q-SERIES AND TAILS OF COLORED JONES POLYNOMIALS

PAUL BEIRNE AND ROBERT OSBURN

Abstract. We extend the table of Garoufalidis, Lê and Zagier concerning conjectural Rogers-Ramanujan type identities for tails of colored Jones polynomials to all alternating knots up to 10 crossings. We then prove these new identities using q-series techniques.

1. Introduction

The colored Jones polynomial $J_N(K;q)$ for a knot K is an important quantum invariant of knots. Here, we use the normalization $J_N(K;q) = 1$ for the unknot K, $J_1(K;q) = 1$ for all knots K and $J_2(K;q)$ is the Jones polynomial of K. The *tail* of $J_N(K;q)$ is a power series whose first N coefficients agree (up to a common sign) with the first N coefficients for $J_N(K;q)$ for all $N \geq 1$. If K is an alternating knot, then the tail exists and equals an explicit q-multisum $\Phi_K(q)$ (see [1], [3], [5]).

Recently, Garoufalidis and Lê (with Zagier) presented a table (see Table 6 in [5]) of 43 conjectural Rogers-Ramanujan type identities between the tails $\Phi_K(q)$ and products of theta functions and/or false theta functions. This table consisted of the following knots K: all alternating knots up to 8_4, the twist knots K_p, $p > 0$ or $p < 0$, the torus knots $T(2,p)$, $p > 0$, each of their mirror knots $-K$ and -8_5. For example, if we define for a positive integer b

$$h_b = h_b(q) = \sum_{n\in\mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd}, \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0 \end{cases}$$

and

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$, then

Date: August 30, 2016.
2010 *Mathematics Subject Classification.* Primary: 33D15; Secondary: 05A30, 57M25.
Key words and phrases. q-series identities, colored Jones polynomial, tails.

$$\Phi_{7_2}(q) = (q)_\infty^7 \sum_{a,b,c,d,e,f,g \geq 0} \frac{q^{3a^2+2a+b^2+bg+ac+ad+ae+af+ag+cd+de+ef+fg+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{b+g}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}} \stackrel{?}{=} h_6. \tag{1.1}$$

Note that $h_1 = 0$, $h_2 = 1$ and $h_3 = (q)_\infty$. In general, h_b is a theta function if b is odd and a false theta function if b is even. Using q-series techniques, Keilthy and the second author [10] proved not only (1.1), but all of the remaining conjectural identities in [5].

The purpose of this paper is to extend the table of Garoufalidis, Lê and Zagier to include all alternating knots up to 10 crossings. This is done in Tables 1 and 2 below. One immediately observes that their table is not "complete" in the sense that there exist knots K such that $\Phi_K(q) \neq \Phi_{K'}(q)$ for any knot K' in Table 6 of [5]. For example, $\Phi_{8_7}(q) = h_3h_5$. Our main result is the following.

Theorem 1.1. *The identities in Tables 1 and 2 are true.*

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
8_6	h_3h_4	h_5	9_6	h_3h_6	h_4	9_{24}	?	?
8_7	h_3h_5	h_3^2	9_7	h_3h_4	h_6	9_{25}	h_3^3	?
8_8	h_3h_5	h_3^2	9_8	h_3h_6	h_3^2	9_{26}	$h_3^2h_4$	h_3^3
8_9	h_3h_4	h_3h_4	9_9	h_4h_5	h_4	9_{27}	h_3^3	$h_3^2h_4$
8_{10}	?	h_3^2	9_{10}	h_4^2	h_5	9_{28}	?	?
8_{11}	h_3h_4	h_3h_4	9_{11}	h_4h_5	h_3^2	9_{29}	?	?
8_{12}	h_3h_4	h_3h_4	9_{12}	h_3h_4	h_3h_5	9_{30}	h_3^3	?
8_{13}	$h_3^2h_4$	h_3^2	9_{13}	h_4^2	h_3h_4	9_{31}	h_3^4	h_3^3
8_{14}	h_3h_4	h_3^3	9_{14}	$h_3^2h_5$	h_3^2	9_{32}	?	?
8_{15}	h_3^3	?	9_{15}	h_3h_4	h_3h_5	9_{33}	?	?
8_{16}	?	?	9_{16}	h_4	?	9_{34}	?	?
8_{17}	?	?	9_{17}	h_3^2	$h_3^2h_5$	9_{35}	?	h_3
8_{18}	?	?	9_{18}	h_3h_4	h_4^2	9_{36}	?	h_3^2
9_1	h_9	1	9_{19}	h_3h_5	h_3^3	9_{37}	h_3^3	?
9_2	h_8	h_3	9_{20}	h_3^2	$h_3h_4^2$	9_{38}	?	?
9_3	h_7	h_4	9_{21}	h_3h_4	$h_3^2h_4$	9_{39}	?	?
9_4	h_6	h_5	9_{22}	?	h_3^2	9_{40}	?	?
9_5	h_3	h_4h_6	9_{23}	h_4^2	h_3^3	9_{41}	?	?

TABLE 1.

Unfortunately, we were unable to find similar identities not only in each case labelled "?" in Tables 1 and 2, but for any alternating knot (or its mirror) from 10_{79} to 10_{123}. This is also the situation for 8_5 where although one has (after q-theoretic simplification or the methods in [8])

$$\Phi_{8_5}(q) = (q)_\infty^2 \sum_{a,b \geq 0} \frac{q^{a^2+a+b^2+b}(q)_{a+b}}{(q)_a^2(q)_b^2}, \tag{1.2}$$

K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$	K	$\Phi_K(q)$	$\Phi_{-K}(q)$
10_1	h_9	h_3	10_{27}	h_3h_5	$h_3^2h_4$	10_{53}	?	h_3^3
10_2	?	h_3	10_{28}	$h_3h_4h_5$	h_3^2	10_{54}	?	h_3^2
10_3	h_7	h_5	10_{29}	$h_3h_4^2$	h_3h_4	10_{55}	?	h_3^3
10_4	?	h_3	10_{30}	$h_3h_4^2$	h_3^3	10_{56}	?	h_3h_4
10_5	h_3h_7	h_3^2	10_{31}	h_3h_5	$h_3^2h_4$	10_{57}	?	$h_3^2h_4$
10_6	h_3h_6	h_5	10_{32}	?	h_3^3	10_{58}	?	h_3^3
10_7	h_3h_6	h_3h_4	10_{33}	?	$h_3^2h_4$	10_{59}	?	h_3^3
10_8	h_3	h_5h_6	10_{34}	h_3h_7	h_3^2	10_{60}	?	h_3^3
10_9	h_3h_6	h_3h_4	10_{35}	h_3h_6	h_3h_4	10_{61}	?	h_3
10_{10}	$h_3^2h_6$	h_3^2	10_{36}	h_3h_6	h_3^3	10_{62}	?	h_3^2
10_{11}	h_4h_5	h_5	10_{37}	h_3h_5	h_3h_5	10_{63}	?	h_3h_4
10_{12}	h_3h_5	h_3h_5	10_{38}	?	h_3^3	10_{64}	?	h_3h_4
10_{13}	h_4h_5	h_3h_4	10_{39}	h_3h_4	$h_3^2h_5$	10_{65}	?	$h_3^2h_4$
10_{14}	$h_3^2h_5$	h_3h_4	10_{40}	?	$h_3^2h_4$	10_{66}	?	?
10_{15}	h_5^2	h_3^2	10_{41}	$h_3h_4^2$	h_3^3	10_{67}	?	h_3^3
10_{16}	h_4h_5	h_3h_4	10_{42}	$h_3^2h_4$	?	10_{68}	?	h_3^2
10_{17}	?	h_3h_5	10_{43}	$h_3^2h_4$	$h_3^2h_4$	10_{69}	?	?
10_{18}	$h_3^2h_5$	h_3h_4	10_{44}	$h_3^3h_4$	h_3^4	10_{70}	?	h_3h_4
10_{19}	$h_3h_4h_5$	h_3^2	10_{45}	h_3^4	h_3^4	10_{71}	?	$h_3^2h_4$
10_{20}	h_7	h_3h_4	10_{46}	?	h_3	10_{72}	h_3h_4	?
10_{21}	h_3h_6	h_3h_4	10_{47}	?	h_3^2	10_{73}	?	$h_3^2h_4$
10_{22}	h_3h_4	h_4h_5	10_{48}	?	h_3h_5	10_{74}	?	h_3h_4
10_{23}	h_3h_5	$h_3^2h_4$	10_{49}	?	$h_3^2h_5$	10_{75}	?	?
10_{24}	h_4h_5	h_3h_4	10_{50}	?	h_3h_4	10_{76}	?	h_5
10_{25}	$h_3h_4^2$	h_3h_4	10_{51}	?	$h_3^2h_4$	10_{77}	?	h_3h_5
10_{26}	$h_3h_4^2$	h_3h_4	10_{52}	?	h_3^3	10_{78}	?	?

TABLE 2.

the modular (or false theta, mock/mixed mock, quantum modular) properties of the double sum in (1.2) are not clear. The difficulty in finding nice identities for these tails is due to the structure of their reduced Tait graphs (see [6]). Another approach to Theorem 1.1 is to utilize the skein-theoretic techniques in [2], [4] and [9]. It would be of considerable interest to investigate the connection between skein theory and q-series to gain a better understanding of these unknown cases and of a general framework.

It would also be desirable to study q-series identities in other settings which arise from knot theory. For example, the q-multisum $\Phi_K(q)$ occurs as the "0-limit" of $J_N(K;q)$ (see Theorem 2 in [5]). Garoufalidis and Lê have also obtained an explicit formula (see Theorem 3 in [5]) for the "1-limit" of $J_N(K;q)$. Finally, do tails exist (in some appropriate sense) for generalizations of $J_N(K;q)$ (see [7], [11]–[13])?

The paper is organized as follows. In Section 2, we recall the necessary background from [10]. In Section 3, we prove Theorem 1.1.

2. Preliminaries

We first recall six q-series identities (see (2.1)–(2.3), Lemma 2.1, (4.3) and the proof of (4.1) in [10]). Namely,

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{n(n-1)/2}}{(q)_n} = (t)_\infty, \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+An}}{(q)_n (q)_{n+A}} = \frac{1}{(q)_\infty} \tag{2.3}$$

for any integer A,

$$\sum_{m,n\geq 0} (-1)^n \frac{q^{m^2+m+mn+\frac{n(n+1)}{2}}}{(q)_m (q)_n} = h_4, \tag{2.4}$$

$$\sum_{l,m,n\geq 0} (-1)^{l+n} \frac{q^{\frac{3l(l+1)}{2}+m^2+m+\frac{n(n+1)}{2}+2lm+ln+mn}}{(q)_l (q)_m (q)_n} = h_5 \tag{2.5}$$

and

$$\sum_{a\geq 0} (-1)^{na} \frac{q^{\frac{na(a+1)}{2}-a+a\sum\limits_{k=1}^{n-1} c_k}}{(q)_a \prod\limits_{k=1}^{n-1} (q)_{a+c_k}} = \frac{1}{(q)_\infty} \sum_{i_1,\dots,i_{n-2}\geq 0} (-1)^{\sum\limits_{k=1}^{n-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{n-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)+\sum\limits_{k=2}^{n-1}\sum\limits_{j=1}^{k-1} c_k i_j}}{\prod\limits_{k=1}^{n-2} (q)_{i_k} \prod\limits_{k=1}^{n-2} (q)_{c_k+\sum\limits_{j=1}^{k} i_j}} \tag{2.6}$$

for any $n > 2$ and integers c_k.

Let K be an alternating knot with c crossings and $\mathcal{T}_K$ its associated Tait graph. The reduced Tait graph $\mathcal{T}_K'$ is obtained from $\mathcal{T}_K$ by replacing every set of two edges that connect the same two vertices by a single edge. The tail $\Phi_K(q)$ is given by

$$\Phi_K(q) = (q)_\infty^c S_K(q) \tag{2.7}$$

where $S_K(q)$ is an explicitly constructed q-multisum (see pages 261–264 in [10]). Now, by Theorem 2 in [2], if $\mathcal{T}_K'$ is the same as $\mathcal{T}_L'$ for two alternating knots K and L, then $\Phi_K(q) = \Phi_L(q)$. Thus, by comparing the reduced Tait graphs for those knots in Table 1 of [10] and Tables 1 and 2 above, it suffices to verify the conjectural identities in the following cases: 8_7, 8_{13}, -9_5, 9_{14}, -9_{17}, -9_{20}, -9_{27}, 9_{31}, 10_5, -10_8, 10_{10}, 10_{15}, 10_{19}, 10_{26}, 10_{28}, 10_{44}. Note that Corollary 2 in [5] is false as stated since $\mathcal{T}_{8_6}' \cong \mathcal{T}_{9_{24}}'$, but $\Phi_{8_6}(q) \neq \Phi_{9_{24}}(q)$.

The strategy for proving Theorem 1.1 is now as follows. For each of the 16 cases, we first compute $S_K(q)$ using the methods from [10]. We then employ (2.1)–(2.6) to reduce this q-multisum to (1.1) or one of the following key identities proven in [10]:

$$S_{5_1}(q) := \sum_{a,b,c,d,e\geq 0} (-1)^a \frac{q^{\frac{a(5a+3)}{2}+ab+ac+ad+ae+bc+cd+de+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{a+b}(q)_{a+c}(q)_{a+d}(q)_{a+e}} = \frac{1}{(q)_\infty^5} h_5, \tag{2.8}$$

$$S_{6_2}(q) := \sum_{a,b,c,d,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(3e+1)}{2}+ab+af+bc+bf+cd+ce+cf+de+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+e}(q)_{c+f}(q)_{d+e}} = \frac{1}{(q)_\infty^5} h_4, \tag{2.9}$$

$$\begin{aligned} S_{7_1}(q) &:= \sum_{a,b,c,d,e,f,g\geq 0} (-1)^a \frac{q^{\frac{a(7a+5)}{2}+ab+ac+ad+ae+af+ag+bc+cd+de+ef+fg+b+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+b}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}} \\ &= \frac{1}{(q)_\infty^7} h_7, \end{aligned} \tag{2.10}$$

$$\begin{aligned} S_{7_4}(q) &:= \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{2f^2+f+2g^2+g+ab+ag+bc+bg+cd+cf+cg+de+df+ef+a+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+g}(q)_{b+g}(q)_{c+f}(q)_{c+g}(q)_{d+f}(q)_{e+f}} \\ &= \frac{1}{(q)_\infty^7} h_4^2, \end{aligned} \tag{2.11}$$

$$\begin{aligned} S_{7_7}(q) &:= \sum_{a,b,c,d,e,f,g\geq 0} (-1)^{e+f+g} \frac{q^{\frac{3e^2}{2}+\frac{e}{2}+\frac{3f^2}{2}+\frac{f}{2}+\frac{3g^2}{2}+\frac{g}{2}+ab+ad+ae+af+bf+cd+cg+de+dg+a+b+c}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+e}(q)_{d+e}(q)_{a+f}(q)_{b+f}(q)_{c+g}} \\ &\times \frac{q^d}{(q)_{d+g}} \\ &= \frac{1}{(q)_\infty^4}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} S_{8_2}(q) &:= \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^b \frac{q^{3a^2+2a+\frac{b(3b+1)}{2}+ad+ae+af+ag+ah+bc+bd+cd+de+ef+fg+gh+c+d+e+f}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{b+c}(q)_{b+d}(q)_{a+d}(q)_{a+e}(q)_{a+f}} \\ &\times \frac{q^{g+h}}{(q)_{a+g}(q)_{a+h}} \\ &= \frac{1}{(q)_\infty^7} h_6 \end{aligned} \tag{2.13}$$

and

$$
\begin{aligned}
S_{-8_4}(q) :=& \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^g \frac{q^{\frac{g(5g+3)}{2}+h(2h+1)+ab+ah+bc+bh+cd+cg+ch+de+dg+ef+eg+fg+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+h}(q)_{b+h}(q)_{c+g}(q)_{c+h}(q)_{d+g}} \\
&\times \frac{q^{e+f}}{(q)_{e+g}(q)_{f+g}} \\
&= \frac{1}{(q)_\infty^8} h_4 h_5.
\end{aligned}
\tag{2.14}
$$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We give full details for 8_7, -9_5 and -10_8. As the remaining cases are handled similarly, we sketch their proofs. For $\Phi_{8_7}(q)$, it suffices to prove

$$
\begin{aligned}
S_{8_7}(q) :=& \sum_{a,b,c,d,e,g,h,i\geq 0} (-1)^{h+i} \frac{q^{\frac{i(5i+3)}{2}+\frac{h(3h+1)}{2}+g^2+ab+ag+ah+bc+bh+bi+cd+ci+de+di+ei+a+b+c}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_g(q)_h(q)_i(q)_{a+g}(q)_{a+h}(q)_{b+h}(q)_{b+i}(q)_{c+i}} \\
&\times \frac{q^{d+e}}{(q)_{d+i}(q)_{e+i}} \\
&= \frac{1}{(q)_\infty^7} h_5.
\end{aligned}
\tag{3.1}
$$

We now have

$$
\begin{aligned}
S_{8_7}(q) =& \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,h,i\geq 0} (-1)^{h+i} \frac{q^{\frac{i(5i+3)}{2}+\frac{h(3h+1)}{2}+ab+ah+bc+bh+bi+cd+ci+de+di+ei+a+b+c}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_h(q)_i(q)_{a+h}(q)_{b+h}(q)_{b+i}(q)_{c+i}(q)_{d+i}} \\
&\times \frac{q^{d+e}}{(q)_{e+i}} \\
&\text{(evaluate the } g\text{-sum with (2.3))} \\
&= \frac{1}{(q)_\infty^2} \sum_{a,b,c,d,e,h,i\geq 0} (-1)^{h+i} \frac{q^{\frac{i(5i+3)}{2}+\frac{h(h+1)}{2}+ab+ah+bc+bi+cd+ci+de+di+ei+a+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_h(q)_i(q)_{b+h}(q)_{b+i}(q)_{c+i}(q)_{d+i}(q)_{e+i}} \\
&\text{(apply (2.6) to the } h\text{-sum with } n=3) \\
&= \frac{1}{(q)_\infty^2} \sum_{b,c,d,e,i\geq 0} (-1)^{i} \frac{q^{\frac{i(5i+3)}{2}+bc+bi+cd+ci+de+di+ei+b+c+d+e}}{(q)_b(q)_c(q)_d(q)_e(q)_i(q)_{b+i}(q)_{c+i}(q)_{d+i}(q)_{e+i}} \\
&\text{(evaluate the } a\text{-sum with (2.1), simplify, then use (2.2) for the } h\text{-sum).}
\end{aligned}
$$

Thus, (3.1) then follows from (2.8) after letting $i \to a$.

For $\Phi_{8_{13}}(q)$, it suffices to prove

$$\begin{aligned} S_{8_{13}}(q) :=& \sum_{a,c,d,e,f,g,h,i\geq 0} (-1)^{g+h} \frac{q^{\frac{g(3g+1)}{2}+\frac{(3h+1)}{2}+i(2i+1)+af+ag+ci+cd+de+di+ef+eh+ei}}{(q)_a(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_{a+g}(q)_{c+i}(q)_{d+i}(q)_{e+i}(q)_{e+h}} \\ &\times \frac{q^{fh+fg+a+c+d+e+f}}{(q)_{f+h}(q)_{f+g}} \\ &= \frac{1}{(q)_\infty^6} h_4. \end{aligned} \tag{3.2}$$

Apply (2.6) with $n = 3$ to the g-sum, (2.1) to the a-sum, then simplify and (2.2) to the g-sum to obtain

$$S_{8_{13}}(q) = \frac{1}{(q)_\infty} \sum_{c,d,e,f,h,i\geq 0} (-1)^h \frac{q^{\frac{h(3h+1)}{2}+i(2i+1)+ci+cd+de+di+ef+eh+ei+fh+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_h(q)_i(q)_{c+i}(q)_{d+i}(q)_{e+i}(q)_{e+h}(q)_{f+h}}.$$

Thus, (3.2) then follows from (2.9) upon $(c, d, e, f, h, i) \to (a, b, c, d, e, f)$.

For $\Phi_{-9_5}(q)$, it suffices to prove

$$\begin{aligned} S_{-9_5}(q) :=& \sum_{a,b,c,d,e,f,g,h,j\geq 0} \frac{q^{h(2h+1)+j(3j+2)+ab+ag+ah+aj+bc+bh+ch+de+dj+ef+ej+fg+fj+gj+a+b+c}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_j(q)_{a+h}(q)_{a+j}(q)_{b+h}(q)_{c+h}(q)_{d+j}} \\ &\times \frac{q^{d+e+f+g}}{(q)_{e+j}(q)_{f+j}(q)_{g+j}} \\ &= \frac{1}{(q)_\infty^9} h_4 h_6. \end{aligned} \tag{3.3}$$

We now have

$$\begin{aligned} S_{-9_5}(q) =& \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f,g,j,s,t\geq 0} \frac{q^{s^2+s+st+\frac{t(t+1)}{2}+bs+c(s+t)+j(3j+2)+ab+ag+aj+bc+de+dj+ef+ej+fg}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_j(q)_s(q)_t(q)_{a+j}(q)_{d+j}(q)_{s+a}} \\ &\times \frac{q^{fj+gj+a+b+c+d+e+f+g}}{(q)_{s+t+b}(q)_{e+j}(q)_{f+j}(q)_{g+j}} \\ &\text{(apply (2.6) to the } h\text{-sum with } n = 4) \\ &= \frac{1}{(q)_\infty^2} \sum_{a,b,d,e,f,g,j,s,t\geq 0} \frac{q^{s^2+s+st+\frac{t(t+1)}{2}+bs+j(3j+2)+ab+ag+aj+de+dj+ef+ej+fg+fj+gj+a+b+d+e}}{(q)_a(q)_b(q)_d(q)_e(q)_f(q)_g(q)_j(q)_s(q)_t(q)_{a+j}(q)_{d+j}(q)_{e+j}(q)_{f+j}(q)_{g+j}} \\ &\times \frac{q^{f+g}}{(q)_{s+a}} \\ &\text{(evaluate the } c\text{-sum with (2.1) and simplify)} \end{aligned}$$

$$= \frac{1}{(q)_\infty^3} h_4 \sum_{a,d,e,g,j\geq 0} \frac{q^{j(3j+2)+ag+aj+de+dj+ef+ej+fg+fj+gj+a+d+e+f+g}}{(q)_a(q)_d(q)_e(q)_f(q)_f(q)_g(q)_j(q)_{a+j}(q)_{d+j}(q)_{e+j}(q)_{f+j}(q)_{g+j}}$$

(evaluate the b-sum with (2.1), simplify, then apply (2.4) to the st-sum).

Now, (3.3) follows from first applying (2.3) the b-sum in (1.1), then letting $(a, d, e, f, g, j) \to (c, g, f, e, d, a)$.

For $\Phi_{9_{14}}(q)$, it suffices to prove

$$\begin{aligned} S_{9_{14}}(q) := &\sum_{a,b,c,d,e,g,h,i,j\geq 0} (-1)^{h+i+j} \frac{q^{\frac{h(3h+1)}{2}+\frac{i(3i+1)}{2}+\frac{j(5j+3)}{2}+ab+ag+ah+ai+bc+bi+bj+cd+cj+de+dj+ej}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_g(q)_h(q)_i(q)_j(q)_{a+h}(q)_{a+i}(q)_{b+i}(q)_{b+j}} \\ &\times \frac{q^{gh+a+b+c+d+e+g}}{(q)_{c+j}(q)_{d+j}(q)_{e+j}(q)_{g+h}} \\ &= \frac{1}{(q)_\infty^7} h_5. \end{aligned} \tag{3.4}$$

First, apply (2.6) with $n = 3$ to the h-sum, (2.1) to the g-sum, simplify and (2.2) to the h-sum, then (2.6) with $n = 3$ to the i-sum, (2.1) to the a-sum, simplify and (2.2) to the i-sum to obtain

$$S_{9_{14}}(q) = \frac{1}{(q)_\infty^2} \sum_{b,c,d,e,j} (-1)^j \frac{q^{\frac{j(5j+3)}{2}+bc+bj+cd+cj+de+dj+ej+b+c+d+e}}{(q)_b(q)_c(q)_d(q)_e(q)_j(q)_{b+j}(q)_{c+j}(q)_{d+j}(q)_{e+j}}.$$

Thus, (3.4) follows from (2.8) after $j \to a$.

For $\Phi_{-9_{17}}(q)$, it suffices to prove

$$\begin{aligned} S_{-9_{17}}(q) := &\sum_{a,b,c,d,e,f,h,i,j\geq 0} (-1)^{h+i+j} \frac{q^{\frac{h(3h+1)}{2}+\frac{i(5i+3)}{2}+\frac{j(3j+1)}{2}+ab+aj+bc+bi+bj+cd+ci+de+di+ef}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_h(q)_i(q)_j(q)_{a+j}(q)_{b+i}(q)_{b+j}(q)_{c+i}} \\ &\times \frac{q^{eh+ei+fh+a+b+c+d+e+f}}{(q)_{d+i}(q)_{e+h}(q)_{e+i}(q)_{f+h}} \\ &= \frac{1}{(q)_\infty^7} h_5. \end{aligned} \tag{3.5}$$

First, apply (2.6) with $n = 3$ to the h-sum, (2.1) to the f-sum, simplify and (2.3) to the h-sum, then (2.6) with $n = 3$ to the j-sum, (2.1) to the a-sum, simplify and (2.3) to the j-sum to get

$$S_{-9_{17}}(q) = \frac{1}{(q)_\infty^2} \sum_{b,c,d,e,i\geq 0} (-1)^i \frac{q^{\frac{i(5i+3)}{2}+bc+bi+cd+ci+de+di+ei+b+c+d+e}}{(q)_b(q)_c(q)_d(q)_e(q)_i(q)_{b+i}(q)_{c+i}(q)_{d+i}(q)_{e+i}}.$$

Thus, (3.5) follows from (2.8) after $i \to a$.

For $\Phi_{-9_{20}}(q)$, it suffices to prove

$$
\begin{aligned}
S_{-9_{20}}(q) :=& \sum_{a,b,c,d,e,f,h,i,j\geq 0} (-1)^h \frac{q^{\frac{h(3h+1)}{2}+i(2i+1)+j(2j+1)+ab+ah+bc+bh+bi+cd+ci+de+di+dj+ef+ej}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_h(q)_i(q)_j(q)_{a+h}(q)_{b+h}(q)_{b+i}(q)_{c+i}} \\
&\times \frac{q^{fj+a+b+c+d+e+f}}{(q)_{d+i}(q)_{d+j}(q)_{e+j}(q)_{f+j}} \\
&= \frac{1}{(q)_\infty^8} h_4^2.
\end{aligned}
\tag{3.6}
$$

Apply (2.6) with $n = 3$ to the h-sum, (2.1) to the a-sum and simplify, then (2.2) to the h-sum to obtain

$$
S_{-9_{20}}(q) = \frac{1}{(q)_\infty} \sum_{b,c,d,e,f,i,j\geq 0} \frac{q^{i(2i+1)+j(2j+1)+bc+bi+cd+ci+de+di+dj+ef+ej+fj+b+c+d+e+f}}{(q)_b(q)_c(q)_d(q)_e(q)_f(q)_i(q)_j(q)_{b+i}(q)_{c+i}(q)_{d+i}(q)_{d+j}(q)_{e+j}(q)_{f+j}}.
$$

Now, (3.6) follows from (2.11) after the substitution $(b, c, d, e, f, i, j) \to (a, b, c, d, e, g, f)$.

For $\Phi_{-9_{27}}(q)$, it suffices to prove

$$
\begin{aligned}
S_{-9_{27}}(q) :=& \sum_{a,b,c,d,e,f,g,h,i\geq 0} (-1)^{f+h} \frac{q^{\frac{f(3f+1)}{2}+g(2g+1)+\frac{h(3h+1)}{2}+i^2+ab+af+bc+bf+bg+cd+cg+de+dg+dh}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_{a+f}(q)_{b+f}(q)_{b+g}(q)_{c+g}} \\
&\times \frac{q^{eh+ei+a+b+c+d+e}}{(q)_{d+g}(q)_{d+h}(q)_{e+h}(q)_{e+i}} \\
&= \frac{1}{(q)_\infty^7} h_4.
\end{aligned}
\tag{3.7}
$$

Apply (2.3) to the i-sum, (2.6) with $n = 3$ to the f-sum, (2.1) to the a-sum, simplify and (2.2) to the f-sum to obtain

$$
S_{-9_{27}} = \frac{1}{(q)_\infty^2} \sum_{b,c,d,e,g,h\geq 0} (-1)^h \frac{q^{g(2g+1)+\frac{h(3h+1)}{2}+bc+bg+cd+cg+de+dg+dh+eh+b+c+d+e}}{(q)_b(q)_c(q)_d(q)_e(q)_g(q)_h(q)_{b+g}(q)_{c+g}(q)_{d+g}(q)_{d+h}(q)_{e+h}}.
$$

Now, (3.7) follows from (2.9) after letting $(b, c, d, e, g, h) \to (a, b, c, d, f, e)$.

For $\Phi_{9_{31}}(q)$, it suffices to prove

$$
\begin{aligned}
S_{9_{31}}(q) &:= \sum_{a,b,c,e,f,g,h,i,j\geq 0} (-1)^{g+h+i+j} \frac{q^{\frac{g(3g+1)}{2}+\frac{h(3h+1)}{2}+\frac{i(3i+1)}{2}+\frac{j(3j+1)}{2}+ab+af+ag+aj+bc+bg+bh}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_{a+g}(q)_{a+j}(q)_{b+g}} \\
&\times \frac{q^{ch+ef+ei+fi+fj+a+b+c+e+f}}{(q)_{b+h}(q)_{c+h}(q)_{e+i}(q)_{f+i}(q)_{f+j}} \\
&= \frac{1}{(q)_\infty^5}.
\end{aligned}
\tag{3.8}
$$

Apply (2.6) with $n = 3$ to the h-sum, (2.1) to the c-sum, simplify and (2.2) to the h-sum to obtain

$$
\begin{aligned}
S_{9_{31}}(q) = \frac{1}{(q)_\infty} &\sum_{a,b,e,f,g,i,j\geq 0} (-1)^{g+i+j} \frac{q^{\frac{g(3g+1)}{2}+\frac{i(3i+1)}{2}+\frac{j(3j+1)}{2}+ab+af+ag+aj+bg+ef+ei+fi+fj+a}}{(q)_a(q)_b(q)_e(q)_f(q)_g(q)_i(q)_j(q)_{a+g}(q)_{a+j}(q)_{b+g}(q)_{e+i}(q)_{f+i}} \\
&\times \frac{q^{b+e+f}}{(q)_{f+j}}.
\end{aligned}
$$

Now, (3.8) follows from (2.12) after letting $(a, b, e, f, g, i, j) \to (a, b, c, d, f, g, e)$.

For $\Phi_{10_5}(q)$, it suffices to prove

$$
\begin{aligned}
S_{10_5}(q) &:= \sum_{a,b,c,d,e,f,g,i,j,k\geq 0} (-1)^{j+k} \frac{q^{\frac{j(3j+1)}{2}+\frac{k(7k+5)}{2}+i^2+ab+ai+aj+bc+bj+bk+cd+ck+de+dk+ef+ek+fg}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_i(q)_j(q)_k(q)_{a+i}(q)_{a+j}(q)_{b+j}} \\
&\times \frac{q^{fk+gk+a+b+c+d+e+f+g}}{(q)_{b+k}(q)_{c+k}(q)_{d+k}(q)_{e+k}(q)_{f+k}(q)_{g+k}} \\
&= \frac{1}{(q)_\infty^9} h_7.
\end{aligned}
\tag{3.9}
$$

Apply (2.3) to the i-sum, (2.6) with $n = 3$ to the j-sum, (2.1) to the a-sum and simplify, then (2.2) to the j-sum to obtain

$$
\begin{aligned}
S_{10_5}(q) = \frac{1}{(q)_\infty^2} &\sum_{b,c,d,e,f,g,k\geq 0} (-1)^k \frac{q^{\frac{k(7k+5)}{2}+bc+bk+cd+ck+de+dk+ef+ek+fg+fk+gk+b+c+d+e+f}}{(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_k(q)_{b+k}(q)_{c+k}(q)_{d+k}(q)_{e+k}(q)_{f+k}} \\
&\times \frac{q^g}{(q)_{g+k}}.
\end{aligned}
$$

Now, (3.9) follows from (2.10) after letting $k \to a$.

For $\Phi_{-10_8}(q)$, it suffices to prove

$$S_{-10_8}(q) := \sum_{a,b,c,d,e,f,g,h,i,k\geq 0} (-1)^i \frac{q^{\frac{i(5i+3)}{2}+k(3k+2)+ab+ae+ai+ak+bc+bi+cd+ci+di+ef+ek+fg+fk+gh}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_k(q)_{a+i}(q)_{a+k}(q)_{b+i}}$$
$$\times \frac{q^{gk+hk+a+b+c+d+e+f+g+h}}{(q)_{c+i}(q)_{d+i}(q)_{e+k}(q)_{f+k}(q)_{g+k}(q)_{h+k}}$$
$$= \frac{1}{(q)_\infty^{10}} h_5 h_6. \tag{3.10}$$

We now have

$$S_{-10_8}(q) = \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f,g,h,i,k,j,l\geq 0} (-1)^{i+l} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{l(l+1)}{2}+2ij+il+jl+k(3k+2)+ab+ae+ak+bc+bi}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_l}$$
$$\times \frac{q^{cd+c(i+j)+d(i+j+l)+ef+ek+fg+fk+gh+gk+hk+a+b+c+d+e+f+g+h}}{(q)_{a+i}(q)_{a+k}(q)_{e+k}(q)_{f+k}(q)_{g+k}(q)_{h+k}(q)_{b+i+j}(q)_{c+i+j+l}}$$

(apply (2.6) to the i-sum with $n = 5$)

$$= \frac{1}{(q)_\infty^4} \sum_{a,e,f,g,h,i,k,j,l\geq 0} (-1)^{i+l} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{l(l+1)}{2}+2ij+il+jl+k(3k+2)+ae+ak+ef+ek+fg+fk+gh+hk}}{(q)_a(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_l(q)_{a+k}(q)_{e+k}(q)_{f+k}}$$
$$\times \frac{q^{a+e+f+g+h}}{(q)_{g+k}(q)_{h+k}}$$

(evaluate the d-sum, c-sum and b-sum with (2.1) and simplify)

$$= \frac{1}{(q)_\infty^4} h_5 \sum_{a,e,f,g,h,k\geq 0} \frac{q^{k(3k+2)+ak+ek+fk+gk+hk+ae+ef+fg+gh+a+e+f+g+h}}{(q)_a(q)_e(q)_f(q)_g(q)_h(q)_k(q)_{a+k}(q)_{e+k}(q)_{f+k}(q)_{g+k}(q)_{h+k}}$$

(evaluate the ijl-sum using (2.5)).

Now, (3.10) follows from (1.1) after applying $(a, e, f, g, h, k) \to (c, d, e, f, g, a)$.

For $\Phi_{10_{10}}(q)$, it suffices to prove

$$S_{10_{10}}(q) := \sum_{a,c,d,e,f,g,h,i,j,k\geq 0} (-1)^{i+j} \frac{q^{\frac{i(3i+1)}{2}+\frac{j(3j+1)}{2}+k(3k+2)+ah+ai+cd+ck+de+dk+ef+ek+fg+fk+gh}}{(q)_a(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_{a+i}(q)_{c+k}(q)_{d+k}}$$
$$\times \frac{q^{gj+gk+hi+hj+a+c+d+e+f+g+h}}{(q)_{e+k}(q)_{f+k}(q)_{g+k}(q)_{g+j}(q)_{h+j}(q)_{h+i}}$$
$$= \frac{1}{(q)_\infty^8} h_6. \tag{3.11}$$

Apply (2.6) with $n = 3$ to the i-sum, (2.1) to the a-sum and simplify, (2.2) to the i and simplify to obtain

$$S_{10_{10}}(q) = \frac{1}{(q)_\infty} \sum_{c,d,e,f,g,h,j,k\geq 0} (-1)^j \frac{q^{\frac{j(3j+1)}{2}+k(3k+2)+cd+ck+de+dk+ef+ek+fg+fk+gh+gj+gk+hj+c}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_j(q)_k(q)_{c+k}(q)_{d+k}(q)_{e+k}(q)_{f+k}} \times \frac{q^{d+e+f+g+h}}{(q)_{g+k}(q)_{g+j}(q)_{h+j}}.$$

Now, (3.11) follows from (2.13) after letting $(c,d,e,f,g,h,j,k) \to (h,g,f,e,d,c,b,a)$.

For $\Phi_{10_{15}}(q)$, it suffices to prove

$$S_{10_{15}}(q) := \sum_{a,b,c,d,e,g,h,i,j,k\geq 0} (-1)^{i+j} \frac{q^{\frac{i(5i+3)}{2}+\frac{j(5j+3)}{2}+k^2+ab+ah+ai+bc+bi+bj+cd+cj+de+dj+ej+gh+gi}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_g(q)_h(q)_i(q)_j(q)_k(q)_{a+i}(q)_{b+i}(q)_{b+j}} \times \frac{q^{gk+hi+a+b+c+d+e+g+h}}{(q)_{c+j}(q)_{d+j}(q)_{e+j}(q)_{g+i}(q)_{g+k}(q)_{h+i}} = \frac{1}{(q)_\infty^{10}} h_5^2. \tag{3.12}$$

Apply (2.3) to the k-sum, (2.6) with $n=5$ to the j-sum, (2.1) to the e-sum and simplify, to the d-sum and simplify and to the c-sum and simplify and (2.5) to obtain

$$S_{10_{15}}(q) = \frac{1}{(q)_\infty^5} h_5 \sum_{a,b,g,h,i\geq 0} (-1)^i \frac{q^{\frac{i(5i+3)}{2}+ab+ah+ai+bi+gh+gi+hi+a+b+g+h}}{(q)_a(q)_b(q)_g(q)_h(q)_i(q)_{a+i}(q)_{b+i}(q)_{g+i}(q)_{h+i}}.$$

Now, (3.12) follows from (2.8) after letting $(a,b,g,h,i) \to (c,b,e,d,a)$.

For $\Phi_{10_{19}}(q)$, it suffices to prove

$$S_{10_{19}}(q) := \sum_{a,c,d,e,f,g,h,i,j,k\geq 0} (-1)^{j+k} \frac{q^{i(2i+1)+\frac{j(3j+1)}{2}+\frac{k(5k+3)}{2}+ah+ai+cd+ck+de+dek+ef+ek+fg+fk}}{(q)_a(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_{a+i}(q)_{c+k}(q)_{d+k}} \times \frac{q^{fj+gh+gi+gj+hi+a+c+d+e+f+g+h}}{(q)_{e+k}(q)_{f+k}(q)_{f+j}(q)_{g+j}(q)_{g+i}(q)_{h+i}} = \frac{1}{(q)_\infty^9} h_4 h_5. \tag{3.13}$$

Apply (2.6) with $n=5$ to the k-sum, (2.1) to the c-sum and simplify, to the d-sum and simplify and to the e-sum and simplify and (2.5) to obtain

$$S_{10_{19}}(q) = \frac{1}{(q)_\infty^4} \sum_{a,f,g,h,i,j\geq 0} (-1)^j \frac{q^{i(2i+1)+\frac{j(3j+1)}{2}+ah+ai+fg+fj+gh+gi+gj+hi+a+f+g+h}}{(q)_a(q)_f(q)_g(q)_h(q)_i(q)_j(q)_{a+i}(q)_{f+j}(q)_{g+j}(q)_{g+i}(q)_{h+i}}.$$

Now, (3.13) follows from (2.9) after letting $(a,f,g,h,i,j) \to (a,d,c,b,f,e)$.

For $\Phi_{10_{26}}(q)$, it suffices to prove

$$
\begin{aligned}
S_{10_{26}}(q) :=& \sum_{a,b,c,e,f,g,h,i,j,k\geq 0} (-1)^i \frac{q^{h(2h+1)+\frac{i(3i+1)}{2}+j^2+k(2k+1)+ab+ag+ah+ai+bc+bh+ch+ef+ek+fg}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_{a+h}(q)_{a+i}(q)_{b+h}} \\
&\times \frac{q^{fk+gi+gj+gk+a+b+c+e+f+g}}{(q)_{c+h}(q)_{e+k}(q)_{f+k}(q)_{g+i}(q)_{g+j}(q)_{g+k}} \\
&= \frac{1}{(q)_\infty^9} h_4^2.
\end{aligned}
\tag{3.14}
$$

Apply (2.3) to the j-sum, (2.6) with $n = 4$ to the k-sum, (2.1) to the e-sum and simplify and to the f-sum and simplify and (2.4) to obtain

$$
S_{10_{26}}(q) = \frac{1}{(q)_\infty^4} h_4 \sum_{a,b,c,g,h,i\geq 0} (-1)^i \frac{q^{h(2h+1)+\frac{i(3i+1)}{2}+ab+ag+ah+ai+bc+bh+ch+gi+a+b+c+g}}{(q)_a(q)_b(q)_c(q)_g(q)_h(q)_i(q)_{a+h}(q)_{a+i}(q)_{b+h}(q)_{c+h}(q)_{g+i}}.
$$

Now, (3.14) follows from (2.9) after letting $(a, b, c, g, h, i) \to (c, b, a, d, f, e)$.

For $\Phi_{10_{28}}(q)$, it suffices to prove

$$
\begin{aligned}
S_{10_{28}}(q) :=& \sum_{a,b,d,e,f,g,h,i,j,k\geq 0} (-1)^{i+j} \frac{q^{\frac{i(3i+1)}{2}+\frac{j(5j+3)}{2}+k(2k+1)+ab+ah+ai+aj+bi+de+dk+ef+ek+fg+fj}}{(q)_a(q)_b(q)_d(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_{a+i}(q)_{a+j}(q)_{b+i}} \\
&\times \frac{q^{fk+gh+gj+hj+a+b+d+e+f+g+h}}{(q)_{d+k}(q)_{e+k}(q)_{f+j}(q)_{f+k}(q)_{g+j}(q)_{h+j}} \\
&= \frac{1}{(q)_\infty^9} h_4 h_5.
\end{aligned}
\tag{3.15}
$$

Apply (2.6) with $n = 3$ to the i-sum, (2.1) to the b-sum and simplify and (2.2) to the i-sum to obtain

$$
\begin{aligned}
S_{10_{28}}(q) =& \frac{1}{(q)_\infty} \sum_{a,d,e,f,g,h,j,k\geq 0} (-1)^j \frac{q^{\frac{j(5j+3)}{2}+k(2k+1)+ah+aj+de+dk+ef+ek+fg+fj+fk+gh+gj+hj}}{(q)_a(q)_d(q)_e(q)_f(q)_g(q)_h(q)_j(q)_k(q)_{a+j}(q)_{d+k}(q)_{e+k}(q)_{f+j}} \\
&\times \frac{q^{a+d+e+f+g+h}}{(q)_{f+k}(q)_{g+j}(q)_{h+j}}.
\end{aligned}
$$

Now, (3.15) follows from (2.14) after letting $(a, d, e, f, g, h, j, k) \to (f, a, b, c, d, e, g, h)$.

For $\Phi_{10_{44}}(q)$, it suffices to prove

$$
\begin{aligned}
S_{10_{44}}(q) &:= \sum_{a,b,c,e,f,g,h,i,j,k\geq 0} (-1)^{h+j+k} \frac{q^{\frac{h(3h+1)}{2}+i(2i+1)+\frac{j(3j+1)}{2}+\frac{k(3k+1)}{2}+ab+ag+ai+aj+bc+bj+bk+ck}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_g(q)_h(q)_i(q)_j(q)_k(q)_{a+i}(q)_{a+j}(q)_{b+j}} \\
&\times \frac{q^{ef+eh+fg+fh+fi+gi+a+b+c+e+f+g}}{(q)_{b+k}(q)_{c+k}(q)_{e+h}(q)_{f+h}(q)_{f+i}(q)_{g+i}} \\
&= \frac{1}{(q)_\infty^7} h_4.
\end{aligned}
\tag{3.16}
$$

Apply (2.6) with $n = 3$ to the h-sum, (2.1) to the e-sum and simplify, (2.2) to the h-sum, (2.6) with $n = 3$ to the k-sum, (2.1) to the c-sum and simplify and (2.2) to the k-sum to obtain

$$
S_{10_{44}}(q) = \frac{1}{(q)_\infty^2} \sum_{a,b,f,g,i,j\geq 0} (-1)^j \frac{q^{i(2i+1)+\frac{j(3j+1)}{2}+ab+ag+ai+aj+bj+fg+fi+gi+a+b+f+g}}{(q)_a(q)_b(q)_f(q)_g(q)_i(q)_j(q)_{a+i}(q)_{a+j}(q)_{b+j}(q)_{f+i}(q)_{g+i}}.
$$

Now, (3.16) follows from (2.9) after letting $(a, b, f, g, i, j) \to (c, d, a, b, f, e)$.

□

Acknowledgements

The authors would like to thank Stavros Garoufalidis, Mustafa Hajij and Jeremy Lovejoy for their suggestions and helpful comments. The second author would like to thank Don Zagier for his question which motivated this paper. This question was posed on August 19, 2014 at the workshop "Low-dimensional topology and number theory" in Oberwolfach. Finally, the second author thanks the organizers (in particular, Frits Beukers) of the conference "Automatic sequences, Number Theory, Aperiodic Order", October 28–30, 2015 at TU Delft for the opportunity to discuss these results.

References

[1] C. Armond, *The head and tail conjecture for alternating knots*, Algebr. Geom. Topol. **13** (2013), no. 5, 2809–2826.

[2] C. Armond, O. Dasbach, *Rogers-Ramanujan type identities and the head and tail of the colored Jones polynomial*, preprint available at `http://arxiv.org/abs/1106.3948`

[3] O. Dasbach, X.-S. Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math. **142** (2006), no. 5, 1332–1342.

[4] M. Elhamdadi, M. Hajij, *Pretzel knots and q-series*, preprint available at `http://arxiv.org/abs/1512.00129`

[5] S. Garoufalidis, T. Lê, *Nahm sums, stability and the colored Jones polynomial*, Res. Math. Sci. **2** (2015), Art. 1, 55pp.

[6] S. Garoufalidis, T. Vuong, *Alternating knots, planar graphs and q-series*, Ramanujan J. **36** (2015), no. 3, 501–527.

[7] S. Gukov, S. Nawata, I. Saberi, M. Stosic and Piotr Sułkowski, *Sequencing BPS spectra*, J. High Energy Phys. 2016, no. 3, 004, front matter+160 pp.

[8] M. Hajij, *The Bubble skein element and applications*, J. Knot Theory Ramifications **23** (2014), no. 14, 1450076, 30pp.

[9] M. Hajij, *The colored Kauffman skein relation and the head and tail of the colored Jones polynomial*, preprint available at `http://arxiv.org/abs/1401.4537`

[10] A. Keilthy, R. Osburn, *Rogers-Ramanujan type identities for alternating knots*, J. Number Theory, **161** (2016), 255–280.
[11] S. Nawata, A. Oblomkov, *Lectures on knot homology*, preprint available at http://arxiv.org/abs/1510.01795
[12] L. Rozansky, *Khovanov homology of a unicolored B-adequate link has a tail*, Quantum Topol. **5** (2014), no. 4, 541–579.
[13] R. van der Veen, *The degree of the colored HOMFLY polynomial*, preprint available at http://arxiv.org/abs/1501.00123

72 Oaktree Road, Merville, Stillorgan, County Dublin

School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland
E-mail address: paul.beirne@ucdconnect.ie
E-mail address: robert.osburn@ucd.ie