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SINUSOIDAL ORDER ESTIMATION USING THE SUBSPACE ORTHOGONALITY AND SHIFT-INVARIANCE PROPERTIES

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ABSTRACT

In this paper, we study and compare a number of subspace-based methods for determining the the number of sinusoids in noise. These are based on the subspace orthogonality and and shift-invariance properties that are known from the MUSIC and ESPRIT frequency estimators. The method based on the orthogonality property has not previously appeared in the literature. We compare, in simulations, the various subspace methods. These show that the subspace methods can estimate the correct order with a high probability for sufficiently high SNRs and number of observations with MUSIC performing the best. Also, unlike the commonly used statistical methods, the subspace methods do not depend on the probability density function of the noise being known.

1. INTRODUCTION

Estimating the order of a model is an important problem. However, most of the literature on parameter estimation assumes prior knowledge of the model order. In many cases, however, the order cannot be known a priori and may change over time. In that case, an adaptive order estimate is desirable. Perhaps the most commonly used approaches for order estimation are the statistical methods (see, e.g., [1]) such as the minimum description length (MDL), the Akaike information criterion (AIC), and the maximum a posteriori (MAP) rule. The problem considered in this paper is the specific problem of finding the number of sinusoids in white noise, which can be defined mathematically as follows. Consider a complex signal consisting of sinusoids having frequencies $\{\omega_l\}$ which is corrupted by an additive noise, $w(n)$, for $n = 0, \ldots, N-1$,

$$x(n) = \sum_{l=1}^{L} A_l e^{j(\omega_l n + \phi_l)} + w(n), \tag{1}$$

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where $A_l > 0$ and ϕ_l are the amplitude and the phase of the l'th sinusoid. Here, $w(n)$, is assumed to be white complex symmetric zero-mean noise. The problem is to estimate the model order, L. In this paper, we compare and study estimation criteria based on subspace methods. Specifically, the criteria considered are based on the orthogonality property used in the MUSIC (MUltiple SIgnal Classification) method [2, 3] and the shift-invariance property of (1) originally applied to parameter estimation in [4] and later extended to order estimation in the SAMOS (Subspace-based Automatic Model Order Selection) [5] and ESTER (ESTimation ERror) [6] methods. The orthogonality property of MUSIC has, to the best of our knowledge, not previously been applied to the order estimation problem, aside from the more specific problem considered in our previous publication [7].

The subspace methods are all based on the following principles: The eigenvectors of the covariance matrix can be partitioned into a set spanning the signal subspace and a set spanning the noise subspace. For the correct partitioning of the eigenvectors (a) the signal subspace model and the noise subspace are orthogonal, and (b) the signal subspace eigenvectors have the shift-invariance property. Based on these observations, the order can be estimated by introducing appropriate metrics. Compared to statistical models for order estimation, the subspace-based order estimation criteria do not require prior knowledge of the probability density function (pdf) of the observation noise. This means that the subspace methods will work in situations where the statistical methods will break down due to the assumed pdf not being a good approximation of the observation noise.

The remaining part of this paper is organized as follows. In Section 2, the fundamentals of the subspace decompositions and methods that form the basis of this paper are briefly reviewed. Then, the order estimation criteria, including the new one, are presented in Section 3. In Section 4, numerical results are presented, while the conclusions are presented in Section 5.

2. SUBSPACE PROPERTIES

We start out this section, in which we will present some fundamental definitions, relations and results, by defining $\mathbf{x}(n)$ as a signal vector containing M samples of the observed signal, i.e., $\mathbf{x}(n) = [\ x(n)\ x(n+1)\ \cdots\ x(n+M-1)\ ]^T$, with $(\cdot)^T$ denoting the transpose. Then, assuming that the phases $\{\phi_l\}$ are independent and uniformly distributed on the interval $(-\pi, \pi]$, the covariance matrix $\mathbf{R} \in \mathbb{C}^{M \times M}$ (with $L < M$) of the signal in (1) can be shown to be

$$\mathbf{R} = \mathrm{E}\left\{\mathbf{x}(n)\mathbf{x}^H(n)\right\} = \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}_M, \quad (2)$$

where $\mathrm{E}\{\cdot\}$ and $(\cdot)^H$ denote the statistical expectation and the conjugate transpose, respectively. The decomposition in (2) does not depend on the noise being Gaussian but only white. Additionally, $\mathbf{P}$ is a diagonal matrix containing the squared amplitudes, i.e., $\mathbf{P} = \mathrm{diag}\left([\ A_1^2\ \cdots\ A_L^2\ ]\right)$, and $\mathbf{A} \in \mathbb{C}^{M \times L}$ a full rank matrix constructed as

$$\mathbf{A} = \left[\ \mathbf{a}(\omega_1)\ \ \cdots\ \ \mathbf{a}(\omega_L)\ \right], \quad (3)$$

where $\mathbf{a}(\omega) = [\ 1\ e^{j\omega}\ \cdots\ e^{j\omega(M-1)}\ ]^T$. Also, σ^2 denotes the variance of the additive noise, $w(n)$, and $\mathbf{I}_M$ is the $M \times M$ identity matrix. We note that $\mathbf{A}\mathbf{P}\mathbf{A}^H$ has rank L. Let

$$\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H \quad (4)$$

be the eigenvalue decomposition (EVD) of the covariance matrix. Then, $\mathbf{Q}$ contains the M orthonormal eigenvectors of $\mathbf{R}$, i.e., $\mathbf{Q} = \left[\ \mathbf{q}_1\ \ \cdots\ \ \mathbf{q}_M\ \right]$ and $\mathbf{\Lambda}$ is a diagonal matrix containing the corresponding eigenvalues, γ_k, with $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_M$. The subspace-based methods are based on a partitioning of the eigenvectors into a set belonging to the signal subspace spanned by the columns of $\mathbf{A}$ and an orthogonal complement known as the noise subspace. Let $\mathbf{S}$ be formed from the eigenvectors corresponding to the L most significant eigenvalues, i.e.,

$$\mathbf{S} = \left[\ \mathbf{q}_1\ \ \cdots\ \ \mathbf{q}_L\ \right]. \quad (5)$$

The subspace that is spanned by the columns of $\mathbf{S}$ we denote $\mathcal{R}(\mathbf{S})$ and is henceforth referred to as the signal subspace. Similarly, let $\mathbf{G}$ be formed from the eigenvectors corresponding to the $M-L$ least significant eigenvalues, i.e.,

$$\mathbf{G} = \left[\ \mathbf{q}_{L+1}\ \ \cdots\ \ \mathbf{q}_M\ \right], \quad (6)$$

where $\mathcal{R}(\mathbf{G})$ is referred to as the noise subspace. Using the EVD in (4), the covariance matrix model in (2) can now be written as $\mathbf{Q}\left(\mathbf{\Gamma} - \sigma^2\mathbf{I}_M\right)\mathbf{Q}^H = \mathbf{A}\mathbf{P}\mathbf{A}^H$. Introducing $\mathbf{\Psi} = \mathrm{diag}\left([\ \gamma_1 - \sigma^2\ \cdots \gamma_L - \sigma^2\ ]\right)$, we can write this as

$$\mathbf{S}\mathbf{\Psi}\mathbf{S}^H = \mathbf{A}\mathbf{P}\mathbf{A}^H. \quad (7)$$

From this equation, it can be seen that the columns of $\mathbf{A}$ span the same space as the columns of $\mathbf{S}$ and that $\mathbf{A}$ therefore also must be orthogonal to $\mathbf{G}$, i.e., $\mathcal{R}(\mathbf{S}) = \mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{G}) \perp \mathcal{R}(\mathbf{A})$, and thus we arrive at the subspace orthogonality property:

$$\mathbf{A}^H\mathbf{G} = \mathbf{0}. \quad (8)$$

Then, by post-multiplication of (7) by $\mathbf{S}$, the following relation between the signal subspace eigenvectors and the Vandermonde matrix can be established (see [1]):

$$\mathbf{S} = \mathbf{A}\mathbf{C} \quad (9)$$

with $\mathbf{C} = \mathbf{P}\mathbf{A}^H\mathbf{S}\mathbf{\Psi}^{-1}$. Next, we define matrices the $\mathbf{A}_1$ and $\mathbf{A}_2$, constructed by removing the last and first rows of $\mathbf{A}$, i.e.,

$$\mathbf{A}_1 = [\ \mathbf{I}_{M-1}\ \mathbf{0}\ ]\,\mathbf{A} \quad \text{and} \quad \mathbf{A}_2 = [\ \mathbf{0}\ \mathbf{I}_{M-1}\ ]\,\mathbf{A}. \quad (10)$$

Similarly, we define from $\mathbf{S}$,

$$\mathbf{S}_1 = [\ \mathbf{I}_{M-1}\ \mathbf{0}\ ]\,\mathbf{S} \quad \text{and} \quad \mathbf{S}_2 = [\ \mathbf{0}\ \mathbf{I}_{M-1}\ ]\,\mathbf{S}. \quad (11)$$

From these definitions and (9), the matrices $\mathbf{S}_1$ and $\mathbf{A}_1$ can be related through the matrix $\mathbf{C}$ as $\mathbf{S}_1 = \mathbf{A}_1\mathbf{C}$. Then, due to the particular structure of $\mathbf{A}$ known as the shift-invariance property, the following can be seen to hold:

$$\mathbf{A}_2 = \mathbf{A}_1\mathbf{D} \quad \text{and} \quad \mathbf{S}_2 = \mathbf{S}_1\mathbf{\Gamma}, \quad (12)$$

with $\mathbf{D} = \mathrm{diag}\left([\ e^{j\omega_1}\ \cdots\ e^{j\omega_L}\ ]\right)$. Then, the matrix relating $\mathbf{S}_1$ to $\mathbf{S}_2$ can be written as follows:

$$\mathbf{\Gamma} = \mathbf{C}^{-1}\mathbf{D}\mathbf{C}. \quad (13)$$

Thus, $\mathbf{\Gamma}$ and $\mathbf{D}$ are related through a similarity transform.

3. ORDER ESTIMATION CRITERIA

For completeness, we will briefly review two other subspace methods for order estimation, namely the ESTER [6] and SAMOS [5] methods that have been proposed recently, before proceeding to introduce the here proposed MUSIC-based order estimation technique. Both methods are derived from the ESPRIT (Estimation of Signal Parameters by Rotational Invariance Techniques) method [4], which is based on $\mathcal{R}(\mathbf{S}) = \mathcal{R}(\mathbf{A})$ and the shift-invariance property of the matrix $\mathbf{A}$. The sinusoidal parameters are found using (12) by constructing the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ as shown in (11) and then solving for $\mathbf{\Gamma}$ in

$$\mathbf{S}_2 \approx \mathbf{S}_1\mathbf{\Gamma}, \quad (14)$$

in some sense. For instance,

$$\widehat{\mathbf{\Gamma}} = \arg\min_{\mathbf{\Gamma}} \|\mathbf{S}_2 - \mathbf{S}_1\mathbf{\Gamma}\|_F^2 \quad (15)$$

$$= \left(\mathbf{S}_1^H\mathbf{S}_1\right)^{-1}\mathbf{S}_1^H\mathbf{S}_2, \quad (16)$$

where the sinusoidal frequencies are found as the eigenvalues of $\widehat{\mathbf{\Gamma}}$ via the relation in (13). The key observation is that equation (14) holds only when the eigenvectors of $\mathbf{R}$ are partitioned into a signal and a noise subspace such that the rank of the signal subspace equals the true number of sinusoids.

An order estimate is obtained using the ESTER method in the following way: First, the sample covariance matrix and its EVD are found. Then, the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ are constructed for each L from the EVD and the matrix $\mathbf{\Gamma}$ is estimated. Then, the order is found by measuring the goodness of the fit in (14) as

$$J(L) = \|\mathbf{S}_2 - \mathbf{S}_1\widehat{\mathbf{\Gamma}}\|_2^2, \tag{17}$$

for various candidate orders L and then picking the one for which the modeling error is minimized in the sense of (17).

The SAMOS method works in a similar way. It is based on the rationale that the extent to which the relation in (14) holds can be measured from the singular values $\{\nu_k\}_{k=1}^{2L}$ of the augmented matrix $\mathbf{\Phi} = [\ \mathbf{S}_1\ \mathbf{S}_2\ ]$ for various L as

$$J(L) = \frac{1}{L}\sum_{k=L+1}^{2L} \nu_k^2, \tag{18}$$

since $\mathbf{\Phi}$ will have rank L when the columns of $\mathbf{S}_1$ can be described accurately as linear combinations of the columns in $\mathbf{S}_2$. The order estimate is then found as the minimizer of the cost function (18). This method requires that a singular value decomposition is calculated for each candidate order L.

The MUSIC algorithm is based on the observation $\mathcal{R}(\mathbf{G}) \perp \mathcal{R}(\mathbf{A})$, i.e., that the Vandermonde matrix is orthogonal to the noise subspace. Specifically, parameters are found as follows: The sample covariance matrix is estimated for a signal segment and the EVD is calculated. Then, the parameters are estimated by finding the rank L model that is orthogonal to the noise subspace, i.e., $\mathbf{A}^H\mathbf{G} = \mathbf{0}$, or is the closest to being so in the sense of a norm, say

$$\{\hat{\omega}_l\} = \arg\min_{\{\omega_l\}} \|\mathbf{A}^H\mathbf{G}\|_F^2, \tag{19}$$

with $\|\cdot\|_F$ denoting the Frobenius norm. Since the squared Frobenius norm is additive over the columns of $\mathbf{A}$, we can find the individual sinusoidal frequencies for $l = 1, \ldots, L$ as

$$\hat{\omega}_l = \arg\min_{\omega_l} \|\mathbf{a}^H(\omega_l)\mathbf{G}\|_F^2. \tag{20}$$

The reciprocal form of the cost function in (20) is sometimes referred to as spectral MUSIC and $1/\|\mathbf{a}^H(\omega_l)\mathbf{G}\|_F^2$ as the pseudo-spectrum from which the L frequencies are obtained as the peaks. A common trait of both MUSIC and ESPRIT is that it is not necessary to solve for the amplitude and phase parameters to find the frequencies. The effects of order estimation errors, i.e., the effect of choosing an erroneous $\mathbf{G}$ in (20), on the parameter estimates obtained using MUSIC has been studied in [8] in a slightly different context and it was concluded that the MUSIC estimator is more sensitive to underestimation of L than overestimation. The statistical properties of MUSIC for a known order have been studied extensively in [9, 10, 11].

Equation (8) can only be expected to hold when the eigenvectors of $\mathbf{R}$ are partitioned into a signal and a noise subspace such that the rank of the signal subspace is equal to the true number of sinusoids. In practice, however, only an estimate of the covariance matrix is available, and the model in (2) can therefore only be expected to hold approximately. The question is then how to measure to what extent the relation (8) holds. We propose to measure this as

$$J(L) = \min_{\{\omega_l\}} \frac{\|\mathbf{A}^H\mathbf{G}\|_F^2}{\|\mathbf{A}\|_F^2\|\mathbf{G}\|_F^2} \tag{21}$$

$$= \frac{1}{ML(M-L)} \min_{\{\omega_l\}} \mathrm{Tr}\left\{\mathbf{A}^H\mathbf{G}\mathbf{G}^H\mathbf{A}\right\}, \tag{22}$$

where $\mathbf{G}$ is the rank $M-L$ noise subspace eigenvectors that are found from the EVD of the sample covariance matrix. The normalization in the denominator is introduced to compensate for the bias of the numerator in (22) with respect to the dimensions of the matrices. Finally, we propose to estimate the model order as

$$\hat{L} = \arg\min_L J(L), \tag{23}$$

The only cases that the proposed subspace-based order estimation criterion cannot differentiate between are $L = 0$ and $L > 0$. This is also the case for the ESTER and SAMOS methods. Additionally, there are certain conditions that must be fulfilled for the subspace methods to result in unique order estimates. The SAMOS method, being based on the singular values of the augmented matrix $\mathbf{\Phi}$, is restricted to candidate orders $1 \leq L \leq \frac{M-1}{2}$. The ESTER method is restricted to candidate orders in the interval $1 \leq L \leq M-2$, while for MUSIC, the interval is $1 \leq L \leq M-1$. This favors the MUSIC and ESTER methods over the SAMOS method. Regarding the computational complexity, the ESTER requires that a least-squares (or total least-squares) problem be solved for every candidate order. Similarly, the SAMOS requires that a singular value decomposition is calculated for each candidate order. In contrast, the MUSIC cost function requires that the matrix products $\mathbf{A}^H\mathbf{G}$ be calculated for different orders. However, this can be done efficiently by first taking an FFT of each of the columns of the eigenvectors in $\mathbf{U}$ from which the minimization in (20) can be performed and $J(L)$ calculated by simply changing the summation limits. Alternatively, the minimization in (20) can be solved by polynomial rooting [12]. We here stress that the MUSIC-based method presented here is more general than those based on the shift-invariance property [6, 5], meaning that the relation (8) can be used for a more general class of signal models.

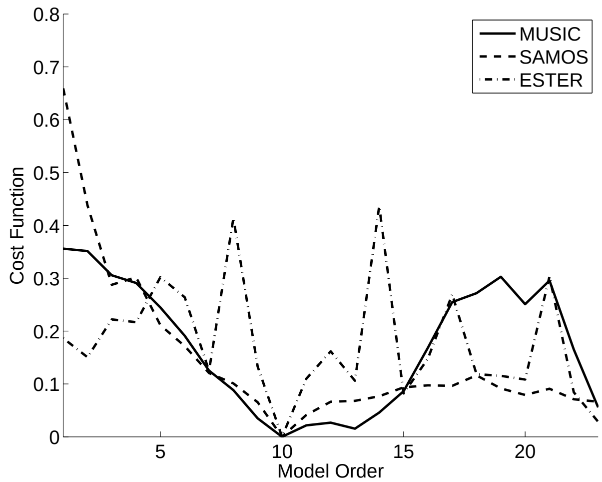

Fig. 1. Example of cost functions for 10 complex sinusoids in white Gaussian noise with $N = 50$, $M = 25$, and an SNR of 40 dB.

4. RESULTS

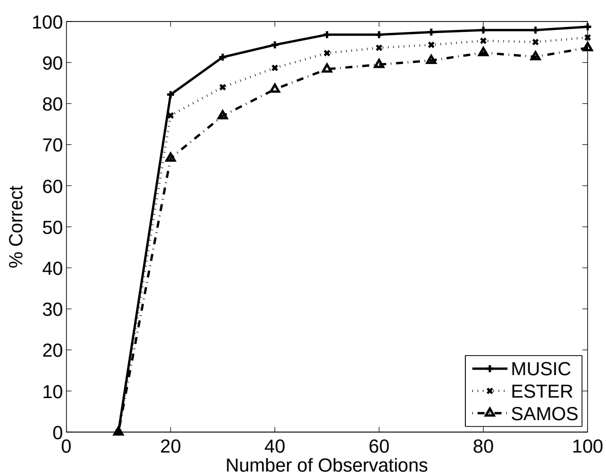

Fig. 2. Percentage of correctly estimated model orders as a function of the number of observations.

We now present some results starting with an illustrative example of the cost functions of the proposed methods. These are shown in Figure 1 for MUSIC and ESPRIT for a true model order of 10 with $N = 50$, $M = 25$ and an SNR of 40 dB. It can be seen from the figure that the three cost functions have a minimum at the true model order. It

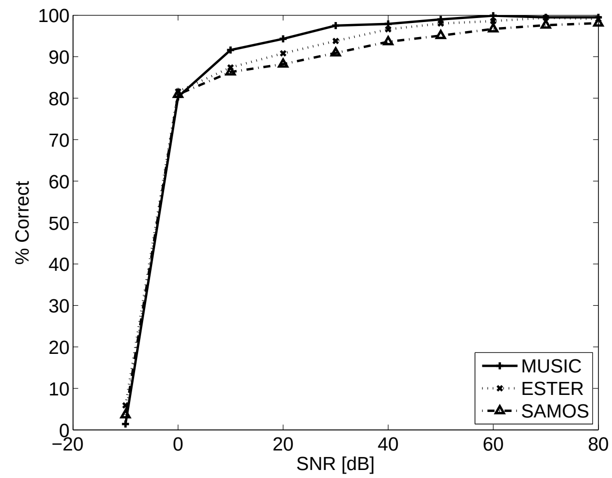

Fig. 3. Percentage of correctly estimated model orders vs. the SNR.

can also be observed that the cost functions are rather complicated containing several minima. Next, we evaluate the performance of the estimators under various conditions using Monte Carlo simulations where we generate signals according to the model in (1) with Gaussian noise. From these signals, we form a persymmetric estimate of the covariance matrix $\mathbf{R}$. This estimate is used for the implementation of MUSIC and ESTER while for SAMOS, following the suggestion of [5], we used a $(N - M + 1) \times M$ Hankel data matrix.

In the experiments to follow, all amplitudes are set to unity, i.e., $A_l = 1$ for all l and the signal-to-noise ratio (SNR) is defined as $SNR = -10 \log_{10} \sigma^2$ [dB]. The sinusoidal phases and frequencies are generated according to a uniform pdf. We will now evaluate the performance in terms of the percentage of correctly estimated orders under various conditions. For each combination of the parameters N, L and SNR, 1000 Monte Carlo simulations were run. First, we will vary the number of observations N while keeping the SNR fixed at 40 dB and then we will keep N fixed at 50 while varying the SNR. The results are shown in Figure 2 and Figure 3. In both cases, a true model order of 5 was used and $M = \frac{N}{2}$. Next, we evaluate the performance as a function of the true model order for $N = 100$, $SNR = 40$ dB and $M = 25$. Note that the choice of M limits the number of possible sinusoids since $M > L$. The results are depicted in Figure 4. First of all, we can see from the figures, that the estimators have the desirable properties that the performance improves as the SNR and/or the number of observations increases, and, as a consequence, that the model order can be determined with high probability based on the subspace methods for a high SNR and/or a high number of observations. Interestingly, MUSIC seems to consistently outperform ESTER and SAMOS. This may be somewhat surprising since ESPRIT is known to produce more accurate results than MUSIC for sinusoidal frequency estimation. Moreover, we observe from Figure 4 that the performance of all the methods deteriorates as the number of parameters approaches N and M.

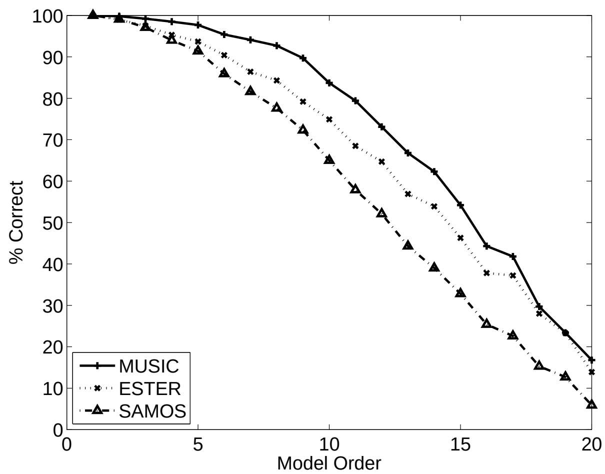

Fig. 4. percentage of correctly estimated model orders as a function of the true model order.

5. CONCLUSION

We have considered three subspace-based methods for finding the number of complex sinusoids in white noise with one new method being based on the subspace orthogonality property known from the MUSIC algorithm. The two other methods are based on the shift-invariance property of the ESPRIT method. The three methods have in common that they are based on the eigenvectors of the covariance matrix rather than the more commonly used eigenvalues. In this sense, the methods are based on the geometry of the subspaces rather than energy distribution. We have compared the three methods analytically and experimentally with the experiments showing that the new method based on the subspace orthogonality property outperform the two others. Aside from having the best performance, the new method is also more general than the other methods, being based on less restrictive model assumptions.

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