain

$$\Big\|\Big\{t_k^s\sum_{\ell\in N(k)}\sum_n s_{\ell,n}\tilde{\psi}_{\ell,n}\Big\}_k\Big\|_{L_p(\ell_q)} \le C\Big\|\Big(\sum_\ell\Big(t_\ell^s\sum_n|s_{\ell,n}||\tilde{\psi}_{\ell,n}|\Big)^q\Big)^{1/q}\Big\|_{L_p}.$$

Fix $0 < r < \min(1, p, q)$. Then Lemma 2.18 and the Fefferman-Stein maximal inequality (2.3) yields

$$\Big\|\Big\{t_k^s \sum_n |s_{k,n}||\tilde{\psi}_{k,n}|\Big\}_k\Big\|_{L_p(\ell_q)} \le C\Big\|\Big\{t_k^{s+\nu/2} M_r^B\Big(\sum_n |s_{k,n}|\chi_{Q(k,n)}\Big)\Big\}_k\Big\|_{L_p(\ell_q)}$$
$$\le C'\Big\|\Big\{t_k^{s+\nu/2} \sum_n |s_{k,n}|\chi_{Q(k,n)}\Big\}_k\Big\|_{L_p(\ell_q)}.$$

The result now follows since the sum over n is locally finite with a uniform bound on the number of non-zero terms, which implies that

$$\Big(\sum_n |s_{k,n}|\chi_{Q(k,n)}\Big)^q \asymp \sum_n |s_{k,n}|^q \chi_{Q(k,n)},$$

uniformly in k. □

We can now finally proceed and prove Theorem 2.16.

Proof of Theorem 2.16. Lemma 2.19 shows that R extends to a bounded operator

$$R : f^s_{p,q}(h) \to F^s_{p,q}(h),$$

while Lemma 2.17 shows that C extends to a bounded operator

$$C : F^s_{p,q}(h) \to f^s_{p,q}(h).$$

Hence RC is bounded on $F_{p,q}(h)$. For $f \in L_2(\mathbb{R}^d)$, we let $Kf := [\hat{f}/G]^\vee$, with G defined by (2.11). It is easy to check that K is self-adjoint and invertible on $L_2(\mathbb{R}^d)$ using the fact that G and G^{-1} are bounded real-valued functions. Hence, for $f \in \mathscr{S}(\mathbb{R}^d) \subset L_2(\mathbb{R}^d)$, we have in L_2-sense,

$$\begin{aligned} f &= KK^{-1}f \\ &= K\sum_{k,n} \langle K^{-1}f, \eta_{k,n}\rangle \eta_{k,n} \\ &= \sum_{k,n} \langle f, K^{-1}\eta_{k,n}\rangle K\eta_{k,n} \\ &= \sum_{k,n} \langle f, \psi_{k,n}\rangle \tilde{\psi}_{k,n} \\ &= RCf. \end{aligned}$$

From the boundedness of R and C it follows that the identify $RCf = f$ also holds true in $F^s_{p,q}(h)$. Hence it holds on the dense subset $\mathscr{S}(\mathbb{R}^d) \subset F^s_{p,q}(h)$, and the result now follows by extension. □

Example 2.20. Let us expand on Example 2.8. Fix $0 < \alpha < 1$. Define $b_k = k|k|^{\alpha/(1-\alpha)}$, $k \in \mathbb{Z}^d \setminus \{0\}$, and let $\mathscr{T} = \{T_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ be given by $T_k\xi = |k|^{\alpha/(1-\alpha)}\xi + b_k$. This type of "polynomial" covering was first considered by Päivärinta and Somersalo in [19] to study pseudodifferential operators, and Gröbner [14] used such coverings to define α-modulation spaces. It is not difficult to verify that on the modulation side we obtain the so-called α-modulation spaces $M^\beta_{p,q}(h_\alpha) = M^{\beta,\alpha}_{p,q}(\mathbb{R}^d)$. Consequently $F^\beta_{p,q}(h_\alpha, \langle\cdot\rangle)$ can be considered a Triebel-Lizorkin equivalent of the α-modulation spaces. The α-Triebel-Lizorkin spaces were first considered in a rather rudimentary form in [2].

In this case we have an explicitly given covering, so we can also write out an explicit norm equivalence that follows immediately from Theorem 2.16. Let $\{\psi_{k,n}\}$ be the frame for $F^{\beta}_{p,q}(h_\alpha,\langle\cdot\rangle)$ given by (2.17). We have

$$\|f\|_{F^{\beta}_{p,q}(h_\alpha)} \asymp \left\| \left\{ \langle k\rangle^{\frac{s}{1-\alpha}+\frac{1}{2}\frac{\alpha d}{1-\alpha}} \Big(\sum_{n\in\mathbb{Z}^d} |\langle f,\psi_{k,n}\rangle|^q \chi_{Q(k,n)} \Big)^{1/q} \right\}_k \right\|_{L_p(\ell_q)},$$

where $Q(k,n)$ is given by (2.21).

3. Compactly supported generators

We now consider a construction of frames for decomposition spaces with a single compactly supported generator. The idea is to obtain a perturbation of the system (2.18) which we control using the notion of almost diagonal matrices for the sequence space $f^s_{p,q}(h)$ developed in [18].

Before we get to the construction of a compactly supported frame generator, we need to introduce some notation and machinery introduced in [18]. First we add a few general restrictions to the moderate function h used to generate admissible coverings:

$$\begin{cases} \text{There exist } \beta, R_1, \rho_1 > 0 \text{ such that } h^{1+\beta} \text{ is moderate and} \\ \quad |\xi-\zeta|_A \le a h(\xi) \text{ for } a \ge \rho_1 \text{ implies } h(\zeta) \le R_1 a h(\xi). \end{cases} \tag{3.1}$$

A rich body of functions h satisfying these conditions can be generated by using functions $s:\mathbb{R}_+\to\mathbb{R}_+$ which satisfy $s(2t)\le Cs(t)$, $t\in\mathbb{R}_+$, and

$$(1+t)^\gamma \le s(t) \le (1+t)^{\frac{1}{1+\beta}}, \qquad t\in\mathbb{R}_+,$$

for some $\gamma>0$. We assign $h = s(|\cdot|_A)$ and use that s is weakly sub-additive to get the results (see [7]). Notice that $s(t)=(1+t)^\alpha, 0\le\alpha<1$, gives Example 2.8 and fulfills the mentioned conditions.

We begin by considering the following notion of a well-localized frame adapted to the general structure of the systems given by Eqs. (2.16), (2.18), and (2.20). The compactly supported frame constructed below will satisfy Definition 3.1.

Definition 3.1. Let $h:\mathbb{R}^d\to[0,\infty)$ be a moderate function satisfying (3.1), and let $\{T_k = \delta_{\rho h(\xi_k)}\cdot+\xi_k\}_{k\in\mathbb{Z}^d}$ be a family of invertible affine transformations on $\mathbb{R}^d$ generated by Lemma 2.9 based on h. Put $t_k := \rho h(\xi_k)$. Assume that the system $\Theta := \{\theta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ is a frame for $L_2(\mathbb{R}^d)$. Let $0<p,q<\infty$ and $s\in\mathbb{R}$. We say that Θ is (p,q,s)-localized, $0<p,q\le\infty$, $s\in\mathbb{R}$, provided there exist constants $C,\delta>0$ such that for any $k,n\in\mathbb{Z}^d$,

$$|\theta_{k,n}(x)| \le C t_k^{\frac{\nu}{2}} (1+t_k|x_{k,n}-x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \tag{3.2}$$

$$|\hat\theta_{k,n}(\xi)| \le C t_k^{-\frac{\nu}{2}} (1+t_k^{-1}|\xi_k-\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)}, \tag{3.3}$$

where $r:=\min(1,p,q)$, and

$$x_{k,n} = \delta^\top_{t_k^{-1}} \frac{\pi}{a} n, \qquad k,n\in\mathbb{Z}^d. \tag{3.4}$$

Closely related to localized frames is the following notion of an almost diagonal matrix for $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$. It turns out that "change-of-frame" matrices between two localized frames are almost diagonal in the following sense.

Definition 3.2. With the same notation as in Definition 3.1, assume that $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $p < \infty$ for $f^s_{p,q}(h)$, and $q < \infty$ for $m^s_{p,q}(h)$. Let $r := \min(1, p, q)$. A matrix $\mathbf{A} := \{a_{(j,m)(k,n)}\}_{j,m,k,n\in\mathbb{Z}^d}$ is called almost diagonal on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$ if there exists $C, \delta > 0$ such that

$$|a_{(j,m)(k,n)}| \leq C[D(x_{k,n}, x_{j,m}\}) + D(x_{k,n}, x_{k,-n})],$$

where

$$D(x,y) := \left(\frac{t_k}{t_j}\right)^{s+\frac{\nu}{2}} \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\frac{\delta}{2}}, \left(\frac{t_k}{t_j}\right)^{\frac{\delta}{2}}\right) c^{\delta}_{jk} \quad \times (1+\min(t_k,t_j)|x-y|_B)^{-\frac{\nu}{r}-\delta},$$

and

$$c^{\delta}_{jk} := \min\left(\left(\frac{t_j}{t_k}\right)^{\frac{\nu}{r}+\delta}, \left(\frac{t_k}{t_j}\right)^{\delta}\right)(1+\max(t_k,t_j)^{-1}|\xi_k - \xi_j|_A)^{-\frac{\nu}{r}-\delta}$$

with $t_k := \rho h(\xi_k)$, $k \in \mathbb{Z}^d$, and $x_{k,n}$ defined by (3.4). We denote the set of almost diagonal matrices on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$ by $\text{ad}^s_{p,q}(h)$.

Remark 3.3. Two important results from [18] that will be needed below are the following.

- Suppose that $A \in \text{ad}^s_{p,q}(h)$. Define the action on a sequence $\mathbf{s} = \{s_{k,n}\}_{k,n\in\mathbb{Z}^d}$ by
$$(A\mathbf{s})_{(j,m)} := \sum_{k,n\in\mathbb{Z}^d} a_{(j,m)(k,n)} s_{k,n}.$$
Then A is bounded on $f^s_{p,q}(h)$ and $m^s_{p,q}(h)$, see [18, Proposition 3.3].
- Suppose the two systems $\{\theta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ and $\{\psi_{j,m}\}_{j,m\in\mathbb{Z}^d}$ defined on $\mathbb{R}^d$ are both (p,q,s)-localized. Then it follows that the "change-of-frame" matrix $[\langle\theta_{k,n}, \psi_{j,m}\rangle]_{(k,n)(j,m)} \in \text{ad}^s_{p,q}(h)$, see [18, Lemma 3.1].

Now suppose that $\Theta := \{\theta_{k,n}\}$ is a (p,q,s)-localized system on $\mathbb{R}^d$. We remark that $\theta_{k,n}$ might not be a Schwartz function – unlike the case of the frame system considered in (2.18) and (2.20). Hence, we have to be careful not to try to use the pairing between $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ to calculate the frame coefficient relative to the system Θ. In fact, for general $f \in \mathscr{S}'(\mathbb{R}^d)$ we cannot calculate the expansion coefficients, but if we restrict our attention to the subclass $f \in F^s_{p,q}(h)$ the situation improves. In this case, we define $[f, \theta_{j,m}]$ as

$$[f,\theta_{j,m}] := \sum_{k,n\in\mathbb{Z}^d} \langle\tilde{\psi}_{k,n}, \theta_{j,m}\rangle\langle f, \psi_{k,n}\rangle, \qquad f \in F^s_{p,q}(h), \tag{3.5}$$

where $\psi_{k,n}$ and $\tilde{\psi}_{k,n}$ are defined in (2.18) and (2.20), respectively. It follows from Remark 3.3 that $[\cdot, \theta_{j,m}]$ is a bounded linear functional on $F^s_{p,q}(h)$; in fact we have

$$\begin{aligned}\sum_{k,n\in\mathbb{Z}^d} |\langle\tilde{\psi}_{k,n}, \theta_{j,m}\rangle||\langle f, \psi_{k,n}\rangle| &\leq \left\| \left\{ \sum_{k,n\in\mathbb{Z}^d} |\langle\tilde{\psi}_{k,n}, \theta_{j,m}\rangle||\langle f, \psi_{k,n}\rangle| \right\}_{j,m\in\mathbb{Z}^d} \right\|_{f^s_{p,q}(h)} \\ &\leq C\|\langle f, \psi_{k,n}\rangle\|_{f^s_{p,q}(h)} \leq C\|f\|_{F^s_{p,q}(h)}.\end{aligned} \tag{3.6}$$

Clearly $[f,\theta_{j,m}]$ coincides with the usual inner product whenever $f \in L_2(\mathbb{R}^d)$. Now let S denote the frame operator given by

$$Sf = \sum_{k,n\in\mathbb{Z}^d} [f,\theta_{k,n}]\theta_{k,n}, \qquad f \in L_2(\mathbb{R}^d).$$

We now turn to the perturbation approach mentioned earlier. Let $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ be the frame given by (2.18), and for given (p,q,s) suppose that $\{\theta_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset L_2(\mathbb{R}^d)$ is a system that is close to $\{\psi_{k,n}\}_{k,n\in\mathbb{Z}^d}$ in the sense that there exists $\varepsilon,\delta > 0$ such that

$$|\psi_{k,n}(x) - \theta_{k,n}(x)| \le \varepsilon t_k^{\frac{\nu}{2}} (1 + t_k |x_{k,n} - x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \tag{3.7}$$

$$|\hat{\psi}_{k,n}(\xi) - \hat{\theta}_{k,n}(\xi)| \le \varepsilon t_k^{-\frac{\nu}{2}} (1 + t_k^{-1} |\xi_k - \xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right) - \frac{2}{\beta}\left(|s| + \frac{2\nu}{r} + \frac{3\delta}{2}\right)}, \tag{3.8}$$

where we have used the notation from Definition 3.2. Notice the dependence on (p,q,s). We can now state a general result on perturbation of the frame given by (2.18).

Proposition 3.4. *Assume that $\{\theta_{k,n}\}_{k,n\in\mathbb{Z}^d} \subset L_2(\mathbb{R}^d)$ is a frame for $L_2(\mathbb{R}^d)$ and satisfies* (3.2) *and* (3.3) *for some $0 < p,q < \infty$ and $s \in \mathbb{R}$. Then there exists $\varepsilon_0 > 0$ such that whenever $\{\theta_{k,n}\}_{k,n\in\mathbb{Z}^d}$ also satisfies* (3.7) *and* (3.8) *for some $0 < \varepsilon \le \varepsilon_0$ there exist $C_1, C_2 > 0$ such that*

$$C_1 \|f\|_{F^s_{p,q}(h)} \le \|[f,\theta_{k,n}]\|_{f^s_{p,q}(h)} \le C_2 \|f\|_{F^s_{p,q}(h)}., \qquad f \in F^s_{p,q}(h). \tag{3.9}$$

Moreover, for $f \in F^s_{p,q}(h)$ we have

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f,\theta_{k,n}\rangle S^{-1}\theta_{k,n}$$

in the sense of $\mathscr{S}'$. A similar result holds for $M^s_{p,q}(h)$ and $m^s_{p,q}(h)$.

Proof. Follows by a straightforward adaptation of Theorem 4.1, Proposition 4.2, and Lemma 4.5 in [18] to the setup with the frame given by (2.18). □

3.1. **Two specific constructions of a compactly supported generator.** With Proposition 3.4 in hand, all that remain is to find a compactly supported generator that satisfy (3.7) and (3.8). Throughtout this section, we assume that the moderate function h is fixed and satisfies (3.1).

We will present two possible approaches. First a straightforward approximation approach with limited flexibility and then a more sophisticated approximation procedure where one can pick the function as a linear combination of translates and dilates of any reasonable function.

The fact that we are looking for a single generator actually simplifies the conditions given in (3.7) and (3.8). In fact, it suffices to find a generator $\tau \in L_2(\mathbb{R}^d)$ which is close enough to ψ considered in (2.20) in the following sense,

$$|\psi(x) - \tau(x)| \le \varepsilon (1 + |x|_B)^{-2\left(\frac{\nu}{r}+\delta\right)}, \tag{3.10}$$

$$|\hat{\psi}(\xi) - \hat{\tau}(\xi)| \le \varepsilon (1 + |\xi|_A)^{-2\left(\frac{\nu}{r}+\delta\right) - \frac{2}{\beta}\left(|s| + \frac{2\nu}{r} + \frac{3\delta}{2}\right)}. \tag{3.11}$$

Then the frame $\{\theta_{k,n}\}$ can be generated from τ in the obvious way adapting the structure of the system (2.18), i.e.,

$$\left\{\theta_{k,n} := t_k^{\frac{\nu}{2}} \tau\left(\delta_{t_k}^\top x - \frac{\pi}{a} n\right) e^{ix\cdot\xi_k}\right\}_{k,n\in\mathbb{Z}^d}. \tag{3.12}$$

Let us consider the following simple approximation of ψ. Take any nonnegative $\eta \in C^\infty(\mathbb{R}^d)$ with compact support satisfying $|\eta| \le 1$ and $\eta(x) = 1$ for $|x|_B \le 1$. Then we define for $j \in \mathbb{N}$,

$$\eta_j(x) = \eta(\delta_{2^{-j}}^\top x) \qquad \text{and} \qquad \tau_j(x) = \eta_j(x)\psi(x).$$

Clearly each τ_j is compactly supported. We have the following result.

Proposition 3.5. *Let $0 < p, q < \infty$ and $s \in \mathbb{R}$. Given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that the compactly supported function τ_j satisfies both* (3.10) *and* (3.11) *for $j \ge n_0$.*

Proof. Clearly $\psi \in \mathscr{S}(\mathbb{R}^d)$ so for fixed $\delta > 0$ there exists $C < \infty$ such that

$$|\psi(x)| \le C(1 + |x|_B)^{-2(\frac{\nu}{r}+\delta)-1}.$$

Hence, $\psi(x) - \eta_j(x)\psi(x) = 0$ for $|x|_B \le 2^j$ and for $|x|_B > 2^j$,

$$|\psi(x) - \eta_j(x)\psi(x)| \le C(1+2^j)^{-1}(1+|x|_B)^{-2(\frac{\nu}{r}+\delta)} \le \varepsilon(1+|x|_B)^{-2(\frac{\nu}{r}+\delta)},$$

whenever $j \ge j_0$ with $2^{j_0} > C\varepsilon^{-1}$.

Now we turn to the estimate on the Fourier side. Notice that $\hat{\eta}_j(\xi) = 2^{j\nu}\hat{\eta}(\delta_{2^j}\xi)$ is an approximation of the identity, and that $\hat{\tau}_j = \hat{\eta}_j * \hat{\psi}$. Since $\eta, \psi \in \mathscr{S}(\mathbb{R}^d)$, we can find C' such that

$$|\hat{\eta}(\xi)|, |\hat{\psi}(\xi)| \le C'(1+|\xi|_A)^{-2(\frac{\nu}{r}+\delta)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)-1}.$$

By standard estimates for convolutions, see [13, Appendix K], there is a constant C'' such that

$$|\hat{\eta}_j * \hat{\psi}(\xi)| \le C''(1+|\xi|_A)^{-2(\frac{\nu}{r}+\delta)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)-1}.$$

As before, we can find $j_1 \ge j_0$ such that for $j \ge j_1$ and $|\xi|_A > 2^{j_1}$ we have,

$$|\hat{\eta}_j * \hat{\psi}(\xi)| \le \varepsilon(1+|\xi|_A)^{-2(\frac{\nu}{r}+\delta)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)}.$$

Now, on the compact set $\{\xi : |\xi|_A \le 2^{j_1}\}$ we have $\hat{\eta}_j * \hat{\psi} \to \hat{\psi}$ uniformly as $j \to \infty$ since $\{\hat{\eta}_j\}_j$ is an approximation of the identity. Then we choose $j \ge j_1$ large enough to ensure that

$$|\hat{\eta}_j * \hat{\psi}(\xi) - \hat{\phi}(\xi)| \le \varepsilon(1+2^{j_1})^{-2(\frac{\nu}{r}+\delta)-\frac{2}{\beta}\left(|s|+\frac{2\nu}{r}+\frac{3\delta}{2}\right)},$$

on $\{\xi : |\xi|_A \le 2^{j_1}\}$. This completes the proof. □

The reader will notice that the proof of Proposition 3.5 is a variation on the proof that compactly supported test functions are dense in $\mathscr{S}(\mathbb{R}^d)$. However, we included the proof to emphasize the constructive nature of the approximation procedure.

Next we turn to a more sophisticated approximation procedure introduced for decomposition spaces in [18]. The result in [18] was in turn inspired by Petrushev and Kyriazis [17,20].

The idea is to take any $g \in C^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, $\hat{g}(0) \ne 0$, which for fixed $N, M > 0$ satisfies

$$|g^{(\kappa)}(x)| \le C(1+|x|_B)^{-N-\alpha_1},\ |\kappa| \le 1, \tag{3.13}$$

$$|\hat{g}(\xi)| \le C(1+|\xi|_A)^{-M-\alpha_2}., \tag{3.14}$$

where α_1 and α_2 are given by (2.1). Then for $m \in \mathbb{N}$, we define $g_m(x) := C_g m^\nu g(\delta_m^\top x)$, where $C_g := \hat{g}(0)^{-1}$. It follows that

$$\begin{aligned} |g_m^{(\kappa)}(x)| &\le C m^{\nu+\alpha_2|\kappa|}(1+m|x|_B)^{-N-\alpha_1}, \quad |\kappa| \le 1, \\ \int_{\mathbb{R}} g_m(x)\,dx &= 1, \\ |\hat{g}_m(\xi)| &\le C m^{M+\alpha_2}(1+|\xi|_A)^{-M-\alpha_2}. \end{aligned} \tag{3.15}$$

To construct a compactly supported generator τ based on g, we will need the set of finite linear combinations of translated and dilates of g,

$$\Theta_{K,m} = \{\eta : \eta(\cdot) = \sum_{i=1}^{K} a_i g_m(\cdot + b_i), a_i \in \mathbb{C}, b_i \in \mathbb{R}^d\}.$$

We are now ready to show that any function with sufficient decay in both direct and frequency space can be approximated to an arbitrary degree by a finite linear combination of another function with similar decay.

Proposition 3.6. *Let $N' > N > \nu$ and $M' > M > \nu$. If $g \in C^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, $\hat{g}(0) \neq 0$, fulfills* (3.13) *and* (3.14) *and $\psi \in C^1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ fulfills*

$$\begin{aligned} |\psi(x)| &\le C(1+|x|_B)^{-N'}, \\ |\psi^{(\kappa)}(x)| &\le C, |\kappa| \le 1, \\ |\hat{\psi}(\xi)| &\le C(1+|\xi|_A)^{-M'}, \end{aligned}$$

then for any $\varepsilon > 0$ there exists $K, m \ge 1$ and $\tau \in \Theta_{K,m}$ such that

$$|\psi(x) - \tau(x)| \le \varepsilon(1+|x|_B)^{-N}, \tag{3.16}$$

$$|\hat{\psi}(\xi) - \hat{\tau}(\xi)| \le \varepsilon(1+|\xi|_A)^{-M}. \tag{3.17}$$

As before, the resulting frame is then given by (3.12).

3.2. **Some concluding remarks on numerical calculations.** There is one major advantage for numerical calculation of the construction given by Proposition 3.6; this is the fact that we can choose the initial function g as we please only subject to some very mild constraints.

We can use this to make the final frame generated by τ from Proposition 3.6 better adapted to numerical calculations. For example, we can choose g to be a compactly supported spline or a compactly supported orthonormal scaling function for a multiresolution analysis. Then we generate τ by Proposition 3.6. We can clearly write

$$\tau = \sum_{i=1}^{K} a_i g(\delta_{m_i} \cdot + b_i).$$

Then for any $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$,

$$\langle f, \tau \rangle = \sum_{i=1}^{K} a_i m_i^{-\tau} \int_{\mathbb{R}^d} f(\delta_{1/m_i}^\top x - \delta_{1/m_i}^\top b_i)\overline{g(x)}\,dx, \tag{3.18}$$

and we can use the properties of g to estimate the integrals in (3.18). The same type of reasoning applies to dilated, translated, and modulated versions of τ, so this way we can estimated any frame coefficient of the system (3.12) generated by τ.

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