Title	Variations of Andrews-Beck type congruences
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Publication date	2021-03-15
Publication information	Chan, Song Heng, Renrong Mao, and Robert Osburn. "Variations of Andrews-Beck Type Congruences" 495, no. 2 (March 15, 2021).
Publisher	Elsevier
Item record/more information	http://hdl.handle.net/10197/25958
Publisher's statement	This is the author's version of a work that was accepted for publication in Journal of Mathematical Analysis and Applications. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Journal of Mathematical Analysis and Applications (495, 2, (2021) https://doi.org/10.1016/j.jmaa.2020.124771
Publisher's version (DOI)	10.1016/j.jmaa.2020.124771

VARIATIONS OF ANDREWS-BECK TYPE CONGRUENCES

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In memory of Freeman Dyson

Abstract. We prove three variations of recent results due to Andrews on congruences for $NT(m,k,n)$, the total number of parts in the partitions of n with rank congruent to m modulo k. We also conjecture new congruences and relations for $NT(m,k,n)$ and for a related crank-type function.

1. Introduction

A partition λ of a natural number n is a non-increasing sequence of positive integers whose sum is n. For example, the 5 partitions of 4 are

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1.$$

In 1944, Dyson [12] introduced the rank of a partition as the largest part $\ell(\lambda)$ minus the number of parts $n(\lambda)$ and, based on numerical evidence, conjectured that this statistic gives a combinatorial explanation of Ramanujan's congruences for the partition function modulo 5 and 7. In 1954, Atkin and Swinnerton-Dyer [5] confirmed Dyson's conjecture by proving explicit formulas for the generating function of rank differences. Recently, Andrews employed these rank differences to prove some intriguing congruences (conjectured by Beck) for $NT(m,k,n)$, the total number of parts in the partitions of n with rank congruent to m modulo k. Specifically, we have (see Theorems 1 and 2 in [3]) for all $n \in \mathbb{N}$

$$NT(1,5,5n+i) - NT(4,5,5n+i) + 2NT(2,5,5n+i) - 2NT(3,5,5n+i) \equiv 0 \pmod 5 \quad (1.1)$$

where $i = 1$ or 4 and

$$\begin{aligned} NT(1,7,7n+i) &- NT(6,7,7n+i) + NT(2,7,7n+i) - NT(5,7,7n+i) \\ &- NT(3,7,7n+i) + NT(4,7,7n+i) \equiv 0 \pmod 7 \quad (1.2) \end{aligned}$$

for $i = 1$ or 5.

It is now well-known that Dyson's rank is a special case of a general notion of rank which is defined on overpartition pairs [7]. Recall that an overpartition λ of n is a partition of n in which the first occurrence of a number may be overlined. For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1,$$
$$\overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

Date: November 5, 2020.

2010 *Mathematics Subject Classification.* 11P81, 05A17.

Key words and phrases. Andrews-Beck type congruences, rank for overpartition pairs, Dyson's rank for overpartitions, M_2-rank for overpartitions, M_2-rank for partitions without repeated odd parts.

An overpartition pair (λ, μ) of n is a pair of overpartitions where the sum of all of the parts is n. We order the parts of (λ, μ) by requiring that for a positive integer k,

$$\overline{k}_\lambda > k_\lambda > \overline{k}_\mu > k_\mu,$$

where the subscript indicates to which of the two overpartitions the part belongs. The rank of an overpartition pair (λ, μ) is

$$\ell((\lambda, \mu)) - n(\lambda) - \overline{n}(\mu) - \chi((\lambda, \mu)), \tag{1.3}$$

where $\overline{n}(\cdot)$ is the number of overlined parts only and $\chi((\lambda, \mu))$ is defined to be 1 if the largest part of (λ, μ) is non-overlined and in μ, and 0 otherwise. When μ is empty and λ has no overlined parts, (1.3) becomes the rank of a partition. To illustrate, the rank of the overpartition pair $((\overline{6}, 6, 5, 4, 4, 4, \overline{3}, \overline{1}), (7, 7, \overline{5}, 2, 2, 2))$ is $7-8-1-1 = -3$, while the rank of the overpartition pair $((4, \overline{3}, 3, \overline{2}, 1), (4, 4, 4, \overline{1}))$ is $4-5-1-0 = -2$. In addition to recovering Dyson's rank, three other special cases of (1.3) have turned out to be of significant interest: the rank of an overpartition [19], the M_2-rank of a partition without repeated odd parts [6], [21] and the M_2-rank of an overpartition [20]. We recall these cases now. First, Dyson's rank extends in an obvious way to overpartitions. Second, the M_2-rank of a partition λ without repeated odd parts is defined as

$$M_2\text{-rank}\,(\lambda) = \left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda).$$

Finally, the M_2-rank of an overpartition π is given by

$$M_2\text{-rank}\,(\pi) = \left\lceil \frac{\ell(\pi)}{2} \right\rceil - n(\pi) + n(\pi_o) - \chi(\pi)$$

where π_o is the subpartition consisting of the odd non-overlined parts and $\chi(\pi) = 1$ if the largest part of π is odd and non-overlined and $\chi(\pi) = 0$ otherwise.

The purpose of this paper is to prove that instances of (1.1) and (1.2) also occur in these three additional situations. Let $\overline{NT2}(b, k, n)$ denote the total number of parts in the overpartitions of n with M_2-rank congruent to b modulo k. Our first result is the following.

Theorem 1.1. *For all $n \in \mathbb{N}$, we have*

$$\overline{NT2}(1, 5, 5n+2) - \overline{NT2}(4, 5, 5n+2) + 2\overline{NT2}(2, 5, 5n+2) - 2\overline{NT2}(3, 5, 5n+2) \equiv 0 \pmod 5. \tag{1.4}$$

If we let $\overline{NT}(b, k, n)$ be the total number of parts in the overpartitions of n with rank congruent to b modulo k, then our second result is as follows.

Theorem 1.2. *For all $n \in \mathbb{N}$, we have*

$$\overline{NT}(1, 3, 3n) - \overline{NT}(2, 3, 3n) \equiv \overline{NT2}(1, 3, 3n) - \overline{NT2}(2, 3, 3n) \pmod 3 \tag{1.5}$$

and

$$\overline{NT}(1, 3, 3n+1) - \overline{NT}(2, 3, 3n+1) \equiv \overline{NT2}(1, 3, 3n+1) - \overline{NT2}(2, 3, 3n+1) \pmod 3. \tag{1.6}$$

Finally, if $NT2(b, k, n)$ is the total number of parts in the partitions of n without repeated odd parts with M_2-rank congruent to b modulo k, then our third result is the following.

Theorem 1.3. *For all $n \in \mathbb{N}$, we have*

$$NT2(1,5,5n+1) - NT2(4,5,5n+1) + 2NT2(2,5,5n+1) - 2NT2(3,5,5n+1) \equiv 0 \pmod 5. \tag{1.7}$$

The paper is organized as follows. In Section 2, we establish the generating function for the rank of an overpartition pair which also keeps track of the total number of parts. Upon appropriate specializations, this result leads to the generating functions for $\overline{NT2}(b,k,n)$, $\overline{NT}(b,k,n)$ and $NT2(b,k,n)$. We also record a key result necessary for the proof of Theorem 1.2. In Section 3, we prove Theorems 1.1–1.3. In Section 4, we make some concluding remarks concerning future directions.

2. Preliminaries

We first recall the standard q-hypergeometric notation

$$(a)_n = (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}),$$
$$(a_1,\ldots,a_m)_n = (a_1,\ldots,a_m;q)_n := (a_1)_n,\cdots(a_m)_n$$

and

$$[a_1,\ldots,a_m]_n = [a_1,\ldots,a_m;q]_n = (a_1,q/a_1,\ldots,a_m,q/a_m)_n,$$

valid for $n \in \mathbb{N}\cup\{\infty\}$.

Let $N(r,s,t,m,n)$ be the number of overpartition pairs (λ,μ) of n with rank m, such that r is the number of overlined parts in λ plus the number of non-overlined parts in μ, s is the number of parts in μ and t is the total number of parts in (λ,μ).

Lemma 2.1. *We have*

$$\mathcal{N}(d,e,x,z;q) := \sum_{\substack{r,s,t,n\geq 0\\ m\in\mathbb{Z}}} N(r,s,t,m,n)d^r e^s x^t z^m q^n = \sum_{n\geq 0}\frac{(-1/d,-1/e)_n(xdeq)^n}{(zq,xq/z)_n}. \tag{2.1}$$

Proof. We follow the proof of Proposition 2.1 in [7]. We split the overpartition pairs into four cases, depending on whether the largest part is overlined or not and whether it is in λ or μ to get four series. For example, the series

$$\sum_{n\geq 1}\frac{exq^n z^{n-1}(-qex/z,-qdx/z)_{n-1}}{(xq/z)_{n-1}(xedq)_n}$$

is the generating function for overpartition pairs whose largest part n is in μ and overlined, where the exponent of q is the number being partitioned, the exponent of z is the rank, the exponent of d is the number of overlined parts in λ plus the number of non-overlined parts in μ, the exponent of e is the number of parts in μ and the exponent of x is the total number of parts in (λ,μ). Combining this with the other three cases, we obtain

$$\sum_{\substack{r,s,t,n\geq 0\\ m\in\mathbb{Z}}} N(r,s,t,m,n)d^r e^s x^t z^m q^n$$

$$
\begin{aligned}
&= 1+\sum_{n\geq 1}\frac{exq^nz^{n-1}(-qex/z,-qdx/z)_{n-1}}{(xq/z)_{n-1}(xedq)_n}+\sum_{n\geq 1}\frac{dexq^nz^{n-1}(-qex/z,-qdx/z)_{n-1}}{(xq/z)_{n-1}(xedq)_n}\\
&\quad +\sum_{n\geq 1}\frac{dxq^nz^{n-1}(-qex/z)_n(-qdx/z)_{n-1}}{(xq/z,xedq)_n}+\sum_{n\geq 1}\frac{xq^nz^{n-1}(-qex/z)_n(-qdx/z)_{n-1}}{(xq/z,xedq)_n}\\
&= 1+(x+dx)\left(\sum_{n\geq 1}\frac{eq^nz^{n-1}(-qex/z,-qdx/z)_{n-1}}{(xq/z)_{n-1}(xedq)_n}+\sum_{n\geq 1}\frac{q^nz^{n-1}(-qex/z)_n(-qdx/z)_{n-1}}{(xq/z,xedq)_n}\right)\\
&= 1+(x+dx)\sum_{n\geq 1}\frac{q^nz^{n-1}(-qex/z,-qdx/z)_{n-1}}{(xq/z)_{n-1}(xedq)_n}\left(e+\frac{1+q^nex/z}{1-q^nx/z}\right)\\
&= 1+(x+dx)(1+e)\sum_{n\geq 1}\frac{q^nz^{n-1}(-qex/z,-qdx/z)_{n-1}}{(xq/z,xedq)_n}\\
&= 1+(x+dx)(1+e)\sum_{n\geq 0}\frac{q^{n+1}z^n(-qex/z,-qdx/z)_n}{(xq/z,xedq)_{n+1}}\\
&= 1+\frac{(x+dx)(1+e)q}{(1-xq/z)(1-xedq)}\sum_{n\geq 0}\frac{q^nz^n(-qex/z,-qdx/z)_n}{(xq^2/z,xedq^2)_n}. \qquad (2.2)
\end{aligned}
$$

Replacing (a,b,c,d,e) by $(q,-qdx/z,-qex/z,q^2x/z,deq^2x)$ in [15, Eq. (3.27)], we find that

$$\sum_{n\geq 0}\frac{q^nz^n(-qex/z,-qdx/z)_n}{(xq^2/z,xedq^2)_n}=\frac{(dexq,q^2z)_\infty}{(dexq^2,qz)_\infty}\sum_{n\geq 0}\frac{(dexq)^n(-q/d,-q/e)_n}{(xq^2/z,q^2z)_n}. \qquad (2.3)$$

Substituting (2.3) into (2.2), we obtain

$$
\begin{aligned}
&\sum_{\substack{r,s,t,n\geq 0\\ m\in\mathbb{Z}}} N(r,s,t,m,n)d^re^sx^tz^mq^n\\
&= 1+\frac{(x+dx)(1+e)q}{(1-xq/z)(1-xedq)}\frac{(dexq,q^2z)_\infty}{(dexq^2,qz)_\infty}\sum_{n\geq 0}\frac{(dexq)^n(-q/d,-q/e)_n}{(xq^2/z,q^2z)_n}\\
&= 1+\frac{(x+dx)(1+e)q}{dexq(1+1/d)(1+1/e)}\sum_{n\geq 0}\frac{(dexq)^{n+1}(-1/d,-1/e)_{n+1}}{(xq/z,qz)_{n+1}}\\
&= \sum_{n\geq 0}\frac{(dexq)^n(-1/d,-1/e)_n}{(xq/z,qz)_n}.
\end{aligned}
$$

□

Next, we prove the following result which generalizes [3, Theorem 3].

Proposition 2.2. *We have*

$$\sum_{n\geq 0}\frac{(-1/d,-1/e)_n(xdeq)^n}{(zq,xq/z)_n}$$

$$= 1 - \frac{(-xqd, -xqe)_\infty}{(xq, xqde)_\infty} \sum_{n\geq 1} \frac{(xq, -1/d, -1/e)_n q^{n(n+3)/2}(-dex)^n}{(q)_{n-1}(-xdq, -xeq)_n} \left(\frac{1}{q^n(1-zq^n)} + \frac{xz^{-1}}{1-xq^n/z} \right). \tag{2.4}$$

Proof. Recall the limiting case of [15, Eq. (2.5.1)]:

$$\sum_{n\geq 0} \frac{(aq/bc, d, e)_n (aq/de)^n}{(q, aq/b, aq/c)_n}$$
$$= \frac{(aq/d, aq/e)_\infty}{(aq, aq/de)_\infty} \sum_{n\geq 0} \frac{(a, \sqrt{a}q, -\sqrt{a}q, b, c, d, e)_n (-a^2q^2)^n q^{n(n-1)/2}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e)_n (bcde)^n}. \tag{2.5}$$

Replacing (a, b, c, d, e) by $(x, x/z, z, -1/d, -1/e)$ in (2.5), we have

$$\sum_{n\geq 0} \frac{(-1/d, -1/e)_n (xdeq)^n}{(zq, xq/z)_n}$$
$$= \frac{(-xqd, -xqe)_\infty}{(xq, xqde)_\infty} \sum_{n\geq 0} \frac{(x, \sqrt{x}q, -\sqrt{x}q, x/z, z, -1/d, -1/e)_n (-xdeq)^n q^{n(n+1)/2}}{(q, \sqrt{x}, -\sqrt{x}, qz, xq/z, -xqd, -xqe)_n}.$$

After simplifying, we find that

$$\sum_{n\geq 0} \frac{(-1/d, -1/e)_n (xdeq)^n}{(zq, xq/z)_n}$$
$$= \frac{(-xqd, -xqe)_\infty}{(xq, xqde)_\infty} \left\{ 1 + \sum_{n\geq 1} \frac{(xq)_{n-1} (-1/d, -1/e)_n (-xde)^n q^{n(n+3)/2} (1-z)(1-x/z)(1-xq^{2n})}{(q, -xdq, -xeq)_n (1-zq^n)(1-xq^n/z)} \right\}. \tag{2.6}$$

Noting that

$$\frac{(1-z)(1-x/z)(1-xq^{2n})}{(1-zq^n)(1-xq^n/z)}$$
$$= -(1-q^n)(1-xq^n) \left(\frac{1}{q^n(1-zq^n)} + \frac{xz^{-1}}{1-xq^n/z} \right) + \frac{1-xq^{2n}}{q^n},$$

we deduce from (2.6) that

$$\sum_{n\geq 0} \frac{(-1/d, -1/e)_n (xdeq)^n}{(zq, xq/z)_n}$$
$$= -\frac{(-xqd, -xqe)_\infty}{(xq, xqde)_\infty} \sum_{n\geq 1} \frac{(xq, -1/d, -1/e)_n (-xde)^n q^{n(n+3)/2}}{(q)_{n-1}(-xdq, -xeq)_n} \left(\frac{1}{q^n(1-zq^n)} + \frac{xz^{-1}}{1-xq^n/z} \right)$$
$$+ \frac{(-xqd, -xqe)_\infty}{(xq, xqde)_\infty} \left\{ 1 + \sum_{n\geq 1} \frac{(xq)_{n-1} (-1/d, -1/e)_n (-xde)^n q^{n(n+1)/2} (1-xq^{2n})}{(q, -xdq, -xeq)_n} \right\}. \tag{2.7}$$

Replacing (a,b,c,d) by $(x,-1/d,-1/e,\infty)$ in [15, Eq. (II.20), p.356], we obtain

$$1+\sum_{n\geq 1}\frac{(xq)_{n-1}\,(-1/d,-1/e)_n\,(-xde)^n q^{n(n+1)/2}(1-xq^{2n})}{(q,-xdq,-xeq)_n}=\frac{(xq,xqde)_\infty}{(-xqd,-xqe)_\infty},$$

which together with (2.7) gives (2.4). □

Let $\overline{NNT}(r,s,b,k,n)$ denote the total number of parts in the overpartition pairs (λ,μ) of n with rank congruent to b modulo k, such that r is the number of overlined parts in λ plus the number of non-overlined parts in μ, s is the number of parts in μ. Proceeding as in the proof of [3, Corollary 4], one can obtain the following generalization of [3, Corollary 4] by applying Proposition 2.2.

Corollary 2.3. *For $1\leq b\leq k-1$, we have*

$$\begin{aligned}&\sum_{r,s,n\geq 0}\left(\overline{NNT}(r,s,b,k,n)-\overline{NNT}(r,s,k-b,k,n)\right)d^r e^s q^n\\&=-\frac{\partial}{\partial x}\bigg|_{x=1}\frac{(-xqd,-xqe)_\infty}{(xq,xqde)_\infty}\sum_{n\geq 1}\frac{(xq,-1/d,-1/e)_n q^{n(n+3)/2}(-dex)^n}{(q)_{n-1}(-xdq,-xeq)_n}\\&\quad\cdot\left(\frac{q^{(b-1)n}-q^{n(k-b-1)}}{1-q^{kn}}+\frac{x^{k-b}q^{(k-1-b)n}-x^b q^{(b-1)n}}{1-x^k q^{kn}}\right).\end{aligned}$$

Finally, we prove a key Lemma required in the proof of Theorem 1.2.

Lemma 2.4. *We have*

$$\begin{aligned}\frac{[q^3;q^9]_\infty^3(q^9;q^9)_\infty^2}{[-q^3;q^9]_\infty^2(-q^9;q^9)_\infty^2}&=2\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+6n}}{1+q^{9n}}-2\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+12n+3}}{1+q^{9n+3}}\\&\quad+4\frac{(-q^9;q^9)_\infty^2}{[-q^3;q^9]_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+18n+9}}{1+q^{9n+6}}.\end{aligned}$$

Proof. Setting $r=1$ and $s=3$ in [9, Theorem 2.1], we see that

$$\begin{aligned}\frac{[a]_\infty(q)_\infty^2}{[b_1,b_2,b_3]_\infty}&=\frac{[a/b_1]_\infty}{[b_2/b_1,b_3/b_1]_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{n(n+1)}}{1-b_1q^n}\left(\frac{ab_1}{b_2b_3}\right)^n\\&\quad+\frac{[a/b_2]_\infty}{[b_1/b_2,b_3/b_2]_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{n(n+1)}}{1-b_2q^n}\left(\frac{ab_2}{b_1b_3}\right)^n\\&\quad+\frac{[a/b_3]_\infty}{[b_1/b_3,b_2/b_3]_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{n(n+1)}}{1-b_3q^n}\left(\frac{ab_3}{b_1b_2}\right)^n.\end{aligned}$$

Replacing q by q^9 and setting $(a,b_1,b_2,b_3)=(q^6,-1,-q^3,-q^6)$, we find that

$$\frac{[q^6;q^9]_\infty(q^9;q^9)_\infty^2}{[-1,-q^3,-q^6;q^9]_\infty}=\frac{[-q^6;q^9]_\infty}{[q^3,q^6;q^9]_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+6n}}{1+q^{9n}}$$

$$+\frac{[-q^3;q^9]_\infty}{[q^{-3},q^3;q^9]_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+12n}}{1+q^{9n+3}}$$
$$+\frac{[-1;q^9]_\infty}{[q^{-3},q^{-6};q^9]_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+18n}}{1+q^{9n+6}}.$$

Multiplying both sides by $2[q^3;q^9]_\infty^2/[-q^3;q^9]_\infty$ and simplifying completes the proof. □

3. Proofs of Theorems 1.1–1.3

We are now in a position to prove Theorems 1.1–1.3.

Proof of Theorem 1.1. We first consider the following special case of (2.1):

$$\mathcal{N}(1,1/q,x,z;q^2)=\sum_{n\geq 0}\frac{(-1,-q;q^2)_n(xq)^n}{(zq,xq/z;q^2)_n}.$$

By [20, Theorem 1.2], the coefficient of $x^tz^bq^n$ in

$$\sum_{n\geq 0}\frac{(-1,-q;q^2)_n(xq)^n}{(zq,xq/z;q^2)_n}$$

is equal to the number of overpartitions of n with t parts and M_2-rank b. By Corollary 2.3, we have for $1\leq b\leq k-1$

$$\sum_{n\geq 0}\left(\overline{NT2}(b,k,n)-\overline{NT2}(k-b,k,n)\right)q^n$$
$$=-\frac{\partial}{\partial x}\bigg|_{x=1}\frac{(-xq)_\infty}{(xq)_\infty}\sum_{n\geq 1}\frac{(xq^2,-1,-q;q^2)_nq^{n(n+2)}(-x)^n}{(q^2;q^2)_{n-1}(-xq^2,-xq;q^2)_n}$$
$$\cdot\left(\frac{q^{2(b-1)n}-q^{2(k-b-1)n}}{1-q^{2kn}}+\frac{x^{k-b}q^{2(k-1-b)n}-x^bq^{2(b-1)n}}{1-x^kq^{2kn}}\right). \tag{3.1}$$

Using (3.1) with $b=1$ and $k=5$, we obtain

$$\sum_{n\geq 0}\left(\overline{NT2}(1,5,n)-\overline{NT2}(4,5,n)\right)q^n$$
$$=-\frac{\partial}{\partial x}\bigg|_{x=1}\frac{(-xq)_\infty}{(xq)_\infty}\sum_{n\geq 1}\frac{(xq^2,-1,-q;q^2)_nq^{n(n+2)}(-x)^n}{(q^2;q^2)_{n-1}(-xq^2,-xq;q^2)_n}\left(\frac{1-q^{6n}}{1-q^{10n}}+\frac{x^4q^{6n}-x}{1-x^5q^{10n}}\right)$$
$$=-\frac{\partial}{\partial x}\bigg|_{x=1}\frac{(-xq)_\infty}{(xq)_\infty}\sum_{n\geq 1}\frac{(xq^2,-1,-q;q^2)_nq^{n(n+2)}(-x)^n}{(q^2;q^2)_{n-1}(-xq^2,-xq;q^2)_n}$$
$$\cdot\frac{(x-1)(q^{2n}-1)(xq^{2n}-1)(xq^{4n}-1)}{(1-q^{10n})(1-x^5q^{10n})}$$
$$\cdot\{1+q^{2n}+q^{4n}+q^{2n}x+2q^{4n}x+q^{6n}x+q^{4n}x^2+q^{6n}x^2+q^{8n}x^2\}.$$

Noting that

$$\frac{\partial}{\partial x}\bigg|_{x=1} (1-x)F(x,q,z) = -F(1,q,z), \tag{3.2}$$

we find

$$\begin{aligned}&\sum_{n\geq 0}\left(\overline{NT2}(1,5,n)-\overline{NT2}(4,5,n)\right)q^n\\&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{n(n+2)}(q^{2n}-1)^3(q^{4n}-1)}{(1+q^{2n})(1-q^{10n})^2}\{1+2q^{2n}+4q^{4n}+2q^{6n}+q^{8n}\}.\end{aligned} \tag{3.3}$$

Similarly, one can prove that

$$\begin{aligned}&\sum_{n\geq 0}\left(\overline{NT2}(2,5,n)-\overline{NT2}(3,5,n)\right)q^n\\&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{n(n+2)}(q^{2n}-1)^3(q^{4n}-1)}{(1+q^{2n})(1-q^{10n})^2}\{2q^{2n}+q^{4n}+2q^{6n}\}.\end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) give

$$\begin{aligned}&\sum_{n\geq 0}\left(\overline{NT2}(1,5,n)-\overline{NT2}(4,5,n)+2\overline{NT2}(2,5,n)-2\overline{NT2}(3,5,n)\right)q^n\\&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{n(n+2)}(q^{2n}-1)^3(q^{4n}-1)}{(1+q^{2n})(1-q^{10n})^2}\{1+6q^{2n}+6q^{4n}+6q^{6n}+q^{8n}\}\\&\equiv\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{n(n+2)}(q^{2n}-1)^2(1-q^{4n})}{(1+q^{2n})(1-q^{10n})}\pmod 5\\&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{n(n+2)}(1-q^{2n})^3}{1-q^{10n}}\\&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\frac{(-1)^nq^{n(n+2)}(1-3q^{2n})}{1-q^{10n}}\\&=\frac{2(-q)_\infty}{(q)_\infty}\left(\bar{S}_2(1)+3\bar{S}_2(3)\right),\end{aligned} \tag{3.5}$$

where

$$\bar{S}_2(b) := \sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\frac{(-1)^nq^{n^2+2bn}}{1-q^{10n}}.$$

By [22, Eq. (4.9)], we have

$$\sum_{n\geq 0}\left\{\overline{N}_2(1,5,n)-\overline{N}_2(2,5,n)\right\}q^n\frac{(q)_\infty}{2(-q)_\infty} = -\bar{S}_2(1)-3\bar{S}_2(3), \tag{3.6}$$

where $\overline{N}_2(b,k,n)$ denotes the number of overpartitions of n whose M_2-rank is congruent to b modulo k. Equations (3.5) and (3.6) imply

$$\begin{aligned}&\sum_{n\geq 0}\left(\overline{NT2}(1,5,n)-\overline{NT2}(4,5,n)+2\overline{NT2}(2,5,n)-2\overline{NT2}(3,5,n)\right)q^n\\&\equiv\sum_{n\geq 0}\left\{\overline{N}_2(2,5,n)-\overline{N}_2(1,5,n)\right\}q^n \pmod 5,\end{aligned}$$

which together with [22, Eq. (1.10)] yields (1.4). □

Proof of Theorem 1.2. Setting $e=0$ and $d=1$ in (2.1), we obtain

$$\mathcal{N}(1,0,x,z;q^2)=\sum_{n\geq 0}\frac{(-1)_n x^n q^{n(n+1)/2}}{(zq,xq/z)_n}.$$

By [19, Proposition 1.1], the coefficient of $x^t z^b q^n$ in

$$\sum_{n\geq 0}\frac{(-1)_n x^n q^{n(n+1)/2}}{(zq,xq/z)_n}$$

is equal to the number of overpartitions of n with t parts and rank b. By Corollary 2.3, we have for $1\leq b\leq k-1$

$$\begin{aligned}&\sum_{n\geq 0}\left(\overline{NT}(b,k,n)-\overline{NT}(k-b,k,n)\right)q^n\\&=-\left.\frac{\partial}{\partial x}\right|_{x=1}\frac{(-xq)_\infty}{(xq)_\infty}\sum_{n\geq 1}\frac{(xq,-1;q)_n q^{n(n+1)}(-x)^n}{(q;q)_{n-1}(-xq;q)_n}\\&\quad\cdot\left(\frac{q^{(b-1)n}-q^{(k-b-1)n}}{1-q^{kn}}+\frac{x^{k-b}q^{(k-1-b)n}-x^b q^{(b-1)n}}{1-x^k q^{kn}}\right).\end{aligned}\tag{3.7}$$

Proceeding as in the proof of Theorem 1.1, after applying (3.1), (3.2) and (3.7), we obtain

$$\begin{aligned}\sum_{n\geq 0}\left(\overline{NT}(1,3,n)-\overline{NT}(2,3,n)\right)q^n&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}(q^n-1)^4}{(1-q^{3n})^2}\\&\equiv\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}(1-q^n)}{1-q^{3n}} \pmod 3\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0}\left(\overline{NT2}(1,3,n)-\overline{NT2}(2,3,n)\right)q^n&=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+2n}(q^{2n}-1)^4}{(1-q^{6n})^2}\\&\equiv\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+2n}(1-q^{2n})}{1-q^{6n}} \pmod 3.\end{aligned}$$

Thus, we have

$$\sum_{n\geq 0}\left(\overline{NT}(1,3,n)-\overline{NT}(2,3,n)-\overline{NT2}(1,3,n)+\overline{NT2}(2,3,n)\right)q^n$$

$$\equiv \frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}(1-2q^n+2q^{3n}-q^{4n})}{1-q^{6n}}$$
$$\equiv \frac{2(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}(1+q^n)}{1+q^{3n}} \pmod 3. \tag{3.8}$$

Now, from [10, Eq. (2.1)], we have

$$\frac{(-q)_\infty}{(q)_\infty} = \frac{(q^{18};q^{18})_\infty^3}{[q^3;q^{18}]_\infty^8 (q^6;q^6)_\infty^4 [q^9;q^{18}]_\infty}\left(1+2q\frac{[q^3;q^{18}]_\infty}{[q^9;q^{18}]_\infty}+4q^2\frac{[q^3;q^{18}]_\infty^2}{[q^9;q^{18}]_\infty^2}\right)$$
$$= \frac{(q^{18};q^{18})_\infty^3}{[q^3;q^{18}]_\infty^8 (q^6;q^6)_\infty^4 [q^9;q^{18}]_\infty}\left(1+2q\frac{(-q^9;q^9)_\infty^2}{[-q^3;q^9]_\infty}+4q^2\frac{(-q^9;q^9)_\infty^4}{[-q^3;q^9]_\infty^2}\right). \tag{3.9}$$

Next, note that

$$\sum_{n\geq 1}\frac{(-1)^n q^{n^2+n}(1+q^n)}{1+q^{3n}} = -\frac{1}{2}+\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}$$
$$= -\frac{1}{2}+\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+6n}}{1+q^{9n}}-\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+12n+3}}{1+q^{9n+3}}+\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+18n+8}}{1+q^{9n+6}}. \tag{3.10}$$

Invoking (3.9) and (3.10) into (3.8) and collecting only terms where the power q is divisible by 3 yields

$$\frac{(q^{18};q^{18})_\infty^3}{[q^3;q^{18}]_\infty^8 (q^6;q^6)_\infty^4 [q^9;q^{18}]_\infty}\Bigg(-1+2\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+6n}}{1+q^{9n}}$$
$$-2\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+12n+3}}{1+q^{9n+3}}+4\frac{(-q^9;q^9)_\infty^2}{[-q^3;q^9]_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{9n^2+18n+9}}{1+q^{9n+6}}\Bigg).$$

Applying Lemma 2.4, the expression in parenthesis then becomes

$$-1+\frac{[q^3;q^9]_\infty^3 (q^9;q^9)_\infty^2}{[-q^3;q^9]_\infty^3(-q^9;q^9)_\infty^2} = -1+\frac{(q^3;q^3)_\infty^3(-q^9;q^9)_\infty}{(-q^3;q^3)_\infty^3(q^9;q^9)_\infty}$$
$$\equiv -1+1=0 \pmod 3.$$

This shows that the coefficients of q^{3n} in (3.8) are divisible by 3, which then implies (1.5). Similarly, congruence (1.6) can be proved in exactly the same way, by showing that the coefficients of q^{3n+1} in (3.8) are also divisible by 3. □

Proof of Theorem 1.3. Replacing (q,d,e) by $(q^2,0,1/q)$ in (2.1), we obtain

$$\mathcal{N}(0,1/q,x,z;q^2) = \sum_{n\geq 0}\frac{(-q;q^2)_n x^n q^{n^2}}{(zq,xq/z;q^2)_n}.$$

Noting that partitions without repeated odd parts correspond to overpartitions in which the odd parts are all overlined and even parts are all non-overlined, we deduce from [20, Theorem

1.2] that the coefficient of $x^tz^bq^n$ in

$$\sum_{n\geq 0}\frac{(-q;q^2)_nx^nq^{n^2}}{(zq,xq/z;q^2)_n}$$

is equal to the number of partitions without repeated odd parts of n with t parts and M_2-rank b. By Corollary 2.3, we have for $1\leq b\leq k-1$

$$\begin{aligned}&\sum_{n\geq 0}\left(NT2(b,k,n)-NT2(k-b,k,n)\right)q^n\\&=-\left.\frac{\partial}{\partial x}\right|_{x=1}\frac{(-xq;q^2)_\infty}{(xq^2;q^2)_\infty}\sum_{n\geq 1}\frac{(xq^2,-q;q^2)_nq^{2n^2+n}(-x)^n}{(q^2;q^2)_{n-1}(-xq;q^2)_n}\\&\quad\cdot\left(\frac{q^{2(b-1)n}-q^{2(k-b-1)n}}{1-q^{2nk}}+\frac{x^{k-b}q^{2(k-1-b)n}-x^bq^{2(b-1)n}}{1-x^kq^{2kn}}\right).\end{aligned}\tag{3.11}$$

Again, proceeding as in the proof of Theorem 1.1, after applying (3.2) and (3.11), we obtain

$$\begin{aligned}&\sum_{n\geq 0}\left(NT2(1,5,n)-NT2(4,5,n)\right)q^n\\&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{2n^2+n}(q^{2n}-1)^3(q^{4n}-1)}{(1-q^{10n})^2}\{1+2q^{2n}+4q^{4n}+2q^{6n}+q^{8n}\}\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}&\sum_{n\geq 0}\left(NT2(2,5,n)-NT2(3,5,n)\right)q^n\\&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{2n^2+n}(q^{2n}-1)^3(q^{4n}-1)}{(1-q^{10n})^2}\{2q^{2n}+q^{4n}+2q^{6n}\}.\end{aligned}\tag{3.13}$$

Equations (3.12) and (3.13) give

$$\begin{aligned}&\sum_{n\geq 0}\left(NT2(1,5,n)-NT2(4,5,n)+2NT2(2,5,n)-2NT2(3,5,n)\right)q^n\\&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{2n^2+n}(q^{2n}-1)^3(q^{4n}-1)}{(1-q^{10n})^2}\{1+6q^{2n}+6q^{4n}+6q^{6n}+q^{8n}\}\\&\equiv\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{2n^2+n}(q^{2n}-1)^2(1-q^{4n})}{(1-q^{10n})}\pmod 5\\&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 1}\frac{(-1)^nq^{2n^2+n}(1-2q^{2n}+2q^{6n}-q^{8n})}{1-q^{10n}}\\&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{\substack{n\in\mathbb{Z}\\n\neq 0}}\frac{(-1)^nq^{2n^2+n}(1-2q^{2n})}{1-q^{10n}}\end{aligned}$$

$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(S_2(1)-2S_2(3)\right), \tag{3.14}$$

where

$$S_2(b) := \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(-1)^n q^{2n^2+bn}}{1-q^{10n}}.$$

By [21, Eq. (5.7)], we have

$$\sum_{n\geq 0}\left\{N_2(1,5,n)-N_2(2,5,n)\right\} q^n \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = -S_2(1)+2S_2(3), \tag{3.15}$$

where $N_2(b,k,n)$ denotes the number of partitions without repeated odd parts of n whose M_2-rank is congruent to b modulo k. Equations (3.14) and (3.15) imply

$$\sum_{n\geq 0}\left(NT2(1,5,n)-NT2(4,5,n)+2NT2(2,5,n)-2NT2(3,5,n)\right)q^n$$
$$\equiv \sum_{n\geq 0}\left\{N_2(2,5,n)-N_2(1,5,n)\right\} q^n \pmod{5},$$

which together with [21, Eq. (1.7)] gives (1.7). □

4. Concluding Remarks

There are several directions for future study. First, Dyson also conjectured in [12] the existence of a partition statistic called the crank which would combinatorially explain Ramanujan's congruences for the partition function modulo 5, 7 and 11. In [4], this statistic was defined and Dyson's conjecture was proven. The crank of a partition is either the largest part, if 1 does not occur, or the difference between the number of parts larger than the number of 1's and the number of 1's, if 1 does occur. Let $M_\omega(m,k,n)$ denote the number of ones in the partitions of n with crank congruent to m modulo k. It appears that there are further congruences and relations for $NT(m,k,n)$ and $M_\omega(m,k,n)$. For example, using the techniques from [3], one can prove that for $i = 1, 3, 4, 5$,

$$NT(1,7,7n+i)-NT(6,7,7n+i)+2NT(3,7,7n+i)-2NT(4,7,7n+i)\equiv 0 \pmod{7} \tag{4.1}$$

and for $i = 0, 1, 5$,

$$NT(2,7,7n+i)-NT(5,7,7n+i)+4NT(3,7,7n+i)-4NT(4,7,7n+i)\equiv 0 \pmod{7}. \tag{4.2}$$

The details are left to the interested reader. We note that for $i = 1, 5$, (4.1) and (4.2) follow from [3, Theorem 2] and [11, Corollary 1.4]. In addition, we make the following

Conjecture 4.1. For all $n \in \mathbb{N}$, we have

$$\sum_{n\geq 0}(NT(1,7,7n+5)-NT(6,7,7n+5)+3NT(2,7,7n+5)-3NT(5,7,7n+5))q^n$$

$$= \frac{-7(q^7;q^7)_\infty^3(q^3,q^4;q^7)_\infty}{(q,q^6;q^7)_\infty(q^2,q^5;q^7)_\infty^2}, \quad (4.3)$$

$$\sum_{n\geq 0}(NT(1,7,7n+4)-NT(6,7,7n+4)+2NT(3,7,7n+4)-2NT(4,7,7n+4))q^n$$
$$= \frac{-7(q^7;q^7)_\infty^3(q^3,q^4;q^7)_\infty^2}{(q,q^6;q^7)_\infty(q^2,q^5;q^7)_\infty^3}, \quad (4.4)$$

$$\begin{aligned}&NT(1,11,11n+6)-NT(10,11,11n+6)+3NT(2,11,11n+6)-3NT(9,11,11n+6)\\&-4NT(3,11,11n+6)+4NT(8,11,11n+6)+3NT(4,11,11n+6)-3NT(7,11,11n+6)\\&+3NT(5,11,11n+6)-3NT(6,11,11n+6)\equiv 0 \pmod{11},\end{aligned} \quad (4.5)$$

$$\begin{aligned}&NT(1,11,11n+1)-NT(10,11,11n+1)-3NT(2,11,11n+1)+3NT(9,11,11n+1)\\&+5NT(3,11,11n+1)-5NT(8,11,11n+1)-2NT(4,11,11n+1)+2NT(7,11,11n+1)\\&+4NT(5,11,11n+1)-4NT(6,11,11n+1)\equiv 0 \pmod{11},\end{aligned} \quad (4.6)$$

$$\begin{aligned}&NT(1,13,13n+1)-NT(12,13,13n+1)+NT(2,13,13n+1)-NT(11,13,13n+1)\\&+6NT(3,13,13n+1)-6NT(10,13,13n+1)+3NT(6,13,13n+1)-3NT(7,13,13n+1)\\&\equiv 0 \pmod{13},\end{aligned} \quad (4.7)$$

$$\begin{aligned}&NT(1,13,13n+3)-NT(12,13,13n+3)+3NT(3,13,13n+3)-3NT(10,13,13n+3)\\&-4NT(4,13,13n+3)+4NT(9,13,13n+3)+NT(5,13,13n+3)-NT(8,13,13n+3)\\&-2NT(6,13,13n+3)+2NT(7,13,13n+3)\equiv 0 \pmod{13},\end{aligned} \quad (4.8)$$

$$M_\omega(1,5,5n+4)-M_\omega(4,5,5n+4)=2M_\omega(3,5,5n+4)-2M_\omega(2,5,5n+4), \quad (4.9)$$

$$M_\omega(1,5,5n+i)-M_\omega(4,5,5n+i)+2NT(2,5,5n+i)-2NT(3,5,5n+i)\equiv 0 \pmod 5 \quad (4.10)$$

for $i=0$, 4,

$$M_\omega(1,5,5n+2)-M_\omega(4,5,5n+2)=2NT(3,5,5n+2)-2NT(2,5,5n+2), \quad (4.11)$$

$$NT(1,5,5n+i)-NT(4,5,5n+i)+2M_\omega(2,5,5n+i)-2M_\omega(3,5,5n+i)\equiv 0 \pmod 5 \quad (4.12)$$

for $i=1$, 2,

$$\sum_{n\geq 0}\big(NT(1,5,5n+4)-NT(4,5,5n+4)+2M_{\omega}(2,5,5n+4)-2M_{\omega}(3,5,5n+4)\big)q^n = \frac{-5(q^5;q^5)_{\infty}^4}{(q)_{\infty}}, \tag{4.13}$$

$$M_{\omega}(1,5,5n+4)-M_{\omega}(4,5,5n+4)=4NT(4,5,5n+4)-4NT(1,5,5n+4), \tag{4.14}$$

$$M_{\omega}(1,7,7n+i)-M_{\omega}(6,7,7n+i)+2M_{\omega}(3,7,7n+i)-2M_{\omega}(4,7,7n+i)\equiv 0 \pmod 7 \tag{4.15}$$

for $i=0,\ 2,\ 5,\ 6$,

$$M_{\omega}(2,7,7n+i)-M_{\omega}(5,7,7n+i)-3M_{\omega}(3,7,7n+i)+3M_{\omega}(4,7,7n+i)\equiv 0 \pmod 7 \tag{4.16}$$

for $i=0,\ 1,\ 4,\ 5$.

Here, (4.9) implies [11, Corollary 1.5]. Second, do similar congruences and/or relations exist for total number of parts functions associated to other partitions statistics, for example, ranks of Durfee symbols [2], the first and second residual crank for overpartitions [8] and dth residual crank for overpartitions [1], the k-rank [13] or the M_d-rank for overpartitions [18], [24]? Finally, a mock modular perspective (such as in [14], [16], [17] or [23]) which explains the occurrences of (1.1), (1.2), (1.4)–(1.7) and (4.3)–(4.16) would be most welcome.

Acknowledgements

The second author was partially supported by the National Natural Science Foundation of China (Grant No. 12071331 and No. 11971341).

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