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DOI

[10.1007/S00220-023-04753-2](https://doi.org/10.1007/S00220-023-04753-2)

Publication date

2023

Document Version

Final published version

Published in

Communications in Mathematical Physics

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[Link to publication](#)

Citation for published version (APA):

Jankowski, J., Kucharski, P., Larraguivel, H., Noshchenko, D., & Sułkowski, P. (2023). Quiver Diagonalization and Open BPS States. *Communications in Mathematical Physics*, 402(2), 1551-1584. <https://doi.org/10.1007/S00220-023-04753-2>

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Quiver Diagonalization and Open BPS States

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Received: 27 December 2022 / Accepted: 29 April 2023
Published online: 21 June 2023 – © The Author(s) 2023

Abstract: We show that motivic Donaldson–Thomas invariants of a symmetric quiver Q , captured by the generating function P_Q , can be encoded in another quiver $Q^{(\infty)}$ of (almost always) infinite size, whose only arrows are loops, and whose generating function $P_{Q^{(\infty)}}$ is equal to P_Q upon appropriate identification of generating parameters. Consequences of this statement include a generalization of the proof of integrality of Donaldson–Thomas and Labastida–Mariño–Ooguri–Vafa invariants that count open BPS states, as well as expressing motivic Donaldson–Thomas invariants of an arbitrary symmetric quiver in terms of invariants of m -loop quivers. In particular, this means that the already known combinatorial interpretation of invariants of m -loop quivers extends to arbitrary symmetric quivers.

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1. Introduction and Summary

Quivers play a pivotal role in mathematics and physics. For mathematicians, an important challenge is to understand the structure of moduli spaces of quiver representations. This structure is characterized by various invariants, in particular motivic Donaldson–Thomas (DT) invariants. In physics, one role of quivers is to characterize BPS states in supersymmetric theories—in this context, nodes of a quiver correspond to certain basic states, while arrows encode how these basic states may form more complicated bound states. Such states are enumerated by (motivic) DT invariants (or appropriate combinations thereof) and their multiplicities provide an important information about a given theory.

In this paper we focus on symmetric quivers, meaning that for each arrow connecting two different nodes, there is also an arrow in the opposite direction. Symmetric quivers are understood better than generic quivers, and in particular it is known how to determine their (motivic) DT invariants [1–6]. Symmetric quivers are especially relevant in the knots-quivers correspondence [7, 8], where they are assigned to knots and encode BPS spectra in associated 3-dimensional $\mathcal{N} = 2$ theories. All this provides an important motivation for this work.

The main result of this paper is the statement that motivic DT invariants of an arbitrary symmetric quiver can be expressed in terms of motivic DT invariants of another quiver, generically of infinite size, whose only arrows are loops (loops are arrows that connect a node to itself). If we encode a structure of a quiver in a matrix C , whose entry C_{ij} is the number of arrows from node i to j , then the matrix representing the latter (generically infinite) quiver is diagonal—this is why we introduce the name diagonal quiver.

Definition 1. We call the quiver diagonal, if all of its arrows are loops, i.e. all nodes are disconnected.

Recall that motivic DT invariants of a symmetric quiver Q are encoded in the factorization of the quiver generating series $P_Q(\mathbf{x}, q)$, whose detailed form is given in (2), and which depends on a quiver matrix C , a number of generating parameters $(x_1, \dots, x_{|Q_0|}) = \mathbf{x}$ with each x_i associated to the i 'th node, and the motivic parameter q . The main result of this paper, i.e. a relation between motivic DT invariants of a symmetric quiver and the corresponding diagonal quiver, follows in fact from the relation between their generating functions, which is given in the following theorem. (Note that matrices C that we consider may also have negative entries; to formalize this feature, we introduce extended quivers, which are objects corresponding to such matrices.)

Theorem 2. *For every symmetric extended quiver Q there exist a diagonal extended quiver $Q^{(\infty)}$ and a set of identifications $\{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(\infty)}}$ such*

that the motivic generating series of Q and $Q^{(\infty)}$ are equal upon the identification of generating parameters given by this set:

$$P_Q(\mathbf{x}, q) = P_{Q^{(\infty)}}(\mathbf{x}^{(\infty)}, q) \Big|_{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)}. \tag{1}$$

This theorem has further interesting consequences. Note that motivic DT invariants of a diagonal quiver are expressed in terms of motivic DT invariants of m -loop quivers, i.e. quivers which consist of one node and m loops. Therefore, our main result relates motivic DT invariants of an arbitrary symmetric quiver to those of m -loop quivers. Since invariants of the m -loop quiver, as well as its Cohomological Hall Algebra, are quite well understood [4], it is of a great advantage to connect them to invariants of an arbitrary symmetric quiver. In particular, it is quite interesting to know the interpretation of the factorization of the motivic generating series of an arbitrary quiver into generating series for m -loop quivers at the level of Cohomological Hall Algebras. On the other hand, taking advantage of this factorization and the knowledge of classical DT invariants for m -loop quivers, it would be of interest to derive expressions for classical DT invariants for an arbitrary symmetric quiver that were proposed in [9]. Similarly, a combinatorial interpretation of invariants of m -loop quivers extends now to an arbitrary symmetric quiver. Relation to m -loop quivers also provides a novel proof of integrality of motivic DT invariants of a symmetric quiver. This relation has also an interesting physical interpretation in the context of 3-dimensional $\mathcal{N} = 2$ theories. We discuss some of these aspects in this paper and leave the rest for elucidation in future work.

The plan of the paper is as follows. In Sect. 2 we summarize relevant results for symmetric quivers and generalize the description of m -loop case (with $m \in \mathbb{N}$) to negative values $-m$. In Sect. 3 we present how to determine a diagonal quiver corresponding to an arbitrary symmetric quiver. In Sect. 4 we discuss how to express (motivic) DT invariants of a symmetric quiver in terms of invariants of the corresponding diagonal quiver. In Sects. 5, 6 and 7 we illustrate our results respectively in simple examples, examples related to the knots-quivers correspondence, and for the variant of this correspondence related to F_K invariants. In Appendix A we discuss combinatorial properties of $(-m)$ -loop quivers.

2. Symmetric Quivers and DT Invariants

In this section we recall relevant aspects of the study of symmetric quivers and (motivic) DT invariants. We also explain the generalization of quiver arrows that allows for the negative entries in the adjacency matrix and show that DT invariants of an $(-m)$ -loop quiver are in one-to-one correspondence with those of the $(m + 1)$ -loop quiver.

2.1. Motivic generating series. Quiver Q is a pair (Q_0, Q_1) , where Q_0 is a set of vertices and Q_1 is a set of arrows $i \rightarrow j$. We denote the number of arrows between vertices i and j by C_{ij} , and assemble these numbers into $|Q_0| \times |Q_0|$ adjacency matrix C . A quiver is called symmetric if for each arrow between two different vertices there is also an arrow in the opposite direction; this also means that $C_{ij} = C_{ji}$.

For a given quiver, it is important to understand the structure of moduli spaces of its representations, i.e. spaces of linear maps $\mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_j}$, where each \mathbb{C}^{d_i} is assigned to vertex i . $\mathbf{d} = (d_1, \dots, d_{|Q_0|}) \in \mathbb{N}^{|Q_0|}$ is called the dimension vector. Basic information about such spaces is encoded in their Betti numbers or their generalizations, which are

captured by motivic DT invariants $\Omega_{(d_1, \dots, d_{|Q_0|}), s} = \Omega_{\mathbf{d}, s}$. For a symmetric quiver, these invariants are encoded in the motivic generating series

$$P_Q(\mathbf{x}, q) = \sum_{\mathbf{d}} (-q)^{d \cdot C \cdot \mathbf{d}} \frac{\mathbf{x}^{\mathbf{d}}}{(q^2; q^2)_{\mathbf{d}}} = \sum_{d_1, \dots, d_{|Q_0|} \geq 0} (-q)^{\sum_{i,j=1}^{|Q_0|} C_{ij} d_i d_j} \prod_{i=1}^{|Q_0|} \frac{x_i^{d_i}}{(q^2; q^2)_{d_i}}, \tag{2}$$

where a generating parameter x_i is assigned to each vertex $i \in Q_0$ and

$$(\alpha; q^2)_n = \prod_{k=0}^{n-1} (1 - \alpha q^{2k}) \tag{3}$$

is the q -Pochhammer symbol. The product decomposition of this series into quantum dilogarithms determines $\Omega_{\mathbf{d}, s}$ as follows.¹

$$\begin{aligned} P_Q(\mathbf{x}, q) &= \prod_{\mathbf{d}, s} (\mathbf{x}^{\mathbf{d}} q^s; q^2)_{\infty}^{\Omega_{\mathbf{d}, s}} \\ &= \prod_{\mathbf{d} \in \mathbb{N}^{|Q_0|} \setminus \mathbf{0}} \prod_{s \in \mathbb{Z}} \prod_{k \geq 0} \left(1 - (x_1^{d_1} \dots x_{|Q_0|}^{d_{|Q_0|}}) q^{2k+s} \right)^{\Omega_{(d_1, \dots, d_{|Q_0|}), s}}. \end{aligned} \tag{4}$$

We also introduce a generating series

$$\Omega(\mathbf{x}, q) = \sum_{\mathbf{d}} \Omega_{\mathbf{d}}(q) \mathbf{x}^{\mathbf{d}} = \sum_{\mathbf{d} \in \mathbb{N}^{|Q_0|} \setminus \mathbf{0}} \sum_{s \in \mathbb{Z}} \Omega_{(d_1, \dots, d_{|Q_0|}), s} x_1^{d_1} \dots x_{|Q_0|}^{d_{|Q_0|}} q^s. \tag{5}$$

A non-trivial fact is that invariants $\Omega_{(d_1, \dots, d_{|Q_0|}), s}$ defined via the decomposition (4) are integer, and multiplied by $(-1)^{j+1}$ become positive [2, 3].

In fact, in this work we consider a larger class of quiver matrices C , which may also have negative entries. Note that such quiver matrices appear in the knots-quivers correspondence [8]. To take such cases into account, we slightly generalize the usual definition of a quiver.

Definition 3. Let C be an arbitrary integer symmetric matrix. We say that it defines an extended quiver $Q = (Q_0, Q_1^+ \cup Q_1^-)$ where Q_0 is a set of nodes, Q_1^+ is a set of arrows and Q_1^- is a set of negative arrows, such that

- every $C_{ii} > 0$ ($C_{ii} < 0$) corresponds to $|C_{ii}|$ loops (negative loops) at the node i ,
- every $C_{ij} > 0$ ($C_{ij} < 0$) corresponds to $|C_{ij}|$ pairs of arrows (negative arrows) between i and j in Q .

As an example, consider extended quiver

$$Q = \bullet \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \bullet \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \bullet, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{6}$$

arrows is denoted by solid lines, whereas the pair of negative arrows is denoted by dashed lines. Moreover, if all $C_{ij} \geq 0$, we call Q a proper quiver (or simply quiver), whereas if all $C_{ij} \leq 0$, we call Q a negative quiver.

¹ Note that in this paper the sign is included in the definition of Ω 's and the product form with quantum dilogarithms serves a basis. In consequence, we have $\sum_{s \in \mathbb{Z}} \Omega_{\mathbf{d}, s} q^s = \sum_{j \in \mathbb{Z}} \Omega'_{\mathbf{d}, j} (-q)^{j+1}$, where $\Omega'_{\mathbf{d}, j}$ is the DT invariant in the notation from [8].

2.2. *Unlinking and linking.* Basing on the study presented in [10], we know that for any symmetric quiver Q with $|Q_0|$ nodes, there exists another symmetric quiver with $|Q_0| + 1$ nodes, given by a removal (or an addition) of a pair of arrows in Q and addition of an extra node.² Importantly, motivic DT invariants for the two quivers are equal after a proper identification of a variable for the new node. We call such operations unlinking and linking, and define them as follows.

Definition 4 (Ekholm, Kucharski, Longhi). Consider a symmetric extended quiver Q together with a set of generating parameters $\{x_i\}_{i \in Q_0}$ and fix $a, b \in Q_0$.

- The unlinking of nodes a, b is defined as a transformation of Q leading to a new quiver $Q^{(\text{unlinked})}$ and a set of identifications $\{x_i^{(\text{unlinked})} = x_i^{(\text{unlinked})}(x_1, \dots, x_{|Q_0|}, q)\}$ such that:
 - There is a new node $n: Q_0^{(\text{unlinked})} = Q_0 \cup \{n\}$.
 - The number of arrows of the new quiver is given by

$$\begin{aligned} C_{ab}^{(\text{unlinked})} &= C_{ab} - 1, & C_{nn}^{(\text{unlinked})} &= C_{aa} + 2C_{ab} + C_{bb} - 1, \\ C_{in}^{(\text{unlinked})} &= C_{ai} + C_{bi} - \delta_{ai} - \delta_{bi}, & C_{ij}^{(\text{unlinked})} &= C_{ij} \quad \text{for all other cases,} \end{aligned} \tag{7}$$

where δ_{ij} is a Kronecker delta.

- The functions encoding the identification of generating parameters are given by

$$\begin{aligned} x_n^{(\text{unlinked})}(x_1, \dots, x_{|Q_0|}, q) &= q^{-1}x_ax_b, \\ x_i^{(\text{unlinked})}(x_1, \dots, x_{|Q_0|}, q) &= x_i \quad \forall i \neq n. \end{aligned} \tag{8}$$

- The linking of nodes a, b is defined as a transformation of Q leading to a new extended quiver $Q^{(\text{linked})}$ such that:
 - There is a new node $n: Q_0^{(\text{linked})} = Q_0 \cup \{n\}$.
 - The number of arrows of the new quiver is given by

$$\begin{aligned} C_{ab}^{(\text{linked})} &= C_{ab} + 1, & C_{nn}^{(\text{linked})} &= C_{aa} + 2C_{ab} + C_{bb}, \\ C_{in}^{(\text{linked})} &= C_{ai} + C_{bi}, & C_{ij}^{(\text{linked})} &= C_{ij} \quad \text{for all other cases.} \end{aligned} \tag{9}$$

- The functions encoding the identification of generating parameters are given by

$$\begin{aligned} x_n^{(\text{linked})}(x_1, \dots, x_{|Q_0|}, q) &= x_ax_b, \\ x_i^{(\text{linked})}(x_1, \dots, x_{|Q_0|}, q) &= x_i \quad \forall i \neq n. \end{aligned} \tag{10}$$

Theorem 5 (Ekholm, Kucharski, Longhi). *The operations of unlinking and linking both preserve the motivic generating function of the quiver:*

$$\begin{aligned} P_Q(\mathbf{x}, q) &= P_{Q^{(\text{unlinked})}}(\mathbf{x}^{(\text{unlinked})}, q) \Big|_{x_i^{(\text{unlinked})} = x_i^{(\text{unlinked})}(x_1, \dots, x_{|Q_0|}, q)}, \\ &= P_{Q^{(\text{linked})}}(\mathbf{x}^{(\text{linked})}, q) \Big|_{x_i^{(\text{linked})} = x_i^{(\text{linked})}(x_1, \dots, x_{|Q_0|}, q)}. \end{aligned} \tag{11}$$

For brevity, in the rest of the paper we will write (un)linking whenever we mean either unlinking or linking.

² Essentially, this procedure is based on the enumerative-geometric interpretation of motivic DT generating series as a generating series for open Gromov-Witten invariants, as noted in [11] and [10]

2.3. *m*-loop quivers. In this section we summarize properties of *m*-loop quivers, i.e. quivers that consist of one vertex and $m \in \mathbb{N}$ loops. In what follows we generalize some properties of *m*-loop quivers to extended quivers (which may be thought of as possibly having also negative number of loops, denoted by $-m$), and show that DT invariants of a $(-m)$ -loop quiver are in one-to-one correspondence with those of $(m + 1)$ -loop quiver.

An adjacency matrix of an *m*-loop quiver consists of a single entry, which counts the number of loops:

$$Q = \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} , C = [m]. \tag{12}$$

The motivic generating series of such a quiver takes form

$$P_{m\text{-loop}}(x, q) = \sum_{d=0}^{\infty} \frac{(-q)^{md^2}}{(q^2; q^2)_d} x^d. \tag{13}$$

From the perspective of this paper, an important feature of *m*-loop quivers is that they play a role of building blocks of diagonal quivers.

The case $m \geq 0$ has been studied in relation to moduli of quiver representations [4, 12], as well as counting of topological strings and quantum knot invariants [7, 8]. A product decomposition (4) for the series (13), which encodes motivic Donaldson–Thomas (DT) invariants $\Omega_{r,s}^m \in \mathbb{Z}$ [1–3], in this case takes form

$$P_{m\text{-loop}}(x, q) = \prod_{r,s} \left(x^r q^s; q^2 \right)_{\infty}^{\Omega_{r,s}^m}. \tag{14}$$

We stress again that in our convention $\Omega_{r,s}^m$ differ from the usual DT invariants considered in the literature by absorbing a minus sign—we do so for future convenience. The generating series of DT invariants (5) in this case takes form

$$\Omega_{m\text{-loop}}(x, q) = \sum_{r=1}^{\infty} \Omega_r^m(q) x^r = \sum_{r,s} \Omega_{r,s}^m x^r q^s. \tag{15}$$

The product (14) is finite only when $m = 0$ and $m = 1$:

$$P_{0\text{-loop}}(x, q) = \frac{1}{(x; q^2)_{\infty}}, \quad P_{1\text{-loop}}(x, q) = (qx; q^2)_{\infty} \tag{16}$$

which corresponds to

$$\Omega_{0\text{-loop}}(x, q) = -x, \quad \Omega_{1\text{-loop}}(x, q) = qx. \tag{17}$$

Otherwise, the combinatorial structure encoded in DT invariants is quite involved and the spectrum of DT invariants is infinite.

In [4] a combinatorial model which captures the behavior of DT invariants for *m*-loop quivers with $m \geq 0$ was introduced. Equivalent model, which is especially convenient for us, has been described in [13]—we discuss it in more detail in Appendix A.

We write the generating function of DT invariants for a few *m*-loop quivers in Table 1.

Table 1. Generating functions of DT invariants for m -loop quivers with $m \geq 0$

m	$\Omega_{m\text{-loop}}(x, q)$
0	$-x$
1	qx
2	$-q^2x + q^4x^2 - q^8x^3 + O(x^4)$
3	$q^3x + q^8x^2 + (q^{11} + q^{13} + q^{17})x^3 + O(x^4)$
4	$-q^4x + (q^8 + q^{12})x^2 - (q^{14} + q^{16} + q^{18} + q^{20} + q^{22} + q^{26})x^3 + O(x^4)$
5	$q^5x + (q^{12} + q^{16})x^2 + (q^{17} + q^{19} + q^{21} + 2q^{23} + q^{25} + q^{27} + q^{29} + q^{31} + q^{35})x^3 + O(x^4)$

One can notice that

$$\Omega_{m\text{-loop}}(x, q) = (-1)^{m-1}q^m x + O(x^2), \tag{18}$$

which is true for any $m \in \mathbb{N}$ (in fact, as we will see in Lemma 6, it is true also for any $m \in \mathbb{Z}$).

Now our goal is to extend (13) to negative integers $-m$. This is important, since an infinite diagonal quiver can have negative entries. We also note that it would be very interesting to understand a possible representation-theoretic meaning of this generalization. We proceed with an important lemma generalizing (14) to $(-m)$ -loop quivers.

Lemma 6. *For any $m \in \mathbb{N}$, the $(-m)$ -loop quiver generating series admits the following form*

$$P_{(-m)\text{-loop}}(x, q) = \prod_{r,s} \left(x^r q^s; q^2\right)_{\infty}^{\Omega_{r,s}^{-m}}, \tag{19}$$

where

$$\Omega_r^{-m}(q) = -q^{2-r} \Omega_r^{m+1}(q^{-1}). \tag{20}$$

We provide two independent proofs of the above lemma. The one presented below is rather elementary and involves a simple q -Pochhammer identity. The second one is included in Appendix A and it shows how this structure is captured by the combinatorial models from [4, 13].

Proof. We use the following q -Pochhammer identity

$$(\alpha; q^2)_d = (-1)^d q^{d^2-d} \alpha^d (\alpha^{-1}; q^{-2})_d, \tag{21}$$

where α is any formal variable, to obtain

$$\begin{aligned} P_{(-m)\text{-loop}}(x, q^{-1}) &= \sum_{d \geq 0} \frac{(-q)^{md^2}}{(q^{-2}; q^{-2})_d} x^d \\ &= \sum_{d \geq 0} \frac{(-q)^{(m+1)d^2}}{(q^2; q^2)_d} (qx)^d = P_{(m+1)\text{-loop}}(qx, q). \end{aligned} \tag{22}$$

Applying this result to $m = 0$ gives

$$(x; q^{-2})_{\infty} = \frac{1}{(q^2x; q^2)_{\infty}}. \tag{23}$$

Table 2. Generating functions of DT invariants for $(-m)$ -loop quivers

$-m$	$\Omega_{(-m)\text{-loop}}(x, q)$
-1	$q^{-1}x + q^{-4}x^2 + q^{-9}x^3 + O(x^4)$
-2	$-q^{-2}x - q^{-8}x^2 - (q^{-18} + q^{-14} + q^{-12})x^3 + O(x^4)$
-3	$q^{-3}x - (q^{-12} + q^{-8})x^2 + (q^{-27} + q^{-23} + q^{-21} + q^{-19} + q^{-17} + q^{-15})x^3 + O(x^4)$
-4	$-q^{-4}x - (q^{-16} + q^{-12})x^2 - (q^{-36} + q^{-32} + q^{-30} + q^{-28} + q^{-26} + 2q^{-24} + q^{-22} + q^{-20} + q^{-18})x^3 + O(x^4)$

The latter in turn can be applied to every q -Pochhammer in (14), yielding the explicit relation between the DT invariants (20). \square

We finish the discussion of $(-m)$ -loop quivers by tabulating DT invariants for some values of $-m$ (Table 2). A complete combinatorial description of these invariants is presented in Appendix A.

3. Quiver Diagonalization

In this section we describe what we call a *quiver diagonalization*. We start from an extended symmetric quiver Q and successively apply (un)linking operations, ending up with a (generically infinite) diagonal extended quiver, denoted $Q^{(\infty)}$. The only arrows of $Q^{(\infty)}$ are loops, and thus its motivic generating series factorizes as a product of $(\pm m)$ -loop quivers. We also discuss the uniqueness of this infinite quiver.

3.1. n -th degree approximation. We will construct $Q^{(\infty)}$ order by order, so we start from the following:

Definition 7. We say that a pair consisting of a symmetric extended quiver $Q^{(n)}$ and a set of identifications $\{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(n)}}$ is an n -th degree approximation of a symmetric extended quiver Q , if

$$P_Q(x, q) - P_{Q^{(n)}}(x^{(n)}, q) \Big|_{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)} = O(\mathbf{x}^{n+1}), \tag{24}$$

i.e. the difference contains only terms proportional to $\mathbf{x}^d = \prod_{i=1}^{|Q_0|} x_i^{d_i}$ with total degree $|d| = \sum_i d_i \geq n + 1$.

Theorem 8. For every symmetric extended quiver Q and $n \in \mathbb{Z}_+$ there exists an n -th degree approximation of Q such that the extended quiver $Q^{(n)}$ is diagonal and finite.

Proof. Let us fix a symmetric quiver Q . Looking at the general expression for the motivic generating series

$$P_Q(x, q) = \sum_d (-q)^{d \cdot C \cdot d} \frac{\mathbf{x}^d}{(q^2; q^2)_d}, \tag{25}$$

we can see that the lowest degree contribution of each diagonal entry of the quiver adjacency matrix to the quiver motivic generating series is

$$(-q)^{C_{ii}} \frac{x_i}{(q^2; q^2)_1}, \tag{26}$$

which is of total degree 1. On the other hand, the lowest degree contribution of each non-diagonal entry to P_Q is

$$(-q)^{2C_{ij}} \frac{x_i x_j}{(q^2; q^2)_1^2}, \tag{27}$$

which is of total degree 2. From [10], we know that the lowest degree contribution of the new diagonal entry $C_{new,new}$ that comes from the unlinking of C_{ij} is

$$(-q)^{C_{new,new}} \frac{x_{new}}{(q^2; q^2)_1} = (-q)^{C_{ii}+C_{jj}+2C_{ij}-1} \frac{q^{-1} x_i x_j}{(q^2; q^2)_1}, \tag{28}$$

whereas the one that comes from linking of C_{ij} is

$$(-q)^{C_{new,new}} \frac{x_{new}}{(q^2; q^2)_1} = (-q)^{C_{ii}+C_{jj}+2C_{ij}} \frac{x_i x_j}{(q^2; q^2)_1}. \tag{29}$$

In both cases it is of total degree 2, since we count the degree in the variables x_i corresponding to the *initial* quiver Q . We can generalize these considerations and write that each non-diagonal entry $C_{r,s}$ and the new entry coming from the unlinking or linking of nodes r and s both have the lowest degree contribution equal to the sum of the lowest degree contributions of nodes r and s . In consequence, the lowest degree of the contribution to P_Q —a positive integer—can be assigned to each entry of the adjacency matrix C and matrices coming from the unlinking or linking procedure.

Now we will recursively construct $\tilde{C}^{(n)}$ and $C^{(n)}$ —the adjacency matrices of quivers $\tilde{Q}^{(n)}$ and $Q^{(n)}$ —assigning to each entry of the matrix the lowest degree of the contribution to P_Q . We start from assigning 1 to each diagonal entry and 2 to each non-diagonal entry of C . In order to avoid confusion with matrix entries, we will write assigned lowest degrees in brackets:

$$\text{lowest_degree}(C) = \begin{bmatrix} (1) & (2) & (2) & \cdots & (2) \\ (2) & (1) & (2) & \ddots & \vdots \\ (2) & (2) & (1) & \ddots & (2) \\ \vdots & \ddots & \ddots & \ddots & (2) \\ (2) & \cdots & (2) & (2) & (1) \end{bmatrix}. \tag{30}$$

Then, we unlink all positive non-diagonal entries of C and link all negative non-diagonal entries. We denote the resulting matrix $\tilde{C}^{(1)}$ and the corresponding change of variables $\tilde{x}_i^{(1)} = \tilde{x}_i^{(1)}(x_1, \dots, x_{|Q_0|}, q)$

$$\text{lowest_degree}(\tilde{C}^{(1)}) = \begin{bmatrix} (1) & & (3) & (3) & \cdots & (3) \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & (1) & (3) & (3) & \cdots & (3) \\ (3) & \cdots & (3) & (2) & (4) & \cdots & (4) \\ (3) & \cdots & (3) & (4) & (2) & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & (4) \\ (3) & \cdots & (3) & (4) & \cdots & (4) & (2) \end{bmatrix}. \tag{31}$$

Moreover, we can see that the $|Q_0| \times |Q_0|$ top-left corner of matrix $\tilde{C}^{(1)}$ is a diagonal matrix which we will call $C^{(1)}$ (and the corresponding quiver $Q^{(1)}$). Since it came from unlinking or linking all non-diagonal entries of the matrix C , we have

$$C^{(1)} = \begin{bmatrix} \tilde{C}_{11}^{(1)} & 0 & \cdots & 0 \\ 0 & \tilde{C}_{22}^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{C}_{|Q_0||Q_0|}^{(1)} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & C_{|Q_0||Q_0|} \end{bmatrix}, \quad x_i^{(1)} = x_i \quad (32)$$

for all $i \in Q_0$, and

$$\begin{aligned} P_Q(\mathbf{x}, q) &= P_{\tilde{Q}^{(1)}}(\tilde{\mathbf{x}}^{(1)}, q) \Big|_{\tilde{x}_i^{(1)} = \tilde{x}_i^{(1)}(x_1, \dots, x_{|Q_0|}, q)} \\ &= 1 + \sum_{i=1}^{|Q_0|} (-q)^{C_{ii}} \frac{x_i}{(q^2; q^2)_1} + O(\mathbf{x}^2) = P_{Q^{(1)}}(\mathbf{x}^{(1)}, q) \Big|_{x_i^{(1)} = x_i} + O(\mathbf{x}^2), \end{aligned} \quad (33)$$

so $(Q^{(1)}, \{x_i^{(1)} = x_i\}_{i \in Q_0})$ is a first degree approximation of Q .

Now we move to the induction step and assume that for some $n \in \mathbb{Z}_+$ there exists an n -th degree approximation of Q formed by a finite diagonal quiver $Q^{(n)}$ and a change of variables $\{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(n)}}$, as well as a finite symmetric quiver $\tilde{Q}^{(n)}$ obtained from the (un)linking of Q , which contains $Q^{(n)}$ as a subquiver.³ In other words

$$\begin{aligned} C^{(n)} &= \begin{bmatrix} \tilde{C}_{11}^{(n)} & 0 & \cdots & 0 \\ 0 & \tilde{C}_{22}^{(n)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{C}_{kk}^{(n)} \end{bmatrix}, \\ \tilde{C}^{(n)} &= \begin{bmatrix} & & \tilde{C}_{1,k+1}^{(n)} & \cdots & \tilde{C}_{1,|\tilde{Q}_0^{(n)}|}^{(n)} \\ & C^{(n)} & \vdots & & \vdots \\ & & \tilde{C}_{k,k+1}^{(n)} & \cdots & \tilde{C}_{k,|\tilde{Q}_0^{(n)}|}^{(n)} \\ \tilde{C}_{1,k+1}^{(n)} & \cdots & \tilde{C}_{k,k+1}^{(n)} & \tilde{C}_{k+1,k+1}^{(n)} & \cdots & \tilde{C}_{k+1,|\tilde{Q}_0^{(n)}|}^{(n)} \\ \vdots & & \vdots & \ddots & \ddots & \vdots \\ \tilde{C}_{1\bar{k}}^{(n)} & \cdots & \tilde{C}_{k,|\tilde{Q}_0^{(n)}|}^{(n)} & \tilde{C}_{k+1,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots & \tilde{C}_{|\tilde{Q}_0^{(n)}|,|\tilde{Q}_0^{(n)}|}^{(n)} \end{bmatrix}, \end{aligned} \quad (34)$$

and we have

$$\begin{aligned} P_Q(\mathbf{x}, q) &= P_{\tilde{Q}^{(n)}}(\tilde{\mathbf{x}}^{(n)}, q) \Big|_{\tilde{x}_i^{(n)} = \tilde{x}_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)} \\ &= P_{Q^{(n)}}(\mathbf{x}^{(n)}, q) \Big|_{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)} + O(\mathbf{x}^{n+1}). \end{aligned} \quad (35)$$

³ The change of variables $\{\tilde{x}_i^{(n)} = \tilde{x}_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(n)}}$ follows from the changes of variables associated to (un)linking, given in Definition 4.

Moreover, from the previous considerations we know that the structure of lowest degrees of contributions to $P_{Q^{(n)}}$ for the entries of $\tilde{C}^{(n)}$ reads

$$\text{lowest_degree}(\tilde{C}^{(n)}) = \begin{bmatrix} (1) & & & (n+2) & \cdots & (n+2) & (n+3) & \cdots \\ & \ddots & & \vdots & \ddots & \vdots & \vdots & \\ & & (n) & (2n) & \cdots & (2n) & (2n+2) & \cdots \\ (n+2) & \cdots & (2n) & (n+1) & \cdots & (2n+2) & (2n+3) & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ (n+2) & \cdots & (2n) & (2n+2) & \cdots & (n+1) & (2n+3) & \cdots \\ (n+3) & \cdots & (2n+2) & (2n+3) & \cdots & (2n+3) & (n+2) & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, in the next step we unlink all positive entries and link all negative entries of $\tilde{C}^{(n)}$ which are above diagonal terms of lowest degree $n + 1$. We denote the resulting quiver $\tilde{Q}^{(n+1)}$ and the change of variables $\tilde{x}_i^{(n+1)} = \tilde{x}_i^{(n+1)}(x_1, \dots, x_{|Q_0|}, q)$. Since unlinking and linking preserve the quiver motivic generating series, we know that

$$P_Q(\mathbf{x}, q) = P_{\tilde{Q}^{(n+1)}}(\tilde{\mathbf{x}}^{(n+1)}, q) \Big|_{\tilde{x}_i^{(n+1)} = \tilde{x}_i^{(n+1)}(x_1, \dots, x_{|Q_0|}, q)}. \tag{36}$$

If we have l terms of lowest degree $n + 1$, the adjacency matrix of $\tilde{Q}^{(n+1)}$ is given by

$$\tilde{C}^{(n+1)} = \begin{bmatrix} \tilde{C}_{11}^{(n)} & 0 & \cdots & 0 & \tilde{C}_{1,k+l+1}^{(n)} & \cdots & \tilde{C}_{1,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots \\ 0 & \tilde{C}_{22}^{(n)} & \ddots & \vdots & \tilde{C}_{2,k+l+1}^{(n)} & \cdots & \tilde{C}_{2,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & \tilde{C}_{k+l,k+l}^{(n)} & \tilde{C}_{k+l,k+l+1}^{(n)} & \cdots & \tilde{C}_{k+l,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots \\ \tilde{C}_{1,k+l+1}^{(n)} & \tilde{C}_{2,k+l+1}^{(n)} & \cdots & \tilde{C}_{k+l,k+l+1}^{(n)} & \tilde{C}_{k+l+1,k+l+1}^{(n)} & \cdots & \tilde{C}_{k+l+1,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ \tilde{C}_{1,|\tilde{Q}_0^{(n)}|}^{(n)} & \tilde{C}_{2,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots & \tilde{C}_{k+l,|\tilde{Q}_0^{(n)}|}^{(n)} & \tilde{C}_{k+l+1,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots & \tilde{C}_{|\tilde{Q}_0^{(n)}|,|\tilde{Q}_0^{(n)}|}^{(n)} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \ddots \end{bmatrix}. \tag{37}$$

The entries denoted by rightmost and downmost dots come from the unlinking and linking of all entries of $\tilde{C}^{(n)}$ which are above diagonal terms of lowest degree $n + 1$ (so $\tilde{C}^{(n+1)}$ is finite). Their lowest degrees are $n + 2, n + 3, \dots, 2n + 2$, so they do not alter $P_{Q^{(n)}}$ up to total degree $n + 1$. The situation is the same for the entries above $\tilde{C}_{k+l+1,k+l+1}^{(n)}, \dots, \tilde{C}_{|\tilde{Q}_0^{(n)}|,|\tilde{Q}_0^{(n)}|}^{(n)}$ —their lowest degrees are bigger than $n + 2$. Summing, up we can write

$$\text{lowest_degree}(\tilde{C}^{(n+1)}) = \begin{bmatrix} (1) & & (n+3) & \cdots \\ & \ddots & \vdots & \\ & & (n+1) & (2n+3) \cdots \\ (n+3) \cdots & (2n+3) & (n+2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{38}$$

and from the analysis above we know that the lowest degree is non-decreasing when we move right or down in $\tilde{C}^{(n+1)}$. Let us denote the $(k+l) \times (k+l)$ top-left corner of matrix $\tilde{C}^{(n+1)}$ as

$$C^{(n+1)} = \begin{bmatrix} \tilde{C}_{11}^{(n)} & 0 & \cdots & 0 \\ 0 & \tilde{C}_{22}^{(n)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{C}_{k+l,k+l}^{(n)} \end{bmatrix}, \tag{39}$$

and the change of variables as $x_i^{(n+1)} = x_i^{(n+1)}(x_1, \dots, x_{|\mathcal{Q}_0|}, q)$. Then, we know that for the corresponding finite diagonal quiver $Q^{(n+1)}$ we have

$$P_Q(\mathbf{x}, q) = P_{Q^{(n+1)}}(\mathbf{x}^{(n+1)}, q) \Big|_{x_i^{(n+1)} = x_i^{(n+1)}(x_1, \dots, x_{|\mathcal{Q}_0|}, q)} + O(\mathbf{x}^{n+2}), \tag{40}$$

so $(Q^{(n+1)}, \{x_i^{(n+1)} = x_i^{(n+1)}(x_1, \dots, x_{|\mathcal{Q}_0|}, q)\}_{i \in \mathcal{Q}_0^{(n)}})$ is an $n+1$ -st degree approximation of the initial quiver Q .

Since we checked the theorem for $n = 1$ and proved the induction step, we know that it is valid for any $n \in \mathbb{Z}_+$. □

Let us formalize the criteria that we used in the proof above, since they will be important in other constructions:

Definition 9. We say that the sequence of linkings and unlinkings follows the rules of diagonalization if:

- For each positive non-diagonal entry C_{ij} we apply unlinking until nodes i and j are disconnected and $C_{ij} = 0$.
- For each negative non-diagonal entry C_{ij} we apply linking until nodes i and j are disconnected and $C_{ij} = 0$.
- We unlink and link nodes of lowest degree $n + 1$ only if all nodes of lowest degree n are disconnected, i.e. $C^{(n)}$ has already been constructed.

3.2. *Infinite diagonal quiver—proof of Theorem 2.* Having discussed the finite approximations of Q , we are ready for a jump to infinity.

Definition 10. For any symmetric extended quiver Q , the recursive construction described in the proof of Theorem 8 enables us to construct an infinite set of pairs

$$\{(Q^{(n)}, \{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|\mathcal{Q}_0|}, q)\}_{i \in \mathcal{Q}_0^{(n)}})\}_{n \in \mathbb{Z}_+}, \tag{41}$$

such that $(Q^{(n)}, \{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|\mathcal{Q}_0|}, q)\}_{i \in \mathcal{Q}_0^{(n)}})$ is the n -th degree approximation of Q and

$$Q^{(n)} \subseteq Q^{(m)}, \tag{42}$$

$$\{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(n)}} \subseteq \{x_i^{(m)} = x_i^{(m)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(m)}} \tag{43}$$

for all $m \geq n$.⁴

The extended quiver $Q^{(\infty)}$ is a union of all extended quivers approximating Q :

$$Q^{(\infty)} = \bigcup_{n=1}^{\infty} Q^{(n)}, \tag{44}$$

and the corresponding identification of the generating parameters is a union of all sets of equations:

$$\{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(\infty)}} = \bigcup_{n=1}^{\infty} \{x_i^{(n)} = x_i^{(n)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(n)}}. \tag{45}$$

We will call $Q^{(\infty)}$ the infinite diagonal quiver. From the definition it is clear that the infinite quantity is strictly speaking the degree of approximation, however in generic case the size of $Q^{(\infty)}$ is also infinite. Exceptions are described in Sect. 5.

Now we are ready to prove the main theorem of this paper, stating that the motivic generating series of Q and $Q^{(\infty)}$ are equal upon the identification of generating parameters given by (45).

Proof of Theorem 2 Assume that $P_Q(\mathbf{x}, q) - P_{Q^{(\infty)}}(\mathbf{x}^{(\infty)}, q) \Big|_{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)}$ is not zero. Then it is proportional to $\mathbf{x}^{\mathbf{d}} = \prod_{i=1}^{|Q_0|} x_i^{d_i}$ for some $d_i \in \mathbb{N}$. However, we know that $(Q^{(\infty)}, \{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(\infty)}})$ is a $|\mathbf{d}|$ -th degree approximation of Q , since

$$Q^{(|\mathbf{d}|)} \subseteq Q^{(\infty)}, \tag{46}$$

$$\{x_i^{(|\mathbf{d}|)} = x_i^{(|\mathbf{d}|)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(|\mathbf{d}|)}} \subseteq \{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)\}_{i \in Q_0^{(\infty)}}, \tag{47}$$

and for all $i \in Q^{(\infty)} \setminus Q^{(|\mathbf{d}|)}$ the total degree of $x_i^{(\infty)}$ is at least $|\mathbf{d}| + 1$. Therefore,

$$P_Q(\mathbf{x}, q) - P_{Q^{(\infty)}}(\mathbf{x}^{(\infty)}, q) \Big|_{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)} = O(\mathbf{x}^{|\mathbf{d}|+1}), \tag{48}$$

which is a contradiction. □

3.3. Uniqueness. In this section we show that if the adjacency matrix of the initial quiver Q satisfies either $C_{ij} \geq 0$ or $C_{ij} \leq 0$, for all $i, j = 1 \dots |Q_0|$, then the entries

⁴ We treat the superscripts (n) and (m) as decorations which make the expressions more readable, but should not interfere with the subset relation. For example, we treat sets $\{x_n^{(2)} = q^{-1}x_ax_b\}$ and $\{x_n^{(3)} = q^{-1}x_ax_b\}$ as equal, so $\{x_n^{(2)} = q^{-1}x_ax_b\} \subseteq \{x_n^{(3)} = q^{-1}x_ax_b\}$ is true.

in the diagonal quiver matrices $C^{(n)}$ are unique, as long as they satisfy the rules of diagonalization from Definition 9.

Lemma 11. *For any proper or negative quiver Q and $n \in \mathbb{Z}_+$ the n -th approximate quiver $Q^{(n)}$ obtained following the rules of diagonalization (Definition 9) is unique (up to permutation corresponding to relabeling of nodes).*

Proof. We use the induction in n and denote the nodes of $Q^{(n)}$ for any n as $i \in \{1, \dots, k\}$, keeping in mind that $Q_0^{(n)} \subseteq Q_0^{(n+1)}$. Without loss of generality we focus on the case of $C_{ij} \geq 0$. Given a symmetric quiver Q , its n -th approximate quiver $Q^{(n)}$ by definition satisfies

$$P_Q(\mathbf{x}, q) - P_{Q^{(n)}}(\mathbf{x}, q) = O(\mathbf{x}^{n+1}). \tag{49}$$

Alternatively, this can be written in terms of DT invariants as

$$\Omega_Q(\mathbf{x}, q) - \Omega_{Q^{(n)}}(\mathbf{x}, q) = O(\mathbf{x}^{n+1}). \tag{50}$$

We have

$$C^{(1)} = \begin{bmatrix} C_{11} & & & \\ & \ddots & & \\ & & C_{|Q_0||Q_0|} & \\ & & & \end{bmatrix}, \quad C^{(n)} = \begin{bmatrix} C_{11} & & & \\ & \ddots & & \\ & & & \\ & & & C_{kk} \end{bmatrix}, \tag{51}$$

where $k > |Q_0|$ and blank space denotes zero entries. The first degree approximation to Q is defined from the main diagonal of C , therefore $C^{(1)}$ is manifestly unique. We further assume that $Q^{(n)}$ is also unique. For $(n + 1)$ we write

$$C^{(n+1)} = \left[\begin{array}{c|ccc} C_{11} & & & \\ & \ddots & & \\ & & C_{kk} & \\ \hline & & & C_{k+1,k+1} \\ & & & \ddots \\ & & & & C_{k+l,k+l} \end{array} \right] = \left[\begin{array}{c|ccc} C^{(n)} & & & \\ \hline & & & R^{(n)} \end{array} \right], \tag{52}$$

where the first block is just $C^{(n)}$ and the rest is a quiver of size l , which we denoted $R^{(n)}$, so that

$$\Omega_{Q^{(n+1)}}(\mathbf{x}, q) = \Omega_{Q^{(n)}}(\mathbf{x}, q) + \Omega_{R^{(n)}}(\mathbf{x}, q). \tag{53}$$

$Q^{(n)}$ is uniquely fixed by induction assumption and it contains all nodes with the lowest degree $\leq n$. The contributions from $R^{(n)}$ to P_Q are of degree $\geq n + 1$, but since we are interested only in the $n + 1$ -st approximation, only the lowest degree contributions matter. Therefore, we need to check whether it is possible to have $R'(n) \neq R(n)$ such that lowest degree terms of $\Omega_{R'(n)}(\mathbf{x}, q)$ are equal to those of $\Omega_{R(n)}(\mathbf{x}, q)$. To clarify that, we study the coefficient in front of the monomial $\mathbf{x}^I = x_1^{i_1} \dots x_{|Q_0|}^{i_{|Q_0|}}$ of total degree $n + 1$ (the lowest degree) in $\Omega_{R(n)}(\mathbf{x}, q)$. Let us assume that this monomial arises from the change of variables associated to the quiver parameters x_{j_1}, \dots, x_{j_s} in $R^{(n)}$. Since \mathbf{x}^I is of total degree $n + 1$, each x_{j_r} ($r \in \{1, \dots, s\}$) originated from n unlinkings and (8) implies that $x_{j_r} = q^{-n} \mathbf{x}^I$. On the other hand, the generating series of DT invariants of each C_{j_r, j_r} -loop quiver reads (18)

$$\Omega_{C_{j_r j_r}\text{-loop}}(x_{j_r}, q) = (-1)^{C_{j_r j_r} - 1} q^{C_{j_r j_r}} x_{j_r} + O(\mathbf{x}^{n+2}). \tag{54}$$

Substituting $x_{j_r} = q^{-n} \mathbf{x}^I$ and summing over r we obtain the explicit form of the coefficient:

$$\text{coeff}_{\mathbf{x}^I} \Omega_{R^{(n)}}(\mathbf{x}, q) = q^{-n} ((-1)^{C_{j_1 j_1} - 1} q^{C_{j_1 j_1}} + \dots + (-1)^{C_{j_s j_s} - 1} q^{C_{j_s j_s}}). \tag{55}$$

It is now evident that the only allowed operations which preserve the coefficient (55) are permutations of the diagonal entries and thus the quiver $R^{(n)}$ is unique. This completes the induction step, and the reasoning for linking negative quivers follows exactly the same path. Note that if linking and unlinking are combined, this argument does not work, since then we do not get the overall factor (like q^{-n} in (55)) and one could smartly trade the number of linkings, unlinkings and loops to get the same power of q . \square

We can easily generalize this result to approximations of an infinite degree.

Theorem 12. *For any proper or negative quiver Q the infinite diagonal quiver $Q^{(\infty)}$ obtained following the rules of diagonalization (Definition 9) is unique (up to permutation corresponding to relabeling of nodes).*

Proof. Given that $Q^{(n)}$ is unique for every $n \in \mathbb{Z}_+$, we make use of the definition of $Q^{(\infty)}$ via set-theoretic union (44), taking into account identifications (45), which proves the statement. \square

It remains unclear whether the infinite diagonal quiver is unique for an extended quiver with both positive and negative entries. This important question will be taken into account in the future. However, since such extended quivers occur frequently in many physical applications, we take a look at a few of them in Sect. 6 and 7.

4. DT Invariants from Infinite Diagonal Quivers

In this section we show how we can apply the knowledge of infinite diagonal quivers to obtain a novel way of analyzing DT invariants. This framework leads to a proof of integrality of DT invariants for any extended quiver, as well as interesting physical and algebraic interpretations.

4.1. New proof of integrality of DT invariants. At this stage of our journey we have discovered two important facts:

- The motivic generating series of any m -loop quiver can be factorized into q -Pochhammers with integer powers (Lemma 6 combined with (14)).
- The motivic generating series of any symmetric extended quiver can be factorized into an infinite product of the motivic generating series of m -loop quivers (Theorem 2).

Combining the above statements we arrive at

Theorem 13. *For any extended symmetric quiver Q (see Definition 3), its motivic DT invariant admits the following decomposition:*

$$\Omega_Q(x_1, \dots, x_{|Q_0|}, q) = \sum_{i \in Q_0^{(\infty)}} \Omega_{C_{ii}^{(\infty)\text{-loop}}}(x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}), q). \tag{56}$$

Moreover, if Q is a proper or negative quiver, then this decomposition is unique.

Proof. Theorem 2 tells us that for every formal symmetric quiver $Q = (Q_0, Q_1)$ there exists an infinite diagonal quiver $Q^{(\infty)}$ such that $P_Q = P_{Q^{(\infty)}}$. Therefore, motivic generating series for Q can be written as a product of m -loop quiver series:

$$P_Q(\mathbf{x}, q) = \prod_{i \in Q_0^{(\infty)}} P_{C_{ii}^{(\infty)\text{-loop}}(x_i^{(\infty)}, q)} \Big|_{x_i^{(\infty)} = x_i^{(\infty)}(x_1, \dots, x_{|Q_0|}, q)}. \tag{57}$$

Each term in the above product corresponds to a diagonal entry in $C^{(\infty)}$, and we can express motivic DT invariants of Q as a linear combination of m -loop quiver invariants, which yields the statement (as uniqueness follows from Theorem 12). \square

Corollary 14. *For any extended quiver Q , its motivic DT invariants are integer.*

Proof. It is left to apply the results from Sect. 2.3, which, combined with Eq. (56), confirm integrality of Ω_Q in terms of DT invariants for m -loop quivers. \square

An important implication of Theorem 13 is that the DT invariants for an arbitrary symmetric quiver obtain a purely combinatorial description. Indeed, Eq. (56) shows that motivic DT invariants of Q are essentially captured by the construction of the DT invariants for m -loop quiver given in [4, 13] along with the data coming from unlinking and linking. In practice, $\Omega_{\mathbf{d},s}$ for a given vector \mathbf{d} can be computed in two steps:

- Finding the $|\mathbf{d}|$ -th approximation quiver $Q^{(|\mathbf{d}|)}$ (see Sect. 3.1), which is diagonal by construction, and whose motivic generating series are given by a product of m -loop quivers.
- Computing the motivic DT invariants $\Omega_{Q^{(|\mathbf{d}|)}}(x_1, \dots, x_{|Q_0|}, q)$ for $Q^{(|\mathbf{d}|)}$ using the formulas for m -loop quivers, .

Note that the definition of approximate quiver implies that

$$\Omega_Q(x_1, \dots, x_{|Q_0|}, q) - \Omega_{Q^{(|\mathbf{d}|)}}(x_1, \dots, x_{|Q_0|}, q) = O(\mathbf{x}^{\mathbf{d}+1}). \tag{58}$$

This allows to identify $\Omega_{\mathbf{d}',s} = \Omega_{\mathbf{d}',s}^{(|\mathbf{d}'|)}$ for any monomial of total degree $|\mathbf{d}'| \leq |\mathbf{d}|$. In particular, since $Q^{(|\mathbf{d}'|)}$ is a finite diagonal quiver, there are only finitely many DT invariants $\Omega_{\mathbf{d}',s} \neq 0$ for a fixed \mathbf{d}' . One can then use the recurrence relations of Ref. [13] to directly compute the $C_{ii}^{(|\mathbf{d}'|)}$ -loop quivers invariants. Afterwards, one unfolds changes of variables used in (un)linkings, and identifies the desired DT invariants in order to find $\Omega_{Q^{(|\mathbf{d}'|)}}$ and thus Ω_Q up to a given order in $x_1, \dots, x_{|Q_0|}$. We discuss these steps in detail in examples in Sect. 5.

Summing up, the above procedure allows for an effective computational approach for the DT invariants of any symmetric quiver. As a useful application one can resort to the knots-quivers correspondence [7, 8] and compute the LMOV invariants of knots, see Sect. 6. Another application, which uses the fact that quivers can be related to the F_K invariants of knot complements [14], is discussed in Sect. 7.

4.2. Physical and algebraic interpretations. Let us study the physical and algebraic interpretations of the results obtained in previous sections.

Following [11], we know that to every symmetric quiver Q we can assign a 3d $\mathcal{N} = 2$ gauge theory $T[Q]$ using the following rules:

- Gauge group: $U(1)^{(1)} \times \dots \times U(1)^{(|Q_0|)}$
- Matter: chiral fields Φ_i charged δ_{ij} under $U(1)^{(j)}$
- Chern-Simons (CS) couplings: $\kappa_{ij}^{\text{eff}} = C_{ij}$
- Fayet-Ilioupoulos (FI) couplings: $\zeta_i = \log((-1)^{C_{ii}} x_i)$

To that end, an m -loop quiver corresponds to a $U(1)$ level m Chern-Simons gauge theory coupled to a single chiral superfield. More concretely, the partition function of this theory is equal to the motivic generating series $P_{m\text{-loop}}(x, q)$ after appropriate identification of variables, as shown in [11].

Theorem 2 implies that with appropriate choice of levels and FI couplings (corresponding to infinite diagonal quiver data), the product of simple $U(1)$ Chern-Simons theories described above can reproduce the partition function of theory $T[Q]$ for any symmetric quiver Q . In other words, the theory $T[Q^{(\infty)}]$ corresponding to infinite diagonal quiver is dual to $T[Q]$. It is possible because the whole structure encoded in the CS couplings connecting different $U(1)$ theories (corresponding to quiver arrows) is translated into FI couplings (corresponding to changes of variables).

The duality between $T[Q]$ and $T[Q^{(\infty)}]$ implies the equality of their BPS spectra, however they are built in different ways. In case of Q , we have basic states corresponding to quiver nodes and the their bound states coming from interactions encoded in the quiver arrows [11]. On the other hand, the BPS spectrum of $Q^{(\infty)}$ consists only of basic states, which are non-interacting from the perspective of CS couplings, but become interdependent via FI couplings. One can also view $T[Q]$ and $T[Q^{(\infty)}]$ as two ends of the chain of dualities between 3d $\mathcal{N} = 2$ theories. Each duality corresponds to unlinking or linking one pair of nodes [10]. Physically, it means that one bound state and the corresponding CS coupling are translated into a basic state and the FI coupling. Since the BPS spectrum of $T[Q]$ is usually infinite, so is the chain of dualities coming from unlinking and linking. However, we can group these dualities into two-level structure.

The first level comes from the fact that whenever we have $C_{ij} > 1$ pairs of arrows connecting nodes i and j with C_{ii} and C_{jj} loops respectively, then unlinking leads to a new node with $C_{ii} + C_{jj} + 2C_{ij} - 1$ loops and a corresponding quiver variable given by $q^{-1}x_i x_j$. However, we still have $C_{ij} - 1 > 0$ pairs of arrows to unlink. The change of variables remains the same, but now there are $C_{ii} + C_{jj} + 2C_{ij} - 3$ loops at the new node. In consequence, unlinking $C_{ij} > 1$ leads to a sequence of C_{ij} new nodes with the number of loops following the double factorial pattern:

$$\begin{aligned}
 &C_{ii} + C_{jj} + 2C_{ij} - 1, \quad C_{ii} + C_{jj} + 2C_{ij} - 3, \\
 &C_{ii} + C_{jj} + 2C_{ij} - 5, \dots, C_{ii} + C_{jj} + 1
 \end{aligned}
 \tag{59}$$

(the corresponding quiver variables are all given by $q^{-1}x_i x_j$). In analogy, linking $C_{ij} < -1$ leads to a sequence of $|C_{ij}|$ new nodes with the number of loops following the double factorial pattern:

$$\begin{aligned}
 &C_{ii} + C_{jj} + 2C_{ij}, \quad C_{ii} + C_{jj} + 2C_{ij} + 2, \quad C_{ii} + C_{jj} + 2C_{ij} + 4, \dots, C_{ii} + C_{jj} - 2
 \end{aligned}
 \tag{60}$$

(the corresponding quiver variables are all given by $x_i x_j$).

The second level of the structure of the chain of dualities is given by the lowest degree of quiver nodes. We start from Q where all nodes are of lowest degree 1. After unlinking (or linking) them, we obtain nodes of lowest degree 2. Unlinking the arrows connecting the new nodes with the old ones produces nodes of lowest degree 3, and so on. This

grouping is clearly visible in the proof of Theorem 8, since the difference between $Q^{(n)}$ and $Q^{(n+1)}$ lies exactly in nodes of lowest degree $n + 1$. We will see these two levels of the structure of the chain of dualities in various examples of quivers in the following sections.

Finally, we mention the intriguing possibility for an algebraic interpretation of Theorem 13. In [15], certain Lie superalgebra \mathfrak{g}_Q , whose Koszul dual is presumably related to Cohomological Hall algebra of Kontsevich and Soibelman [1], has been defined for every symmetric quiver. \mathfrak{g}_Q has generators $b_{i,k}$ for $i = 1 \dots |Q_0|$, subject to relations for all $i, j = 1 \dots |Q_0|$ and $k, l \in \mathbb{N}$:

$$\begin{aligned}
 [b_{i,k}, b_{i,l}] &= 0, \quad C_{ii} = 0, \\
 \sum_{p=0}^{C_{ii}-1} (-1)^p \binom{C_{ii}-1}{p} [b_{i,k-p}, b_{i,l+p}] &= 0, \quad C_{ii} \geq 1, \\
 \sum_{p=0}^{C_{ij}} (-1)^p \binom{C_{ij}}{p} [b_{i,k-p}, b_{j,l+p}] &= 0, \quad i \neq j.
 \end{aligned} \tag{61}$$

The main outcome of this construction is that the Poincare series for \mathfrak{g}_Q coincides with the motivic DT invariant of Q , up to an overall factor. In this context (56) suggests that \mathfrak{g}_Q admits a decomposition into “elementary” Lie algebras associated to m -loop quivers. Furthermore, we expect a relation between (un)linking and the Lie bracket. These considerations deserve a thorough study, especially in relation to negative quivers (for example, in the context of the vertex operator description in [15]), and we leave it for future work.

5. Simple Examples

In the following part of the paper we illustrate our results in explicit examples. To start with, in this section we consider simple quivers with one or two pairs of arrows connecting different nodes. In the next sections we discuss examples related to the knots-quivers correspondence.

While examples considered in this paper are relatively uncomplicated and we are able to analyze them explicitly, in analysis of more intricate cases one could take advantage of prequivers [16], which encode a part of the spectrum and simplify computations by dividing it in steps.⁵

5.1. Symmetric A_2 quiver. The simplest example of the quiver diagonalization corresponds to the unlinking of the single pair of arrows in the symmetrization of the A_2 Dynkin quiver. The quiver and its adjacency matrix are given by

$$Q = \bullet \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \bullet, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{62}$$

and we will call it a symmetric A_2 quiver.

⁵ The idea of this approach follows from the observation that (un)linking and splitting of prequiver nodes commute. One way to understand this is to notice that unlinking does not affect the diagonal terms, while splitting acts solely on them.

Let us follow Sect. 3.1 and construct all n -th degree approximations of Q . From (32) we immediately find that

$$C^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{diag}(0, 0), \quad x_1^{(1)} = x_1, \quad x_2^{(1)} = x_2, \tag{63}$$

and Eq. (33) takes a very simple form:

$$P_Q(x, q) = 1 + \frac{x_1}{1 - q^2} + \frac{x_2}{1 - q^2} + O(x^2). \tag{64}$$

We can obtain the matrix $\tilde{C}^{(1)}$ and the change of variables $\tilde{x}_i^{(1)} = \tilde{x}_i^{(1)}(x_1, x_2, q)$ by applying the unlinking operation described in Definition 4 and Theorem 5:

$$\tilde{C}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}(0, 0, 1), \quad \tilde{x}_1^{(1)} = x_1, \quad \tilde{x}_2^{(1)} = x_2, \quad \tilde{x}_3^{(1)} = q^{-1}x_1x_2. \tag{65}$$

We can see that $Q^{(1)}$ is a diagonal subquiver of $\tilde{Q}^{(1)}$ and the lowest degrees of contributions to $P_{\tilde{Q}^{(1)}}$ for entries of $\tilde{C}^{(1)}$ are given by

$$\text{lowest_degree}(\tilde{C}^{(1)}) = \begin{bmatrix} (1) & (2) & (3) \\ (2) & (1) & (3) \\ (3) & (3) & (2) \end{bmatrix}. \tag{66}$$

In the next step, we would normally unlink the new node from the initial ones, but we can see that $\tilde{Q}^{(1)}$ is already diagonal. In consequence, we have

$$C^{(n)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}\left(0, 0 \middle| 1\right), \quad x_1^{(n)} = x_1, \quad x_2^{(n)} = x_2, \quad x_3^{(n)} = q^{-1}x_1x_2 \tag{67}$$

for all $n > 1$, where the dashed line separates the nodes which belong to n -th and $n + 1$ -st approximations. Following the Definition 10, it leads to

$$C^{(\infty)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}\left(0, 0 \middle| 1\right), \quad x_1^{(\infty)} = x_1, \quad x_2^{(\infty)} = x_2, \quad x_3^{(\infty)} = q^{-1}x_1x_2. \tag{68}$$

Having found $Q^{(\infty)}$, we can construct the BPS spectrum of Q using our knowledge about m -loop quivers, described in Sect. 4. In the case of symmetric A_2 quiver, Eq. (57) takes form

$$P_Q(x, q) = P_{0\text{-loop}}(x_1, q)P_{0\text{-loop}}(x_2, q)P_{1\text{-loop}}(q^{-1}x_1x_2, q). \tag{69}$$

From Table 1 we can immediately read off that

$$P_Q(x, q) = (x_1; q^2)_{\infty}^{-1}(x_2; q^2)_{\infty}^{-1}(x_1x_2; q^2)_{\infty}^+1, \tag{70}$$

Table 3. $Q^{(\infty)}$ data for symmetric A_2 quiver

m	Unlinkings
0	x_1, x_2
1	(x_1, x_2)

so DT invariants are given by

$$\Omega(\mathbf{x}, q) = -x_1 - x_2 + x_1x_2 \Leftrightarrow \Omega_{(1,0),0} = -1, \Omega_{(0,1),0} = -1, \Omega_{(1,1),0} = +1. \tag{71}$$

The symmetric A_2 quiver is exceptional in a twofold sense. First, $Q^{(\infty)}$ is finite, which comes from the fact that the pair $(Q^{(2)}, x_i^{(2)} = x_i^{(2)}(x_1, x_2, q))$ is not only a second degree approximation of Q , but also gives an exact expression for the motivic generating series:

$$P_Q(\mathbf{x}, q) = P_{Q^{(2)}}(\mathbf{x}^{(2)}, q) \Big|_{x_i^{(2)} = x_i^{(2)}(x_1, x_2, q)}. \tag{72}$$

Second, $Q^{(\infty)}$ is not only diagonal, but all nodes have either zero or one loop. This leads to a finite BPS spectrum consisting of 3 states. In this simple case we can see clearly that after diagonalization the bound state arising from the interaction of nodes 1 and 2 (represented by the pair of arrows) becomes a basic state associated to the node 3.

To summarize, the unlinking data which generate the infinite diagonal quiver in this example can be compactly presented in the table form (see Table 3).

The first row encodes two copies of 0-loop quiver, each corresponding to x_i ; the second row is one copy of 1-loop quiver and (x_1, x_2) denotes the node created at unlinking of the initial pair of nodes. Therefore, by looking at this table one can immediately write down the factorized form (69). This phenomenon is closely related to the pentagon relation [1], which was noticed in [10]. Analogous features can be stated for more complicated quivers, as we will discuss in the upcoming paper [17].

5.2. Symmetric A_3 quiver. The next example we consider is a symmetrization of Dynkin quiver A_3 :

$$Q = \bullet \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \bullet \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \bullet, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{73}$$

Unlike the A_2 case, here the set of DT invariants is infinite, and so $Q^{(\infty)}$ has infinitely many nodes. Nevertheless, we can take advantage of the construction of n -th degree approximation quivers $Q^{(n)}$ to compute $Q^{(\infty)}$ up to any given order.

Let us begin with the first degree approximation. We can consecutively unlink (x_1, x_2) and (x_2, x_3) (pairs of arrows that will be unlinked are highlighted in red)

$$C = \begin{bmatrix} \boxed{0 & 1 & 0} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{0 & 0 & 0} & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{0 & 1 & 0} & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{0 & 0 & 0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \boxed{0 & 0 & 0} & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \tilde{C}^{(1)}, \tag{74}$$

where the last two nodes coming from unlinking correspond to quiver variables $q^{-1}x_1x_2$ and $q^{-1}x_2x_3$. This gives the first degree approximation

$$C^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_1^{(1)} = x_1, \quad x_2^{(1)} = x_2, \quad x_3^{(1)} = x_3. \tag{75}$$

In order to get the second degree approximation, we start from $\tilde{C}^{(1)}$ and perform unlinking at $(x_3, (x_1, x_2))$, i.e. we unlink the node corresponding to x_3 and the node that came from unlinking x_1 and x_2 . In the second step⁶ we perform unlinking at $(x_3, (x_1, x_2))$, which gives $\tilde{C}^{(2)}$:

$$\tilde{C}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 \end{bmatrix} = \tilde{C}^{(2)}. \tag{76}$$

The top-left 5×5 subquiver is diagonal and contains all entries of lowest degree ≤ 2 , therefore it provides the second degree approximation of Q :

$$C^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{aligned} x_1^{(2)} &= x_1, & x_2^{(2)} &= x_2, & x_3^{(2)} &= x_3, \\ x_4^{(2)} &= q^{-1}x_1x_2, & x_5^{(2)} &= q^{-1}x_2x_3. \end{aligned} \tag{77}$$

Likewise, to get the cubic approximation we unlink all nodes of lowest degree ≤ 3 . They are inside the dashed submatrix of $\tilde{C}^{(2)}$ (the bottom-right entry 3 corresponding to $\tilde{x}_7^{(2)}$ is of lowest degree 4):

$$\tilde{C}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 \end{bmatrix} = \tilde{C}^{(3)}. \tag{78}$$

⁶ The ordering of the steps corresponding to the same degree is arbitrary.

This gives

$$C^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{aligned} x_1^{(3)} &= x_1, & x_2^{(3)} &= x_2, & x_3^{(3)} &= x_3, \\ x_4^{(3)} &= q^{-1}x_1x_2, & x_5^{(3)} &= q^{-1}x_2x_3, & x_6^{(3)} &= q^{-2}x_1x_2x_3. \end{aligned} \tag{79}$$

This allows us to write the cubic approximation (79) as

$$\begin{aligned} P_{Q^{(3)}}(\mathbf{x}, q) &= P_{0\text{-loop}}(x_1, q) P_{0\text{-loop}}(x_2, q) P_{0\text{-loop}}(x_3, q) P_{1\text{-loop}}(q^{-1}x_1x_2, q) \\ &\quad \times P_{1\text{-loop}}(q^{-1}x_2x_3, q) P_{2\text{-loop}}(q^{-2}x_1x_2x_3, q) \\ &= (x_1; q^2)_\infty^{-1} (x_2; q^2)_\infty^{-1} (x_3; q^2)_\infty^{-1} (x_1x_2; q^2)_\infty \\ &\quad \times (x_2x_3; q^2)_\infty [(x_1x_2x_3; q^2)_\infty^{-1} \times (1 + O(\mathbf{x}^4))], \end{aligned} \tag{80}$$

where $(1 + O(\mathbf{x}^4))$ factor comes from infinite q -Pochhammers with the first entry of total degree 4 or higher. The generating series of corresponding DT invariant reads

$$\Omega(\mathbf{x}, q) = -x_1 - x_2 - x_3 + x_1x_2 + x_2x_3 - x_1x_2x_3 + O(\mathbf{x}^4). \tag{81}$$

Summing up, by taking the union of n -th degree approximations we obtain the infinite diagonal quiver:

$$\begin{aligned} C^{(\infty)} &= \text{diag} (0, 0, 0 \mid 1, 1 \mid 2 \mid 3 \mid 4, 4 \mid 5, 5, \dots) \\ \mathbf{x}^{(\infty)} &= (x_1, x_2, x_3 \mid q^{-1}x_1x_2, q^{-1}x_2x_3 \mid q^{-2}x_1x_2x_3 \mid q^{-3}x_1x_2^2x_3 \mid \\ &\quad q^{-4}x_1x_2^2x_3^2, q^{-4}x_1x_2^2x_3^2 \mid q^{-5}x_1^2x_2^3x_3, q^{-5}x_1x_2^3x_3^2, \dots) \end{aligned} \tag{82}$$

Last but not least, in Table 4 we provide a slightly different presentation of the infinite diagonal quiver (82), where various contributions from one-vertex quivers up to $C_{ii}^{(\infty)} \leq 8$ are segregated. We believe that this data can be ultimately related to the poset of generators of the Lie algebra \mathfrak{g}_Q (see Sect. 4.2), and leave it for future research.

6. Quivers Corresponding to Knots

The knots-quivers correspondence, as its name indicates, expresses various characteristics of knots in terms of those of corresponding symmetric quivers. In particular, it relates the motivic generating series and motivic DT invariants of a quiver to the generating series of HOMFLY-PT polynomials and Labastida–Mariño–Ooguri–Vafa (LMOV) invariants of a knot. The appearance of symmetric quivers in this context is an important motivation for our work. Understanding properties of such quivers enables to understand properties of corresponding knots, as well as properties of brane systems in which knots and quivers can be engineered and related to each other.

The knots-quivers correspondence was discovered in [7,8]. Further developments and elucidations were presented in [9–11, 16, 18–20]. In [21,22] the correspondence was generalized to toric Calabi-Yau manifolds other than the conifold, and in [14,23]

Table 4. $Q^{(\infty)}$ data for symmetric A_3 quiver up to $C_{ii}^{(\infty)} \leq 8$. The rows encode all $C_{ii}^{(\infty)}$ -loop quivers contributing to the infinite factorized form (57), and the variable x for each copy of such a quiver is given by the bracket expression in the second column from identification $(x_i, x_j) \simeq q^{-1}x_i x_j$

m	unlinkings
0	x_1, x_2, x_3
1	$(x_1, x_2), (x_2, x_3)$
2	$(x_3, (x_1, x_2))$
3	$((x_1, x_2), (x_2, x_3))$
4	$((x_1, x_2), (x_3, (x_1, x_2))), ((x_2, x_3), (x_3, (x_1, x_2)))$
5	$((x_1, x_2), ((x_1, x_2), (x_2, x_3))), ((x_2, x_3), ((x_1, x_2), (x_2, x_3)))$
6	$((x_3, (x_1, x_2)), ((x_1, x_2), (x_2, x_3))), ((x_1, x_2), ((x_1, x_2), (x_3, (x_1, x_2))))$ $((x_2, x_3), ((x_1, x_2), (x_3, (x_1, x_2))))$, $((x_2, x_3), ((x_2, x_3), (x_3, (x_1, x_2))))$
7	$((x_1, x_2), ((x_1, x_2), ((x_1, x_2), (x_2, x_3))))$, $((x_2, x_3), ((x_1, x_2), ((x_1, x_2), (x_2, x_3))))$ $((x_2, x_3), ((x_2, x_3), ((x_1, x_2), (x_2, x_3))))$, $((x_3, (x_1, x_2)), ((x_1, x_2), (x_3, (x_1, x_2))))$ $((x_3, (x_1, x_2)), ((x_2, x_3), (x_3, (x_1, x_2))))$
8	$((x_3, (x_1, x_2)), ((x_1, x_2), (x_2, x_3))), ((x_3, (x_1, x_2)), ((x_1, x_2), ((x_1, x_2), (x_2, x_3))))$ $((x_3, (x_1, x_2)), ((x_2, x_3), ((x_1, x_2), (x_2, x_3))))$, $((x_1, x_2), (x_2, x_3)), ((x_1, x_2), (x_3, (x_1, x_2))))$, $((x_1, x_2), (x_2, x_3)), ((x_2, x_3), (x_3, (x_1, x_2))))$

its version for 3-manifolds that are knot complements was proposed. Diagonalization of quivers discussed in this paper may be of interest in all these contexts.

For a knot $K \subset S^3$, the HOMFLY-PT polynomial $P_K(a, q)$ [24,25] is a topological invariant which can be calculated via the skein relation. More generally, colored HOMFLY-PT polynomials $P_{K,R}(a, q)$ are similar polynomial knot invariants that depend also on a representation R of the Lie algebra $\mathfrak{u}(N)$. (In this setting, the original HOMFLY-PT polynomials correspond to the fundamental representation.) From the physical point of view, $P_{K,R}(a, q)$ is the expectation value of the knot viewed as a Wilson line in $U(N)$ Chern-Simons gauge theory [26]. The HOMFLY-PT generating series is given by

$$P_K(\lambda, a, q) = \sum_{r=0}^{\infty} P_{K,r}(a, q)\lambda^{-r}, \tag{83}$$

where $P_{K,r}(a, q)$ are HOMFLY-PT polynomials colored by the totally symmetric representations S^r . The unusual expansion variable with the negative power comes from the necessity of resolving the clash of different conventions present in the literature and avoiding the confusion with the quiver variables. For more detailed discussion of all conventions see [14, Sect. 2.1] and references therein.

The LMOV invariants $N_{r,i,j}$ [27–29] are numbers that give the following expression for the HOMFLY-PT generating series:⁷

$$P_K(\lambda, a, q) = \prod_{r \geq 1} \prod_{i,j \in \mathbb{Z}} (\lambda^{-r} a^i q^j; q^2)_{\infty}^{N_{r,i,j}} \tag{84}$$

We can assemble $N_{r,i,j}$ into the LMOV generating series $N(\lambda, a, q) = \sum_{r,i,j} N_{r,i,j} \lambda^{-r} a^i q^j$. From the physical point of view, LMOV invariants count BPS states in the effective 3d $\mathcal{N} = 2$ theory on the world-volume of M5-brane wrapped on the knot conormal inside the resolved conifold [27].

⁷ Note that in this paper the product form with quantum dilogarithms serves as a basis. In consequence, we have $\sum_{j \in \mathbb{Z}} N_{r,i,j} q^j = \sum_{k \in \mathbb{Z}} N'_{r,i,k} q^{k+1}$, where $N'_{r,i,k}$ is the LMOV invariant in the notation from [8].

The knots-quivers correspondence [7, 8] is a conjecture that for each knot K there exist a quiver Q and integers $n_i, a_i, l_i, i \in Q_0$, such that

$$P_K(\lambda, a, q) = P_Q(\mathbf{x}, q)|_{x_i = \lambda^{n_i} a^{a_i} q^{l_i}}. \tag{85}$$

If we substitute (84) and (4), we obtain the knots-quivers correspondence at the level of LMOV and DT invariants:

$$N(\lambda, a, q) = \Omega(\mathbf{x}, q)|_{x_i = \lambda^{n_i} a^{a_i} q^{l_i}}. \tag{86}$$

Since DT invariants are integer, this equation implies integrality of $N_{r,i,j}$ —this means that the knots-quivers correspondence automatically proves the LMOV conjecture.

In the rest of this section we apply the framework of infinite diagonal quivers and m -loop quivers to the knots-quivers correspondence. Quivers corresponding to knots are in general much more complicated than examples from the previous section. In consequence, we will focus on the reduced normalization⁸ and simplest nontrivial examples, namely the trefoil and figure-eight. Moreover, even in these cases the matrix $\tilde{C}^{(2)}$ exceeds 50×50 entries. Therefore, we will compute $C^{(2)}$ and, if possible, $C^{(3)}$ in the most efficient way, (un)liking only those non-diagonal entries that are necessary.

6.1. Trefoil knot. In this case the adjacency matrix and the change of variables are given by [8]

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda^{-1} a^2 q^{-2} \\ \lambda^{-1} a^2 \\ \lambda^{-1} a^4 q^{-3} \end{bmatrix}. \tag{87}$$

The first degree approximation is immediate, as it coincides with the diagonal of the initial quiver (87) with $\mathbf{x}^{(1)} = (x_1, x_2, x_3)$. The second order approximaton comes from unlinking all arrows in (87), which gives

$$\tilde{C}^{(1)} = \begin{bmatrix} \overline{0} & \overline{0} & \overline{0} & 0 & 0 & 0 & 0 \\ 0 & \overline{2} & \overline{0} & 2 & 2 & 3 & 2 \\ \overline{0} & \overline{0} & \overline{3} & 3 & 3 & 4 & 3 \\ \overline{0} & \overline{2} & \overline{3} & 3 & 3 & 5 & 5 \\ 0 & 2 & 3 & 3 & 4 & 5 & 5 \\ 0 & 3 & 4 & 5 & 5 & 8 & 7 \\ 0 & 2 & 3 & 5 & 5 & 7 & 6 \end{bmatrix}. \tag{88}$$

Since the seven diagonal entries of $\tilde{C}^{(1)}$ are the only ones which contribute to quadratic terms $x_i x_j$ in $P_Q(\mathbf{x}, q)$, we do not have to write the whole $\tilde{C}^{(2)}$, but we can conclude that

$$C^{(2)} = \text{diag}(0, 2, 3 | 3, 4, 8, 6) \tag{89}$$

$$\mathbf{x}^{(2)} = \left(x_1, x_2, x_3 | q^{-1} x_1 x_2, q^{-1} x_1 x_3, q^{-1} x_2 x_3, q^{-1} x_2 x_3 \right).$$

⁸ For the discussion of the normalizations of HOMFLY-PT polynomials see [8, Sect. 4.5]. The fact that we use the reduced normalization implies that the BPS spectrum we consider is a subset of the BPS spectrum of the unreduced case built from the half of the generators. We do it to avoid very tedious computations, but all our methods can be applied also in the unreduced normalization.

Following Sect. 4.1 and expanding the product of C_{ii} -loop quivers, we obtain

$$\Omega(\mathbf{x}, q) = -x_1 - q^2x_2 + q^3x_3 + q^2x_1x_2 - q^3x_1x_3 + q^4x_2^2 - q^5x_2x_3 - q^7x_2x_3 + q^8x_3^2 + O(\mathbf{x}^3). \tag{90}$$

The application of the change of variables (87) translates DT invariants into LMOV invariants:

$$N(\lambda, a, q) = -a^2q^{-2} (1 + a^2q^5 + q^6) \lambda^{-1} + a^4q (a^2 + q) (1 + q^6 + a^2q^7) \lambda^{-2} + O(\lambda^{-3}).$$

In order to find the third degree approximation, we have to unlink all arrows between the four new nodes corresponding to diagonal entries (3, 4, 8, 6) and the initial triple corresponding to (0, 2, 3). For clarity, we highlight the unlinked non-diagonal entries in red:

$$\tilde{C}^{(1)} = \begin{bmatrix} \begin{array}{|ccc|ccc} \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 2 & 2 & 3 & 2 \\ \hline 0 & 0 & 3 & 3 & 3 & 4 & 3 \\ \hline 0 & 2 & 3 & 3 & 3 & 5 & 5 \\ \hline 0 & 2 & 3 & 3 & 4 & 5 & 5 \\ \hline 0 & 3 & 4 & 5 & 5 & 8 & 7 \\ \hline 0 & 2 & 3 & 5 & 5 & 7 & 6 \\ \hline \end{array} \end{bmatrix}. \tag{91}$$

This procedure results in 29×29 matrix which is too large to write here. However, its main diagonal and the corresponding change of variables gives us the third order approximation:

$$C^{(3)} = \text{diag}(0, 2, 3, 3, 4, 8, 6, 8, 6, 9, 7, 15, 13, 11, 11, 9, 11, 9, 7, 12, 10, 8, 18, 16, 14, 12, 14, 12, 10),$$

$$\mathbf{x}^{(3)} = (x_1, x_2, x_3, q^{-1}x_1x_2, q^{-1}x_1x_3, \{q^{-1}x_2x_3\}_2, \{q^{-2}x_1x_2^2\}_2, \{q^{-2}x_1x_2x_3\}_2, \{q^{-2}x_2^2x_3\}_3, \{q^{-2}x_2^2x_3\}_2, \{q^{-2}x_1x_2x_3\}_3, \{q^{-2}x_1x_2^2\}_3, \{q^{-2}x_2x_3^2\}_4, \{q^{-2}x_2x_3^2\}_3), \tag{92}$$

where we denote $\{\alpha\}_n = \overbrace{\alpha, \dots, \alpha}^{n \text{ times}}$ for any monomial α .

In turn, using Theorem 13 we compute

$$\Omega(\mathbf{x}, q) = -x_1 - q^2x_2 + q^3x_3 + q^2x_1x_2 - q^3x_1x_3 + q^4x_2^2 - q^5x_2x_3 + q^8x_3^2 - (q^4 + q^6)x_1x_2^2 - (q^6 + q^8)x_1x_3^2 + (2q^5 + 2q^7 + q^9)x_1x_2x_3 + O(\mathbf{x}^4) \tag{93}$$

The generating series of the corresponding LMOV invariants reads

$$N(\lambda, a, q) = -a^2q^{-2} (1 + a^2q^5 + q^6) \lambda^{-1} + a^4q (a^2 + q) (1 + q^6 + a^2q^7) \lambda^{-2} - a^6q^{11} (a^6 + a^6q^2 + q^3 + a^6q^6) \lambda^{-3} + O(\lambda^{-4}). \tag{94}$$

6.2. *Figure-eight knot.* This is the first example in our selection where we encounter both ordinary and negative arrows in Q . This indicates necessity to use both unlinking and linking in order to compute $Q^{(n)}$, following the rules of diagonalization given in Definition 9. The adjacency matrix and the change of variables are given by

$$C = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 \\ -1 & -2 & -2 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \lambda^{-1} \\ \lambda^{-1}a^{-2}q^2 \\ \lambda^{-1}q^{-1} \\ \lambda^{-1}q \\ \lambda^{-1}a^2q^{-2} \end{bmatrix}. \tag{95}$$

The first degree approximation is immediately given by $C^{(1)} = \text{diag}(0, -2, -1, 1, 2)$, $\mathbf{x}^{(1)} = (x_1, \dots, x_5)$. $C^{(1)}$ is a diagonal submatrix of

$$\tilde{C}^{(1)} = \begin{bmatrix} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & -2 & \boxed{0} & \boxed{0} & \boxed{0} & -3 & -2 & -4 & -3 & -3 & -1 & 0 \\ \boxed{0} & \boxed{0} & -1 & \boxed{0} & \boxed{0} & -3 & -2 & -3 & -2 & -2 & -2 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & 1 & \boxed{0} & -1 & -1 & -2 & 0 & 0 & 0 & 1 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ -1 & -3 & -3 & -1 & 0 & -4 & -4 & -6 & -6 & -4 & -4 & -1 \\ -1 & -2 & -2 & -1 & 0 & -4 & -3 & -4 & -4 & -3 & -3 & -1 \\ 0 & -4 & -3 & -2 & 0 & -6 & -4 & -7 & -7 & -6 & -5 & -2 \\ 0 & -3 & -2 & 0 & 0 & -6 & -4 & -7 & -5 & -5 & -3 & 0 \\ 0 & -3 & -2 & 0 & 1 & -4 & -3 & -6 & -5 & -3 & -2 & 1 \\ 0 & -1 & -2 & 0 & 1 & -4 & -3 & -5 & -3 & -2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & -1 & -2 & 0 & 1 & 1 & 4 \end{bmatrix}, \tag{96}$$

obtained from C by the successive application of (un)linking. The diagonal of $\tilde{C}^{(1)}$ together with the change of variables coming from (un)linking gives us the second degree approximation (the matrix $\tilde{C}^{(2)}$ is of size 125×125):

$$C^{(2)} = \text{diag} (0, -2, -1, 1, 2 \mid -4, -3, -7, -5, -3, -2, 4). \tag{97}$$

$$\mathbf{x}^{(2)} = (x_1, x_2, x_3, x_4, x_5 \mid x_1x_2, x_1x_3, \{x_2x_3\}_2, x_2x_4, x_3x_4, q^{-1}x_4x_5).$$

This comes from the fact that all nodes whose leading term is of the form $x_i x_j, i \neq j$, come from (un)linking of the nodes of Q . Using this data and Theorem 13, we can write the generating seires of DT invariants up to quadratic terms:

$$\begin{aligned} \Omega(\mathbf{x}, q) = & -x_1 - q^{-2}x_2 + q^{-1}x_3 + qx_4 - q^2x_5 - q^{-4}x_1x_2 + q^{-3}x_1x_3 \\ & - q^{-8}x_2^2 + (q^{-7} + q^{-5})x_2x_3 + q^{-3}x_2x_4 - q^{-4}x_3^2 - q^{-2}x_3x_4 \\ & - q^3x_4x_5 + q^4x_5^2 + O(\mathbf{x}^3). \end{aligned} \tag{98}$$

As a consistency check, note that terms of the form $x_i x_j$ with $i \neq j$ precisely correspond to seven extra nodes in $C^{(2)}$. The application of the change of variables (95) gives the following LMOV invariants:

$$N(\lambda^{-1}, a, q) = -a^{-2}q^{-3} \left(a^2 + a^4q + a^2q^3 + q^5 + a^2q^6 \right) \lambda^{-1} - a^{-4}q^{-9} \left(1 + a^2q \right) \left(a^6 + a^4q + a^4q^4 + a^4q^7 - q^{13} \right) \lambda^{-2} + O(\lambda^{-3}). \tag{99}$$

7. Quivers Corresponding to Knot Complements

F_K invariants were introduced in [30] as knot complement versions of \hat{Z} invariants [31,32]:

$$F_K = \hat{Z}(S^3 \setminus K). \tag{100}$$

From the physical point of view, the F_K invariant arises from the reduction of 6d $\mathcal{N} = (0, 2)$ theory describing M5-branes on the 3-manifold with the topology of the knot complement [30]. The a -deformed F_K invariants were introduced in [33]. In [14] the knots-quivers correspondence was generalized to knot complements using the a -deformed version of F_K (which for simplicity we will call just F_K invariant). More specifically, the author conjectured—and proved for the complements of $(2, 2p+1)$ torus knots—that after appropriate change of variables the F_K invariant can be expressed as a motivic generating series of some quiver Q :

$$\begin{aligned}
 F_K(\mu, a, q) &= \sum_{d_1, \dots, d_{|Q_0|} \geq 0} (-q)^{\sum_{i,j=1}^{|Q_0|} c_{ij} d_i d_j} \prod_{i=1}^{|Q_0|} \frac{\mu^{n_i d_i} a^{a_i d_i} q^{l_i d_i}}{(q^2; q^2)_{d_i}} \\
 &= P_Q(\mathbf{x}, q) \Big|_{x_i = \mu^{n_i} a^{a_i} q^{l_i}}.
 \end{aligned} \tag{101}$$

As a consequence, the knot complement analogues of LMOV invariants $N(\mu, a, q)$ were introduced basing on DT invariants of corresponding quivers:

$$N(\mu, a, q) = \sum_{r,i,j} N_{r,i,j} \mu^r a^i q^j = \Omega(\mathbf{x}, q) \Big|_{x_i = \mu^{n_i} a^{a_i} q^{l_i}}. \tag{102}$$

Further developments, including explicit formulas for many knot complements and generalisation to different branches of A -polynomials were presented in [23].

In the rest of this section we apply the framework of infinite diagonal quivers and m -loop quivers to the simplest nontrivial cases of quivers corresponding to the trefoil and figure-eight knot complements.⁹

7.1. Trefoil knot complement. In case of the trefoil knot complement, F_{3_1} can be written in the form (101) for the following quiver and the change of variables:¹⁰ [14]

$$C = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mu^2 a^2 \\ \mu^2 q^3 \\ \mu a^2 q^{-1} \\ \mu q^2 \end{bmatrix}. \tag{103}$$

In this example we explicitly present two steps of the calculation—the second and third degree approximations, which are computable in reasonable time. After linking all non-diagonal entries, we obtain the second degree approximation:

$$\begin{aligned}
 C^{(2)} &= \text{diag}(0, -1, 1, 0 \mid -3, -2, -3), \\
 \mathbf{x}^{(2)} &= (x_1, x_2, x_3, x_4 \mid x_1 x_2, x_1 x_4, x_2 x_4).
 \end{aligned} \tag{104}$$

⁹ More precisely, we consider quivers corresponding to the F_K invariants associated to the abelian branches of the A -polynomials for these knots. For more details see [23].

¹⁰ The difference between the change of variables (103) and [14] comes from the conventional switch $q \leftrightarrow q^2$ and $a \leftrightarrow a^2$.

Following the procedure of expanding the product of C_{ii} -loop quivers, described in Sect. 4.1, we get

$$\begin{aligned} \Omega(\mathbf{x}, q) = & -x_1 + q^{-1}x_2 + qx_3 - x_4 \\ & + q^{-3}x_1x_2 - q^{-4}x_2^2 - q^{-2}x_1x_4 + q^{-3}x_2x_4 + O(\mathbf{x}^3). \end{aligned} \tag{105}$$

Combining the above result with the definition (102) and change of variables (103), we obtain the trefoil knot complement analogues of LMOV invariants:

$$N(\mu, a, q) = (a^2 - q^2)\mu - (a^2 - q^2)\mu^2 + O(\mu^3). \tag{106}$$

Following analogous steps, we obtain the third degree approximation:

$$\begin{aligned} C^{(3)} = & \text{diag}(0, -1, 1, 0 \mid -3, -2, -3 \mid -5, -4, -8, -6, \\ & -5, -8, -6, -7, -5, -4, -5), \\ \mathbf{x}^{(3)} = & (x_1, x_2, x_3, x_4 \mid x_1x_2, x_1x_4, x_2x_4 \mid x_1^2x_2, x_1^2x_4, \{x_1x_2\}_2, x_1x_2x_4, \{x_2^2x_4\}_2, \\ & \{x_1x_2x_4\}_2, x_1x_4^2, x_2x_4^2). \end{aligned} \tag{107}$$

This allows to refine the expression for the generating series given in (105):

$$\begin{aligned} \Omega(\mathbf{x}, q) = & -x_1 + q^{-1}x_2 + qx_3 - x_4 \\ & + q^{-3}x_1x_2 - q^{-2}x_1x_4 - q^{-4}x_2^2 + q^{-3}x_2x_4 \\ & + q^{-5}x_1^2x_2 - q^{-4}x_1^2x_4 - (q^{-8} + q^{-6})x_1x_2^2 + (q^{-7} + 2q^{-5})x_1x_2x_4 \\ & - q^{-4}x_1x_4^2 + q^{-9}x_2^3 - (q^{-8} + q^{-6})x_2^2x_4 + q^{-5}x_2x_4^2 + O(\mathbf{x}^4). \end{aligned} \tag{108}$$

After specialization of x_i , given by (103), they take form

$$N(\mu, a, q) = (a^2 - q^2)\mu - (a^2 - q^2)\mu^2 - (a^2 - q^2)\mu^3 + O(\mu^4). \tag{109}$$

The presence of $(a^2 - q^2)$ factor in the expression for $N(\mu, a, q)$ is a trace of trivialization $F_K(\mu, a = q, q) = 1$ conjectured in [33] for any knot complement. However, the lack of other dependence on a and q suggests that the analogues of LMOV invariants for the complement of 3_1 satisfy the following property

$$N_{S^3 \setminus 3_1}(\mu, a^{-1}, q^{-1}) = (-a^{-2}q^{-2})N_{S^3 \setminus 3_1}(\mu, a, q). \tag{110}$$

7.2. *Figure-eight knot complement.* For the figure-eight knot complement, one can write F_{4_1} in the form (101) using the following quiver and the change of variables:¹¹ [23]

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \mu q^2 \\ \mu q^2 \\ \mu q^2 \\ \mu a^2 q^{-1} \\ \mu a^2 q^{-1} \\ \mu a^2 q^{-1} \end{bmatrix}. \tag{111}$$

¹¹ The difference between the change of variables (111) and [23] comes from the conventional switch $q \leftrightarrow q^2$ and $a \leftrightarrow a^2$.

The second degree approximation obtained after (un)linking all non-diagonal entries reads

$$\begin{aligned}
 C^{(2)} &= \text{diag}(0, 0, 0, 1, 1, 1 \mid 2, -2, -1, 3) , \\
 \mathbf{x}^{(2)} &= (x_1, x_2, x_3, x_4, x_5, x_6 \mid q^{-1}x_1x_4, x_2x_3, x_2x_4, q^{-1}x_4x_5) .
 \end{aligned}
 \tag{112}$$

After the application of Theorem 13, this leads to

$$\begin{aligned}
 \Omega(\mathbf{x}, q) &= -x_1 - x_2 - x_3 + qx_4 + qx_5 + qx_6 \\
 &\quad - qx_1x_4 - q^{-2}x_2x_3 + q^{-1}x_2x_4 + q^2x_4x_5 + O(\mathbf{x}^3) .
 \end{aligned}
 \tag{113}$$

If we put this equation together with (103) into (102), we obtain

$$N(\mu, a, q) = 3(a^2 - q^2)\mu + (1 + a^2)(a^2 - q^2)\mu^2 + O(\mu^3) .
 \tag{114}$$

Following similar steps, we get the third degree approximation:

$$\begin{aligned}
 C^{(3)} &= \text{diag}(0, 0, 0, 1, 1, 1 \mid 2, -2, -1, 3 \mid 0, -4, -3, -4, 4, -3, 5, 4, 1, 5) , \\
 \mathbf{x}^{(3)} &= (x_1, x_2, x_3, x_4, x_5, x_6 \mid q^{-1}x_1x_4, x_2x_3, x_2x_4, q^{-1}x_4x_5 \mid q^{-1}x_1x_2x_4, x_2^2x_3, x_2^2x_4, \\
 &\quad x_2x_3^2, q^{-2}x_1x_4^2, x_2x_3x_4, q^{-2}x_4^2x_5, q^{-2}x_1x_4x_5, q^{-1}x_2x_4x_5, q^{-2}x_4x_5^2) .
 \end{aligned}
 \tag{115}$$

This allows us to compute more DT invariants, extending (113) to

$$\begin{aligned}
 \Omega(\mathbf{x}, q) &= -x_1 - x_2 - x_3 + qx_4 + qx_5 + qx_6 \\
 &\quad - qx_1x_4 - q^{-2}x_2x_3 + q^{-1}x_2x_4 + q^2x_4x_5 + q^2x_4x_5 - q^{-1} \\
 &\quad \times x_1x_2x_4 - q^2x_1x_4^2 - q^2x_1x_4x_5 - q^{-4}x_2^2x_3 + q^{-3}x_2^2x_4 - q^{-4}x_2x_3^2 \\
 &\quad + q^{-3}x_2x_3x_4 + x_2x_4x_5 + q^3x_4^2x_5 + q^3x_4x_5^2 + O(\mathbf{x}^4) .
 \end{aligned}$$

Finally, we perform the change of variables indicated in (111) to derive the analogue of LMOV invariants of F_{4_1} :

$$\begin{aligned}
 N(\mu, a, q) &= 3(a^2 - q^2)\mu + (1 + a^2)(a^2 - q^2)\mu^2 \\
 &\quad + (2 + a^2 + 2a^4)(a^2 - q^2)\mu^3 + O(\mu^4) .
 \end{aligned}
 \tag{116}$$

Similarly to the LMOV invariants for the complement of 3_1 (which are reciprocal polynomials, see Eq. (110)), we conjecture that the above expression enjoys the property

$$N_{S^3 \setminus 4_1}(\mu, a^{-1}, q^{-1}) = (-q^{-2}) N_{S^3 \setminus 4_1}(a^2\mu, a, q) .
 \tag{117}$$

Acknowledgements This work has been supported by the TEAM programme of the Foundation for Polish Science co-financed by the European Union under the European Regional Development Fund (POIR.04.04.00-00-5C55/17-00), Vidi grant (number 016.Vidi.189.182) from the Dutch Research Council (NWO), and Excellence Initiative: Research University programme financed by the Polish Ministry of Education and Science (01/IDUB/2019/94). P.S. also acknowledges the support of Fulbright STEM Impact Award and hospitality of the High Energy Theory Group at Harvard University. The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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Appendix A: Combinatorial Model for $(-m)$ -Loop Quivers

In Sect. 2.3 we proved Lemma 6 using a simple q -Pochhammer identity. However, we can also show how the relation between DT invariants for $(-m)$ - and $(m + 1)$ -loop quivers is realized at the combinatorial level. The base for this construction is a combinatorial model for m -loop quivers introduced in [4]. In [13] it was connected to the quantum version of extremal A -polynomials and extremal LMOV invariants of knots [34]. The quantum A -polynomials annihilate HOMFLY-PT polynomials [35–38] and have quiver analogues that annihilate motivic generating series [10,39,40].

The reasoning of the proof below can be summarized in the following way. The relation between m -loop quiver model [4] and the model associated to extremal quantum A -polynomial with m ’th q -power [13] was described in [13, App. B] for $m \geq 0$. However, extremal quantum A -polynomials can be considered also with a q -power of $(-m)$ (see Eq. (120)). Then, following the steps of [13, App. B], we can relate the case of $(-m)$ to the one corresponding to $m + 1$.

Combinatorial proof of Lemma 6. In order to follow [13, App. B] and the conventions used in that paper, we rescale our variable x by q^{m+1} and sum over r instead of d :

$$P_{(-m)\text{-loop}}(x'q^{m+1}, q) = \sum_{r=0}^{\infty} \frac{(-q)^{(-m)r^2} q^{(m+1)r}}{(q^2; q^2)_r} (x')^r. \tag{118}$$

Then, we define

$$Y(x', q) = \frac{P_{(-m)\text{-loop}}(x'q^{m+3}, q)}{P_{(-m)\text{-loop}}(x'q^{m+1}, q)}, \tag{119}$$

which satisfies the following A -polynomial equation:

$$1 - Y(x', q) + (-1)^{m+1} q x' \prod_{i=1}^m Y(q^{-2i} x', q)^{-1} = 0. \tag{120}$$

Using the series expansion, $Y(x', q) = \sum_{n=0}^{\infty} Y_n(q)(x')^n$, Eq. (120) implies that

$$Y_n(q) = - \sum_{\substack{a+b=n \\ 0 < a < b}} Y_a(q) \sum_{k_1 + \dots + k_m = b} \prod_{i=1}^m q^{-2ik_i} Y_{k_i}(q), \tag{121}$$

with $Y_0(q) = 1$ and $Y_1(q) = (-1)^{m+1}q$.¹² For example, for $n = 2$ we have only one option $a = 1$ and $b = 1$, whence

$$Y_2(q) = -Y_1(q) \sum_{i=1}^m q^{-2i} Y_1(q) = -Y_1(q)^2 \sum_{i=1}^m q^{-2i} = -q^2 \sum_{i=1}^m q^{-2i} . \quad (122)$$

Using Eq. (121), analogously to the construction from Ref. [13], we can represent $Y_n(q)$ as a signed list $\phi = (-1)^{m+1}[\phi_1, \dots, \phi_n]$. In order to do it, we start from

$$T_0 = \{[]\} , \quad T_1 = \{(-1)^{m+1}[1]\} , \quad (123)$$

and define sets T_2, T_3, \dots recursively by

$$T_n = \left\{ -\phi_{(a)} * (\phi_{(k_m)} - 2m) * \dots * (\phi_{(k_1)} - 2) \right. \\ \left. \mid \forall (a + b = n \wedge 0 < ab) \wedge (\forall k_1 + \dots + k_m = b \wedge \phi_{(k_i)} \in T_{k_i}, i = 1, \dots, m) \right\} , \quad (124)$$

where $*$ is a concatenation of lists:

$$\phi_{(1)} * \phi_{(2)} = [\phi_{(1)}, \phi_{(2)}] = [\phi_{(1),1}, \dots, \phi_{(1),n_1}, \phi_{(2),1}, \dots, \phi_{(2),n_2}] , \quad (125)$$

and subtracting a number from a list means a subtraction for each entry. All powers in Eq. (121) sum up to n and eventually all terms are expressed as powers of $Y_1(q) = (-1)^{m+1}q$. Since each Y_* term contributes a (-1) , then the sign of a list of length n is $(-1)^{n(m+1)+n+1} = (-1)^{1+(2+m)n} = (-1)^{1+mn}$. The weight of a list is $\text{wt}(\phi) = \sum_{i=1}^n \phi_i$, so that

$$Y_n(q) = \sum_{\phi \in T_n} \text{sgn}(\phi) q^{\text{wt}(\phi)} . \quad (126)$$

Note that Eq. (124) implies that for $n > 0$ we have

$$\phi_0 = 1 , \quad \phi_i \leq \phi_0 \quad \text{for} \quad i = 1, \dots, n - 1 , \quad |\phi_{i+1} - \phi_i| \leq 2m . \quad (127)$$

Now we are ready to define a map φ from $(-m)$ -loop case to $(m + 1)$ -loop case:

$$\varphi(\phi) = -[2 - \phi_1, 2 - \phi_2, \dots, 2 - \phi_n] . \quad (128)$$

Comparing Eq. (124) with the rules of creating primary lists¹³ given in [13, Eq. (4.18)], one can check that φ is a bijection from T_n of $(-m)$ -loop case to T_n^0 of $(m + 1)$ -loop

¹² Note that on the right-hand side there are $m + 1$ factors of Y , whereas in [13, Eq. (4.13)] there are m of them. This is a structural manifestation of the Lemma 6.

¹³ We say that the list is primary if it cannot be expressed as a concatenation of smaller lists.

case. Combining it with the construction described in [13, Sect. 3.2]¹⁴ we learn that

$$\begin{aligned}
 \Omega_{(-m)\text{-loop}}(x', q) &= \sum_{r \geq 0} \left(\sum_{\phi \in (T_r^{L,+})_{(-m)\text{-loop}}} \text{sgn}(\phi) q^{\text{wt}(\phi)} \right) \frac{(x')^r}{[r]_{q^2}} \\
 &= \sum_{r \geq 0} \left(\sum_{\varphi \in (T_r^{L,+})_{(m+1)\text{-loop}}} -\text{sgn}(\varphi) q^{2r - \text{wt}(\varphi)} \right) \frac{(x')^r}{[r]_{q^2}} \\
 &= \sum_{r \geq 0} \left(\sum_{\varphi \in (T_r^{L,+})_{(m+1)\text{-loop}}} -\text{sgn}(\varphi) q^{2 - \text{wt}(\varphi)} \right) \frac{(x')^r}{[r]_{q^2}} \\
 &= -q^2 \Omega_{(m+1)\text{-loop}}(x', q^{-1}),
 \end{aligned}
 \tag{129}$$

where we used

$$\begin{aligned}
 \text{wt}(\varphi) &= \sum_{i=1}^n \varphi_i = 2n - \sum_{i=1}^n \phi_i = 2n - \text{wt}(\phi), \\
 [r]_{q^2} &= \frac{1 - q^{2r}}{1 - q^2}.
 \end{aligned}
 \tag{130}$$

In Ref. [13] $\sum_{\varphi \in (T_r^{L,+})_{(m+1)\text{-loop}}} \text{sgn}(\varphi) q^{-\text{wt}(\varphi)}$ was linked to the $Q_r(q^{-2})$ of Ref. [12], where $Q_r(q^2)$ was proven to be divisible by $[r]_{q^2}$ and that implies that $Q_r(q^{-2})$ is also divisible by $[r]_{q^2}$. This happens because if ζ is a root of unity then ζ^{-1} is also a root of unity.

Now we only need to take into account that the relation between x' and x for the $(m + 1)$ -loop quiver with inverted q reads $x = x' q^m$, therefore

$$\Omega_{(-m)\text{-loop}}(x, q) = -q^2 \Omega_{(m+1)\text{-loop}}(q^{-1}x, q^{-1}),
 \tag{131}$$

which is consistent with (20). □

References

1. Kontsevich, M., Soibelman, Y.: Stability structures, motivic Donaldson–Thomas invariants and cluster transformations, [arXiv:0811.2435](#)
2. Kontsevich, M., Soibelman, Y.: Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants. *Commun. Num. Theor. Phys.* **5**, 231–352 (2011). [arXiv:1006.2706](#)
3. Efimov, A.I.: Cohomological Hall algebra of a symmetric quiver. *Compos. Math.* **148**(4), 1133–1146 (2012). [arXiv:1103.2736](#)
4. Reineke, M.: Degenerate cohomological hall algebra and quantized Donaldson–Thomas invariants for m-loop quivers. *Doc. Math.* **17**, 1 (2012). [arXiv:1102.3978](#)
5. Franzen, H., Reineke, M.: Semi-stable Chow–Hall algebras of quivers and quantized Donaldson–Thomas invariants. *Algebra Number Theory* **12**(5), 1001–1025 (2018). [arXiv:1512.03748](#)

¹⁴ In [13] LMOV invariants are considered instead of corresponding DT invariants, so $N_r(q)$ is used instead of $\Omega_r(q)$. The definition of the sets of modified Lyndon lists $T_r^{L,+}$ is given in [13, Eq. (3.32)]

6. Meinhardt, S., Reineke, M.: Donaldson–Thomas invariants versus intersection cohomology of quiver moduli, [arXiv:1411.4062](#)
7. Kucharski, P., Reineke, M., Stosic, M., Sulkowski, P.: BPS states, knots and quivers. *Phys. Rev. D* **96**(12), 121902 (2017), [arXiv:1707.02991](#)
8. Kucharski, P., Reineke, M., Stosic, M., Sulkowski, P.: Knots-quivers correspondence. *Adv. Theor. Math. Phys.* **23**(7), 1849–1902 (2019). [arXiv:1707.04017](#)
9. Panfil, M., Sulkowski, P., Stosic, M.: Donaldson–Thomas invariants, torus knots, and lattice paths. *Phys. Rev. D* **98**(026022), 2 (2018). [arXiv:1802.04573](#)
10. Ekholm, T., Kucharski, P., Longhi, P.: Multi-cover skeins, quivers, and 3d $\mathcal{N} = 2$ dualities. *JHEP* **02**, 018 (2020). [arXiv:1910.06193](#)
11. Ekholm, T., Kucharski, P., Longhi, P.: Physics and geometry of knots-quivers correspondence. *Commun. Math. Phys.* **379**(2), 361–415 (2020). [arXiv:1811.03110](#)
12. Reineke, M.: Cohomology of quiver moduli, functional equations, and integrality of Donaldson–Thomas type invariants. *Compos. Math.* **147**(5), 943–964 (2011). [arXiv:0903.0261](#)
13. Kucharski, P., Sulkowski, P.: BPS counting for knots and combinatorics on words. *JHEP* **11**, 120 (2016). [arXiv:1608.06600](#)
14. Kucharski, P.: Quivers for 3-manifolds: the correspondence, BPS states, and 3d $\mathcal{N} = 2$ theories. *JHEP* **09**, 075 (2020). [arXiv:2005.13394](#)
15. Dotsenko, V., Feigin, E., Reineke, M.: Koszul algebras and Donaldson–Thomas invariants. *Lett. Math. Phys.* **112**(106), 5 (2021). [arXiv:2111.07588](#)
16. Jankowski, J., Kucharski, P., Larraguivel, H., Noshchenko, D., Sulkowski, P.: Permutohedra for knots and quivers. *Phys. Rev. D* **104**(086017), 8 (2021). [arXiv:2105.11806](#)
17. Jankowski, J., Kucharski, P., Larraguivel, H., Noshchenko, D., Sulkowski, P.: To appear
18. Stosic, M., Wedrich, P.: Rational links and DT invariants of quivers. *Int. Math. Res. Not. rny***289** (2019) [arXiv:1711.03333](#)
19. Stosic, M., Wedrich, P.: Tangle addition and the knots-quivers correspondence, [arXiv:2004.10837](#)
20. Ekholm, T., Kucharski, P., Longhi, P.: Knot homologies and generalized quiver partition functions, [arXiv:2108.12645](#)
21. Panfil, M., Sulkowski, P.: Topological strings, strips and quivers. *JHEP* **01**, 124 (2019). [arXiv:1811.03556](#)
22. Kimura, T., Panfil, M., Sugimoto, Y., Sulkowski, P.: Branes, quivers and wave-functions. *SciPost Phys.* **10**, 51 (2021). [arXiv:2011.06783](#)
23. Ekholm, T., Gruen, A., Gukov, S., Kucharski, P., Park, S., Stosic, M., Sulkowski, P.: Branches, quivers, and ideals for knot complements. *J. Geom. Phys.* **177**, 104520 (2022). [arXiv:2110.13768](#)
24. Hoste, J., Ocneanu, A., Millett, K., Freyd, P.J., Lickorish, W.B.R., Yetter, D.N.: A new polynomial invariant of knots and links. *Bull. Am. Math. Soc.* **12**(2), 239–246 (1985)
25. Przytycki, J., Traczyk, P.: Invariants of links of Conway type. *Kobe J. Math.* **4**, 115–139 (1987)
26. Witten, E.: Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121**(3), 351–399 (1989)
27. Ooguri, H., Vafa, C.: Knot invariants and topological strings. *Nucl. Phys. B* **577**, 419–438 (2000). [hep-th/9912123](#)
28. Labastida, J.M.F., Marino, M.: Polynomial invariants for torus knots and topological strings. *Commun. Math. Phys.* **217**(2), 423–449 (2001). [hep-th/0004196](#)
29. Labastida, J.M.F., Marino, M., Vafa, C.: Knots, links and branes at large N . *JHEP* **11**, 007 (2000). [hep-th/0010102](#)
30. Gukov, S., Manolescu, C.: A two-variable series for knot complements. *Quantum Topol.* **12**(1), 1–109 (2021). [arXiv:1904.06057](#)
31. Gukov, S., Putrov, P., Vafa, C.: Fivebranes and 3-manifold homology. *JHEP* **07**, 071 (2017). [arXiv:1602.05302](#)
32. Gukov, S., Pei, D., Putrov, P., Vafa, C.: BPS spectra and 3-manifold invariants. *J. Knot Theor. Ramifications* **29**(2040003), 02 (2020). [arXiv:1701.06567](#)
33. Ekholm, T., Gruen, A., Gukov, S., Kucharski, P., Park, S., Sulkowski, P.: $\widehat{\mathcal{Z}}$ at large N : from curve counts to quantum modularity. *Commun. Math. Phys.* **396**(1), 143–186 (2022). [arXiv:2005.13349](#)
34. Garoufalidis, S., Kucharski, P., Sulkowski, P.: Knots, BPS states, and algebraic curves. *Commun. Math. Phys.* **346**, 75–113 (2016). [arXiv:1504.06327](#)
35. Garoufalidis, S.: On the characteristic and deformation varieties of a knot. *Geom. Topol. Monographs* **7**, 291–304 (2004). ([math/0306230](#))
36. Gukov, S.: Three-dimensional quantum gravity, Chern–Simons theory, and the A-polynomial. *Commun. Math. Phys.* **255**(3), 577–627 (2005). [hep-th/0306165](#)
37. Aganagic, M., Vafa, C.: Large N duality, mirror symmetry, and a Q -deformed A-polynomial for knots, [arXiv:1204.4709](#)
38. Garoufalidis, S., Lauda, A.D., Le, T.T.Q.: The colored HOMFLYPT function is q -holonomic. *Duke Math. J.* **167**(3), 397–447 (2018). [arXiv:1604.08502](#)

39. Larraguivel, H., Noshchenko, D., Panfil, M., Sułkowski, P.: Nahm sums, quiver A-polynomials and topological recursion. *JHEP* **07**, 151 (2020). [arXiv:2005.01776](https://arxiv.org/abs/2005.01776)
40. Noshchenko, D.: Combinatorics of Nahm sums, quiver resultants and the K-theoretic condition. *JHEP* **03**, 236 (2021). [arXiv:2007.15398](https://arxiv.org/abs/2007.15398)

Communicated by S. Gukov