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Incremental source coding using a Gaussian test channel and MSE distortion

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***Abstract*—The additive rate-distortion function (ARDF) is defined as the minimum mutual information over an additive test channel followed by estimation. We consider the special case of quadratic distortion and where the noise in the test channel is Gaussian distributed and show that unconditional incremental refinement, i.e., where each refinement is encoded independently of the other refinements, is ARDF optimal in the limit of low resolution, independently of the source distribution.**

I. Introduction

The additive rate-distortion function (ARDF) consists of an additive test channel followed by estimation. We are interested in analyzing the ARDF at low resolutions and consider the special case where the test channel's noise is Gaussian and the distortion measure is the MSE. We establish a link to the mutual information – minimum mean squared estimation (I-MMSE) relation of Guo et al. [8] and show that unconditional incremental refinement, i.e., where each refinement is encoded independently of the other refinements, is ARDF optimal in the limit of low resolution, independently of the source distribution.

II. Results

We refer the reader to [1] for the proofs of the lemmas presented in this section.

Lemma 1. *Let $Y_i = \sqrt{\gamma}X + N_i, i = 0,\dots,k-1$, where $N_i \perp\!\!\!\perp X, \forall i$. Moreover, let X be arbitrarily distributed with variance σ_X^2 and let $N_0,\dots,N_{k-1}$, be zero-mean unit-variance i.i.d. Gaussian distributed. Then*

$$\lim_{\gamma\to 0}\frac{1}{\gamma}I(X;Y_0,\dots,Y_{k-1}) = k\lim_{\gamma\to 0}\frac{1}{\gamma}I(X;\sqrt{\gamma}X+N_0) = \frac{k\log_2(e)}{2}\sigma_X^2$$

and

$$\lim_{\gamma\to 0}\frac{1}{\gamma}\left[\frac{1}{\mathrm{var}(X|Y_1,\dots,Y_{k-1})}-\frac{1}{\sigma_X^2}\right] = k,$$

where $\mathrm{var}(X|Y)$ denotes the MMSE due to estimating X from Y.

Lemma 2. *Let $Y_i = \sqrt{\gamma}X + N_i, i = 0,\dots,k-1$, where $N_i \perp\!\!\!\perp X, \forall i$, and $N_0,\dots,N_{k-1}$. Let X be arbitrarily distributed with variance σ_X^2 and let $N_0,\dots,N_{k-1}$, be zero-mean unit-variance i.i.d. Gaussian distributed. Let Z be arbitrarily distributed and correlated with X but independent of $N_i, \forall i$. Then*

$$\lim_{\gamma\to 0}\frac{1}{\gamma}I(X;Y_0,\dots,Y_{k-1}|Z) = \frac{k\log_2(e)}{2}\mathrm{var}(X|Z).$$

Remark 1. *Lemmas 1 and 2 basically extend the I-MMSE relation of Guo et al. [2] to the case of several variables and side information, respectively. By doing this, an interesting connection to source coding is made. In particular, let an arbitrarily distributed source X be encoded into k representations $Y_i = \sqrt{\gamma}X + N_i$ where $\{N_i\}, i = 1,\dots,k$, are mutually independent, Gaussian distributed, and independent of X. Then Lemma 1 reveals that $I(X;Y_1,\dots,Y_k) \approx \sum_i I(X;Y_i)$ at low rates. This is interesting since conditional source coding is generally more complicated than unconditional source coding, i.e., creating descriptions that are individually optimal and at the same time jointly optimal is a long standing problem in information theory, where it is known as the multiple descriptions problem [3]. Furthermore, if side information Z, where Z is independent of $N_i, i = 1,\dots,k$, but arbitrarily jointly distributed with X, is available both at the encoder and decoder, then Lemma 2 shows that $I(X;Y_1,\dots,Y_k|Z) \approx \sum_i I(X;Y_i|Z)$.*

References

[1] J. Østergaard and R. Zamir, "Incremental refinement using a Gaussian test channel," in *IEEE Int. Symposium on Information Theory (ISIT)*, August 2011, pp. 2233 – 2237.

[2] D. Guo, S. Shamai, and S. Verdú, "Mutual information and minimum mean-square error in Gaussian channels," *IEEE Trans. Inform. Theory*, vol. 51, pp. 1261 – 1282, April 2005.

[3] A. A. E. Gamal and T. M. Cover, "Achievable rates for multiple descriptions," *IEEE Trans. Inform. Theory*, vol. IT-28, no. 6, pp. 851 – 857, Nov. 1982.